# On the Gevrey Regularity of Solutions of a Class of Semilinear Elliptic Degenerate Equations on the Plane

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**Abstract.** We investigate the Gevrey regularity (in particular, the analyticity) of solutions of semilinear elliptic degenerate equations on the plane. The method is based on constructing explicit formulas for fundamental solutions and the Friedman effect near the boundary.

## §1. Introduction

In this paper we deal with the Gevrey regularity (in particular, the analyticity) of solutions of semilinear elliptic degenerate equations of Grushin's type on $\mathbb{R}^2$. We confine with consideration of a model equation, but it is our belief that the method can be applied to treat more general equations. Recently we have used this method to achieve some progress in studying the Gevrey regularity of solutions of semilinear subelliptic partial differential equations, see [1], [2], [3]. First let us define a space generalizing the space of analytic functions (see for example [4]). Let $L_n$ and $\bar{L}_n$ be two sequences of positive numbers, satisfying the monotonicity condition $\binom{n}{i}L_iL_{n-i} \leq AL_n (i = 1, 2...; n = 1, 2...)$, where $A$ is a positive constant. A function $F(x, v)$, defined for $x = (x_1, x_2)$ and for $v = (v_1, ..., v_\mu)$ in a $\mu-$dimensional open set $E$, is said to belong to the class $C\{L_{n-a}; \Omega|\bar{L}_{n-a}; E\}$ (a is an integer) if and only if $F(x, v)$ is infinitely differentiable and to every pair of compact subsets $\Omega_0 \subset \Omega$ and $E_0 \subset E$ there correspond constants $A_1$ and $A_2$ such that for $x \in \Omega_0$ and $v \in E_0$

$$\left| \frac{\partial^{j+k} F(x, v)}{\partial x_1^{j_1} \partial x_2^{j_2} \partial v_1^{k_1} ... \partial v_\mu^{k_\mu}} \right| \leq A_1 A_2^{j+k} L_{j-a} \bar{L}_{k-a},$$

$$\Big(j_1 + j_2 = j, \sum_{i=1}^{\mu} k_i = k; j, k = 0, 1, 2...\Big).$$

We use the notation $L_{-i} = 1 (i = 0, 1, 2, ...)$. If $F(x, v) = f(x)$, we simply write $f(x) \in C\{L_{n-a}; \Omega\}$. Note that $C\{n!; \Omega\}, (C\{n!^s; \Omega\})$ is the space of all analytic functions (s-Gevrey functions), respectively, in $\Omega$. Now we introduce some notations used in the paper

$$\Xi_t = \{(\alpha, \beta, \gamma) \in \mathbb{Z}_+^3 : \alpha + \beta \leq t, kt \geq \gamma \geq \alpha + (1 + k)\beta - t\}.$$

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For a function $f(x,y)$ on $\mathbb{R}^2$ we write $\partial_1^\alpha f, \partial_2^\beta f, \partial_{1,2}^{\alpha,\beta} f, \ {}_\gamma\partial_{\alpha,\beta} f$ for $\frac{\partial^\alpha f(x,y)}{\partial x^\alpha}, \frac{\partial^\beta f(x,y)}{\partial y^\beta}, \frac{\partial^{\alpha+\beta} f(x,y)}{\partial x^\alpha \partial y^\beta}, x^\gamma \frac{\partial^{\alpha+\beta} f(x,y)}{\partial x^\alpha \partial y^\beta}$, respectively. We will consider the following equation

$$G_{k,\lambda} f + \Psi\Big(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\Big) = 0 \ \text{ in a domain } \ \Omega \subset \mathbb{R}^2, \tag{1}$$

where

$$G_{k,\lambda} = \frac{\partial^2}{\partial x^2} + x^{2k}\frac{\partial^2}{\partial y^2} + i\lambda x^{k-1}\frac{\partial}{\partial y}$$

with $(x,y) \in \Omega \subset \mathbb{R}^2, \lambda \in \mathbb{C}, i = \sqrt{-1}$ and $k$ is a positive integer. Since our consideration is purely local we can assume that $\Omega$ is a bounded domain in $\mathbb{R}^2$. Let us define the following quantities

$$R = \left(x^{k+1} + u^{k+1}\right)^2 + (k+1)^2(y-v)^2, p = \frac{4x^{k+1}u^{k+1}}{R},$$

$$A_+ = x^{k+1} + u^{k+1} + i(k+1)(y-v), A_- = x^{k+1} + u^{k+1} - i(k+1)(y-v),$$

$$M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}},$$

here we take $z_1^{z_2} = e^{z_2 \ln z_1}$ for $z_1, z_2 \in \mathbb{C}$ and if $z_1 = re^{i\varphi}, -\pi < \varphi \le \pi$ then $\ln z_1 = \ln r + i\varphi$. Next we rewrite $G_{k,\lambda}$ as $X_2X_1 + i(\lambda+k)x^{k-1}\frac{\partial}{\partial y}$ where $X_1 = \frac{\partial}{\partial x} - ix^k\frac{\partial}{\partial y}, X_2 = \frac{\partial}{\partial x} + ix^k\frac{\partial}{\partial y}$. We will find the uniform fundamental solution of $G_{k,\lambda}$, that is

$$G_{k,\lambda}F_{k,\lambda}(x,y,u,v) = \delta(x-u, y-v),$$

in the following form

$$F_{k,\lambda}(x,y,u,v) = F(p)M.$$

After some computations we arrive at

$$G_{k,\lambda}F_{k,\lambda} = 16(k+1)^2u^{2k+2}x^{2k}\left[\left(u^{k+1} - x^{k+1}\right)^2 + (k+1)^2(y-v)^2\right]MR^{-3}F''(p) +$$
$$+4(k+1)x^{k-1}u^{k+1}[k\left(x^{2k+2} + u^{2k+2} + (k+1)^2(y-v)^2\right) - (6k+4)x^{k+1}u^{k+1}]MR^{-2}F'(p) +$$
$$+ (\lambda^2 - k^2)x^{k-1}u^{k+1}MR^{-1}F(p).$$

Therefore if $F(p)$ satisfies the following hypergeometric equation (see [5], p. 56)

$$p(1-p)F''(p) + [c - (1+a+b)p]F'(p) - abF(p) = 0 \tag{2}$$

with $a = \frac{k+\lambda}{2k+2}, b = \frac{k-\lambda}{2k+2}, c = \frac{k}{k+1}$, then formally we will have

$$G_{k,\lambda}F_{k,\lambda} = 0.$$

The general solution of (2) is

$$F(p) = C_1 F\Big(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\Big) + C_2 p^{\frac{1}{k+1}} F\Big(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\Big),$$

where $F(a,b,c,p)$ is the Gauss hypergeometric function and $C_1, C_2$ are some complex constant (see [5], p. 74). Now we will separately consider the case $k$ is odd and $k$ is even.

## §2. Case $k$ is odd.

Since $k$ is odd we note that $0 \le p \le 1$. Moreover $p = 1$ if and only if $x = \pm u \neq 0, y = v$. If $u = 0$ then $p = 0$ therefore from the result of [6] we should choose

$$C_1 = -\frac{\Gamma\Big(\frac{k+\lambda}{2k+2}\Big)\Gamma\Big(\frac{k-\lambda}{2k+2}\Big)}{2^{2+\frac{1}{k+1}}\pi\Gamma\Big(\frac{k}{k+1}\Big)}.$$

If $u \neq 0$ then the singularities of $F_{k,\lambda}(x,y,u,v)$ will be located at the one of $F(p)$. On the other hand, $F(p)$ with $0 \le p \le 1$ has singularity only when $p = 1$. As $p \to 1$ we have the following asymptotic expansions (see [5], p. 74)

$$F_1(p) := F\Big(\frac{k+\lambda}{2k+2}, \frac{k-\lambda}{2k+2}, \frac{k}{k+1}, p\Big) = -\frac{\Gamma\Big(\frac{k}{k+1}\Big)}{\Gamma\Big(\frac{k+\lambda}{2k+2}\Big)\Gamma\Big(\frac{k-\lambda}{2k+2}\Big)}\log(1-p) + O(1),$$

$$F_2(p) := F\Big(\frac{k+2+\lambda}{2k+2}, \frac{k+2-\lambda}{2k+2}, \frac{k+2}{k+1}, p\Big) = -\frac{\Gamma\Big(\frac{k+2}{k+1}\Big)}{\Gamma\Big(\frac{k+2+\lambda}{2k+2}\Big)\Gamma\Big(\frac{k+2-\lambda}{2k+2}\Big)}\log(1-p) + O(1).$$

We expect that $F_{k,\lambda}(x,y,u,v)$ has singularity only when $x = u, y = v$. Since $p^{\frac{1}{k+1}} \to -1$ when $(x,y) \to (-u,v)$, we should choose

$$C_2 = -\frac{\Gamma\Big(\frac{k+2+\lambda}{2k+2}\Big)\Gamma\Big(\frac{k+2-\lambda}{2k+2}\Big)}{2^{2+\frac{1}{k+1}}\pi\Gamma\Big(\frac{k+2}{k+1}\Big)}$$

such that $F(p)$ has no singularity at $x = -u, y = v$. Note that the following conditions

(3)

$$\lambda \neq \pm[2N(k+1)+k], \lambda \neq \pm[2N(k+1)+k+2], \text{ where } N \text{ is a non-negative integer,}$$

guarantee that $C_1, C_2 < \infty$ and hence $F(p)$ has a logarithm growth (if $u \neq 0$) at $(x,y) = (u,v)$.

**Definition.** *The parameter $\lambda$ is called admissible if $\lambda$ satisfies the condition (3).*

*Remark 1.* Comparing with the well-known results (see [7], [8]) we see that $\lambda$ is admissible if and only if $G_{k,\lambda}$ is hypoelliptic (analytic hypoelliptic).

Therefore if $\lambda$ is admissible then we expect that the function $F(p)M$, or

$$F_{k,\lambda}(x,y,u,v) = -\frac{\Gamma\left(\frac{k+\lambda}{2k+2}\right)\Gamma\left(\frac{k-\lambda}{2k+2}\right)F\left(\frac{k+\lambda}{2k+2},\frac{k-\lambda}{2k+2},\frac{k}{k+1},p\right)}{2^{2+\frac{1}{k+1}}\pi\Gamma\left(\frac{k}{k+1}\right)A_+^{\frac{k+\lambda}{2k+2}}A_-^{\frac{k-\lambda}{2k+2}}} - \\ -\frac{xu\Gamma\left(\frac{k+2+\lambda}{2k+2}\right)\Gamma\left(\frac{k+2-\lambda}{2k+2}\right)F\left(\frac{k+2+\lambda}{2k+2},\frac{k+2-\lambda}{2k+2},\frac{k+2}{k+1},p\right)}{2^{2-\frac{1}{k+1}}\pi\Gamma\left(\frac{k+2}{k+1}\right)A_+^{\frac{k+2+\lambda}{2k+2}}A_-^{\frac{k+2-\lambda}{2k+2}}},$$

will be our desired uniform fundamental solution. Indeed we have

**Theorem 1.** *Assume that $\lambda$ is admissible. Then*

$$G_{k,\lambda}F_{k,\lambda}(x,y,u,v) = \delta(x-u,y-v).$$

*Proof.* We begin by fixing $(u,v)\in\mathbb{R}^2$. First assume that $u\neq 0$. Then $A_+^\alpha, A_-^\beta\in C^\infty$ for every $\alpha$ and $\beta$. Let us introduce the following polar coordinate

$$x = u + r\cos\varphi, y = v + r\sin\varphi$$

It is easy to see that $F_{k,\lambda}(\cdot,\cdot,u,v)\in L^q_{loc}(\mathbb{R}^2)$ for every $1\leq q<\infty$. Let $B_\varepsilon(u,v) = \{(x,y)|r<\varepsilon\}$ and $\mathbb{R}^2_\varepsilon(u,v) = \mathbb{R}^2\backslash B_\varepsilon(u,v) = \{(x,y)\in\mathbb{R}^2|r\geq\varepsilon\}$. By applying Green's formula we have for every $w(x,y)\in C_0^\infty(\mathbb{R}^2)$

$$\text{(4)}\quad \int_{\mathbb{R}^2_\varepsilon(u,v)} F_{k,\lambda}(x,y,u,v)G_{k,-\lambda}w(x,y)\,dxdy = \int_{\mathbb{R}^2_\varepsilon(u,v)} V(F_{k,\lambda},w,k,\lambda)\,dxdy + \\ + \int_{r=\varepsilon} F_{k,\lambda}B_1(w,k,\lambda)\,ds - \int_{r=\varepsilon} w(x,y)B_2(F_{k,\lambda},k)\,ds,$$

where

$$V(F_{k,\lambda},w,k,\lambda) = wG_{k,\lambda}F_{k,\lambda}, B_2(F_{k,\lambda},k) = (\nu_1 + ix^k\nu_2)X_1F_{k,\lambda}, \\ B_1(w,k,\lambda) = (\nu_1 - ix^k\nu_2)X_2w - i(\lambda+k)x^{k-1}\nu_2 w,$$

and $\nu = (\nu_1,\nu_2)$ is the unit outward normal to $\partial\mathbb{R}^2_\varepsilon(u,v)$. The first integral in the right side of (4) vanishes. We now compute the second and the third integral in the right side of (4). When $(x,y)$ tends to $(u,v)$ it is easy to check that

$$M = (2u^{k+1}+o(1))^{-\frac{k+\lambda}{2k+2}}(2u^{k+1}+o(1))^{-\frac{k-\lambda}{2k+2}} = 2^{-\frac{k}{k+1}}u^{-k}+o(1),$$
$$x^{k+1} = u^{k+1}+(k+1)u^k r\cos\varphi + o(r),\, x^k = u^k + ku^{k-1}r\cos\varphi + o(r),$$
$$x^{k-1} = u^{k-1}+(k-1)u^{k-2}r\cos\varphi+o(r),\, \nu_1\big|_{\partial B_\varepsilon(u,v)} = -\cos\varphi,$$
$$ix^k\nu_2\big|_{\partial B_\varepsilon(u,v)} = -iu^k\sin\varphi + o(1),\, R = 4u^{2k+2}+o(1),$$
$$1-p = \frac{(k+1)^2\big(\cos^2\varphi u^{2k}+\sin^2\varphi\big)}{4u^{2k+2}}r^2 + o(r^2),$$
$$X_1 p = \frac{(k+1)^2u^{-k-2}(-u^k\cos\varphi + i\sin\varphi)}{2}r + o(r).$$

Moreover using the asymptotic expansions (see [5], p. 75)

$$F'\Big(\frac{k+\lambda}{2k+2},\frac{k-\lambda}{2k+2},\frac{k}{k+1},p\Big) = \frac{\Gamma\Big(\frac{k}{k+1}\Big)}{\Gamma\Big(\frac{k+\lambda}{2k+2}\Big)\Gamma\Big(\frac{k-\lambda}{2k+2}\Big)(1-p)} + o((1-p)^{-1}),$$

$$F'\Big(\frac{k+2+\lambda}{2k+2},\frac{k+2-\lambda}{2k+2},\frac{k+2}{k+1},p\Big) = \frac{\Gamma\Big(\frac{k+2}{k+1}\Big)}{\Gamma\Big(\frac{k+2+\lambda}{2k+2}\Big)\Gamma\Big(\frac{k+2-\lambda}{2k+2}\Big)(1-p)} + o((1-p)^{-1})$$

we deduce immediately that

$$-\int_{r=\varepsilon} w(x,y)B_2(F_{k,\lambda},k)\,ds \to \frac{w(u,v)}{2\pi}\int_0^{2\pi}\frac{u^k d\varphi}{u^{2k}\cos^2\varphi+\sin^2\varphi} = w(u,v),$$

(5)

$$\text{and}\quad \int_{r=\varepsilon} F_{k,\lambda}B_1(w,k,\lambda)\,ds \to 0 \quad\text{as}\quad \varepsilon\to 0.$$

Now from (4), (5) we have

(6)
$$(G_{k,\lambda}F_{k,\lambda}, w(x,y)) = (F_{k,\lambda}, G_{k,-\lambda}w(x,y)) =$$
$$= \lim_{\varepsilon\to 0}\int_{r\geq\varepsilon} F_{k,\lambda}G_{k,-\lambda}w(x,y)dxdy = w(u,v).$$

From (6) we see that

(7)
$$G_{k,\lambda}F_{k,\lambda} = \delta(x-u, y-v).$$

Now if $u=0$ we can follow the proof in [6], or take the limit in (7) when $u\to 0$. This completes the proof of Theorem 1. □

*Remark 2.* A similar expression for $F_{k,0}$ is also given in [9], [10].

It is obvious that we also have the following

**Theorem 2.** *$G_{k,\lambda}$ is hypoelliptic if and only if the hypergeometric equation (2) has no bounded solution on the interval* $[0,1]$.

Let us denote $X_1' = \frac{\partial}{\partial u} - iu^k \frac{\partial}{\partial v}, X_2' = \frac{\partial}{\partial u} + iu^k \frac{\partial}{\partial v}$, and $G_{k,\lambda}' = X_2' X_1' + i(\lambda + k)u^{k-1}\frac{\partial}{\partial v}$. Noting that $F_{k,\lambda}(x,y,u,v) = F_{k,-\lambda}(u,v,x,y)$, from Theorem 1 we can easily deduce the following

**Proposition 1 (Representation formula).** *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with piece-wise smooth boundary, $f \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $\lambda$ is admissible then we have*

(8)

$$f(x,y) = \int_\Omega F_{k,\lambda}(x,y,u,v) G_{k,\lambda}' f(u,v)dudv - \int_{\partial\Omega} F_{k,\lambda}(x,y,u,v) B_1'(f(u,v),k,-\lambda)ds + \\ + \int_{\partial\Omega} f(u,v) B_2'(F_{k,\lambda}(x,y,u,v),k)ds,$$

*where*

$$B_1'(f(u,v),k,-\lambda) = \left(\nu_1 - iu^k\nu_2\right) X_2' f(u,v) - i(-\lambda + k)u^{k-1}\nu_2 f(u,v),$$
$$B_2'(F_{k,\lambda}(x,y,u,v),k) = \left(\nu_1 + iu^k\nu_2\right) X_1' F_{k,\lambda}(x,y,u,v).$$

*and $\nu = (\nu_1, \nu_2)$ is the unit outward normal vector on $\partial\Omega$.*

For $m \in \mathbb{Z}^+$ let us denote by $\mathbb{H}^m_{loc}(\Omega)$ the space of all function $f \in L^2_{loc}(\Omega)$ such that for any compact $K$ of $\Omega$ we have $\sum_{(\alpha,\beta,\gamma)\in\Xi_m} \|{}_\gamma\partial_{\alpha,\beta} f\|_{L^2(K)} < \infty$. Now we are able to formulate the main theorem of this section

**Theorem 3.** *Assume that $m \geq 2k^2 + 6k + 5$. Let $f$ be a $\mathbb{H}^m_{loc}(\Omega)$ solution of the equation (1) and $\Psi \in C\{L_{n-a-2}; \Omega | L_{n-a-2}; \mathbb{R}^3\}$ for every $a \in [0, 2k+2]$. Then $u \in C\{L_{n-2k-4}; \Omega\}$. In particular, if $\Psi$ is a $G^s-$function (or analytic) of its arguments then so is $f$.*

*Proof.* The proof of Theorem 3 consists of Theorem 4 and Theorem 5.□

**Theorem 4.** *Let $\Psi$ be a $C^\infty-$function of its arguments and $m \geq 2k^2 + 6k + 5$. Assume that $f \in \mathbb{H}^m_{loc}(\Omega)$ is a solution of the equation (1) then $f \in C^\infty(\Omega)$.*

*Proof.* We begin with establishing the following

**Proposition 2.** *Let $m \geq 2k^2 + 6k + 5$. Assume that $f \in \mathbb{H}^m_{loc}(\Omega)$. Then $\Psi\big(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\big) \in \mathbb{H}^{m-1}_{loc}(\Omega)$.*

*Proof.* It is sufficient to prove that ${}_\gamma\partial_{\alpha,\beta}\Psi\big(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\big) \in L^2_{loc}(\Omega)$ for every $(\alpha,\beta,\gamma) \in \Xi_{m-1}$. Denote by $w_1, w_2, w_3$, respectively, $f, \frac{\partial f}{\partial x}, x^k\frac{\partial f}{\partial y}$. Since $m \geq 2k^2 + 6k + 5$, by a theorem of Sobolev we deduce that $w_1, w_2, w_3 \in C(\Omega)$. Using the Faà di

Bruno formula we see that ${}_{\gamma}\partial_{\alpha,\beta}\Psi(x,y,w_1,w_2,w_3)$ is a linear combination of terms of the form

$$\frac{\partial^{|\vartheta|}\Psi(x,y,w_1,w_2,w_3)}{\partial x^{\vartheta_1}\partial y^{\vartheta_2}\partial w_1^{\vartheta_3}\partial w_2^{\vartheta_4}\partial w_3^{\vartheta_5}} x^{\gamma}\prod_q\prod_{j=1}^{3}(\partial_1^{\alpha_{q,j}}\partial_2^{\beta_{q,j}}w_j)^{\zeta_{q,j}},$$

where $q$ belongs to a finite set, $\vartheta=(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5)$, $|\vartheta|\le\alpha+\beta$, $\alpha_{q,j}+\beta_{q,j}>0$, $\sum_{q,j}\zeta_{q,j}=\vartheta_3+\vartheta_4+\vartheta_5$, $\sum_{q,j}(\alpha_{q,j},\beta_{q,j})\zeta_{q,j}=(\alpha-\vartheta_1,\beta-\vartheta_2)$, Therefore the theorem is proved if we can show this general terms are in $L^2_{loc}(\Omega)$. If all $\zeta_{q,j}$ vanish then it is immediate that $\frac{\partial^{|\vartheta|}\Psi(x,y,\xi_1,\xi_2,\xi_3)}{\partial x^{\vartheta_1}\partial y^{\vartheta_2}\partial\xi_1^{\vartheta_3}\partial\xi_2^{\vartheta_4}\partial\xi_3^{\vartheta_5}}\in C(\Omega)$, since $\Psi\in C^{\infty}$, $\xi_1,\xi_2,\xi_3\in C(\Omega)$. Therefore we can assume that there exists at least one of $\zeta_{q,j}$ that differs from 0. Choose $q_0,j_0$ such that $\zeta_{q_0,j_0}\ge 1$ and

$$\alpha_{q_0,j_0}+(k+1)\beta_{q_0,j_0}=\max_{\substack{q,j\\ \zeta_{q,j}\ge 1}}(\alpha_{q,j}+(k+1)\beta_{q,j}).$$

Consider the following possibilities

I) $\zeta_{q_0,j_0}\ge 2$. We then have $\alpha_{q,j}+\beta_{q,j}\le m-1-(2k+2)$ for all $q,j$ such that $\zeta_{q,j}\ge 1$. Indeed, if $q\ne q_0$ or $j\ne j_0$ and $\alpha_{q,j}+\beta_{q,j}>m-1-(2k+2)$ then $\alpha_{q_0,j_0}+\beta_{q_0,j_0}\le k$. Therefore

$$\begin{aligned}m-1-(2k+2)&<\alpha_{q,j}+\beta_{q,j}\le\alpha_{q,j}+(k+1)\beta_{q,j}\le\\ &\le\alpha_{q_0,j_0}+(k+1)\beta_{q_0,j_0}\le(k+1)(\alpha_{q_0,j_0}+\beta_{q_0,j_0})\le k(k+1).\end{aligned}$$

Thus $m<k^2+3k+3$, a contradiction.

If $q=q_0, j=j_0$ and $\alpha_{q_0,j_0}+\beta_{q_0,j_0}>m-1-(2k+2)$ then we have

$$m-1\ge\alpha+\beta\ge 2(\alpha_{q_0,j_0}+\beta_{q_0,j_0})>2(m-1-(2k+2))+1.$$

Therefore $m<4(k+1)$, a contradiction.

Next, for $q,j$ such that $\zeta_{q,j}\ge 1$, set

$$\gamma_{q,j}=\max\{0,\alpha_{q,j}+(k+1)\beta_{q,j}+1+(2k+2)-m\}.$$

Since $0\le\alpha_{q,j}+\beta_{q,j}\le m-1-(2k+2)$, we have

$$\gamma_{q,j}\le\max\{0,k\beta_{q,j}\}\le k(m-1-(2k+2)).$$

From all above arguments we deduce that $(\alpha_{q,j},\beta_{q,j},\gamma_{q,j})\in\Xi_{m-1-(2k+2)}$ for all $q,j$ such that $\zeta_{q,j}\ge 1$. Next we claim that $\sum_{q,j}\gamma_{q,j}\zeta_{q,j}\le\gamma$. Indeed, if $\sum_{q,j}\gamma_{q,j}\zeta_{q,j}>\gamma$ then we deduce that

$$\alpha+(k+1)\beta-2(m-1-(2k+2))\ge\sum_{q,j}\gamma_{q,j}\zeta_{q,j}>\gamma\ge\alpha+(k+1)\beta-(m-1).$$

Therefore $m < 4k+5$, a contradiction.
Now we have

$$x^{\gamma}\prod_{q}\prod_{j=1}^{\mu}\left(\partial_1^{\alpha_{q,j}}\partial_2^{\beta_{q,j}}w_j\right)^{\zeta_{q,j}} = x^{\tilde{\gamma}}\prod_{q}\prod_{j=1}^{\mu}\left(x_1^{\gamma_{q,j}}\partial_1^{\alpha_{q,j}}\partial_2^{\beta_{q,j}}w_j\right)^{\zeta_{q,j}} \in C(\Omega), \tilde{\gamma}\geq 0,$$

since $x^{\gamma_{q,j}}\partial_1^{\alpha_{q,j}}\partial_2^{\beta_{q,j}}w_j \in \mathbb{H}_{loc}^{2k+2}(\Omega)\subset C(\Omega)$.
II) $\zeta_{q_0,j_0}=1$ and $\zeta_{q,j}=0$ for $q\neq q_0$ or $j\neq j_0$. We have

$$x^{\gamma}\prod_{q}\prod_{j=1}^{\mu}\left(\partial_1^{\alpha_{q,j}}\partial_2^{\beta_{q,j}}w_j\right)^{\zeta_{q,j}} = x^{\gamma}\partial_1^{\alpha_{q_0,j_0}}\partial_2^{\beta_{q_0,j_0}}w_{j_0}\in L^2_{loc}(\Omega).$$

III) $\zeta_{q_0,j_0}=1$ and there exists $(q_1,j_1)\neq(q_0,j_0)$ such that $\zeta_{q_1,j_1}\neq 0$. Define

$$\bar{\gamma}_{q_0,j_0}=\max\{0,\alpha_{q_0,j_0}+(k+1)\beta_{q_0,j_0}+1-m\}.$$

As in part I) we can prove that $(\alpha_{q,j},\beta_{q,j},\gamma_{q,j})\in\Xi_{m-1-(2k+2)}$ for $q\neq q_0$ or $j\neq j_0$ and $(\alpha_{q_0,j_0},\beta_{q_0,j_0},\bar{\gamma}_{q_0,j_0})\in\Xi_{m-1}$. Therefore $x^{\gamma_{q,j}}\partial_1^{\alpha_{q,j}}\partial_2^{\beta_{q,j}}w_j\in\mathbb{H}_{loc}^{2k+2}(\Omega)\subset C(\Omega)$ for $(q,j)\neq(q_0,j_0)$ and $x^{\bar{\gamma}_{q_0,j_0}}\partial_1^{\alpha_{q_0,j_0}}\partial_2^{\beta_{q_0,j_0}}w_{j_0}\in L^2_{loc}(\Omega)$. We also have $\bar{\gamma}_{q_0,j_0}+\sum_{(q,j)\neq(q_0,j_0)}\gamma_{q,j}\zeta_{q,j}\leq\gamma$ as in part I). Now the desired result follows from the decomposition of the general terms. □

(continuing the proof of Theorem 4) $f\in\mathbb{H}^m_{loc}(\Omega), m\geq 2k^2+6k+5\Longrightarrow\Psi\in\mathbb{H}^{m-1}_{loc}(\Omega)$ (by Proposition 2). Therefore by a theorem of Grushin we deduce that $f\in\mathbb{H}^{m+1}_{loc}(\Omega)$. Repeat the argument again and again we finally arrive at $f\in\mathbb{H}^{m+t}_{loc}(\Omega)$ for every positive $t$, i. e. $f\in C^\infty(\Omega)$.□

Now put $r_0=2k+2$. For $r\in\mathbb{Z}_+$ let $\Gamma_r$ denote the set of pairs of multi-indices $(\alpha,\beta)$ such that $\Gamma_r=\Gamma_r^1\cup\Gamma_r^2$ where

$$\Gamma_r^1=\{(\alpha,\beta):\alpha\leq r_0, 2\alpha+\beta\leq r\}, \Gamma_r^2=\{(\alpha,\beta):\alpha\geq r_0,\alpha+\beta\leq r-r_0\}.$$

For a pair $(\alpha,\beta)$ we denote by $(\alpha,\beta)^*$ the minimum of $r$ such that $(\alpha,\beta)\in\Gamma_r$. Next define the following norm

$$|f,\Omega|_r=\max_{(\alpha,\beta)\in\Gamma_r}|\partial_1^\alpha\partial_2^\beta f,\Omega| + \max_{\substack{(\alpha,\beta)\in\Gamma_r\\ \alpha\geq 1,\beta\geq 1}}\max_{(x,y)\in\bar{\Omega}}|\partial_1^{\alpha+2}\partial_2^\beta f|,$$

where $|f,\Omega|=\max_{(x,y)\in\bar{\Omega}}\left|f+\frac{\partial f}{\partial x}+x^k\frac{\partial f}{\partial y}\right|$.
The next lemma is due to Friedman (see [4]).

**Lemma 1.** *There exists a constant $C_3$ such that if $g(z)$ is a positive monotone decreasing function, defined in the interval $0 \le z \le 1$ and satisfying*

$$g(z) \le \frac{1}{10^{12}} g\Big(z\Big(1-\frac{4}{n}\Big)\Big) + \frac{C}{z^{n-r_0-1}} \quad (n \ge r_0+2, C>0),$$

*then $g(z) < CC_3/z^{n-r_0-1}$.*

**Theorem 5.** *Let $f$ be a $C^\infty$ solution of the equation (1) and $\Psi \in C\{L_{n-a-2};\Omega|L_{n-a-2};\mathbb{R}^3\}$ for every $a \in [0,r_0]$. Then $f \in C\{L_{n-r_0-2};\Omega\}$. In particular, if $\Psi$ is a $G^s$−function (or analytic) of its arguments then so is $f$.*

*Proof.* We begin with the following

**Proposition 3.** *Assume that $\Psi \in C\{L_{n-a-2};\Omega|L_{n-a-2};\mathbb{R}^3\}$ for every $a \in [0,r_0]$. Then there exist constants $C_4, C_5$ such that for every $H_0 \ge 1, H_1 \ge C_4 H_0^{2k+3}$ if*

$$|f,\Omega|_d \le H_0 H_1^{(d-r_0-2)} L_{d-r_0-2}, \quad 0 \le d \le N+1, r_0+2 \le N$$

*then*

$$\max_{(x,y)\in\bar\Omega} \Big|\partial_1^\alpha \partial_2^\beta \Psi\Big(x,y,f,\frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\Big)\Big| \le C_5 H_0 H_1^{N-r_0-1} L_{N-r_0-1}$$

*for every $(\alpha,\beta) \in \Gamma_{N+1}$.*

*Proof.* For reason of convenience from now on we shall use the following notations

$$\theta(z) = \begin{cases} 1 & \text{if} \quad z \ge 0, \\ 0 & \text{if} \quad z < 0, \end{cases} \quad \text{and} \quad z^n = \begin{cases} z^n & \text{if} \quad n \ge 1, \\ 1 & \text{if} \quad n \le 0. \end{cases}$$

All the constants used in the proof of this Proposition are taken to be greater than 1. Now it is sufficient to prove the estimate when $(\alpha,\beta) \in \Gamma_{N+1}\backslash\Gamma_N$. As in Proposition 2, $\partial_1^\alpha \partial_2^\beta \Psi(x,y,f,\frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y})$ is a linear combination with positive coefficients of terms of the form

$$\text{(9)} \qquad \frac{\partial^{|\vartheta|}\Psi(x,y,w_1,w_2,w_3)}{\partial x^{\vartheta_1}\partial y^{\vartheta_2}\partial w_1^{\vartheta_3}\partial w_2^{\vartheta_4}\partial w_3^{\vartheta_5}} \prod_q \prod_{j=1}^3 (\partial_1^{\alpha_{q,j}}\partial_2^{\beta_{q,j}} w_j)^{\zeta_{q,j}}.$$

Substituting $w_j$ by one of the terms $f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}$ we obtain

$$\partial_1^{\alpha_{q,1}}\partial_2^{\beta_{q,1}}(w_1) = \partial_1^{\alpha_{q,1}}\partial_2^{\beta_{q,1}} f, \qquad \partial_1^{\alpha_{q,2}}\partial_2^{\beta_{q,2}}(w_2) = \partial_1^{1+\alpha_{q,2}}\partial_2^{\beta_{q,2}} f,$$

$$\partial_1^{\alpha_{q,3}}\partial_2^{\beta_{q,3}}(w_3) = \sum_{m=0}^{\alpha_{q,3}} \binom{\alpha_{q,3}}{m} k\cdots(k-m+1)\theta(k-m+1)\ {}_{(k-m)}\partial_{\alpha_{q,3}-m,1+\beta_{q,3}} f.$$

Hence, for $j=1,2,3$, we can decompose ${}_{(k-m)}\partial_{\alpha_{q,j}-m,1+\beta_{q,j}} f$ into ${}_{(k-m)}\partial_{\alpha_2,\beta_2}\left(\partial_1^{\alpha_3}\partial_2^{\beta_3} f\right)$ with $(\alpha_2,\beta_2,k-m)\in\Xi_1$ and $(\alpha_3,\beta_3)\in\Gamma_{(\alpha_{q,j},\beta_{q,j})^*-m}$. Put $S=N+1-\alpha-\beta$. Define $R=r_0-S$. It is easy to see that $0\le R\le r_0$. Since $\alpha_{q,j}\le\alpha$ we deduce that $(\alpha_{q,j},\beta_{q,j})\in\Gamma_{(\alpha_{q,j}+\beta_{q,j}+S)}$. Using the inductive assumption we have

$$\left|\binom{\alpha_{q,j}}{m} k\cdots(k-m+1)\theta(k-m+1)\ {}_{(k-m)}\partial_{\alpha_{q,j}-m,1+\beta_{q,j}} f\right| \le$$
$$\le C_6\binom{\alpha_{q,j}}{m} k\cdots(k-m+1)\theta(k-m+1) H_0 H_1^{\alpha_{q,j}+\beta_{q,j}-m-R-2} L_{\alpha_{q,j}+\beta_{q,j}-m-R-2}\le$$
$$\le C_7 H_0 H_1^{\alpha_{q,j}+\beta_{q,j}-m-R-2} L_{\alpha_{q,j}+\beta_{q,j}-R-2}.$$

Therefore we deduce that

$$\left|\partial_1^{\alpha_{q,j}}\partial_2^{\beta_{q,j}}\left(x^k\frac{\partial f}{\partial y}\right)\right|\le$$
$$\left|\sum_{m=0}^{\alpha_{q,j}}\binom{\alpha_{q,j}}{m} k\cdots(k-m+1)\theta(k-m+1)\ {}_{(k-m)}\partial_{\alpha_{q,j}-m,1+\beta_{q,j}} f\right|\le$$
$$\le C_8 H_0 H_1^{\alpha_{q,j}+\beta_{q,j}-R-2} L_{\alpha_{q,j}+\beta_{q,j}-R-2}.$$

and the general terms can be estimated by

$$\prod_{j=1}^{3} C_8 H_0 H_1^{\alpha_{q,j}+\beta_{q,j}-R-2} L_{\alpha_{q,j}+\beta_{q,j}-R-2}.$$

Since $\Psi\in C\{L_{n-a};\Omega|L_{n-a};\mathbb{R}^3\}$, there exist constants $C_9, C_{10}$ such that

$$\left|\frac{\partial^{|\vartheta|}\Psi(x,y,w_1,w_2,w_3)}{\partial x^{\vartheta_1}\partial y^{\vartheta_2}\partial w_1^{\vartheta_3}\partial w_2^{\vartheta_4}\partial w_3^{\vartheta_5}}\right|_{(x,y)\in\bar\Omega}\le C_9 C_{10}^{\tilde\vartheta-R} L_{\tilde\vartheta-R-2} C_{10}^{\tilde{\tilde\vartheta}} L_{\tilde{\tilde\vartheta}-R-2},$$
$$(\tilde\vartheta=\vartheta_1+\vartheta_2,\ \tilde{\tilde\vartheta}=\vartheta_3+\vartheta_4+\vartheta_5).$$

Set $p=\alpha+\beta$. Now for $\xi\in\mathbb{R}, v=v(\xi):\mathbb{R}\longrightarrow\mathbb{R}$ we take $Z(\xi)=Z_1(v(\xi))\cdot Z_2(\xi)$ where

$$Z_1(\xi)=Z_1(v(\xi))=C_9\sum_{i=0}^{p}\frac{C_{11}^i L_{i-R-2} v^i(\xi)}{i!},\ \ Z_2(\xi)=\sum_{i=0}^{p}\frac{C_{10}^{i-R} L_{i-R-2}\xi^i}{i!},$$

and

$$v(\xi)=C_8 H_0\sum_{i=1}^{p}\frac{H_1^{i-R-2} L_{i-R-2}\xi^i}{i!}.$$

By comparing terms of the form (9) with the corresponding terms in $\frac{d^p}{d\xi^p}Z(\xi)$ it follows that

$$\left|\partial_1^\alpha \partial_2^\beta \Psi\Big(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\Big)\right|\Bigg|_{(x,y)\in\bar\Omega} \le \frac{d^p}{d\xi^p} Z(\xi)\Bigg|_{\xi=0}.$$

Next we introduce the following notation (see [4]): $v(\xi) \ll h(\xi)$ if and only if $v^{(j)}(0) \le h^{(j)}(0)$ for $1 \le j \le p$. It is not difficult to check that, there exists a constant $C_{12}$ (independent of $p$ ) such that

$$v^2(\xi) \ll (C_8 H_0)^2 C_{12} \sum_{i=2}^{p} \frac{H_1^{i-R-3} L_{i-R-3} \xi^i}{(i-1)!}.$$

And by induction we have

$$v^j(\xi) \ll (C_8 H_0)^j C_{12}^{j-1} \sum_{i=j}^{p} H_1^{i-j-R-1} \frac{L_{i-j-R-1}\xi^i}{(i-j+1)!}.$$

Next, it is easy to verify that $Z_1(0) = C_9$, $\left.\frac{dZ_1(\xi)}{d\xi}\right|_{\xi=0} \le C_8 C_9 C_{11} H_0$, and $\left.\frac{d^j Z_2(\xi)}{d\xi^j}\right|_{\xi=0} = C_{10}^{j-R} L_{j-R-2}$. We now compute $\left.\frac{d^j Z_1(\xi)}{d\xi^j}\right|_{\xi=0}$ when $2 \le j \le p$.
If $2 \le j \le R+2$ then using that fact that $0 \le R \le r_0$ we deduce that

$$\left.\frac{d^j Z_1(\xi)}{d\xi^j}\right|_{\xi=0} \le C_9 \sum_{i=1}^{j} C_{11}^i L_{i-R-2} \left.\frac{d^j v^i(\xi)}{i! d\xi^j}\right|_{\xi=0} \le C_9 \sum_{i=1}^{j} C_{11}^i (C_8 H_0)^i C_{12}^{i-1} \frac{j!}{(j-i+1)! i!} \le$$

(10)

$$\le (R+2)! C_9 C_{11} C_8 H_0 \sum_{i=1}^{j} (C_8 C_{11} C_{12} H_0)^{i-1} \le C_{13} H_0 H_1$$

provided $H_1 \ge (C_8 C_{11} C_{12} H_0)^{r_0+1}$.
If $R+3 \le j \le 2R+4$ then

$$\left.\frac{d^j Z_1(\xi)}{d\xi^j}\right|_{\xi=0} \le C_9 \sum_{i=1}^{j} C_{11}^i L_{i-R-2} \left.\frac{d^j v^i(\xi)}{i! d\xi^j}\right|_{\xi=0} \le$$

$$\le C_9 \sum_{i=1}^{j} C_{11}^i (C_8 H_0)^i C_{12}^{i-1} L_{i-R-2} H_1^{j-i-R-1} \frac{L_{j-i-R-2}\, j!}{(j-i+1)! i!} \le$$

(11)

$$\le C_{14} \sum_{i=1}^{j} (C_8 C_{11} C_{12} H_0 H_1^{-1})^{i-1} H_0 H_1^{j-R-2} L_{j-R-2} \le C_{15} H_0 H_1^{j-R-2} L_{j-R-2}$$

provided $H_1 \geq C_8 C_{11} C_{12} H_0$.
If $j \geq 2R+5$, we have

$$\left.\frac{d^j Z_1(\xi)}{d\xi^j}\right|_{\xi=0} \leq C_9 \left( \sum_{i=1}^{R+2} \frac{C_{11}^i (C_8 H_0)^i C_{12}^{i-1} H_1^{j-i-R-1} L_{j-i-R-1} j!}{i!(j-i+1)!} + \right.$$

$$(12) \qquad + \sum_{i=R+3}^{j-R-2} \frac{C_{11}^i (C_8 H_0)^i C_{12}^{i-1} H_1^{j-i-R-1} L_{i-R-2} L_{j-i-R-1} j!}{i!(j-i+1)!} +$$

$$\left. + \sum_{i=j-R-1}^{j} \frac{C_{11}^i (C_8 H_0)^i C_{12}^{i-1} L_{i-R-2} j!}{i!(j-i+1)!} \right).$$

The first sum in (12) is estimated by

$$(13) \qquad C_{16} H_0 H_1^{j-R-2} L_{j-R-2}$$

provided $H_1 \geq C_8 C_{11} C_{12} H_0$ as for $R+3 \leq j \leq 2R+4$.
By using the monotonicity condition on $L_n$ the second sum in (12) is estimated by

$$C_9 \sum_{i=R+3}^{j-R-2} \frac{C_{11}^i (C_8 H_0)^i C_{12}^{i-1} H_1^{j-i-R-1} L_{i-R-2} L_{j-i-R-1} j!}{i!(j-i+1)!} \leq$$

$$\leq \frac{C_{17} H_0 H_1^{j-R-2} j! L_{j-2R-3}}{(j-2R-3)!} \sum_{i=R+3}^{j-R-2} \frac{1}{i \cdots (i-R-1)} \frac{1}{(j-i+1)\cdots(j-i-R)} \leq$$

$$(14)$$

$$\leq C_{18} H_0 H_1^{j-R-2} L_{j-R-2},$$

provided $H_1 \geq C_8 C_{11} C_{12} H_0$.
For the third sum we see that

$$C_9 \sum_{i=j-R-1}^{j} \frac{C_{11}^i (C_8 H_0)^i C_{12}^{i-1} L_{i-R-2} j!}{i!(j-i+1)!} \leq$$

$$(15)$$

$$\leq C_8 C_9 C_{11} H_0 H_1^{j-R-2} j! \sum_{i=j-R-1}^{j} \frac{L_{i-R-2}}{i!(j-i+1)!} \leq C_{19} H_0 H_1^{j-R-2} L_{j-R-2},$$

if $H_1 \geq (C_8 C_{11} C_{12} H_0)^2$.

By (10)-(15) and taking $H_1 \geq (C_8 C_{10} C_{11} C_{12} H_0)^{r_0+1} = C_4 H_0^{2k+3}$ we obtain

$$\left.\frac{d^p Z(\xi, v)}{d\xi^p}\right|_{\xi=0} \leq C_{20} \sum_{j=0}^{R-2} \binom{p}{j} H_0 H_1 C_{10}^{p-j-R} L_{p-j-R-2} +$$

$$+ C_{21} \sum_{j=R+3}^{p} \binom{p}{j} H_0 H_1^{j-R-2} L_{j-R-2} C_{10}^{p-j-R} L_{p-j-R-2} \leq C_{22} H_0 H_1^{p-R-2} L_{p-R-2}.$$

Hence

$$\max_{x \in \bar{\Omega}} \left| \partial_1^\alpha \partial_2^\beta \Psi\left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\right)\right| \leq C_{22} H_0 H_1^{p-R-2} L_{p-R-2} \leq$$

$$\leq C_5 H_0 H_1^{N-r_0-1} L_{N-r_0-1}. \square$$

*Remark 3.* The constants $C_4, C_5$ depend on $\Omega$ increasingly in the sense that with the same $C_4, C_5$, Proposition 3 remains valid if we substitute $\Omega$ by any $\Omega^{'} \subset \Omega$.

**Corollary 1.** *Under the same hypotheses of the Proposition with $d \leq N+1$ replaced by $d \leq N$, then*

$$\max_{x \in \bar{\Omega}} \left| \partial_1^\alpha \partial_2^\beta \Psi\left(x, y, f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}\right)\right| \leq C_3 \left( |f, \Omega|_{N+1} + H_0 H_1^{N-r_0-1} L_{N-r_0-1} \right)$$

*for every $(\alpha, \beta) \in \Gamma_{N+1}$.*

*Proof.* Indeed, as in the proof of the Proposition 3 all typical terms, except $\frac{\partial \Psi}{\partial w_{j_0}} \partial_1^\alpha \partial_2^\beta w_{j_0}$ can be estimated by $|\cdot|_N$. Replacing $w_{j_0}$ by one of $f, \frac{\partial f}{\partial x}, x^k \frac{\partial f}{\partial y}$, we have

$$\frac{\partial \Psi}{\partial w_1} \partial_1^\alpha \partial_2^\beta (w_1) = \frac{\partial \Psi}{\partial w_1} \partial_1^\alpha \partial_2^\beta f, \qquad \frac{\partial \Psi}{\partial w_2} \partial_1^\alpha \partial_2^\beta (w_2) = \frac{\partial \Psi}{\partial w_2} \partial_1^{1+\alpha} \partial_2^\beta f,$$

$$\frac{\partial \Psi}{\partial w_3} \partial_1^\alpha \partial_2^\beta (w_3) = \frac{\partial \Psi}{\partial w_3}\ {}_k\partial_{\alpha, 1+\beta} f +$$

$$\frac{\partial \Psi}{\partial w_3} \sum_{m=1}^{\alpha} \binom{\alpha}{m} k \cdots (k-m+1) \theta(k-m+1)\ {}_{(k-m)}\partial_{\alpha-m, 1+\beta} f.$$

The terms $\frac{\partial \Psi}{\partial w_1} \partial_1^\alpha \partial_2^\beta f, \frac{\partial \Psi}{\partial w_2} \partial_1^{1+\alpha} \partial_2^\beta f, \frac{\partial \Psi}{\partial w_3}\ {}_k\partial_{\alpha, 1+\beta} f$ are estimated by $C_3 |f, \Omega|_{N+1}$. The last sum is majorized as in Proposition 3. $\square$

(continuing the proof of Theorem 5) Since $G_{k,\lambda}$ is elliptic if $x \neq 0$ it suffices to consider the case $(0,0) \in \Omega$ and $\Omega$ is a small neighborhood of $(0,0)$. Let us define a distance

$$\rho\big((u,v),(x,y)\big) = \begin{cases} \max\left\{|x^{k+1} - u^{k+1}|, (k+1)|y-v|\right\}, & \text{for } xu \geq 0 \\ \max\left\{x^{k+1} + u^{k+1}, (k+1)|y-v|\right\}, & \text{for } xu \leq 0. \end{cases}$$

For two sets $S_1, S_2$ the distance between them is defined as

$$\rho(S_1, S_2) = \inf_{(x,y)\in S_1,(u,v)\in S_2} \rho((x,y),(u,v)).$$

Let $V^T (T \leq 1)$ be the cube with edges of size (in the $\rho$ metric) $2T$, which are parallel to the coordinate axes and centered at $(0,0)$. Denote by $V_\delta^T$ the subcube which is homothetic with $V^T$ and such that the distance between its boundary and the boundary of $V^T$ is $\delta$. We shall prove by induction that if $T$ is small enough then there exist constants $H_0, H_1$ with $H_1 \geq C_4 H_0^{2k+3}$ such that

$$|f, V_\delta^T|_n \leq H_0 \text{ for } 0 \leq n \leq 6k+4, \tag{16}$$

and

$$|f, V_\delta^T|_n \leq H_0 \Big(\frac{H_1}{\delta}\Big)^{n-r_0-2} L_{n-r_0-2} \text{ for } n \geq 6k+4, \text{ and } \delta \text{ sufficiently small.} \tag{17}$$

Hence the desired conclusion follows. (16) follows easily from the $C^\infty$ smoothness assumption on $f$. Assume that (17) holds for $n = N$. We shall prove it for $n = N+1$. Put $\delta' = \delta(1-1/N), \delta'' = \delta(1-4/N)$. Fix $(x,y) \in V_\delta^T$ and then define $\sigma = \rho((x,y), \partial V^T)$ and $\tilde\sigma = \sigma/N$. Let $V_{\tilde\sigma}(x,y)$ denote the cube with center at $(x,y)$ and edges of length $2\tilde\sigma$ which are parallel to the coordinate axes, and $S_{\tilde\sigma}(x,y)$ the boundary of $V_{\tilde\sigma}(x,y)$. Note that $\sigma \geq \delta$, and $V_{\tilde\sigma}(x,y) \subset V_{\delta'}^T$. Let $E_1, E_3 (E_2, E_4)$ be edges of $S_{\tilde\sigma}(x,y)$ which are parallel to $Ox (Oy)$ respectively. We have to estimate $\max_{(x,y)\in V_\delta^T} \big|{}_\gamma\partial_{\alpha,\beta}\big(\partial_1^{\alpha_1}\partial_2^{\beta_1} f\big)\big|$ for all $(\alpha,\beta,\gamma) \in \Xi_1, (\alpha_1,\beta_1) \in \Gamma_{N+1}$, and $\max_{(x,y)\in V_\delta^T} \big|\big(\partial_1^{2+\alpha_1}\partial_2^{\beta_1} f\big)\big|$ for all $(\alpha_1,\beta_1) \in \Gamma_{N+1}, \alpha_1 \geq 1, \beta_1 \geq 1$. But when $(\alpha_1,\beta_1) \in \Gamma_N$ we have already the desired estimate. Hence it suffices to obtain the estimate only for $(\alpha_1,\beta_1) \in \Gamma_{N+1}\backslash\Gamma_N$. Let us abbreviate $\frac{\partial^\alpha}{\partial u^\alpha}, \frac{\partial^\beta}{\partial v^\beta}, \frac{\partial^{\alpha+\beta}}{\partial u^\alpha \partial v^\beta}$ as $\partial_u^\alpha, \partial_v^\beta, \partial_u^\alpha\partial_v^\beta$, respectively.

**Lemma 2.** *Assume that* $(\alpha,\beta,\gamma) \in \Xi_1$ *and* $(\alpha_1,\beta_1) \in \Gamma_{N+1}$. *Then if* $\alpha_1 \geq 1, \beta_1 \geq 1$ *there exists a constant* $C_{23}$ *such that*

$$\max_{(x,y)\in V_\delta^T} \big|{}_\gamma\partial_{\alpha,\beta}\big(\partial_1^{\alpha_1}\partial_2^{\beta_1} f(x,y)\big)\big| \leq$$
$$\leq C_{23}\Bigg(T^{\frac{1}{k+1}}\big|f, V_{\delta'}^T\big|_{N+1} + H_0\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1} L_{N-r_0-1}\Big(T^{\frac{1}{k+1}} + \frac{1}{H_1}\Big)\Bigg).$$

*Proof.* Differentiating the equation (1) $\alpha_1$ times in $x$ and $\beta_1$ times in $y$ then applying the representation formula (8) for $\Omega = V_{\tilde{\sigma}}(x,y)$ we have

$$\partial_1^{\alpha_1}\partial_2^{\beta_1} f(x,y) = \int_{V_{\tilde{\sigma}}(x,y)} F_{k,\lambda}(x,y,u,v)(A(u,v)+B(u,v))dudv- \\ - \int_{S_{\tilde{\sigma}}(x,y)} F_{k,\lambda}(x,y,u,v)B_1'\big(\partial_u^{\alpha_1}\partial_v^{\beta_1} f(u,v),k,-\lambda\big)ds+ \\ + \int_{S_{\tilde{\sigma}}(x,y)} \partial_u^{\alpha_1}\partial_v^{\beta_1} f(u,v)B_2'(F_{k,\lambda}(x,y,u,v),k)ds,$$

where

$$A(u,v) = -\sum_{m=1}^{\min\{2k,\alpha_1\}} \binom{\alpha_1}{m} 2k\cdots(2k+1-m)u^{2k-m}\partial_u^{\alpha_1-m}\partial_v^{\beta_1+2} f- \\ - i\lambda \sum_{m=1}^{\min\{k-1,\alpha_1\}} \binom{\alpha_1}{m}(k-1)\cdots(k-m)u^{k-1-m}\partial_u^{\alpha_1-m}\partial_v^{\beta_1+1} f,$$

and

$$B(u,v) = -\partial_u^{\alpha_1}\partial_v^{\beta_1}\Psi\Big(u,v,f,\frac{\partial f}{\partial u},u^k\frac{\partial f}{\partial v}\Big).$$

Therefore differentiating ${}_\gamma\partial_{\alpha,\beta}$ gives

$${}_\gamma\partial_{\alpha,\beta}\big(\partial_1^{\alpha_1}\partial_2^{\beta_1} f(x,y)\big) = \int_{V_{\tilde{\sigma}}(x,y)} {}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}(x,y,u,v)(A(u,v)+B(u,v))dudv- \\ - \int_{S_{\tilde{\sigma}}(x,y)} {}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}(x,y,u,v)B_1'\big(\partial_u^{\alpha_1}\partial_v^{\beta_1} f(u,v),k,-\lambda\big)ds+ \\ + \int_{S_{\tilde{\sigma}}(x,y)} \partial_u^{\alpha_1}\partial_v^{\beta_1} f(u,v)\,{}_\gamma\partial_{\alpha,\beta}B_2'(F_{k,\lambda}(x,y,u,v),k)ds =: \\ =: I_1 + I_2 + I_3. \tag{18}$$

It is not difficult to show that

$$\big|{}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}\big|_{V^T} \le C_{24}\Big[\big|x^{k+1}-u^{k+1}\big|^2 + (k+1)^2\big|y-v\big|^2\Big]^{-\frac{1}{2}} =: C_{24}R_1^{-\frac{1}{2}}. \tag{19}$$

Indeed, (19) follows from the set of estimates on $V_T$

$$0 \le p \le 1,\ \max\big\{|F_1'(p)|,|F_2'(p)|\big\} \le C_{25}(1-p)^{-1} \le C_{26}RR_1^{-1},$$

$$\max\big\{|F_1(p)|,|F_2(p)|\big\} \le C_{27}R^{\frac{1}{4}}R_1^{-\frac{1}{4}},\ 0 \le R_1 \le R \le C_{28},$$

$$|M| = R^{-\frac{k}{2k+2}},\ \max\Big\{\Big|\frac{\partial M}{\partial x}\Big|,\Big|x^k\frac{\partial M}{\partial y}\Big|\Big\} \le C_{29}R^{-\frac{1}{2}},$$

(20)

$$\max\Big\{\Big|\frac{\partial p}{\partial x}\Big|,\Big|x^k\frac{\partial p}{\partial y}\Big|\Big\} \le C_{30}R_1^{\frac{1}{2}}R^{-\frac{k+2}{2k+2}},\ \max\Big\{\Big|\frac{\partial p^{\frac{1}{k+1}}}{\partial x}\Big|,\Big|x^k\frac{\partial p^{\frac{1}{k+1}}}{\partial y}\Big|\Big\} \le C_{31}R^{-\frac{1}{2k+2}}.$$

Next we estimate $A(u,v)$. Consider three cases
I) $\alpha_1 = 1$. We have $u^{2k-1}\partial_v^{\beta_1+2} = u^{2k-1}\partial_v^1(\partial_v^{\beta_1+1})$ with $(0,\beta_1+1)\in\Gamma_N$. Therefore we have

$$|A(u,v)|\Big|_{V_{\tilde\sigma}(x,y)} \le C_{32}H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-2}L_{N-r_0-1}.$$

II) $2\le\alpha_1\le 4k+4$. In this case we have $\beta_1\le N-3$. If $m=\alpha_1$ then

$$\begin{aligned}
&\big|u^{2k-m}\partial_v^{\beta_1+2}f\big|_{V_{\tilde\sigma}(x,y)} \le C_{33}H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-2}L_{N-r_0-2},\\
&\big|u^{k-m-1}\partial_v^{\beta_1+1}f\big|_{V_{\tilde\sigma}(x,y)} \le C_{34}H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-2}L_{N-r_0-2}
\end{aligned}\tag{21}$$

since $(0,\beta_1+2),(0,\beta_1+1)\in\Gamma_{N-1}$. If $2\le\alpha_1\le 2k+2$ and $m<\alpha_1$ then it is easy to verify that $(\alpha_1-m-1,\beta_1+2),(\alpha_1-m-1,\beta_1+1)\in\Gamma_N$. Hence by the inductive assumptions we have on $V_{\tilde\sigma}(x,y)$

$$\begin{aligned}
&\big|u^{2k-m}\partial_u^{\alpha_1-m}\partial_v^{\beta_1+2}f\big| \le \big|\partial_u^1\big(\partial_u^{\alpha_1-m-1}\partial_v^{\beta_1+2}f\big)\big| \le C_{35}H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-2}L_{N-r_0-2},\\
&\big|u^{k-m-1}\partial_u^{\alpha_1-m}\partial_v^{\beta_1+1}f\big| \le \big|\partial_u^1\big(\partial_u^{\alpha_1-m-1}\partial_v^{\beta_1+1}f\big)\big| \le C_{36}H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-2}L_{N-r_0-2}.
\end{aligned}\tag{22}$$

If $2k+3\le\alpha_1\le 4k+4$ and $1<m<\alpha_1$ we have the same estimates for $u^{2k-m}\partial_u^{\alpha_1-m}\partial_v^{\beta_1+2}f, u^{k-m-1}\partial_u^{\alpha_1-m-1}\partial_v^{\beta_1+1}f$ as in (22), since $(\alpha_1-m-1,\beta_1+2),(\alpha_1-m-1,\beta_1+1)\in\Gamma_N$.
If $2k+3\le\alpha_1\le 4k+4$ and $m=1$ then $\alpha_1-m\ge 4$ and $(\alpha_1-3,\beta_1+2),(\alpha_1-3,\beta_1+1)\in\Gamma_N$, with $\alpha_1-3\ge 2,\beta_1+1\ge 2$. Therefore if we writes $u^{2k-1}\partial_u^{\alpha_1-1}\partial_v^{\beta_1+2}f, u^{k-2}\partial_u^{\alpha_1-1}\partial_v^{\beta_1+1}f$ as $u^{2k-1}\partial_u^2\big(\partial_u^{\alpha_1-3}\partial_v^{\beta_1+2}f\big), u^{k-2}\partial_u^2\big(\partial_u^{\alpha_1-3}\partial_v^{\beta_1+1}f\big)$, respectively, and use the inductive assumptions we still have the estimate (22) for $u^{2k-1}\partial_u^{\alpha_1-1}\partial_v^{\beta_1+2}f, u^{k-2}\partial_u^{\alpha_1-1}\partial_v^{\beta_1+1}f$.
Therefore combining (21), (22) yields

$$|A(u,v)|\Big|_{V_{\tilde\sigma}(x,y)} \le C_{37}H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-2}L_{N-r_0-1}.$$

III) If $\alpha_1\ge 4k+5$ then we can write

$$\begin{aligned}
u^{2k-m}\partial_u^{\alpha_1-m}\partial_v^{\beta_1+2}f(u,v) &= u^{2k-m}\partial_u^2\Big(\partial_u^{\alpha_1-m-2}\partial_v^{\beta_1+2}f(u,v)\Big),\\
u^{k-m-1}\partial_u^{\alpha_1-m}\partial_v^{\beta_1+1}f(u,v) &= u^{k-m-1}\partial_u^2\Big(\partial_u^{\alpha_1-m-2}\partial_v^{\beta_1+1}f(u,v)\Big),
\end{aligned}$$

where $(\alpha_1-m-2,\beta_1+2),(\alpha_1-m-2,\beta_1+1)\in\Gamma_{N+1-m}$ and $\alpha_1-m-2\geq 1,\beta_1\geq 1$. Therefore by inductive assumptions we see that

$$|A(u,v)|\Big|_{V_{\tilde\sigma}(x,y)}\leq C_{38}\sum_{m=1}^{\min\{2k,\alpha_1\}}\binom{\alpha_1}{m}H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-m-1}L_{N-r_0-m-1}\leq$$
$$\leq C_{39}H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-2}L_{N-r_0-1}.$$

Hence in both cases we have

$$|A(u,v)|\Big|_{V_{\tilde\sigma}(x,y)}\leq C_{40}H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-2}L_{N-r_0-1}. \tag{23}$$

On the other hand, from Proposition 3 and the inductive assumption we have

$$|B(u,v)|\Big|_{V_{\tilde\sigma}(x,y)}\leq C_5\Big(|f,V^T_{\delta'}|_{N+1}+H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-1}L_{N-r_0-1}\Big). \tag{24}$$

Combining (19), (23), (24) we obtain

$$\begin{aligned}|I_1|\leq C_{41}\Big(|f,V^T_{\delta'}|_{N+1}+H_0\Big(\frac{H_1}{\delta'}\Big)^{N-r_0-1}L_{N-r_0-1}\Big)\times\\ \times\int_{V_{\tilde\sigma}(x,y)}\Big[\big|x^{k+1}-u^{k+1}\big|^2+(k+1)^2\big|y-v\big|^2\Big]^{-\frac12}dudv\leq\\ \leq C_{42}T^{\frac{1}{k+1}}\Big(|f,V^T_{\delta'}|_{N+1}+H_0\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}\Big).\end{aligned} \tag{25}$$

A) Let us now estimate the third integral in (18). Consider two cases:
1) $|x|\leq(2\tilde\sigma)^{\frac{1}{k+1}}$. It is clear that $|u|\leq 3\tilde\sigma^{\frac{1}{k+1}}$ on $S_{\tilde\sigma}(x,y)$. We have the following estimate

$$\big|{}_\gamma\partial_{\alpha,\beta}X_1'F_{k,\lambda}\big|\Big|_{S_{\tilde\sigma}(x,y)}\leq\frac{C_{43}}{\tilde\sigma^{\frac{k+2}{k+1}}}. \tag{26}$$

Indeed, (26) results from the following set of estimates on $S_{\tilde\sigma}(x,y)$

$$\begin{aligned}&\max\{|F(p)|,|F'(p)|,|F''(p)|\}\leq C_{44}\ \text{ if }\ xu\leq 0,\\ &0\leq p\leq C_{45}<1\ \text{ if }\ xu\geq 0\Longrightarrow\max\{|F(p)|,|F'(p)|,|F''(p)|\}\leq C_{46}\ \text{ if }\ xu\geq 0,\\ &\max\Big\{\Big|\frac{\partial M}{\partial u}\Big|,\Big|u^k\frac{\partial M}{\partial v}\Big|,\Big|\frac{\partial M}{\partial x}\Big|,\Big|x^k\frac{\partial M}{\partial y}\Big|\Big\}\leq C_{47}R^{-\frac12},\\ &\max\Big\{\Big|\frac{\partial p}{\partial u}\Big|,\Big|u^k\frac{\partial p}{\partial v}\Big|,\Big|\frac{\partial p}{\partial x}\Big|,\Big|x^k\frac{\partial p}{\partial y}\Big|\Big\}\leq C_{48}R_1^{\frac12}R^{-\frac{k+2}{2k+2}},\\ &|M|=R^{-\frac{k}{2k+2}},\ \ C_{49}\tilde\sigma^2\leq R,\ \ 0\leq R_1\leq R\leq C_{50}\tilde\sigma^2,\\ &\max\Big\{\Big|\frac{\partial^2 M}{\partial x\partial u}\Big|,\Big|u^k\frac{\partial^2 M}{\partial x\partial v}\Big|,\Big|x^k\frac{\partial^2 M}{\partial y\partial u}\Big|,\Big|x^ku^k\frac{\partial^2 M}{\partial y\partial v}\Big|\Big\}\leq C_{51}R^{-\frac{k+2}{2k+2}},\\ &\max\Big\{\Big|\frac{\partial^2 p}{\partial x\partial u}\Big|,\Big|u^k\frac{\partial^2 p}{\partial x\partial v}\Big|,\Big|x^k\frac{\partial^2 p}{\partial y\partial u}\Big|,\Big|x^ku^k\frac{\partial^2 p}{\partial y\partial v}\Big|\Big\}\leq C_{52}R^{-\frac{1}{k+1}}.\end{aligned} \tag{27}$$

Since $\alpha_1 \geq 1$ we can write $\partial_u^{\alpha_1}\partial_v^{\beta_1} f(u,v) = \partial_u^1\big(\partial_u^{\alpha_1-1}\partial_v^{\beta_1} f(u,v)\big)$. Therefore we have

$$|I_3| \leq \frac{C_{53}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-1-r_0} L_{N-r_0-1}. \tag{28}$$

2) $|x| \geq (2\tilde{\sigma})^{\frac{1}{k+1}}$. We then have $xu > 0, |u| \geq 2^{-\frac{1}{k+1}}|x| \geq \tilde{\sigma}^{\frac{1}{k+1}}$ on $S_{\tilde{\sigma}}(x,y)$. In this case $p$ is not bounded away from 1 as in (27) hence the estimates in (27) for $F(p), F'(p), F''(p)$ are not necessarily true. But we use the following set of estimates

$$\begin{gathered} C_{54}R^{\frac{1}{2k+2}} \leq |u| \leq C_{55}R^{\frac{1}{2k+2}},\ C_{56}\tilde{\sigma}^2 \leq R_1 \leq C_{57}\tilde{\sigma}^2,\ C_{58}\tilde{\sigma}^2 \leq R, \\ |F(p)| \leq C_{59}R^{\frac{1}{2}}R_1^{-\frac{1}{2}},\ |F'(p)| \leq C_{60}RR_1^{-1},\ |F''(p)| \leq C_{61}R^2R_1^{-2}. \end{gathered} \tag{29}$$

and the estimates in (27) (except the estimates for $p, F(p), F'(p), F''(p)$) to deduce that

$$\Big|\frac{{}_\gamma\partial_{\alpha,\beta}X_1'F_{k,\lambda}}{u^k}\Big|\Big|_{S_{\tilde{\sigma}}(x,y)} \leq \frac{C_{62}}{\tilde{\sigma}^2}. \tag{30}$$

It implies that

$$\begin{aligned} \Big|\,\big(\partial_u^{\alpha_1}\partial_v^{\beta_1} f(u,v)\big)\,{}_\gamma\partial_{\alpha,\beta}X_1'F_{k,\lambda}\Big|_{S_{\tilde{\sigma}}(x,y)}\Big| &= \Big|\,u^k\big(\partial_u^{\alpha_1}\partial_v^{\beta_1} f(u,v)\big)\frac{{}_\gamma\partial_{\alpha,\beta}X_1'F_{k,\lambda}}{u^k}\Big|_{S_{\tilde{\sigma}}(x,y)}\Big| \leq \\ &\leq \frac{C_{63}}{\tilde{\sigma}^2} \times \Big|\,\big(u^k\partial_u^{\alpha_1}\partial_v^{\beta_1} f(u,v)\big)\Big|_{S_{\tilde{\sigma}}(x,y)}. \end{aligned}$$

On the other hand $u^k\partial_u^{\alpha_1}\partial_v^{\beta_1} f(u,v) = u^k\partial_v^1\big(\partial_u^{\alpha_1}\partial_v^{\beta_1-1} f(u,v)\big)$. It follows that

$$|I_3| \leq C_{64}\frac{|f,V_{\delta'}^T|_N}{\tilde{\sigma}^2}\Big|\int_{S_{\tilde{\sigma}}(x,y)}(\nu_1 + iu^k\nu_2)ds\Big| \leq \frac{C_{65}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-1-r_0} L_{N-r_0-1}. \tag{31}$$

B) We now estimate the second integral in (18).
1) We first estimate the integrals along the edges $E_2$ and $E_4$. Integrating by parts gives

$$\begin{aligned} &\Big|\int_{E_2\cup E_4} {}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}B_1'\big(\partial_u^{\alpha_1}\partial_v^{\beta_1} f, k, -\lambda\big)dv\Big| \leq \\ &\leq C_{66}\bigg(\Big|\int_{E_2\cup E_4}\partial_v^1\big({}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}\big)B_1'\big(\partial_u^{\alpha_1}\partial_v^{\beta_1-1} f, k, -\lambda\big)dv\Big| + \\ &+ \Big|\,{}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}B_1'\big(\partial_u^{\alpha_1}\partial_v^{\beta_1-1} f, k, -\lambda\big)\Big|_{\partial E_2\cup\partial E_4}\Big|\bigg) \leq \\ &\leq C_{67}\frac{|f,V_{\delta'}^T|_N}{(\delta-\delta')} \leq \frac{C_{68}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1} L_{N-r_0-1}. \end{aligned}$$

2) We now estimate the integrals along the edges $E_1$ and $E_3$. Integrating by parts gives

$$\Big|\int_{E_1\cup E_3} {}_{\gamma}\partial_{\alpha,\beta}F_{k,\lambda}B_1'\big(\partial_u^{\alpha_1}\partial_v^{\beta_1}f,k,-\lambda\big)du\Big| \le \int_{E_1\cup E_3}|\lambda-k|\big|{}_{\gamma}\partial_{\alpha,\beta}F_{k,\lambda}\big|\big|u^{k-1}\partial_u^{\alpha_1}\partial_v^{\beta_1}f\big|du+$$

$$+\Big|\int_{E_1\cup E_3} {}_{\gamma}\partial_{\alpha,\beta}F_{k,\lambda}\big(iu^k\partial_u^{\alpha_1+1}\partial_v^{\beta_1}f-u^{2k}\partial_u^{\alpha_1}\partial_v^{\beta_1+1}f\big)du\Big|\le$$

$$\le C_{69}\Bigg(\Big|\int_{E_1\cup E_3}\partial_u^1\big({}_{\gamma}\partial_{\alpha,\beta}F_{k,\lambda}\big)\big(iu^k\partial_u^{\alpha_1}\partial_v^{\beta_1}f-u^{2k}\partial_u^{\alpha_1-1}\partial_v^{\beta_1+1}f\big)du\Big|+$$

$$+\int_{E_1\cup E_3}\big|{}_{\gamma}\partial_{\alpha,\beta}F_{k,\lambda}\big|\big(\big|u^{k-1}\partial_u^{\alpha_1}\partial_v^{\beta_1}f\big|+\big|u^{2k-1}\partial_u^{\alpha_1-1}\partial_v^{\beta_1+1}f\big|\big)du+$$

$$+\Big|{}_{\gamma}\partial_{\alpha,\beta}F_{k,\lambda}\big(iu^k\partial_u^{\alpha_1}\partial_v^{\beta_1}f-u^{2k}\partial_u^{\alpha_1-1}\partial_v^{\beta_1+1}f\big)\Big|_{\partial E_1\cup\partial E_3}\Big|\Bigg)\le$$

$$\le C_{70}\frac{|f,V_{\delta'}^T|_N}{(\delta-\delta')}\le\frac{C_{71}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}.$$

Therefore we deduce that

$$|I_2|\le\frac{C_{72}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}. \tag{32}$$

We complete the proof of Lemma 2 by combining the estimates (18), (25), (28), (31), (32). □

**Lemma 3.** *Assume that $(\alpha,\beta,\gamma)\in\Xi_1$. Then there exists a constant $C_{73}$ such that*

$$\max_{(x,y)\in V_\delta^T}\big|{}_{\gamma}\partial_{\alpha,\beta}\big(\partial_2^{N+1}f(x,y)\big)\big|\le$$

$$\le C_{73}\Bigg(T^{\frac{1}{k+1}}\big|f,V_{\delta''}^T\big|_{N+1}+H_0\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}\Big(T^{\frac{1}{k+1}}+\frac{1}{H_1}\Big)\Bigg).$$

*Proof.* I) $|x|\le(2\tilde\sigma)^{\frac{1}{k+1}}$. Instead of $V_{\tilde\sigma}(x,y)$ we take the cube $V_{4\tilde\sigma}(x,y)$. As in Lemma 2 the following formula holds

$$\begin{aligned}{}_{\gamma}\partial_{\alpha,\beta}\big(\partial_2^{N+1}f(x,y)\big)&=\int_{V_{4\tilde\sigma}(x,y)}{}_{\gamma}\partial_{\alpha,\beta}F_{k,\lambda}(x,y,u,v)B(u,v)dudv-\\&-\int_{S_{4\tilde\sigma}(x,y)}{}_{\gamma}\partial_{\alpha,\beta}F_{k,\lambda}(x,y,u,v)B_1'\big(\partial_v^{N+1}f(u,v),k,-\lambda\big)ds+\\&+\int_{S_{4\tilde\sigma}(x,y)}\partial_v^{N+1}f(u,v){}_{\gamma}\partial_{\alpha,\beta}B_2'(F_{k,\lambda}(x,y,u,v),k)ds=:\\&=:I_4+I_5+I_6,\end{aligned} \tag{33}$$

where

$$B(u,v) = -\frac{\partial^{N+1}}{\partial v^{N+1}}\Psi\Big(u,v,f,\frac{\partial f}{\partial u},u^k\frac{\partial f}{\partial v}\Big).$$

Since $V_{4\tilde{\sigma}}(x,y) \subset V^T_{\delta''}$, as in Lemma 2, we have

$$|I_4| \le C_{74}T^{\frac{1}{k+1}}\Big(|f,V^T_{\delta''}|_{N+1} + H_0\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}\Big).$$

We now estimate $I_5, I_6$ in (33).
A) First consider the integrals along $\tilde{E}_2$ and $\tilde{E}_4$, where $\tilde{E}_2, \tilde{E}_4$ denote the edges of $S_{4\tilde{\sigma}}$, which are parallel to $Oy$. We note that $|u| \ge (2\tilde{\sigma})^{\frac{1}{k+1}}$ on $\tilde{E}_2, \tilde{E}_4$.
1) To estimate $I_6$, as in Lemma 2, we note that

$$\Big|\ \big(\partial_v^{N+1}f\big)\ {}_\gamma\partial_{\alpha,\beta}B_2'F_{k,\lambda}\Big|_{\tilde{E}_2\cup\tilde{E}_4}\Big| \le \frac{C_{75}}{\tilde{\sigma}^2}\times\Big|u^k\partial_v\big(\partial_v^N f\big)\Big|_{\tilde{E}_2\cup\tilde{E}_4},$$

with $(0,1,k)\in\Xi_1$ and $(0,N)\in\Gamma_N$. Therefore

$$\Big|\int_{\tilde{E}_2\cup\tilde{E}_4}\partial_v^{N+1}f\ {}_\gamma\partial_{\alpha,\beta}B_2'(F_{k,\lambda},k)dv\Big| \le \frac{C_{76}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-1-r_0}L_{N-r_0-1}.$$

2) As in Lemma 2, integrating by parts gives the following estimate for $I_6$:

$$\Big|\int_{\tilde{E}_2\cup\tilde{E}_4}{}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}B_1'\big(\partial_v^{N+1}f,k,-\lambda\big)dv\Big| \le$$

$$C_{77}\frac{N|f,V^T_{\delta''}|_N}{\delta} \le \frac{C_{78}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}.$$

B) Consider now integrals along the edges $\tilde{E}_1$ and $\tilde{E}_3$, where $\tilde{E}_1, \tilde{E}_3$ denote the edges of $S_{4\tilde{\sigma}}$, which are parallel to $Ox$.
1) To estimate $I_6$, as in Lemma 2, we have

$$\big|\ {}_\gamma\partial_{\alpha,\beta}X_1'F_{k,\lambda}\big|\Big|_{\tilde{E}_1\cup\tilde{E}_3} \le \frac{C_{79}}{\tilde{\sigma}^{\frac{k+2}{k+1}}} \quad\text{and}\quad |u|\Big|_{\tilde{E}_1\cup\tilde{E}_3} \le C_{80}\tilde{\sigma}^{\frac{1}{k+1}}.$$

Hence we get

$$\Big|\int_{\tilde{E}_1\cup\tilde{E}_3}\partial_v^{N+1}f_\gamma\partial_{\alpha,\beta}B_2'F_{k,\lambda}du\Big| = \Big|\int_{\tilde{E}_1\cup\tilde{E}_3}u^k\partial_v^1\big(\partial_v^N f\big)\ {}_\gamma\partial_{\alpha,\beta}X_1'F_{k,\lambda}du\Big| \le$$

$$\le C_{81}\frac{\big|f,V^T_{\delta''}\big|_N}{(\delta-\delta'')^{\frac{k+2}{k+1}}}\int_{\tilde{E}_1\cup\tilde{E}_3}du \le \frac{C_{82}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}.$$

2) To estimate $I_5$, we first note that

$$u^{2k}\partial_v^{N+2}f + i\lambda u^{k-1}\partial_v^{N+1}f = G'_{k,\lambda}(\partial_v^N f) - \partial_u^2\partial_v^N f = \partial_v^N(G'_{k,\lambda}f) - \partial_u^2\partial_v^N f =$$
$$= -\partial_v^N\Psi\Big(u, v, f, \frac{\partial f}{\partial u}, u^k\frac{\partial f}{\partial v}\Big) - \partial_u^2\partial_v^N f.$$

It implies that

(34)

$$\left|\int_{\tilde E_1\cup\tilde E_3} {}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}B'_1(\partial_v^{N+1}f, k, -\lambda)du\right| \le \left|\int_{\tilde E_1\cup\tilde E_3} u^k\partial_v^1(\partial_v^N f)\,{}_\gamma\partial_{\alpha,\beta}\partial_u^1F_{k,\lambda}du\right| +$$
$$\left|\int_{\tilde E_1\cup\tilde E_3}\partial_y^N\Psi\,{}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}du\right| + \left|u^k\partial_v^1(\partial_v^N f)\,{}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}\Big|_{\partial E_1\cup\partial E_3}\right| +$$
$$+\left|\int_{\tilde E_1\cup\tilde E_3}\partial_u^1(\partial_u^1\partial_v^N f)\,{}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}du\right|.$$

By Proposition 3 and the inductive assumptions the first three terms in the right hand side of (34) can be estimated by

$$\frac{C_{83}|f, V_{\delta''}^T|_N}{\delta-\delta''} \le \frac{C_{84}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}.$$

To estimate the last integral in the right hand side of (34) we again integrate by parts:

$$\left|\int_{\tilde E_1\cup\tilde E_3}\partial_u^1(\partial_u^1\partial_v^N f)\,{}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}du\right| \le \left|\int_{\tilde E_1\cup\tilde E_3}\partial_u^1\partial_v^N f\,{}_\gamma\partial_{\alpha,\beta}\partial_u^1F_{k,\lambda}du\right| +$$
$$+\left|\partial_u^1\partial_v^N f\,{}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}\Big|_{\partial\tilde E_1\cup\partial\tilde E_3}\right| \le C_{85}|f, V_{\delta''}^T|_N\left(\frac{N}{\sigma} + \frac{N^{1+\frac{1}{k+1}}}{\sigma^{1+\frac{1}{k+1}}}\left|\int_{\tilde E_1\cup\tilde E_3}du\right|\right) \le$$
$$\le \frac{C_{86}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}.$$

II) $|x| \ge (2\tilde\sigma)^{\frac{1}{k+1}}$. We now consider the representation formula for $\partial_2^{N+1}f(x,y)$ in $V_{\tilde\sigma}(x,y)$. As above, the volume integral can be estimated by

$$C_{87}T^{\frac{1}{k+1}}\Big(|f, V_{\delta'}^T|_{N+1} + \Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}\Big).$$

For the boundary integral we again split into 2 cases

A) First we estimate the integral involving $B_2'$. Since $|u| \geq \tilde{\sigma}^{\frac{1}{k+1}}$, on $S_{\tilde{\sigma}}(x,y)$, as in Lemma 2 we obtain the following estimate

$$\Big|\int_{S_{\tilde{\sigma}}(x,y)} \partial_v^{N+1} f_\gamma \partial_{\alpha,\beta} B_2'(F_{k,\lambda},k)ds\Big| \leq \frac{C_{88}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1} L_{N-r_0-1}.$$

B) Now we estimate the integral involving $B_1'$. Along $E_2$ and $E_4$ we can estimate exactly as in Lemma 2.

Consider now the boundary integral along $E_1$ and $E_3$. As in this Lemma I) B) 2) we have

$$\left|\int_{E_1\cup E_3} {}_\gamma\partial_{\alpha,\beta} F_{k,\lambda} B_1'\big(\partial_v^{N+1} f, k, -\lambda\big)du\right| \leq \frac{C_{89}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1} L_{N-r_0-1}.$$

This completes the proof of Lemma 3.□

**Lemma 4.** *Assume that* $(\alpha,\beta,\gamma)\in\Xi_1$. *Then there exists a constant* $C_{90}$ *such that*

$$\max_{(x,y)\in V_\delta^T}\big|\,{}_\gamma\partial_{\alpha,\beta}\big(\partial_1^{N-r_0+1} f(x,y)\big)\big| \leq$$
$$\leq C_{90}\left(T^{\frac{1}{k+1}}\big|f, V_{\delta'}^T\big|_{N+1} + H_0\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1} L_{N-r_0-1}\Big(T^{\frac{1}{k+1}} + \frac{1}{H_1}\Big)\right).$$

*Proof.* We have the following representation formula for ${}_\gamma\partial_{\alpha,\beta}\big(\partial_1^{N-r_0+1} f(x,y)\big)$ in $V_{\tilde{\sigma}}(x,y)$

$$\begin{aligned}
{}_\gamma\partial_{\alpha,\beta}\big(\partial_1^{N-r_0+1} f(x,y)\big) = &\int_{V_{\tilde{\sigma}}(x,y)} {}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}(x,y,u,v)(A(u,v)+B(u,v))dudv-\\
&-\int_{S_{\tilde{\sigma}}(x,y)} {}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}(x,y,u,v)B_1'\big(\partial_u^{N-r_0+1} f(u,v),k,-\lambda\big)ds+\\
&+\int_{S_{\tilde{\sigma}}(x,y)} \partial_u^{N-r_0+1} f(u,v)\,{}_\gamma\partial_{\alpha,\beta}B_2'(F_{k,\lambda}(x,y,u,v),k)ds =:\\
&=: I_7+I_8+I_9,
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
A(u,v) = &-\sum_{m=1}^{\min\{2k,N-r_0+1\}} \binom{N-r_0+1}{m} 2k\cdots(2k+1-m)u^{2k-m}\partial_u^{N-r_0+1-m}\partial_v^2 f-\\
&-i\lambda\sum_{m=1}^{\min\{k-1,N-r_0+1\}} \binom{N-r_0+1}{m}(k-1)\cdots(k-m)u^{k-1-m}\partial_u^{N-r_0+1-m}\partial_v^1 f,
\end{aligned}$$

and

$$B(u,v) = -\partial_u^{N-r_0+1}\Psi\Big(u,v,f,\frac{\partial f}{\partial u},u^k\frac{\partial f}{\partial v}\Big).$$

Therefore, as in Lemma 2, the first integral in (35) is estimated by

$$C_{91}T^{\frac{1}{k+1}}\Big(|f,V_{\delta'}^T|_{N+1}+H_0\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}\Big).$$

I) For $I_9$, as in Lemma 2 A), we have the following estimate

$$\left|\int_{S_{\tilde\sigma}(x,y)}\partial_u^{N-r_0+1}f\ {}_\gamma\partial_{\alpha,\beta}B_2'(F_{k,\lambda},k)ds\right|\le\frac{C_{92}H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}.$$

II) Now we estimate $I_8$.

A) We first estimate the integrals along $E_2$ and $E_4$. Note that

$$\begin{aligned}&(\partial_u^1+iu^k\partial_v^1)(\partial_u^{N-r_0+1}f)=(\partial_u^1+iu^k\partial_v^1)(\partial_u^1-iu^k\partial_v^1+iu^k\partial_v^1)(\partial_u^{N-r_0}f)=\\&\partial_v^1(iu^k\partial_u^{N-r_0+1}-u^{2k}\partial_u^{N-r_0}\partial_v^1)f+G_{k,\lambda}'(\partial_u^{N-r_0}f)-i\lambda u^{k-1}\partial_u^{N-r_0}\partial_v^1f=\\&-\partial_u^{N-r_0}\Psi-\sum_{m=1}^{\min\{2k,N-r_0\}}\binom{N-r_0}{m}2k(2k-1)\cdots(2k-m+1)u^{2k-m}\partial_u^{N-r_0-m}\partial_v^2f-\\&i\lambda\sum_{m=1}^{\min\{k-1,N-r_0\}}\binom{N-r_0}{m}(k-1)(k-2)\cdots(k-m)u^{k-m-1}\partial_u^{N-r_0-m}\partial_v^1f+\\&\partial_v^1(iu^k\partial_u^{N-r_0+1}-u^{2k}\partial_u^{N-r_0}\partial_v^1)f-i\lambda u^{k-1}\partial_u^{N-r_0}\partial_v^1f=:\\&=:\partial_v^1(iu^k\partial_u^{N-r_0+1}-u^{2k}\partial_u^{N-r_0}\partial_v^1)f+I_{10}.\end{aligned}$$

By the inductive assumptions, Proposition 3, and the condition $N\ge 6k+4$, it is easily seen that

$$|I_{10}|\Big|_{E_2\cup E_4}\le C_{93}\frac{H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}.$$

Therefore integrating by parts gives

$$\begin{aligned}&\left|\int_{E_2\cup E_4}{}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}B_1'(\partial_u^{N-r_0+1}f,k,-\lambda)dv\right|\le C_{94}\frac{H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}+\\&\qquad\left|\int_{E_2\cup E_4}(iu^k\partial_u^1(\partial_u^{N-r_0}f)-u^{2k}\partial_u^1(\partial_u^{N-r_0-1}\partial_v^1f))\,{}_\gamma\partial_{\alpha,\beta+1}F_{k,\lambda}dv\right|+\\&\left|(iu^k\partial_u^{N-r_0+1}-u^{2k}\partial_u^{N-r_0}\partial_v^1)f\ {}_\gamma\partial_{\alpha,\beta}F_{k,\lambda}\Big|_{\partial E_2\cup\partial E_4}\right|\le C_{95}\frac{H_0}{H_1}\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1}L_{N-r_0-1}.\end{aligned}$$

B) The integrals along $E_1$ and $E_3$ now can be estimated in the same manner as in Lemma 2. □

**Lemma 5.** *Assume that* $(\alpha_1, \beta_1) \in \Gamma_{N+1} \backslash \Gamma_N, \alpha_1 \geq 1, \beta_1 \geq 1$. *Then there exists a constant* $C_{96}$ *such that*

$$\max_{(x,y)\in V_\delta^T} \left| \left(\partial_1^{\alpha_1+2}\partial_2^{\beta_1} f(x,y)\right)\right| \leq$$
$$\leq C_{96}\left(T^{\frac{1}{k+1}}\left|f, V_{\delta''}^T\right|_{N+1} + H_0\Big(\frac{H_1}{\delta}\Big)^{N-r_0-1} L_{N-r_0-1}\Big(T^{\frac{1}{k+1}} + \frac{1}{H_1}\Big)\right).$$

*Proof.* Differentiating $\partial_1^{\alpha_1}\partial_2^{\beta_1}$ the equation (1) we obtain

$$(36)\quad \partial_1^2\big(\partial_1^{\alpha_1}\partial_2^{\beta_1} f(x,y)\big) = -\partial_1^{\alpha_1}\partial_2^{\beta_1}\Psi(x,y,f,\partial_1 f, x^k\partial_2 f) -$$
$$\sum_{m=0}^{\min\{2k,\alpha_1\}} \binom{\alpha_1}{m} 2k(2k-1)\cdots(2k-m+1)x^{2k-m}\partial_1^{\alpha_1-m}\partial_2^{\beta_1+2} f(x,y) -$$
$$i\lambda \sum_{m=0}^{\min\{k-1,\alpha_1\}} \binom{\alpha_1}{m}(k-1)(k-2)\cdots(k-m)x^{k-m-1}\partial_1^{\alpha_1-m}\partial_2^{\beta_1+1} f(x,y) =: -J_1 - J_2 - J_3.$$

By Lemmas 2-4 we see that

$$(37)\quad |J_1| \leq C_{97}\left(T^{\frac{1}{k+1}}\left|f, V_{\delta''}^T\right|_{N+1} + H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}\Big(T^{\frac{1}{k+1}} + \frac{1}{H_1}\Big)\right).$$

To estimate $J_2, J_3$ we consider two cases

I) $\alpha_1 \leq r_0$.

A) For $m < \alpha_1$ the typical terms $x^{2k-m}\partial_{1,2}^{\alpha_1-m,\beta_1+2} f(x,y), x^{k-m-1}\partial_{1,2}^{\alpha_1-m,\beta_1+1} f(x,y)$ in (36) can be rewritten as $x^{2k-m}\partial_1^1\big(\partial_{1,2}^{\alpha_1-m-1,\beta_1+2} f(x,y)\big), x^{k-m-1}\partial_1^1\big(\partial_{1,2}^{\alpha_1-m-1,\beta_1+1} f(x,y)\big)$ with $(\alpha_1-m-1, \beta_1+2), (\alpha_1-m-1,\beta_1+1) \in \Gamma_{N+1}$. Hence by the inductive assumptions and Lemmas 2-4 we have

$$\left|\sum_{m=0}^{\min\{2k,\alpha_1-1\}} \binom{\alpha_1}{m} 2k(2k-1)\cdots(2k-m+1)x^{2k-m}\partial_1^{\alpha_1-m}\partial_2^{\beta_1+2} f(x,y)\right| +$$
$$\left|i\lambda \sum_{m=0}^{\min\{k-1,\alpha_1-1\}} \binom{\alpha_1}{m}(k-1)(k-2)\cdots(k-m)x^{k-m-1}\partial_1^{\alpha_1-m}\partial_2^{\beta_1+1} f(x,y)\right| \leq$$

(38)

$$C_{98}\left(T^{\frac{1}{k+1}}\left|f, V_{\delta''}^T\right|_{N+1} + \frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}\right).$$

B) $m = \alpha_1$. By Lemma 3 we have

$$\max\{\left|x^{2k-m}\partial_2^{\beta_1+2}f(x,y)\right|, \left|x^{k-m-1}\partial_2^{\beta_1+1}f(x,y)\right|\} \leq$$

$$C_{99}\left(T^{\frac{1}{k+1}}\left|f, V_{\delta''}^T\right|_{N+1} + \frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}\right) \tag{39}$$

since $(0, \beta_1+2), (0, \beta_1+1) \in \Gamma_{N+1}$.
Combining (38) and (39) we deduce that

$$|J_2| + |J_3| \leq C_{100}\left(T^{\frac{1}{k+1}}\left|f, V_{\delta''}^T\right|_{N+1} + \frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}\right). \tag{40}$$

II)$\alpha_1 \geq 2k+3$. In this case we have $\alpha_1 - m \geq 3$.
A)$m = 0$, then $x^{2k}\partial_{1,2}^{\alpha_1,\beta_1+2}f(x,y), x^{k-1}\partial_{1,2}^{\alpha_1,\beta_1+1}f(x,y)$ in (36) can be rewritten as $x^{2k}\partial_1^2\big(\partial_{1,2}^{\alpha_1-2,\beta_1+2}f(x,y)\big), x^{k-1}\partial_1^2\big(\partial_{1,2}^{\alpha_1-2,\beta_1+1}f(x,y)\big)$ with $(\alpha_1-2, \beta_1+2) \in \Gamma_{N+1}, (\alpha_1-2, \beta_1+1) \in \Gamma_N$ and $\alpha_1 - 2 \geq 1, \beta_1 \geq 1$. Hence

$$\left|x^{k-1}\partial_{1,2}^{\alpha_1,\beta_1+1}f(x,y)\right| = \left|x^{k-1}\partial_1^2\big(\partial_{1,2}^{\alpha_1-2,\beta_1+1}f(x,y)\big)\right| \leq C_{101}\frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1},$$

(41)

$$\left|x^{2k}\partial_{1,2}^{\alpha_1,\beta_1+2}f(x,y)\right| \leq T^{\frac{1}{k+1}}\left|\partial_1^2\big(\partial_{1,2}^{\alpha_1-2,\beta_1+2}f(x,y)\big)\right|.$$

B)$m \geq 1, \alpha_1 \leq 4k+3$, then $x^{2k-m}\partial_{1,2}^{\alpha_1-m,\beta_1+2}f(x,y), x^{k-m-1}\partial_{1,2}^{\alpha_1-m,\beta_1+1}f(x,y)$ in (36) can be rewritten as $x^{2k-m}\partial_1^2\big(\partial_{1,2}^{\alpha_1-m-2,\beta_1+2}f(x,y)\big), x^{k-m-1}\partial_1^2\big(\partial_{1,2}^{\alpha_1-m-2,\beta_1+1}f(x,y)\big)$ with $(\alpha_1-m-2, \beta_1+2), (\alpha_1-m-2, \beta_1+1) \in \Gamma_N$ and $\alpha_1-m-2 \geq 1, \beta_1 \geq 1$. Hence by the inductive assumptions we obtain

$$\Big|\sum_{m=1}^{\min\{2k,\alpha_1\}}\binom{\alpha_1}{m}2k(2k-1)\cdots(2k-m+1)x^{2k-m}\partial_1^{\alpha_1-m}\partial_2^{\beta_1+2}f(x,y)\Big| +$$

$$\Big|i\lambda\sum_{m=1}^{\min\{k-1,\alpha_1\}}\binom{\alpha_1}{m}(k-1)(k-2)\cdots(k-m)x^{k-m-1}\partial_1^{\alpha_1-m}\partial_2^{\beta_1+1}f(x,y)\Big| \leq$$

(42)

$$C_{102}\frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.$$

C)$m \geq 1, \alpha_1 \geq 4k+4$, then $x^{2k-m}\partial_{1,2}^{\alpha_1-m,\beta_1+2}f(x,y), x^{k-m-1}\partial_{1,2}^{\alpha_1-m,\beta_1+1}f(x,y)$ in (36) can be rewritten as $x^{2k-m}\partial_1^2\big(\partial_{1,2}^{\alpha_1-m-2,\beta_1+2}f(x,y)\big), x^{k-m-1}\partial_1^2\big(\partial_{1,2}^{\alpha_1-m-2,\beta_1+1}f(x,y)\big)$

with $(\alpha_1-m-2,\beta_1+2),(\alpha_1-m-2,\beta_1+1)\in\Gamma_{N+1-m}$ and $\alpha_1-m-2\geq 1,\beta_1\geq 1$. Hence by the inductive assumptions we have

$$\Big|\sum_{m=1}^{\min\{2k,\alpha_1\}}\binom{\alpha_1}{m}2k(2k-1)\cdots(2k-m+1)x^{2k-m}\partial_1^{\alpha_1-m}\partial_2^{\beta_1+2}f(x,y)\Big|+$$

$$\Big|i\lambda\sum_{m=1}^{\min\{k-1,\alpha_1\}}\binom{\alpha_1}{m}(k-1)(k-2)\cdots(k-m)x^{k-m-1}\partial_1^{\alpha_1-m}\partial_2^{\beta_1+1}f(x,y)\Big|\leq$$

(43)

$$C_{103}\frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}.$$

Combining (41), (42) and (43) we see that

(44)

$$|J_2|+|J_3|\leq C_{104}\left(T^{\frac{1}{k+1}}\left|\partial_1^2\left(\partial_{1,2}^{\alpha_1-2,\beta_1+2}f(x,y)\right)\right|+\frac{H_0}{H_1}\left(\frac{H_1}{\delta}\right)^{N-r_0-1}(N-r_0-1)!\right).$$

By (36), (37), (40), (44) we deduce that

$$\max_{\substack{(\alpha_1,\beta_1)\in\Gamma_{N+1}\setminus\Gamma_N\\ \alpha_1\geq1,\beta_1\geq1}}\ \max_{(x,y)\in V_\delta^T}\left|\ \partial_1^2\left(\partial_1^{\alpha_1}\partial_2^{\beta_1}f(x,y)\right)\right|\leq$$

$$\leq C_{105}T^{\frac{1}{k+1}}\max_{\substack{(\alpha_1,\beta_1)\in\Gamma_{N+1}\setminus\Gamma_N\\ \alpha_1\geq1,\beta_1\geq1}}\ \max_{(x,y)\in V_\delta^T}\left|\ \partial_1^2\left(\partial_1^{\alpha_1}\partial_2^{\beta_1}f(x,y)\right)\right|+$$

(45)

$$+C_{106}\left(T^{\frac{1}{k+1}}\left|f,V_{\delta''}^T\right|_{N+1}+H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}\Big(T^{\frac{1}{k+1}}+\frac{1}{H_1}\Big)\right).$$

Finally, in (45) choosing $T\leq\left(\frac{1}{2C_{106}}\right)^{k+1}$ yields

$$\max_{\substack{(\alpha_1,\beta_1)\in\Gamma_{N+1}\setminus\Gamma_N\\ \alpha_1\geq1,\beta_1\geq1}}\ \max_{(x,y)\in V_\delta^T}\left|\ \partial_1^2\left(\partial_1^{\alpha_1}\partial_2^{\beta_1}f(x,y)\right)\right|\leq$$

$$\leq C_{107}\left(T^{\frac{1}{k+1}}\left|f,V_{\delta''}^T\right|_{N+1}+H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}\Big(T^{\frac{1}{k+1}}+\frac{1}{H_1}\Big)\right).$$

This completes the proof of Lemma 5. □

(Continuing the proof of the Theorem 5) Put $|f,V_\delta^T|_{N+1}=g(\delta)$. Combining Lemmas 2-5 gives

$$g(\delta)\leq C_{108}\left(T^{\frac{1}{k+1}}g\Big(\delta\Big(1-\frac{4}{N}\Big)\Big)+H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1}L_{N-r_0-1}\Big(T^{\frac{1}{k+1}}+\frac{1}{H_1}\Big)\right).$$

Choosing $T \leq \left(1/10^{12}C_{108}\right)^{k+1}$ then by Lemma 1 we deduce that

$$g(\delta) \leq C_{109}H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}\Big(T^{\frac{1}{k+1}}+\frac{1}{H_1}\Big).$$

If $T$ is chosen to be small enough such that $T \leq \left(1/2C_{109}\right)^{k+1}$ and choosing $H_1 \geq 2C_{109}$ (in addition to $H_1 \geq C_4 H_0^{2k+3}$ ) we arrive at

$$g(\delta) \leq H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.$$

That means

$$|f, V_\delta^T|_{N+1} \leq H_0\left(\frac{H_1}{\delta}\right)^{N-r_0-1} L_{N-r_0-1}.$$

The proof of Theorem 5 is therefore completed. □

## §3. Case $k$ is even.

In this case we will prove a similar result as in §2 for $\lambda = 2N(k+1)$, where $N$ is an integer, by establishing the explicit fundamental solutions of $G_{k,2N(k+1)}$. Let us maintain the notations used for $p, A_+, A_-, M, F_{k,\lambda}, \ldots$ from the very beginning of the paper (now, of course, with an even $k$). If $(u,v) \neq (0,v)$ is fixed then the real parts of $A_+, A_-$ change sign when $(x,y)$ passes through $(-u,v)$. Therefore $M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}}$ may have singularities alone the half-line $(x,v)$ with $x \leq -u$ for an arbitrary complex number $\lambda$. But if $\lambda = 2N(k+1)$, then it is not difficult to see that $M = A_+^{-\frac{k+\lambda}{2k+2}} A_-^{-\frac{k-\lambda}{2k+2}}$ is smooth alone the half-line $(x,v)$ with $x < -u$, that is $M(\cdot,\cdot,u,v) \in C^\infty(\mathbb{R}^2 \backslash \{(u,v),(-u,v)\})$. Moreover when $k$ is even and $u \neq 0$ we have $-\infty \leq p \leq 1$. More precisely, $p \to 1$ when $(x,y) \to (u,v)$, and $p \to -\infty$ when $(x,y) \to (-u,v)$. If $N < 0$ and $p \to -\infty$ then we have the following asymptotic expansions (see [5], p. 63 )

$$F_1(p) = F\Big(\frac{k}{2k+2}-N, \frac{k}{2k+2}+N, \frac{k}{k+1}, p\Big) = \frac{\Gamma\Big(\frac{k}{k+1}\Big)}{\Gamma\Big(\frac{k}{2k+2}+N\Big)}\Bigg\{\frac{(-p)^{-\frac{k}{2k+2}-N}}{\Gamma\Big(\frac{k}{2k+2}+N\Big)}\times$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma\Big(\frac{k}{2k+2}+N+n\Big)\Gamma\Big(\frac{k+2}{2k+2}+N+n\Big)}{\Gamma\Big(\frac{k}{2k+2}-N\Big)\Gamma\Big(\frac{k+2}{2k+2}-N\Big)n!(n+2N)!}p^{-n}[\log(-p)+a_n]+$$
$$+(-p)^{-\frac{k}{2k+2}+N}\sum_{n=0}^{2N-1}\frac{\Gamma\Big(\frac{k}{2k+2}-N+n\Big)\Gamma(2N-n)}{\Gamma\Big(\frac{k}{2k+2}-N\Big)\Gamma\Big(\frac{k}{2k+2}+N-n\Big)n!}p^{-n}\Bigg\},$$

$$F_2(p) = F\Big(\frac{k+2}{2k+2}-N, \frac{k+2}{2k+2}+N, \frac{k+2}{k+1}, p\Big) = \frac{\Gamma\Big(\frac{k+2}{k+1}\Big)}{\Gamma\Big(\frac{k+2}{2k+2}+N\Big)}\Bigg\{\frac{(-p)^{-\frac{k+2}{2k+2}-N}}{\Gamma\Big(\frac{k+2}{2k+2}+N\Big)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma\Big(\frac{k+2}{2k+2}+N+n\Big)\Gamma\Big(\frac{k}{2k+2}+N+n\Big)}{\Gamma\Big(\frac{k+2}{2k+2}-N\Big)\Gamma\Big(\frac{k}{2k+2}-N\Big)n!(n+2N)!}p^{-n}[\log(-p)+b_n]+$$

$$+(-p)^{-\frac{k+2}{2k+2}+N}\sum_{n=0}^{2N-1}\frac{\Gamma\Big(\frac{k+2}{2k+2}-N+n\Big)\Gamma(2N-n)}{\Gamma\Big(\frac{k+2}{2k+2}-N\Big)\Gamma\Big(\frac{k+2}{2k+2}+N-n\Big)n!}p^{-n}\Bigg\},$$

where

$$a_n = \psi(1+2N+n)+\psi(1+n)-\psi\Big(\frac{k}{2k+2}+2N+n\Big)-\psi\Big(\frac{k}{2k+2}-2N-n\Big),$$

$$b_n = \psi(1+2N+n)+\psi(1+n)-\psi\Big(\frac{k+2}{2k+2}+2N+n\Big)-\psi\Big(\frac{k+2}{2k+2}-2N-n\Big),$$

and $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is the polygamma function. Therefore if we choose the expressions for constants $C_1, C_2$ as in the beginning of the paper (with $\lambda$ replaced by $2N(k+1)$), we will have $F_{k,2N(k+1)}(x,y,u,v)\in C^{\infty}(\mathbb{R}^2\backslash(u,v))$, with $F_{k,2N(k+1)}(-u,v,u,v)=0$. Similar conclusions hold for $F_{k,2N(k+1)}(x,y,u,v)$ when $N>0$. If $N=0$, then

$$F_1(p) = F\Big(\frac{k}{2k+2}, \frac{k}{2k+2}, \frac{k}{k+1}, p\Big) = \frac{\Gamma\Big(\frac{k}{k+1}\Big)(-p)^{-\frac{k}{2k+2}}}{\Gamma^2\Big(\frac{k}{2k+2}\Big)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma\Big(\frac{k}{2k+2}+n\Big)\Gamma\Big(\frac{k+2}{2k+2}+n\Big)}{\Gamma\Big(\frac{k+2}{2k+2}\Big)\Gamma\Big(\frac{k}{2k+2}\Big)(n!)^2}p^{-n}[\log(-p)+c_n],$$

$$F_2(p) = F\Big(\frac{k+2}{2k+2}, \frac{k+2}{2k+2}, \frac{k+2}{k+1}, p\Big) = \frac{\Gamma\Big(\frac{k+2}{k+1}\Big)(-p)^{-\frac{k+2}{2k+2}}}{\Gamma^2\Big(\frac{k+2}{2k+2}\Big)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma\Big(\frac{k+2}{2k+2}+n\Big)\Gamma\Big(\frac{k}{2k+2}+n\Big)}{\Gamma\Big(\frac{k+2}{2k+2}\Big)\Gamma\Big(\frac{k}{2k+2}\Big)(n!)^2}p^{-n}[\log(-p)+d_n],$$

where

$$c_n = 2\psi(1+n)-\psi\Big(\frac{k}{2k+2}+n\Big)-\psi\Big(\frac{k}{2k+2}-n\Big),$$

$$d_n = 2\psi(1+n)-\psi\Big(\frac{k+2}{2k+2}+n\Big)-\psi\Big(\frac{k+2}{2k+2}-n\Big).$$

Hence $F_{k,0}(x,y,u,v)\in C^{\infty}(\mathbb{R}^2\backslash(u,v))$, with $F_{k,0}(-u,v,u,v)=-\frac{\cot\frac{k\pi}{2k+2}}{4u^k}$.

**Theorem 6.** *Let $\Psi \in C\{L_{n-a-2};\Omega|L_{n-a-2};\mathbb{R}^3\}$ for every $a \in [0, r_0]$. Assume that $m \geq 2k^2+6k+5, \lambda = 2N(k+1)$, and $f$ is a $\mathbb{H}^m_{loc}(\Omega)$ solution of the equation (1). Then $f \in C\{L_{n-r_0-2};\Omega\}$. In particular, if $\Psi$ is $G^s-$function (or analytic) of its arguments then so is $f$.*

*Proof.* Almost all the arguments used for the case when $k$ is odd can be applied here. Therefore we only give the sketch of the proof. Instead of the distance $\rho$ in §2 we use the following metric

$$\rho_1\big((u,v),(x,y)\big) = \max\big\{|x^{k+1}-u^{k+1}|,(k+1)|y-v|\big\}.$$

To establish (19) we consider 3 cases:
If $0 \leq p \leq 1$ then we use (20) to deduce (19).
If $-1 \leq p < 0$ then we have the following set of estimates

$$\begin{aligned}
&xu<0,\ |4x^{k+1}u^{k+1}| \leq R,\ \max\{|x|,|u|\} \leq C_{110}R^{\frac{1}{2k+2}},\ |M| = R^{-\frac{k}{2k+2}},\\
&2^{-1}R_1 \leq R \leq R_1,\ \max\left\{\left|\frac{\partial p}{\partial x}\right|,\left|x^k\frac{\partial p}{\partial y}\right|\right\} \leq C_{111}R^{-\frac{1}{2k+2}},\\
&\max\{|F(p)|,|F'(p)|\} \leq C_{112},\ \max\left\{\left|\frac{\partial M}{\partial x}\right|,\left|x^k\frac{\partial M}{\partial y}\right|\right\} \leq C_{113}R^{-\frac{1}{2}}.
\end{aligned}$$

If $p \leq -1$ then by using asymptotic expansions of $F(p)$ we have

$$\begin{aligned}
&xu<0,\ R \leq |4x^{k+1}u^{k+1}|,\ 2^{-1}|u| \leq |x| \leq 2|u|,\ |M| = R^{-\frac{k}{2k+2}},\\
&C_{114}R_1^{\frac{1}{2k+2}} \leq \min\{|x|,|u|\} \leq \max\{|x|,|u|\} \leq C_{115}R_1^{\frac{1}{2k+2}},\\
&|F_{k,2N(k+1)}| \leq C_{116}x^{-k}, \max\left\{\left|\frac{\partial F_{k,2N(k+1)}}{\partial x}\right|,\left|x^k\frac{\partial F_{k,2N(k+1)}}{\partial y}\right|\right\} \leq C_{117}x^{-k-1}.
\end{aligned}$$

Next we note that the estimate (26) is still true. Indeed, if $0 \leq p \leq 1$ then the part of (27), which relates to the case $0 \leq xu$, can be used to deduce (26). If $-1 \leq p \leq 0$ then we use $\tilde{\sigma}^2 \leq R_1 \leq 2R \leq C_{118}\tilde{\sigma}^2$ together with the part of (27), which relates to the case $xu < 0$, to obtain (26). If $p \leq -1$ then we have

$$\begin{aligned}
&xu<0,\ C_{119}R_1^{\frac{1}{2k+2}} \leq \min\{|x|,|u|\} \leq \max\{|x|,|u|\} \leq C_{120}R_1^{\frac{1}{2k+2}},\\
&|F_{k,2N(k+1)}| \leq C_{121}u^{-k}, \max\left\{\left|\frac{\partial F_{k,2N(k+1)}}{\partial u}\right|,\left|u^k\frac{\partial F_{k,2N(k+1)}}{\partial v}\right|\right\} \leq C_{122}u^{-k-1},\\
&\max\left\{\left|\frac{\partial^2 F_{k,2N(k+1)}}{\partial x\partial u}\right|,\left|u^k\frac{\partial^2 F_{k,2N(k+1)}}{\partial x\partial v}\right|\right\} \leq C_{123}u^{-k-2},\ C_{124}\tilde{\sigma}^2 \leq R_1 \leq C_{125}\tilde{\sigma}^2,\\
&\max\left\{\left|x^k\frac{\partial^2 F_{k,2N(k+1)}}{\partial y\partial u}\right|,\left|x^k u^k\frac{\partial^2 F_{k,2N(k+1)}}{\partial y\partial v}\right|\right\} \leq C_{126}u^{-k-2}.
\end{aligned}$$

Finally the set of estimates (29) and therefore the estimate (30) remain unchanged since we have $0 \leq xu$ (or $0 \leq p \leq 1$.)$\square$

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