Averaging formula for Nielsen numbers

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Abstract

We will show that the averaging formula for Nielsen numbers holds for continuous maps on infra-nilmanifolds: Let M be an infra-nilmanifold with a holonomy group Φ and $f: M \to M$ be a continuous map. Then

$$N(f) = \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(A_* - f_*)|.$$

Here, A_*, f_* are natural linear maps.

Outline

1. Nielsen fixed point class
   - Lefschetz fixed point theorem
   - Fixed point class
   - Some history
2. Averaging formula
   - Flat manifolds
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3. Proof of Averaging formula
   - Induced homomorphisms
   - Comparison of fixed point classes
   - Averaging formula in general
   - Averaging formula on infra-nilmanifolds

Lefschetz Fixed Point Theorem

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map. The Lefschetz number $L(f)$ of f is defined by

$$L(f) = \sum_k (-1)^k \mathrm{trace}\{(f_*)_k : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})\}.$$

Then the Lefschetz number $L(f)$ is a homotopy invariant.

Theorem (Lefschetz Fixed Point Theorem)

If $L(f) \neq 0$, then every map homotopic to f has a fixed point x.

A fixed point of f is a point x such that $f(x) = x$.

Lefschetz Fixed Point Theorem-Example

Let $f : S^1 \to S^1$ be given by $f(z) = z^k$. Then

$$f_* : H_0(S^1) \to H_0(S^1), 1 \mapsto 1$$
$$f_* : H_1(S^1) \to H_1(S^1), 1 \mapsto k$$

Hence $L(f) = 1 - k$.
If $k = 1$ then f is the identity; rotate f a little, and then there is no fixed point.
If $k \neq 1$, then f has a fixed point. But HOW MANY fixed points does f (up to homotopy) have? For example, take $k = 3$, solve $f(z) = z^3 = z$ for z and get two fixed points ± 1.

Our interest is to find

$$\min\{|\mathrm{Fix}(g)| \mid g \simeq f\}.$$

Nielsen Theory-Fixed point class

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map. Let

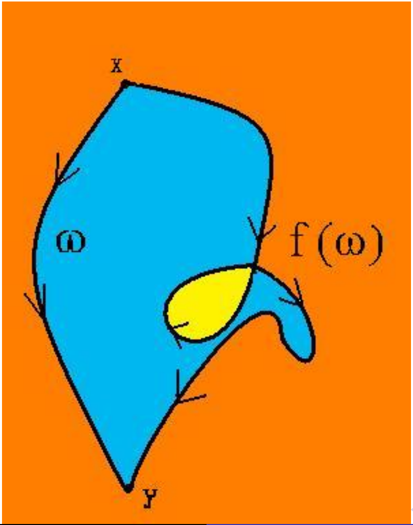
$$\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}.$$

Definition (for nonempty fixed point class)

Two fixed points $x, y \in \mathrm{Fix}(f)$ are in the same fixed point class $\Longleftrightarrow$ there is a path ω (called a Nielsen path) from x to y such that $\omega \simeq f(\omega)$ rel endpoints.

This is an equivalence relation on $\mathrm{Fix}(f)$. The equivalence classes are called fixed point classes.

Fixed point class



Fixed point class (Algebraic Description)

Let $p : \tilde{X} \to \tilde{X}$ be the universal covering of X, with group π of covering transformations.
Let $\tilde{f} : \tilde{X} \to \tilde{X}$ be a lifting of f (this always exists), i.e., have a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{f} & X \end{array}$$

If $\tilde{f}'$ is another lifting of f, then $\tilde{f}' = \alpha\tilde{f}$ for some $\alpha \in \pi$.
In what follows, we shall fix a lifting $\tilde{f}$ of f once and for all. The set of all liftings of f is $\{\alpha\tilde{f} \mid \alpha \in \pi\}$.

Fixed point class (Algebraic Description)

For any $\alpha \in \pi$, $\tilde{f}\alpha$ is a lifting of f and so we have

$$\alpha'\tilde{f} = \tilde{f}\alpha \text{ for some } \alpha' \in \pi.$$

This defines a homomorphism $\varphi : \pi \to \pi$ given by $\varphi(\alpha) = \alpha'$.
[This is nothing but $f_* : \pi_1(X) \to \pi_1(X)$.]

Define the Reidemeister action of π on π as follows:

$$\pi \times \pi \longrightarrow \pi, \ (\gamma, \alpha) \mapsto \gamma\alpha\varphi(\gamma)^{-1}.$$

This defines an equivalence relation whose equivalence classes are called the Reidemeister classes. The set of the Reidemeister classes determined by φ is denoted by $\mathcal{R}[\varphi] = \{[\alpha] \mid \alpha \in \pi\}$.

Fixed point class (Algebraic Description)

Theorem

1. *For $\alpha \in \pi$, if $p(\mathrm{Fix}(\alpha\tilde{f}))$ is non-empty, then it is a fixed point class, and vice versa.*
2. $p(\mathrm{Fix}(\alpha\tilde{f})) = p(\mathrm{Fix}(\alpha'\tilde{f}))$ *iff* $[\alpha] = [\alpha']$.

Hence the fixed point classes $p(\mathrm{Fix}(\alpha\tilde{f}))$ are labeled by the Reidemeister classes $[\alpha]$. That is,

$$\mathrm{Fix}(f) = \coprod_{[\alpha]\in\mathcal{R}[\varphi]} p(\mathrm{Fix}(\alpha\tilde{f})).$$

Index of a fixed point class

Definition

The index of a fixed point class $\mathbf{F}$ is defined by the fixed point index (winding number)

$$\operatorname{ind}(\mathbf{F}) := \operatorname{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \operatorname{ind}(f, x).$$

The summation is meant for $\mathbf{F}$ consisting of isolated fixed points. Empty fixed point classes have $\operatorname{ind} = 0$.

Nielsen number

Definition

Let $f : X \to X$ be a self-map.
A fixed point class $\boldsymbol{F}$ of f is called essential if $\operatorname{ind} \boldsymbol{F} \neq 0$.
The Nielsen number $N(f)$ of f is defined to be

$$N(f) := \text{the number of essential fixed point classes of } f$$

The Nielsen number $N(f)$ is also a homotopy invariant with the property that

$$\min\{|\mathrm{Fix}(g)| \mid g \simeq f\} \geq N(f).$$

The Nielsen number gives more precise information concerning the existence of fixed points than the Lefschetz number, but its computation when compared with that of the Lefschetz number is in general much more difficult.

Example again

Recall that the map $f : z \in S^1 \mapsto z^k \in S^1$ has $|k-1|$ fixed points, and $L(f) = 1-k$. Each fixed point has index(=winding number) $+1$ or -1 according as $k > 0$ or $k < 0$. Thus the Nielsen number is $N(f) = |k-1|$.

Notice $N(f) = |L(f)|$.

History

Several attempts to find some relations between these two numbers.

- [BBPT 75] For a continuous map $f : T^s \to T^s$ on the torus T^s,
$$|L(f)| = N(f) = |\det(I - f_*)|,$$
where $f_* : \mathbb{Z}^s \to \mathbb{Z}^s$ or its extension $f_* : \mathbb{R}^s \to \mathbb{R}^s$.

- [Anosov 85] Extended [BBPT 75] to nilmanifolds $\Gamma\backslash L$. Here $f_* : \Gamma \to \Gamma$ and its extension $f_* : L \to L$, and then the differential $f_*; \mathfrak{L} \to \mathfrak{L}$. Then
$$|L(f)| = N(f) = |\det(I - f_*)|.$$

History

- [KL 88; M 01] If M is an infra-nilmanifold, and f is homotopically periodic or more generally virtually unipotent, then $L(f) = N(f)$.
- [KM 95] Extended [Anosov] to solvmanifolds of type (NR).
- [DDM 04, 07, 07], [DDP, 06] Anosov relation holds (used averaging formula).

Flat manifolds

This classification was done by W. Killing and H. Hopf.

Let M be a Riemannian manifold of dimension n. Then M is complete, connected of constant curvature 0 if and only if it is isometric to $\pi\backslash\mathbb{R}^n$ with $\pi \subset \mathbb{R}^n \rtimes O(n)$ a torsion-free discrete cocompact subgroup.

The study of flat manifolds is to study the
<u>Euclidean space forms</u>

$$\pi \subset \mathbb{R}^n \rtimes O(n) \text{ a torsion-free discrete cocompact subgroup}$$

Bieberbach Theorems

The following (three) theorems have been proven by Bieberbach.

Theorem

Let $\pi \subset \mathbb{R}^n \rtimes O(n)$ be a discrete cocompact subgroup. Then $\Gamma = \pi \cap \mathbb{R}^n$ is a discrete cocompact subgroup of $\mathbb{R}^n$ and Γ has finite index in π.

Theorem

Let $\pi, \pi' \subset \mathbb{R}^n \rtimes O(n)$ be discrete cocompact subgroups. Then every isomorphism $\theta : \pi \to \pi'$ is conjugate by an element of $\mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$.

Due to the first Bieberbach Theorem, if $\pi \subset \mathbb{R}^n \rtimes O(n)$ be a torsion-free discrete cocompact subgroup, then we have

$$\mathbb{R}^n$$
$$\downarrow$$
$$\Gamma\backslash\mathbb{R}^n = \mathbb{Z}^n\backslash\mathbb{R}^n \qquad \text{a torus}$$
$$\downarrow \text{ a finite covering}$$
$$\pi\backslash\mathbb{R}^n \qquad \text{a flat manifold with holonomy group } \pi/\Gamma$$

Example-the Klein bottle

Let $\alpha = (a, A)$ and $t_i = (e_i, I_2)$ be elements of $\mathbb{R}^2 \rtimes \mathrm{Aut}(\mathbb{R}^2)$, where

$$a = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \; A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then A has period 2, $(a, A)^2 = (a + Aa, I_2) = (\begin{bmatrix} 0 \\ 1 \end{bmatrix}, I_2) = t_2$, and $t_1\alpha = \alpha t_1^{-1}$. Let

$$\Gamma = \langle e_1, e_2 \rangle = \mathbb{Z}^2 \subset \mathbb{R}^2 \rtimes \mathrm{Aut}(\mathbb{R}^2),$$
$$\pi = \langle \Gamma, (a, A) \rangle \subset \mathbb{R}^2 \rtimes \mathrm{Aut}(\mathbb{R}^2)$$

Then $\pi = \langle t_1, t_2, \alpha \mid [t_1, t_2] = 1, \alpha t_1 \alpha^{-1} = t_1^{-1}, \alpha^2 = t_2 \rangle$ is the Klein bottle group. Thus $\Gamma\backslash\mathbb{R}^2 \to \pi\backslash\mathbb{R}^2$.

Nielsen number on the torus

Let $f : \Gamma\backslash\mathbb{R}^2 \to \Gamma\backslash\mathbb{R}^2$. Then f has a lifting $\tilde{f}$ so that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\tilde{f}} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \Gamma\backslash\mathbb{R}^2 & \xrightarrow{f} & \Gamma\backslash\mathbb{R}^2 \end{array}$$

$\tilde{f}$ induces a homo $\varphi : \Gamma \to \Gamma$ given by the rule $\varphi(\gamma)\tilde{f} = \tilde{f}\gamma$ for all $\gamma \in \Gamma$. This homo φ extends uniquely to a homo $F : \mathbb{R}^2 \to \mathbb{R}^2$.

By [BBPT 75], $N(f) = |L(f)| = |\det(I - F)|$.

Nielsen number on the Klein bottle

See the handouts!

We will see how difficult the computation of the Nielsen numbers in general.

The purpose of this talk is to introduce an algebraic computation formula, which is a workable formula for the difficult number $N(f)$.

Introduction

Let G be a Lie group and let $\mathrm{Aut}(G)$ be the group of continuous automorphisms of G. The group $\mathrm{Aff}(G)$ is the semi-direct product $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$ with multiplication

$$(a, \alpha) \cdot (b, \beta) = (a \cdot \alpha(b), \alpha\beta).$$

It has a Lie group structure and acts on G by

$$(a, \alpha) \cdot x = a \cdot \alpha(x)$$

for all $x \in G$. A lattice of a Lie group G is a discrete cocompact subgroup of G.

For $G = \mathbb{R}^n$,
$\mathrm{Aut}(G) = \mathrm{GL}(n, \mathbb{R})$, $\mathrm{Aff}(G) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$,
$O(n)$ is a maximal compact subgroup of $\mathrm{Aut}(G)$

Generalization of Bieberbach theorems

All three Bieberbach theorems have been generalized to the situation where G is a simply connected, connected nilpotent Lie groups ([Auslander], [Lee-Raymond]).

Let G be a simply connected, connected nilpotent Lie groups and let C be a maximal compact subgroup of $\mathrm{Aut}(G)$. Then:

Theorem

Let $\pi \subset G \rtimes C$ be a lattice. Then $\Gamma = \pi \cap G$ is a lattice of G and Γ has finite index in π.

Theorem

Let $\pi, \pi' \subset G \rtimes C$ be lattices. Then every isomorphism $\theta : \pi \to \pi'$ is conjugate by an element of $\mathrm{Aff}(G)$.

Nilmanifolds, Infra-nilmanifolds

If $\pi \subset G \rtimes C \subset G \rtimes \mathrm{Aut}(G)$ is a torsion-free lattice, then

$$\begin{array}{ll} G & \\ \downarrow & \\ \Gamma\backslash G & \text{a nilmanifold} \\ \downarrow \text{ a finite covering} & \\ \pi\backslash G & \text{an infra-nilmanifold} \\ & \text{with holonomy group } \pi/\Gamma \end{array}$$

Almost flat manifolds

Let M be a compact Riemannian manifold. Due to Gromov, M is ϵ-flat if its sectional curvature K and diameter d satisfy $|K|d^2 \leq \epsilon$.

If M is as above and of dimension n, then there exists a constant $\epsilon = \epsilon(n) > 0$ such that $|K|d^2 < \epsilon$ implies M is diffeomorphic to an infra-nilmanifold $\pi\backslash G$.

Results

- [KLL 05] Averaging formula for Nielsen numbers of continuous maps on infra-nilmanifolds. Given

$$\begin{array}{ccc} \Lambda\backslash L & \stackrel{\bar f}{\longrightarrow} & \Lambda\backslash L \quad \text{a nilmanifold} \\ \downarrow & & \downarrow \\ \pi\backslash L & \stackrel{f}{\longrightarrow} & \pi\backslash L \text{ an infra-nilmanifold} \end{array}$$

we have

$$L(f) = \frac{1}{|\pi:\Lambda|} \sum_{\bar\alpha\in\pi/\Lambda} L(\bar\alpha\bar f), \text{ (by Jiang)}$$

$$N(f) = \frac{1}{|\pi:\Lambda|} \sum_{\bar\alpha\in\pi/\Lambda} N(\bar\alpha\bar f).$$

Results

- [LL 06] Computation formula for Nielsen numbers of continuous maps on infra-nilmanifolds in terms of the holonomy:

$$L(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} \frac{\det(A_* - f_*)}{\det(A_*)},$$

$$N(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} |\det(A_* - f_*)|.$$

Results

- [LL 09] Computation formula for Nielsen numbers of continuous maps on infra-solvmanifolds of type (R) in terms of the holonomy:

$$L(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} \frac{\det(A_* - f_*)}{\det(A_*)},$$

$$N(f) = \frac{1}{|\Psi|} \sum_{A \in \Psi} |\det(A_* - f_*)|.$$

Results

Natural generalization from fixed point theory to coincidence theory!

The fixed point theory concerns with the self-maps $f : M \to M$, and the coincidence theory concerns with the pair of maps $f, g : M \to N$. If $M = N$ and $g = \mathrm{id}$, then the coincidence theory is just the fixed point theory. However, the coincidence theory is in general much difficult than the fixed point theory.

- [KL 05] Anosov theorem for coincidences on nilmanifolds
- [KL 06] Universal factorization property of certain polycyclic groups
- [KL 07] Averaging formula for Nielsen coincidence numbers
- [HLP 09] Anosov theorem for coincidences on special solvmanifolds of type (R)

Beginning

Let X be a compact connected space and let π be the group of covering transformations for the universal covering projection $p:\tilde{X}\to X$. Let Λ be a finite index normal subgroup of π. Write $\bar{X}=\Lambda\backslash\tilde{X}$. Then $p':\tilde{X}\to\bar{X}$ and $\bar{p}:\bar{X}\to X$ be the covering projections.

Assume

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow p' & & \downarrow p' \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{X} \\ \downarrow \bar{p} & & \downarrow \bar{p} \\ X & \xrightarrow{f} & X \end{array}$$

Induced homomorphisms

The lifting $\tilde{f}$ of f and the lifting $\bar{f}$ of f induce homomorphisms

$$\varphi : \pi \longrightarrow \pi \text{ defined by } \varphi(\alpha)\tilde{f} = \tilde{f}\alpha,$$
$$\bar{\varphi} : \pi/\Lambda \longrightarrow \pi/\Lambda \text{ defined by } \bar{\varphi}(\bar{\alpha})\bar{f} = \bar{f}\bar{\alpha}$$

so that $\varphi' = \varphi|_\Lambda : \Lambda \to \Lambda$ is induced and the diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Lambda & \stackrel{i}{\longrightarrow} & \pi & \stackrel{q}{\longrightarrow} & \pi/\Lambda & \longrightarrow & 1 \\ & & \downarrow \varphi' & & \downarrow \varphi & & \downarrow \bar{\varphi} & & \\ 1 & \longrightarrow & \Lambda & \stackrel{i}{\longrightarrow} & \pi & \stackrel{q}{\longrightarrow} & \pi/\Lambda & \longrightarrow & 1 \end{array}$$

Induced homomorphisms

For any $\alpha \in \pi$, $\alpha\tilde{f}$ is a lifting of f and $\bar{\alpha}\bar{f}$ is a lifting of f, and their induced homomorphisms are $\tau_\alpha\varphi$ and $\tau_{\bar{\alpha}}\bar{\varphi}$.
[Here, $\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}$.]
So, the diagram is commutative

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Lambda & \xrightarrow{i_\alpha} & \pi & \xrightarrow{q_\alpha} & \pi/\Lambda & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \tau_\alpha\varphi'} & & \downarrow{\scriptstyle \tau_\alpha\varphi} & & \downarrow{\scriptstyle \tau_{\bar{\alpha}}\bar{\varphi}} & & \\
1 & \longrightarrow & \Lambda & \xrightarrow{i_\alpha} & \pi & \xrightarrow{q_\alpha} & \pi/\Lambda & \longrightarrow & 1
\end{array}$$

an exact sequence of groups:

$$1 \longrightarrow \mathrm{fix}(\tau_\alpha\varphi') \xrightarrow{i_\alpha} \mathrm{fix}(\tau_\alpha\varphi) \xrightarrow{q_\alpha} \mathrm{fix}(\tau_{\bar{\alpha}}\bar{\varphi})$$

an exact sequence of sets:

$$\mathcal{R}[\tau_\alpha\varphi'] \xrightarrow{\hat{i}_\alpha} \mathcal{R}[\tau_\alpha\varphi] \xrightarrow{\hat{q}_\alpha} \mathcal{R}[\tau_{\bar{\alpha}}\bar{\varphi}] \longrightarrow 1$$

Comparison of fixed point classes

Given

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\alpha\tilde{f}} & \tilde{X} \\ \downarrow p' & & \downarrow p' \\ \bar{X} & \xrightarrow{\bar{\alpha}\bar{f}} & \bar{X} \\ \downarrow \bar{p} & & \downarrow \bar{p} \\ X & \xrightarrow{f} & X \end{array}$$

Recall

$$\mathrm{Fix}(f) = \coprod_{[\alpha]\in\mathcal{R}[\varphi]} p(\mathrm{Fix}(\alpha\tilde{f}))$$

Applying to $\bar{\alpha}\bar{f}$, we get

$$\mathrm{Fix}(\bar{\alpha}\bar{f}) = \coprod p'(\mathrm{Fix}(\lambda(\alpha\tilde{f})))$$

Comparison of fixed point classes

Essential is to compare the fixed point classes $p'(\mathrm{Fix}(\lambda(\alpha\tilde{f})))$ of $\bar{\alpha}\bar{f}$ to those $p(\mathrm{Fix}(\alpha\tilde{f}))$ of f.
Recalling

$$\mathrm{Fix}(f) = \coprod_{[\alpha]\in\mathcal{R}[\varphi]} p(\mathrm{Fix}(\alpha\tilde{f}))$$

$$\mathrm{Fix}(\bar{\alpha}\bar{f}) = \coprod_{[\lambda]\in\mathcal{R}[\tau_\alpha\varphi']} p'(\mathrm{Fix}(\lambda(\alpha\tilde{f})))$$

we do this by re-labeling the fixed point classes of f and $\bar{\alpha}\bar{f}$.

General formula

Compariing fixed point classes yields:

Theorem

$$N(f) \geq \frac{1}{[\pi : \Lambda]} \sum_{\bar{\alpha} \in \pi/\Lambda} N(\bar{\alpha}\bar{f})$$

and equality holds if and only if $\forall \lambda \in \Lambda$ *and* $\forall \alpha \in \pi$ *with* $p(\mathrm{Fix}(\lambda(\alpha\tilde{f})))$ *essential,* $q_{\lambda\alpha}(\mathrm{fix}(\tau_{\lambda\alpha}\varphi))$ *is the trivial group.*

Recall an exact sequence of groups:

$$1 \longrightarrow \mathrm{fix}(\tau_\alpha\varphi') \xrightarrow{i_\alpha} \mathrm{fix}(\tau_\alpha\varphi) \xrightarrow{q_\alpha} \mathrm{fix}(\tau_{\bar{\alpha}}\bar{\varphi})$$

General formula-Example

If π is finite with $\Lambda = 1$ (so $\tilde{X} = \bar{X}$), then $N(f) \geq \frac{1}{|\pi|} \sum_{\alpha \in \pi} N(\alpha\tilde{f})$.
For instance, consider

$$\begin{array}{ccc} S^2 & \xrightarrow{\tilde{f}=\mathrm{id}} & S^2 \\ \downarrow & & \downarrow \\ \mathbb{R}P^2 & \xrightarrow{f=\mathrm{id}} & \mathbb{R}P^2 \end{array}$$

$$\pi = \{1, \alpha\} = \mathbb{Z}_2$$

$$\alpha = \text{antipodal map}$$

Observe

1. the induced homo $\varphi : \pi \to \pi$ is the identity homo; $\mathcal{R}[\varphi] = \{[1], [\alpha]\}$; $\mathrm{Fix}(f) = p(\mathrm{Fix}(\tilde{f})) \coprod p(\mathrm{Fix}(\alpha\tilde{f}))$.
2. $\alpha\tilde{f} = \alpha$ is fixed point free; $p(\mathrm{Fix}(\alpha\tilde{f}))$ is inessential and $N(\alpha\tilde{f}) = 0$.
3. $L(\tilde{f}) = L(\mathrm{id}) = \chi(S^2) = 2$; $L(\tilde{f}) = |\mathrm{fix}(\varphi)| \cdot \mathrm{ind}(f, p(\mathrm{Fix}(\tilde{f})))$; $p(\mathrm{Fix}(\tilde{f}))$ is essential $\Rightarrow N(f) = 1$; $\mathrm{Fix}(\tilde{f})$ is essential and $N(\tilde{f}) = 1$

Step I: Existence of Commutative diagram

Given $f : \pi\backslash G \to \pi\backslash G$ on the infra-nilmanifold $\pi\backslash G$, one can a finite index, **fully invariant** subgroup Λ of π so that $\Lambda \subset G$. This yields a nilmanifold $\Lambda\backslash G$ and a finite regular covering $\Lambda\backslash G \to \pi\backslash G$. Further,

$$\begin{array}{ccc} G & \xrightarrow{\alpha\tilde{f}} & G \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p'} \\ \Lambda\backslash G & \xrightarrow{\bar{\alpha}\bar{f}} & \Lambda\backslash G \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle \bar{p}} \\ \pi\backslash G & \xrightarrow{f} & \pi\backslash G \end{array}$$

Step II: Homotopy Lifting

The lifting $\tilde{f}$ of f induces homomorphisms $\varphi : \pi \to \pi$ and $\varphi' : \Lambda \to \Lambda$ and then $\bar{\varphi} : \pi/\Lambda \to \pi/\Lambda$.
By the Bieberbach-Lee theorem, there exists an affine map (d, D) on G so that

$$\varphi(\alpha) \cdot (d, D) = (d, D) \cdot \alpha \quad (\text{recall } \varphi(\alpha) \cdot \tilde{f} = \tilde{f} \cdot \alpha)$$

This implies that f “has” an affine lifting (d, D).
On the other hand, we have

$$\varphi'(\lambda) = d \cdot D(\lambda) \cdot d^{-1} = \tau_d \circ D(\lambda).$$

This induces that $\bar{f}$ “has” an endomorphism lifting $\tau_d \circ D$.

Step III: Ad(d)

Summing up,

$$\begin{array}{ccc} G & \xrightarrow{\lambda\alpha(d,D)} & G \\ \downarrow p & & \downarrow p \\ \pi\backslash G & \xrightarrow{f} & \pi\backslash G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\tau_d \circ D} & G \\ \downarrow p' & & \downarrow p' \\ \Lambda\backslash G & \xrightarrow{\bar f} & \Lambda\backslash G \end{array}$$

Then

$$\begin{aligned} N(\bar f) &= |\det(I - (\tau_g \circ D)_*)| \quad \text{by Anosov} \\ &= |\det(I - \mathrm{Ad}(d)D_*)| \\ &= |\det(I - D_*)| \quad \text{by some effort} \end{aligned}$$

Hence $N(\bar f) \neq 0$ iff $\mathrm{fix}(D_*) = \{0\}$ iff $\mathrm{Fix}(D) = \{e\}$.

Step IV: Equality condition

We show that if $p(\mathrm{Fix}(\lambda\alpha\tilde{f}))$ is essential for some $\lambda \in \Lambda$ and $\alpha \in \pi \subset \mathrm{Aff}(G)$, then $q_{\lambda\alpha}(\mathrm{fix}(\tau_{\lambda\alpha}\varphi))$ is the trivial group. This implies that

$$N(f) = \frac{1}{[\pi : \Lambda]} \sum_{\bar{\alpha}\in\pi/\Lambda} N(\bar{\alpha}\bar{f})$$

Let $\lambda \in \Lambda$ and $\alpha = (a, A) \in \pi \subset \mathrm{Aff}(G)$. Then $\lambda\alpha\tilde{f} = (\lambda, I)(a, A)(d, D) = (\lambda \cdot a \cdot A(d), AD)$ and hence $N(\bar{\alpha}\bar{f}) = |\det(I - A_* D_*)|$. By the previous slide, $N(\bar{\alpha}\bar{f}) \neq 0$ iff $\mathrm{Fix}(\lambda\alpha\tilde{f})$ has only one point for all $\lambda \in \Lambda$. In this case, the group $\mathrm{fix}(\tau_{\lambda\alpha}\varphi)$ is already trivial. On the other hand if $N(\bar{\alpha}\bar{f}) = 0$, then $p(\mathrm{Fix}(\lambda\alpha\tilde{f}))$ is inessential.

Step V: Final formula

Given an infra-nilmanifold $\pi\backslash G$, we have $\Lambda \subset \Gamma = \pi \cap G$ and $\Phi = \pi/\Gamma \subset \mathrm{Aut}(G)$. Thus naturally $\bar{\alpha} \in \pi/\Lambda \mapsto A \in \Phi$ where $\alpha = (a, A)$.

Consequently,

$$
\begin{aligned}
N(f) &= \frac{1}{[\pi : \Lambda]} \sum_{\bar{\alpha} \in \pi/\Lambda} N(\bar{\alpha}\bar{f}) \\
&= \frac{1}{[\pi : \Lambda]} \sum_{\bar{\alpha} \in \pi/\Lambda} |\det(I - A_* D_*)| \\
&= \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(A_* - D_*)|
\end{aligned}
$$