

# On Motivic Transcendence Theory in Positive Characteristics

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# Algebraic relations

We are interested in understanding transcendental invariants which arise naturally in mathematics. Understanding means that we want to determine all the algebraic relations among these very special values.

Let  $A$  be an abelian variety over  $\overline{\mathbb{Q}}$  of dimension  $d$ , and let  $P$  be the period matrix of  $A$ . Grothendieck in 1960's made the **conjecture** :

$$\mathrm{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(P) = \dim \mathrm{MT}(A),$$

where  $\mathrm{MT}(A)$  is the Mumford-Tate group of  $A$  and is an algebraic subgroup of  $\mathrm{GL}_{2d} \times \mathbb{G}_m$ . This Mumford-Tate group is the motivic Galois group of the motive  $h_1(A) \oplus \mathbb{Q}(1)$ .

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# Euler relations

As example, consider the following values from arithmetic of  $\mathbb{Q}$  :

$$S = \{2\pi i, \zeta(2), \zeta(3), \dots, \zeta(m) \dots \},$$

$$\zeta(m) := \sum_{n=1}^{\infty} n^{-m}.$$

The value of Riemann zeta function at positive integer  $m > 1$ .

For  $m$  *even*, one knows from Euler the relations:

$$\zeta(m) = \frac{-(2\pi\sqrt{-1})^m B_m}{2m!}.$$

where  $B_m$  are the Bernoulli numbers:

$$\frac{Z}{e^Z - 1} = \sum_{m=0}^{\infty} B_m \frac{Z^m}{m!}.$$

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## Mixed Tate motives

One **conjectures** that these relations generate all the algebraic relations among numbers from  $S$  over the field of algebraic numbers  $\overline{\mathbb{Q}}$ . In particular, all the zeta values  $\zeta(m)$  for odd integer  $m > 1$  should be transcendental, and algebraically independent from each other, as well as algebraically independent from  $\pi$ .

This follows from the period conjecture for Mixed Tate motives.

Analogues of Grothendieck's period conjecture can be proved in the world of positive characteristic, after two decades works by Anderson, Brownawell, Papanikolas, Thakur, and Yu. This enables us to answer many questions about algebraic relations, in particular about algebraic independances.

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## Carlitz zeta values

In the world of positive characteristic  $p$ , let  $\theta$  be a variable. Let  $\mathbb{F}_q$  be finite field. The problem of Carlitz zeta values is to consider, for integer  $m \geq 1$ ,

$$\zeta_C(m) = \sum_{a \in \mathbb{F}_q[\theta]_+} \frac{1}{a^m} \in \mathbb{F}_q\left(\left(\frac{1}{\theta}\right)\right),$$

where the summation are taken over all monic polynomials  $a$ .

One has the obvious Frobenius relations among these special zeta values in characteristic  $p$  :

$$\zeta_C(m)^p = \zeta_C(mp).$$

If  $m$  is *even*, i.e.  $m \equiv 0 \pmod{q-1}$  because the ring  $\mathbb{F}_q[\theta]$  has  $q-1$  signs, one also has the Euler-Carlitz relation,

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# Euler-Carlitz relations

$$\zeta_C(m) = \frac{\tilde{\pi}^m \tilde{B}_m}{\Gamma_{m+1}},$$

where  $\tilde{\pi}$  is a *fundamental period* of the Carlitz exponential for  $\mathbb{F}_q$  :

$$\tilde{\pi} = \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1},$$

which is transcendental over  $\mathbb{F}_q(\theta)$ . The  $\Gamma_m$  are Carlitz factorials :

- setting  $D_0 = 1$ , and  $D_i = (\theta^{q^i} - \theta^{q^{i-1}}) \dots (\theta^{q^i} - \theta)$ , for  $i \geq 1$ ,
- writing down the  $q$ -adic expansion  $\sum_{i=0}^{\infty} n_i q^i$  of  $n$ , and let

$$\Gamma_{n+1} = \prod_{i=0}^{\infty} D_i^{n_i}.$$

# Bernoulli-Carlitz

The  $\widetilde{B}_m \in \mathbb{F}_q(\theta)$  are the Bernoulli-Carlitz “numbers” given by

$$\frac{z}{\exp_C(z)} = \sum_{m=0}^{\infty} \widetilde{B}_m \frac{z^m}{\Gamma_{m+1}}.$$

Here Carlitz exponential is the series

$$\exp_C(z) = \sum_{h=0}^{\infty} \frac{z^{q^h}}{D_h} = z \prod_{a \neq 0 \in \mathbb{F}_q[\theta]} \left(1 - \frac{z}{a\tilde{\pi}}\right).$$

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# Theorems

For arbitrary  $q$ , interested in following set of special values :

$$S_q = \{\tilde{\pi}, \zeta_C(1), \zeta_C(2), \dots, \zeta_C(m), \dots\}$$

Yu 1991 proves that all these are transcendental over the field  $\bar{k} = \overline{\mathbb{F}_q(\theta)}$ . Chang-Yu 2005 proves that the Euler-Carlitz relations and the Frobenius relations generate all the algebraic relations among these “numbers” over the field  $\bar{k}$ .

In particular the transcendence degree of the set

$$S_q^{(n)} = \{\tilde{\pi}, \zeta_C(1), \dots, \zeta_C(n)\}$$

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$$n - \lfloor n/p \rfloor - \lfloor n/(q-1) \rfloor + \lfloor n/(p(q-1)) \rfloor + 1.$$

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## $t$ -motives

By an Anderson  $t$ -motive we mean a left  $\bar{k}[t, \sigma]$ -module  $M$  which is free and finitely generated both as left  $\bar{k}[t]$ -module and left  $\bar{k}[\sigma]$ -module, and satisfying that for  $N$  sufficiently large

$$(t - \theta)^N M \subset \sigma M.$$

Here  $\bar{k}[t, \sigma]$  is the polynomial ring s.t. for all  $c \in \bar{k}$  :  
 $ct = tc$ ,  $\sigma t = t\sigma$ ,  $\sigma c = c^{1/q}\sigma$ .

- Given  $t$ -motive  $M$ . Let  $\mathbf{m} \in \text{Mat}_{r \times 1}(M)$  be a  $\bar{k}[t]$ -basis of  $M$ . Multiplication by  $\sigma$  on  $M$  is then described by  $\sigma(\mathbf{m}) = \Phi \cdot \mathbf{m}$ , for some matrix  $\Phi \in \text{Mat}_r(\bar{k}[t]) \cap \text{GL}_r(\bar{k}(t))$ .
- Tensoring with  $\bar{k}(t)$  we arrive at abelian category of pre- $t$ -motives which consists of  $\bar{k}(t)[\sigma, \sigma^{-1}]$ -modules finite dimensional over  $\bar{k}(t)$ .



## Periods of $t$ -motives

Let  $\mathbb{K}$  be the completion of a fixed algebraic closure of  $\mathbb{F}_q((1/\theta))$ .  
 Let  $\mathbb{T} = \mathbb{K}\{t\} \subset \mathbb{K}[[t]]$  be the Tate algebra of power series  
 converging on the “unit” disc. Let  $\mathbb{L}$  be the field of fractions of  $\mathbb{K}$ .  
 Let  $\sigma$  be the automorphism on  $\mathbb{L}$  defined by

$$\sigma(f) = \sum_i a_i^{1/q} t^i, \quad \text{if } f = \sum_i a_i t^i \in \mathbb{T}.$$

Then  $\mathbb{L}^\sigma = \bar{k}(t)^\sigma = \mathbb{F}_q(t)$  is the fixed subfield.

The motive described by  $\Phi$  is said to be rigid analytically trivial if  
 there exists  $\Psi = (\Psi_{ij}) \in \text{GL}_r(\mathbb{L})$  s.t.

$$\sigma(\Psi) = \Phi \cdot \Psi.$$

If all entries of  $\Psi$  are analytic at  $t = \theta$ , then  $\Psi(\theta)^{-1}$  is called a  
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## Method

To verify the transcendence degree of a set  $S$  following Papanikolas 2007. This amounts to finding a motive, in other words, a system of  $\sigma$ -semilinear equations :

$$\sigma(\Psi) = \Phi \cdot \Psi,$$

where  $\Phi = \Phi(S) \in \text{Mat}_r(\bar{k}[t])$  is certain matrix defining the motive and  $\Psi = (\Psi_{ij}) \in \text{GL}_r(\mathbb{L})$  gives a “fundamental” solution.

We want entries  $\Psi_{ij}$  of the solution matrix be analytic at  $t = \theta$  and satisfying

$$\text{trdeg}_{\bar{k}} \bar{k}(\Psi_{ij}(\theta)) = \text{trdeg}_{\bar{k}} \bar{k}(S).$$

## System of $\sigma$ -semilinear equations

By developing a theory analogous to the differential Galois theory of Picard-Vessiot, one can attach to this system an algebraic group  $\Gamma_\Psi$  over  $\mathbb{F}_q(t)$  as Galois group. Theorem basing on previous works of Anderson-Brownawell-Papanikolas and Yu then asserts

$$\dim \Gamma_\Psi = \text{trdeg}_{\bar{k}}(\Psi_{ij}(\theta)).$$

The counting of transcendence degree is therefore reduced to the computation of the dimension of an algebraic Galois group, provided one can successfully construct the motive and the analytic solution of  $\sigma$ -semilinear system in question.

# Tannakian Duality

Any Anderson  $t$ -motive  $M$  defines a Tannakian category of pre- $t$ -motives which is equivalent to the category of finite dimensional representations of a linear algebraic group  $\Gamma_M$  over  $\mathbb{F}_q(t)$ . This motivic Galois group  $\Gamma_M$  turns out to be isomorphic to the group  $\Gamma_\Psi$  of the  $\sigma$ -semilinear system in question.

**Example.** The Carlitz motive  $C$ . Let  $C = \bar{k}[t]$  with  $\sigma$ -action:  $\sigma f = (t - \theta)f^{(-1)}$ ,  $f \in C$ . Here  $\Phi = (t - \theta)$  and  $r = 1$ ,  $f^{(-1)}$  is obtained from  $f$  by raising all coefficients to  $q^{-1}$ -th power. Analytic solution  $\Psi$  of the system is

$$\Psi_C(t) = (-\theta)^{-q/(q-1)} \prod_{i=1}^{\infty} (1 - t/\theta^{q^i}).$$

Note  $\Psi_C(\theta) = \frac{-1}{\tilde{\pi}}$ ,  $\tilde{\pi}$  is Carlitz period. Galois group is  $\Gamma_C = \mathbb{G}_m$ .

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# Logarithms

**Example.**  $t$ -motives for Carlitz polylogarithms. Given  $\alpha_1, \dots, \alpha_m \in \bar{k}^\times$  satisfying  $|\alpha_i|_\infty < |\theta|_\infty^{sq/q-1}$ , where  $s$  is positive integer. Set

$$\Phi_s(\alpha_1, \dots, \alpha_m) = \begin{pmatrix} (t - \theta)^s & 0 & \cdots & 0 \\ \sigma(\alpha_1)(t - \theta)^s & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(\alpha_m)(t - \theta)^s & 0 & \cdots & 1 \end{pmatrix}$$

Analytic solution to the system is given by:

$$\Psi_s(\alpha_1, \dots, \alpha_m) = \begin{pmatrix} (\Psi_C)^s & 0 & \cdots & 0 \\ L_{\alpha_1, s}(\Psi_C)^s & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{\alpha_m, s}(\Psi_C)^s & 0 & \cdots & 1 \end{pmatrix},$$

# Polylogarithms

where

$$L_{\alpha_i, s}(t) = \alpha_i + \sum_{j=1}^{\infty} \frac{\alpha_i^{q^j}}{(t - \theta q)^s \cdots (t - \theta q^j)^s}.$$

One has

$$\sigma(L_{\alpha, s}) = \sigma(\alpha) + \frac{L_{\alpha, s}}{(t - \theta)^s}.$$

Substituting  $\theta$  for  $t$ ,  $L_{\alpha, s}(\theta)$  is the  $s$ -polylogarithm of  $\alpha$ .

Suppose  $L_{\alpha_1, s}(\theta), \dots, L_{\alpha_m, s}(\theta)$  and  $\tilde{\pi}^s$  are  $k$ -linearly independent. Then we compute the Galois group  $\Gamma_{\Phi_s}$  as an extension of  $\mathbb{G}_a^m$  by  $\mathbb{G}_m$  and  $\dim \Gamma_{\Phi_s} = m + 1$ . It follows that  $L_{\alpha_1, s}(\theta), \dots, L_{\alpha_m, s}(\theta)$  and  $\tilde{\pi}$  are algebraically independent over  $\bar{k}$ .

This leads to *algebraic independence* of  $\zeta_G(s)$  and  $\tilde{\pi}$  when  $q - 1 \nmid s$ .



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This leads to *algebraic independence* of  $\zeta_C(s)$  and  $\tilde{\pi}$  when  $q - 1 \nmid s$ .

# Zeta values and polylogarithms

Connecting tensor powers of the Carlitz motive  $C^{\otimes n}$  with the value  $\zeta_C(n)$ , Anderson-Thakur 1990 establishes :

$$\Gamma_n \zeta_C(n) = \sum_{i=0}^{\ell} h_{n,i} L_{\theta^i, n}(\theta),$$

where  $h_{n,i} \in \mathbb{F}_q[\theta] \forall i$ , and  $\ell < nq/(q-1)$ . These  $h_{n,i}$  comes from generating function identity:

$$\left(1 - \sum_{j=0}^{\infty} \frac{\prod_{i=1}^j (\theta^{q^j} - y^{q^i})}{D_j} x^{q^j}\right)^{-1} := \sum_{n=0}^{\infty} \frac{H_n(y)}{\Gamma_{n+1}} x^n,$$

$$H_{n-1}(y) := \sum_i h_{n,i} y^i.$$

## Gamma Function for Function Field

Let  $\mathbb{K}$  be the completion of a fixed algebraic closure of  $\mathbb{F}_q((1/\theta))$ .

Let  $\mathbf{A}_+$  be the set of all monics in  $\mathbf{A} = \mathbb{F}_q[\theta]$ .

Thakur's gamma function 1991, for  $s \in \mathbb{K} - \{-\mathbf{A}_+\}$  :

$$\Gamma(s) = \frac{1}{s} \prod_{a \in \mathbf{A}_+} \left(1 + \frac{s}{a}\right)^{-1}.$$

*Translation* formula for this gamma, for  $a \neq 0 \in \mathbf{A}$ :

$$\frac{\Gamma(s)}{\Gamma(s+a)} = R_a(s) \in \mathbb{F}_q(\theta)(s).$$

*Reflection* formula :

$$\prod_{\xi \in \mathbb{F}_q^\times} \Gamma(\xi s) = \frac{\tilde{\pi}}{s^{q-2} \exp_C(\tilde{\pi} s)}.$$

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# Gamma values in positive characteristic

Also Multiplication formula, for  $g \in \mathbf{A}_+$ ,  $d = \deg g$  :

$$\prod_{a \in \mathbf{A} \atop \deg a < d} \left(\frac{s+a}{g}\right) \Gamma\left(\frac{s+a}{g}\right) = s \Gamma(s) \tilde{\pi}^{\frac{q^d-1}{q-1}} ((-1)^d g)^{\frac{q^d}{1-q}} \prod_{\deg a < d} (s+a).$$

Consider the following special values for fixed  $q$  :

$$\{\tilde{\pi}\} \cup \Gamma(\mathbb{F}_q(\theta) - \{-\mathbf{A}_+\}).$$

When  $q = 2$ , all these values are algebraic multiples of  $\tilde{\pi}$ .

For arbitrary  $q$ , these special values are transcendental over  $\mathbb{F}_q(\theta)$ ,

Sinha 1997, by Yu's method.

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# Theorems

2004 Anderson-Brownawell-Papanikolas. All the algebraic relations among these special gamma values taken at rationals come from the translation relations, the reflection relation, and the multiplication relations (Analogue of Rohrlich-Lang conjecture).

2007 C.-Y. Chang, M. Papanikolas, and Yu. No further algebraic relations among the Carlitz zeta values at positive integers and Thakur gamma values at rationals taken together. Indeed

$$\begin{aligned} & \text{trdeg}_{\bar{k}} \bar{k}(\{\tilde{\pi}, \zeta_C(1), \zeta_C(2), \dots, \zeta_C(n)\} \cup \{\Gamma(a/f) \mid a \in \mathbf{A}, \deg a < \deg f\}) \\ &= n - \lfloor n/p \rfloor - \lfloor n/(q-1) \rfloor + \lfloor n/(p(q-1)) \rfloor + 1 + \frac{q-2}{q-1} \#(\mathbf{A}/f\mathbf{A})^\times. \end{aligned}$$

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## Special Gamma values

For  $f \in \mathbf{A}_+$  of positive degree.

Let  $\mathcal{A}_f$  be the free abelian group generated by  $f^{-1}\mathbf{A}/\mathbf{A}$ .

The group  $(\mathbf{A}/f\mathbf{A})^\times$  acts on  $\mathcal{A}_f$  by

$$a \star [x] = [ax], \quad \text{for } [x] \in f^{-1}\mathbf{A}/\mathbf{A}$$

Define  $\Pi(s) := s\Gamma(s)$  and consider  $\Pi$ -monomials of level  $f$ , i.e. images of the homomorphism

$$\mathcal{A}_f \longrightarrow \mathbb{K}^\times, \quad \mathbf{a} \mapsto \Pi(\mathbf{a}),$$

where  $\Pi([x]) = \Pi(x)$  if  $x \in f^{-1}\mathbf{A}/\mathbf{A}$  represented by  $|x|_\infty < 1$ .

Given non-trivial effective  $\mathbf{a} \in \mathcal{A}_f$ . Set

$$S_{\mathbf{a}} = \Pi((\mathbf{A}/f\mathbf{A})^\times \star \mathbf{a}).$$

## Motives for the Gamma values

- For each  $\mathbf{a} \in \mathcal{A}_f$ , Anderson-Brownawell-Papanikolas constructs  $\sigma$ -semilinear system  $(\Phi_{\mathbf{a}}, \Psi_{\mathbf{a}})$  and  $t$ -motive  $M_{\mathbf{a}}$  of rank  $\ell = \#(\mathbf{A}/f\mathbf{A})^\times$ , with the set  $S_{\mathbf{a}}$  and entries of inverse period matrix  $\Psi(\theta)$  span the same  $\bar{k}$ -linear space.
- The construction of motive  $M_{\mathbf{a}}$  starts from the cyclotomic function field  $L_f$  of level  $f$ . This extension has degree  $\ell$  over  $\mathbb{F}_q(t)$ . It follows that  $M_{\mathbf{a}}$  comes with “complex multiplications” by  $L_f$  which forces the motivic Galois group to be a torus.

Extending scalars from  $\mathbb{F}_q$  to  $\bar{k}$ .

Let  $\mathbf{X} = \mathbf{X}_f$  be nonsingular projective model of  $L_f$  containing affine curve  $\mathbf{U}_f$ , and let  $\mathbf{L} = H^0(\mathbf{U}_f, \mathcal{O}_{\mathbf{X}})$ .

# Geometric cyclotomy

Given non-trivial effective  $\mathbf{a} \in \mathcal{A}_f$ . The key point is to choose effective divisor  $\mathcal{D}_{\mathbf{a}}$  on  $\mathbf{U}_{\mathbf{a}}$ . Then put

$$M_{\mathbf{a}} := H^0(\mathbf{U}_f, \mathcal{O}_{\mathbf{X}}(\mathcal{D}_{\mathbf{a}})).$$

Making  $M_{\mathbf{a}}$  to be  $\bar{k}[\sigma, t]$ -module by setting

$$\sigma \cdot h = g_{\mathbf{a}} h^{(-1)},$$

where  $h \mapsto h^{(N)}$  is the  $N$ -th Frobenius power acting on  $L_f$ , and  $g_{\mathbf{a}}$  is called the Coleman function which has to do with the  $f$ -division values of the Carlitz exponential.

The divisor  $\mathcal{D}_{\mathbf{a}}$  is chosen to match the divisor of  $g_{\mathbf{a}}$ .

## Coleman functions

The motives  $M_{\mathbf{a}}$  is connected to the special values in  $S_{\mathbf{a}}$  because of the following episode happens.

For  $a \in \mathbf{A}$  prime to  $f$ , let  $\xi_a$  be the special point  $(\theta, \exp_C(\tilde{\pi}a/f))$  on the affine curve  $\mathbf{U}_f$ . The Coleman function  $g_{\mathbf{a}}$  taking at these special points is related to special Gamma values :

$$\Pi(a \star \mathbf{a})^{-1} = \prod_{N=1}^{\infty} g_{\mathbf{a}}^{(N)}(\xi_a).$$

Concerning our special zeta values, the motivic Galois groups are always extensions of unipotent groups by  $\mathbb{G}_m$ , whereas concerning the above special Gamma values, the Galois groups in question are always tori.

## Gamma values versus zeta values

Given  $f \in \mathbf{A}_+$  of positive degree. Let  $\mathbf{a} = [1/f] \in \mathcal{A}_f$ . Also given positive integer  $s$ , with  $q - 1 \nmid s$ . Want to prove the algebraic independence of  $\Gamma(1/f)$  with  $\zeta_C(s)$ .

Consider the direct sum motive  $\Phi = \Phi_s \oplus \Phi_{\mathbf{a}}$ , with analytic trivialization  $\Psi$ . Also let  $\Phi' = C^{\otimes s} \oplus \Phi_{\mathbf{a}}$ . The motivic Galois group of  $C^{\otimes s}$  is always  $\mathbb{G}_m$  whose period leads to  $\tilde{\pi}^s$ . The Galois groups of these motives are described by the following diagram :

$$\begin{array}{ccccccc} 1 & \rightarrow & V & \rightarrow & \Gamma_{\Psi} & \rightarrow & \Gamma_{\Psi'} & \rightarrow & 1 \\ & & & & \downarrow & & & & \\ 1 & \rightarrow & V_s & \rightarrow & \Gamma_{\Psi_s} & \rightarrow & \mathbb{G}_m & \rightarrow & 1 \end{array}$$

where  $V = V_s$  is unipotent (vector) group, and  $\Gamma_{\Psi'}$  is torus. A transcendence basis of  $\bar{k}(\Psi(\theta))$  contains both  $\Gamma(1/f)$  and  $\zeta_C(s)$ .

The End. Thank You.