On spherical dual width

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Subsets in Q-polynomial scheme and sphere

Q-polynomial scheme

- ▷ Design (Delsarte,1973)
- ▷ Dual width (Brouwer-Godsil-Koolen-Martin,2003)

Sphere

- ▷ Spherical design (Delsarte-Goethals-Seidel,1977)
- ▷ Spherical dual width (in this talk)

Symmetric Association Schemes

Definition (Symmetric Association Schemes)

Let X be a finite set and $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ be a set of non-empty subsets of $X \times X$.
Let A_i be the adjacency matrix of the graph (X, R_i).
$(X, \mathcal{R})$ is a symmetric association scheme if

(1) A_0 is the identity matrix;

(2) $\sum_{i=0}^{d} A_i = J$, where J is the all ones matrix;

(3) $A_i^T = A_i$ for $1 \leq i \leq d$;

(4) $A_i A_j$ is a linear combination of $A_0, A_1, \ldots, A_d$ for $0 \leq i, j \leq d$.

The vector space $\mathcal{A}$ spanned by the A_i is an algebra.
$\mathcal{A}$ is called the Bose-Mesner algebra of $(X, \mathcal{R})$.

Parameters of Bose-Mesner algebra

Since $\mathscr{A}$ is commutative and is closed under borh entrywise and ordinary multiplication, there exist two canonical bases of $\mathscr{A}$:

$$\{A_0 = I, A_1, \ldots, A_d\} \qquad \{E_0 = \frac{1}{|X|}J, E_1, \ldots, E_d\}$$

$$A_i \circ A_j = \delta_{i,j} A_i \qquad E_i E_j = \delta_{i,j} E_i$$

$$A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k \qquad E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{i,j}^k E_k$$

$$A_i = \sum_{j=0}^{d} P_{j,i} E_j \qquad E_i = \frac{1}{|X|} \sum_{j=0}^{d} Q_{j,i} A_j$$

P-,Q-polynomiality

Definition (P-polynomial)

$(X,\mathcal{R})$ is P-polynomial with respect to the ordering $\{A_i\}_{i=0}^d$ if for each i there is a polynomial v_i with degree i such that $A_i = v_i(A_1)$.

- P-polynomial scheme $=$ Distance-regular graph.

Definition (Q-polynomial)

$(X,\mathcal{R})$ is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$ if for each i there is a polynomial v_i^* with degree i such that $E_i = v_i^* \circ (E_1)$.

Notation: $f \circ (M)$ is matrix obtained by applying f to each entry.

Design and dual width in Q-polynomial scheme

Let $(X, \mathcal{R})$ be a Q-polynomial scheme with respect to the ordering $E_0, E_1, \ldots, E_d$.
Let Y be a subset of X, and χ be the characteristic vector of Y.
We define $b_i = \frac{|Y|}{|X|}\chi^T E_i \chi$ for $0 \leq i \leq d$.

Definition (Design)

Y is a t-design if $b_1 = \cdots = b_t = 0 \neq b_{t+1}$.

Definition (Dual width)

Y has a dual width w^* if $b_{w^*} \neq 0 = b_{w^*+1} = \cdots = b_d$.

Well known results

Let $(X, \mathcal{R})$ be a Q-polynomial scheme.
Let Y be a subset in X which is t-design with dual width w^* and degree s of Y is defined by

$$s = |\{j \mid \chi^T A_j \chi \neq 0\}|.$$

Theorem(Delsarte,1973)

If $2s - 2 \leq t$, then Y induces a Q-polynomial scheme.

Theorem(Brouwer-Godsil-Koolen-Martin,2003)

$w^* \geq d - s$ holds. If equality holds, Y induces a Q-polynomial scheme.

Spherical design

Definition (Spherical t-design)

$X \subset S^{d-1}$ is a spherical t-design if

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)d\sigma(x)$$

for all $f(x) \in \bigoplus_{l=1}^{t} \bigoplus_{k=0}^{[\frac{l}{2}]} (x_1^2 + \cdots + x_d^2)^k \mathrm{Harm}_{l-2k}(\mathbb{R}^d)$.

Spherical design is generalized as follows;

(1) sphere is replaced by another space;(E.g.) Euclidean design

(2) add integration to weight function; (E.g.) Weighted spherical design

(3) the vector space $\mathbb{R}[x_1, \ldots, x_d]_{\leq t}$ is replaced by another vector space;(E.g.) in this talk

Spherical dual width

Definition (Spherical (w^*, t)-design)

$X \subset S^{d-1}$ is a spherical (w^*, t)-design if

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)d\sigma(x)$$

for all $f(x) \in \bigoplus_{l=1}^{t} \bigoplus_{k=0}^{[\frac{l}{2}]} (x_1^2 + \cdots + x_d^2)^k \mathrm{Harm}_{w^*+l-2k}(\mathbb{R}^d)$.

- A spherical $(0, t)$-design coincides with a spherical t-design.

Definition (Spherical dual width)

w^* is a spherical dual width of X if X is a spherical (w^*, t)-design and is not a spherical $(w^* - 1, t)$-design.

Characterization of spherical designs

Let X be finite set in S^{m-1}.
We define $b_k = \sum_{x,y \in X} Q_k(\langle x, y \rangle)$ for $k \in \mathbb{N}$.
$Q_0(x) = 1$, $Q_1(x) = x$, $\frac{k+1}{m+2k} Q_{k+1}(x) = xQ_k(x) - \frac{m+k-3}{m+2k-4} Q_{k-1}(x)$.

Proposition

The following are equivalent;

(1) X is a spherical t-design;

(2) $b_1 = \cdots = b_t = 0$.

Proposition

The following are equivalent;

(1) X is a spherical (w^*, t)-design ;

(2) $b_{w^*+1} = \cdots = b_{w^*+t} = 0$.

Main result

We define inner product set $A(X) := \{\langle x, y\rangle \mid x, y \in X, x \neq y\}$.
Let $d = |A(X)|$ and $A(X) = \{\alpha_1, \dots, \alpha_d\}$, $\alpha_0 = 1$.
$R_k = \{(x, y) \in X \times X \mid \langle x, y\rangle = \alpha_k\}$. Let X be a (w^*, t)-design.

Theorem(Delsarte-Goethals-Seidel,1977)

Assume $w^* = 0$. If $2d - 2 \leq t$, then $(X, \{R_k\}_{k=0}^d)$ is a Q-polynomial association scheme.

Theorem(S)

Assume $w^* \geq 1$. If $2d - 1 \leq t$, then $(X, \{R_k\}_{k=0}^d)$ is a Q-polynomial association scheme.