# Geometric transversal theory

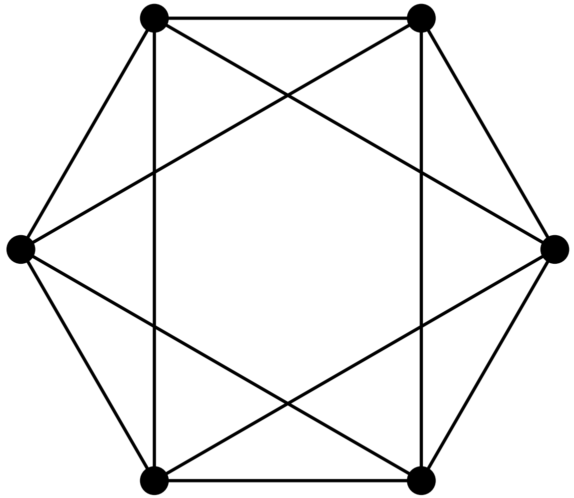
Andreas F. Holmsen
KAIST

# Some basic definitions

$F = \{S_1, S_2, \ldots, S_m\}$ , $S_i \subset X$.

A *transversal* to $F$ is a subset $T \subset X$ such that $T \cap S_i \neq \emptyset$ for all $1 \leq i \leq m$.

## Example:



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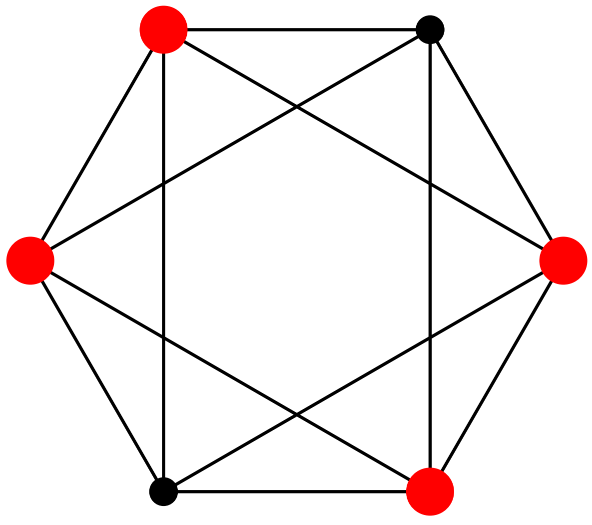

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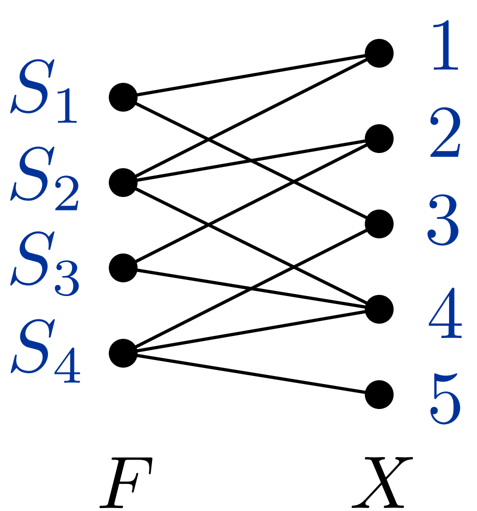
The *transversal number*, $\tau(F)$, of a hypergraph $F$ is the minimum cardinality of a transversal of $F$.

A *system of distinct representatives* is a transversal $T = \{x_1, x_2, \ldots, x_m\}$ such that $x_i \in S_i$, $1 \leq i \leq m$ and $x_i \neq x_j$ whenever $i \neq j$.

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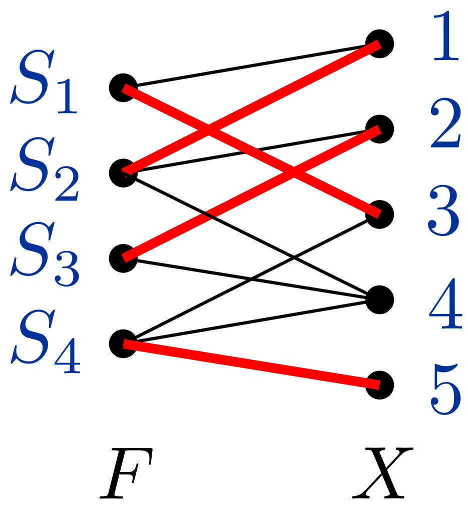


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$$T = \{3, 1, 2, 5\}$$

# Hall's marriage theorem

**Theorem.** (Hall, 1935)
Let $F = \{S_1, S_2, \ldots, S_m\}$ be a collection of finite sets. $F$ has a system of distinct representatives if and only if for every $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ we have

$$|S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k}| \geq k$$

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$F = \{S_1, S_2, \ldots, S_m\}$ , $S_i$ *convex sets* in $\mathbb{R}^d$.

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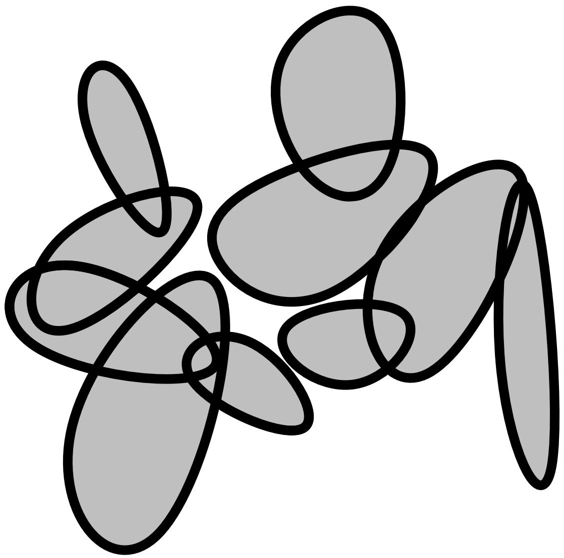

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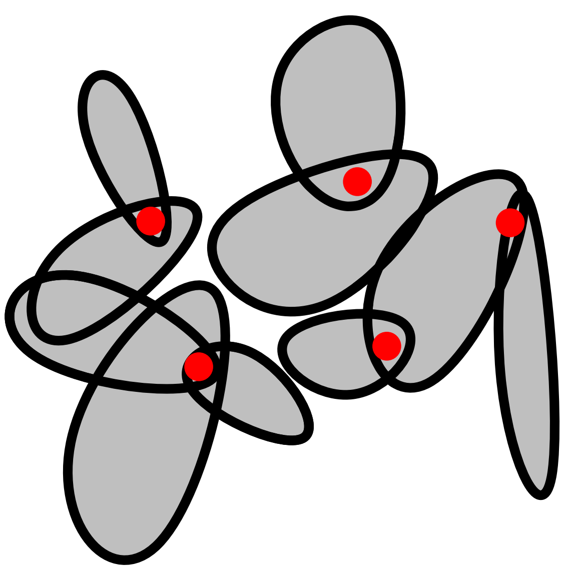

*point transversals*

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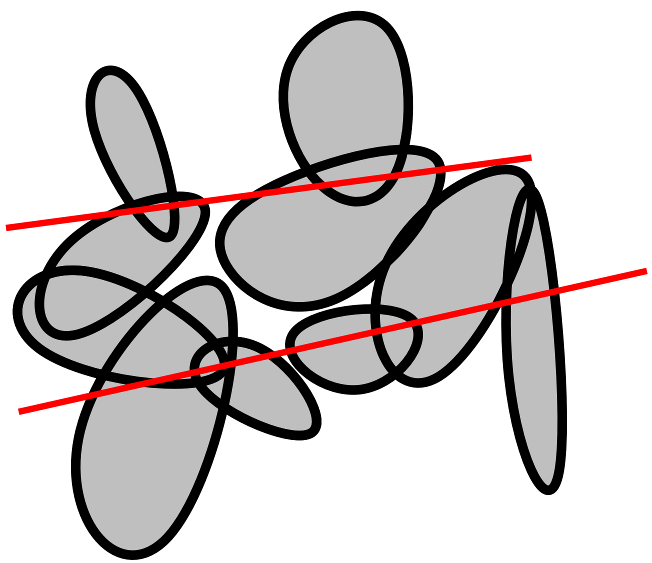
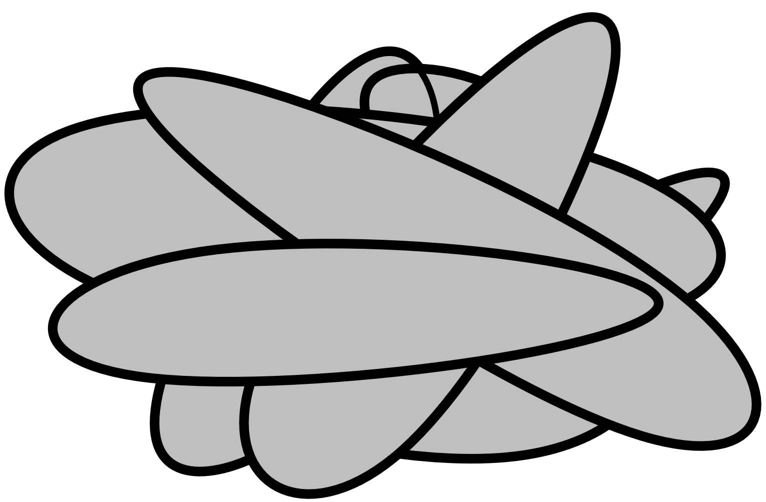


*line transversals*

# Helly's theorem

**Theorem.** (Helly, 1913)

A family of compact convex sets in $\mathbb{R}^d$ has a point transversal if and only if every subfamily of size at most $d+1$ members has a point transversal.



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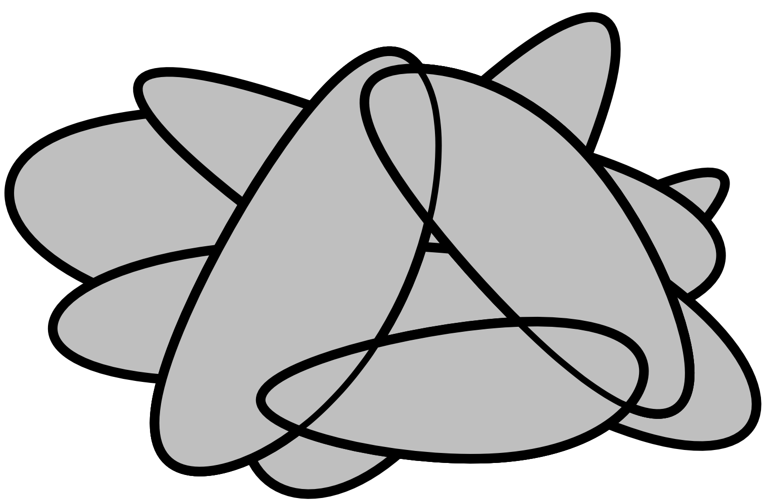
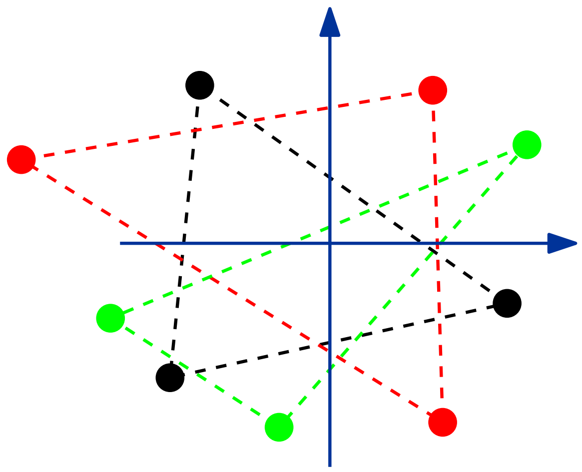


# Colorful Carathéodory Theorem

**Theorem.** (Bárány, 1982)

Let $A_1, A_2, \ldots, A_{d+1}$ finite subsets of $\mathbb{R}^d$. If $0 \in \text{conv}(A_i)$ for all $1 \leq i \leq d+1$, then $0 \in \text{conv}(Y)$ for some $Y$ such that $|Y \cap A_i| = 1$. ($Y$ is an SDR)



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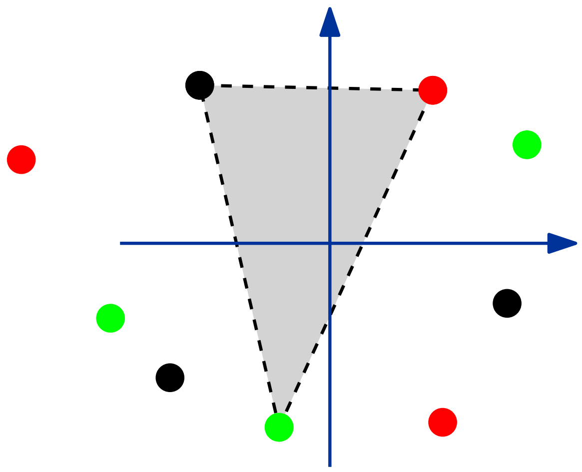
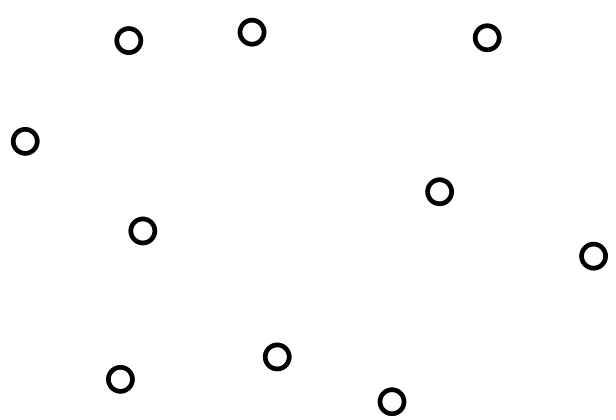


# Tverberg's theorem

**Theorem.** (Tverberg, 1966)
Let $S \subset \mathbb{R}^d$ with $|S| = (d+1)(k-1)+1$. Then $S$ can be partitioned into $k$ non-empty parts $S = S_1 \cup \cdots \cup S_k$ such that $\text{conv}(S_1) \cap \cdots \cap \text{conv}(S_k) \neq \emptyset$

Example:

$d = 2, k = 4$



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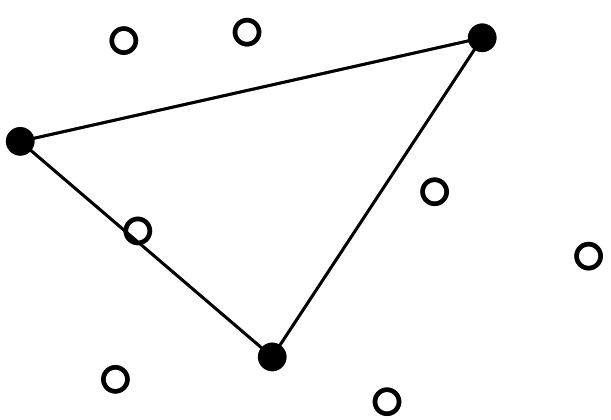

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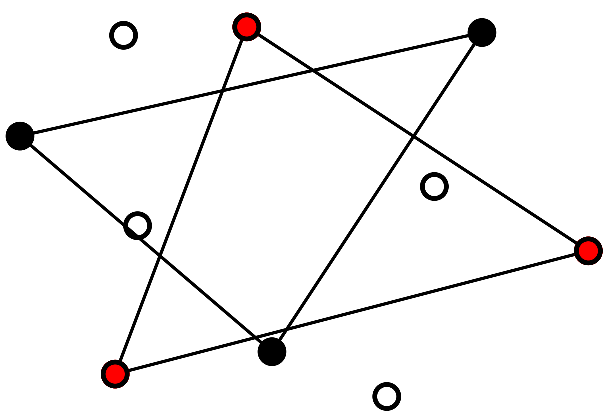

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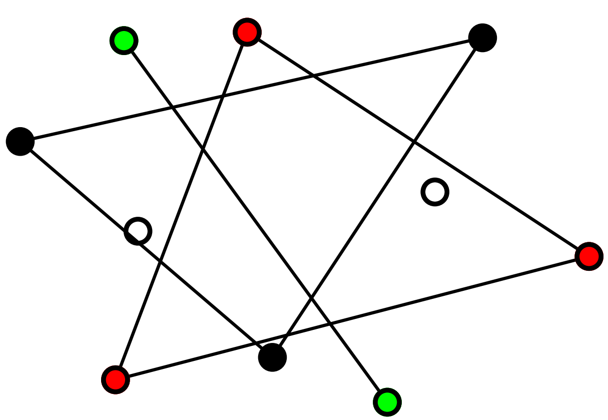

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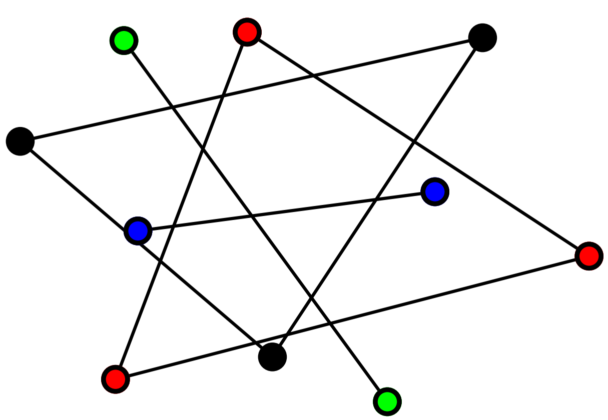

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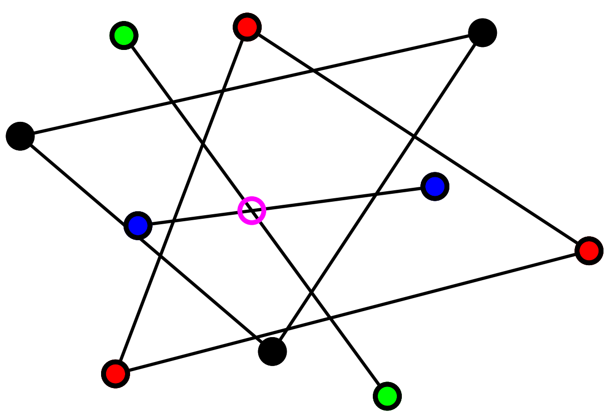
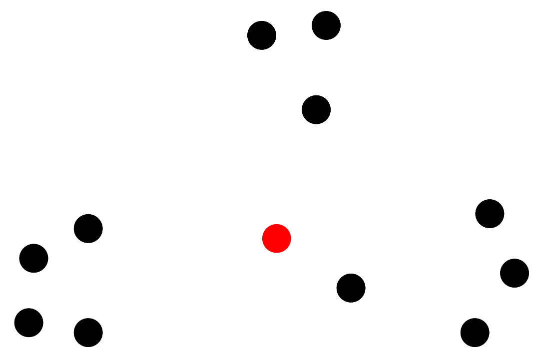


# Weak $\epsilon$ - net theorem for convex sets

$\mu$ - probability measure on $\mathbb{R}^d$.
$0 < \epsilon < 1$.
$F_\epsilon$ - family of all convex sets $S$ such that $\mu(S) > \epsilon$.

From Helly's theorem we have the following:

For every $\mu$ and $\epsilon \geq \frac{d}{d+1}$ the family $F_\epsilon$ has a point transversal. (Rado's centerpoint theorem)



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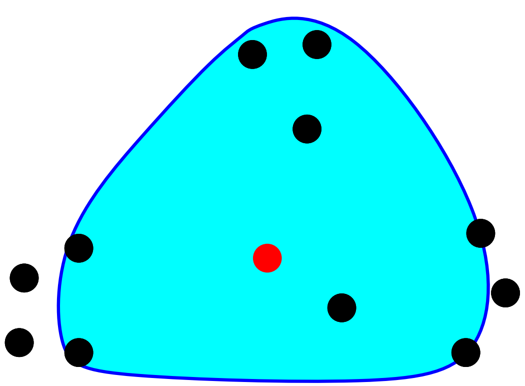

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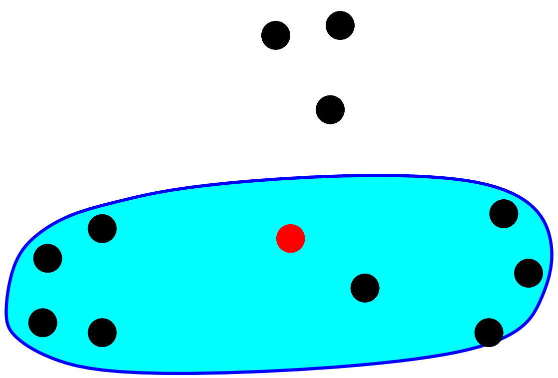

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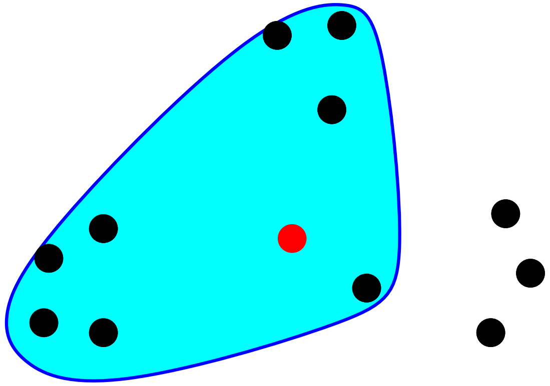

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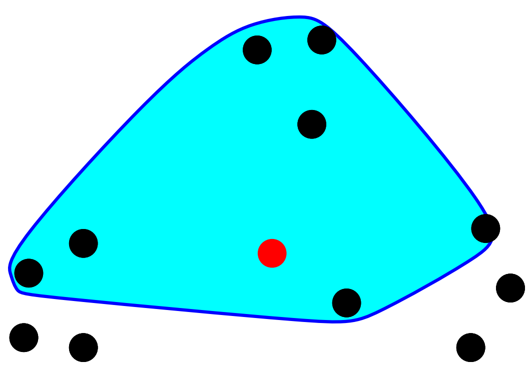

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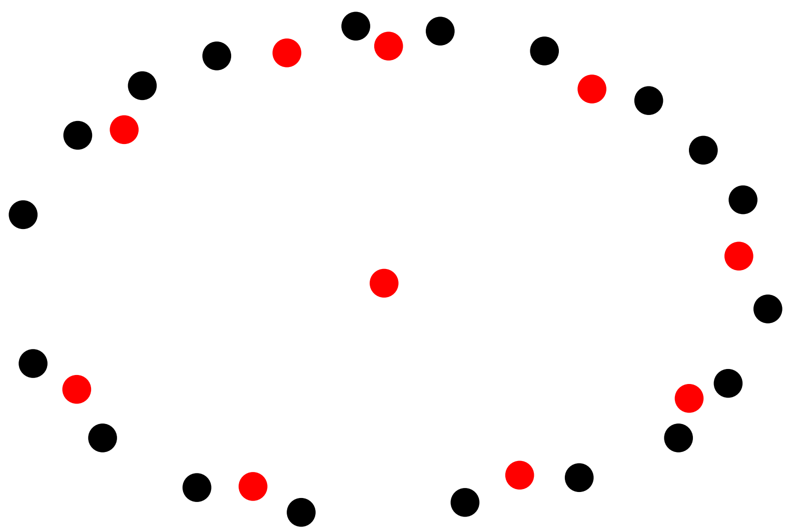
**Theorem.**
For every $0 < \epsilon < 1$ and every positive integer $d$ there exists a minimum positive integer $n(\epsilon, d)$ such that the following holds:
For any probability measure $\mu$ on $\mathbb{R}^d$ there exists a set $N(\mu)$ of at most $n(\epsilon, d)$ points such that $N(\mu)$ is a transversal to $F_\epsilon$.



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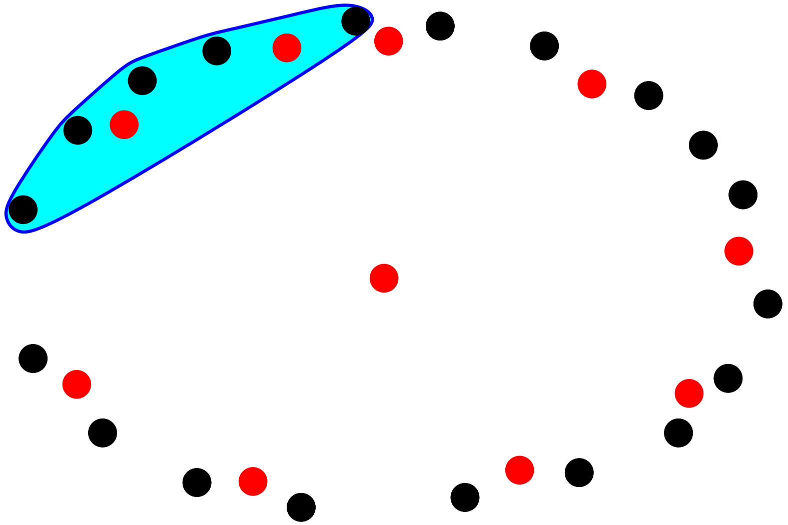

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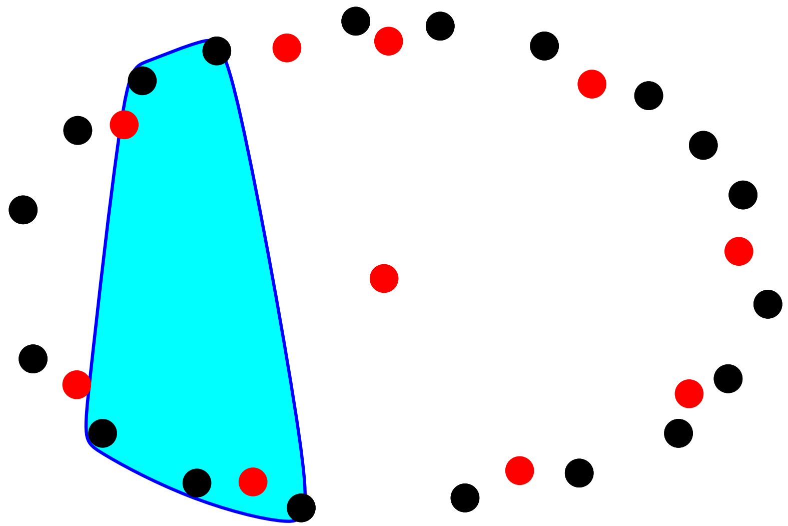

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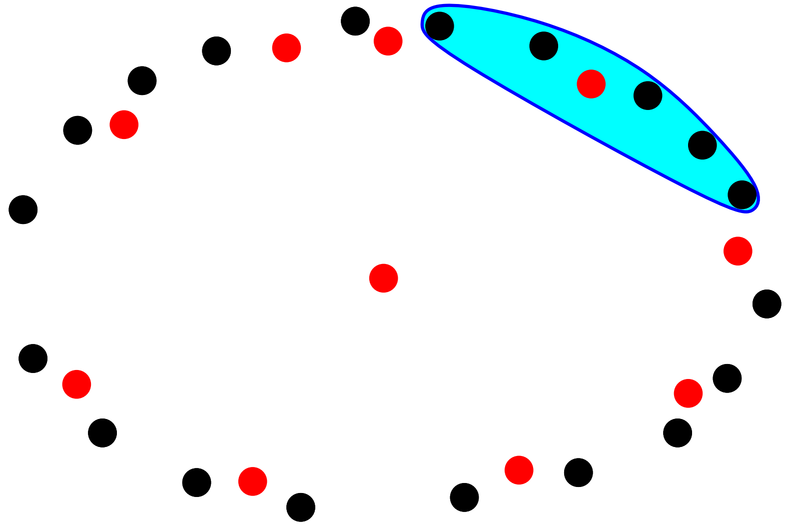

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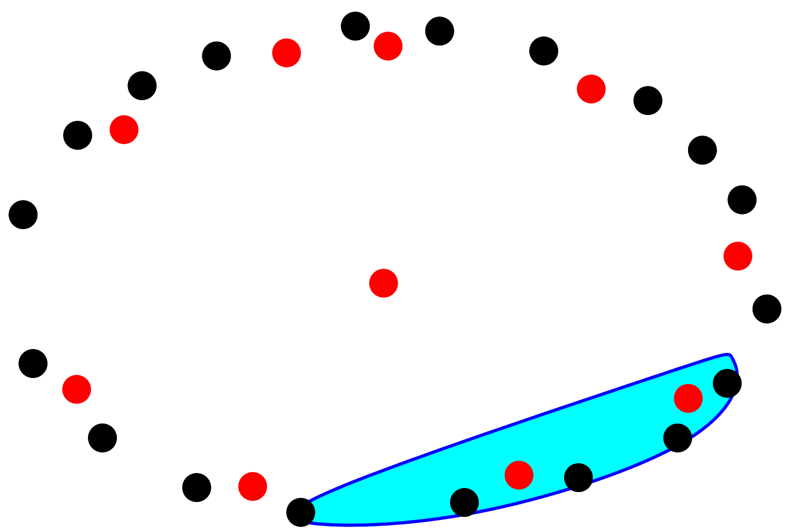

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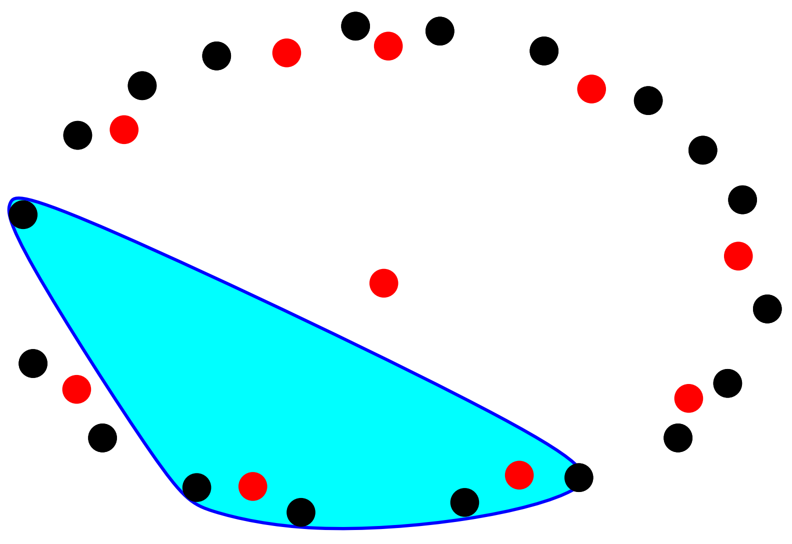

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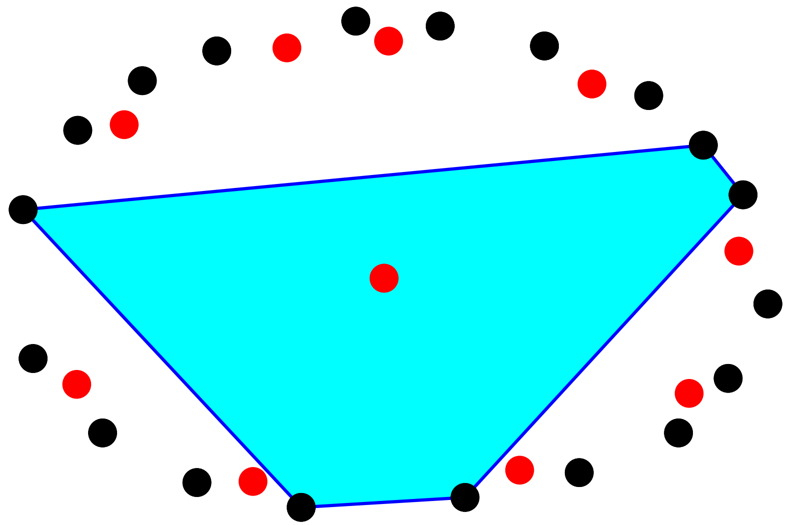

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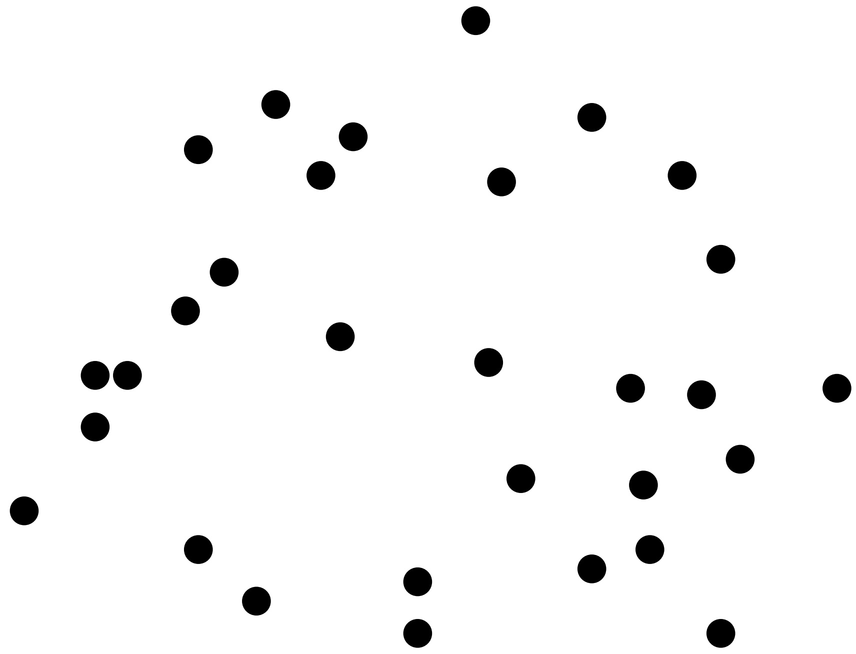
**(1)** For every positive integer $d$ there exists a positive constant $c_d$ such that for any set $X$ of $n$ points in $\mathbb{R}^d$ there exists a point contained in at least $c_d n^{d+1}$ simplices spanned by $X$.



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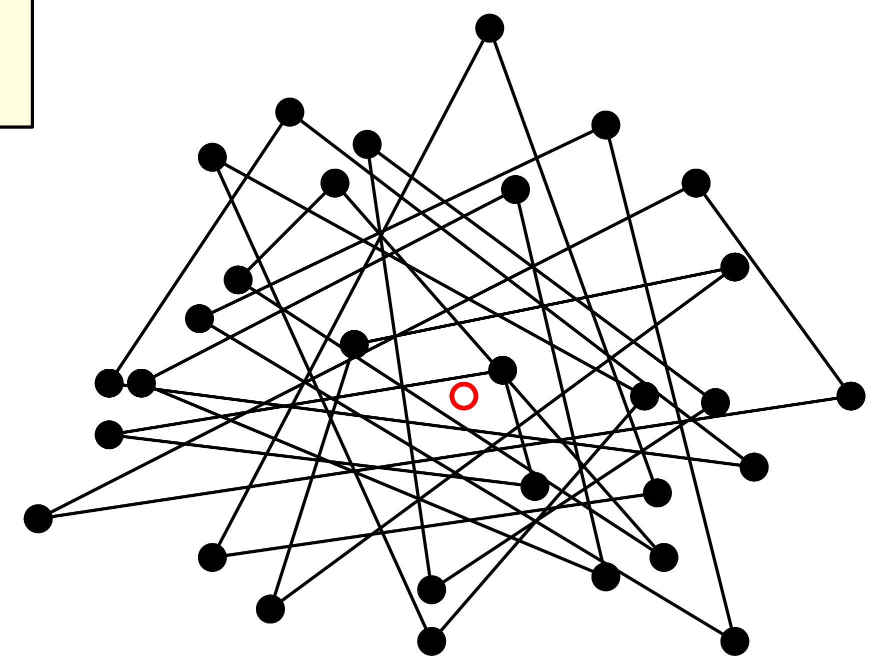
The point set can be partitioned (roughly) into $\frac{n}{d+1}$ *simplices* ($(d+1)$-tuples) that share a common point, $p$. (Tverberg's theorem)



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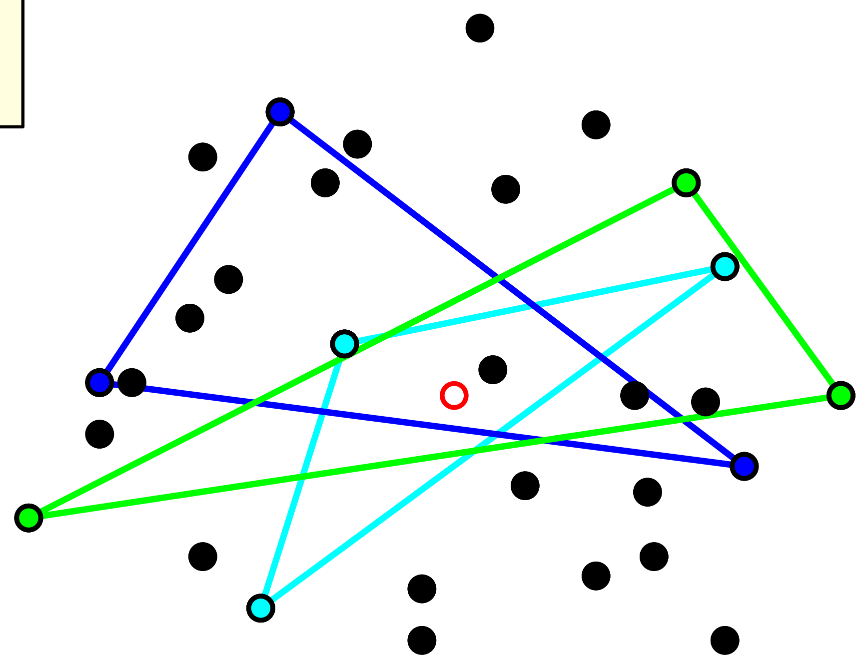

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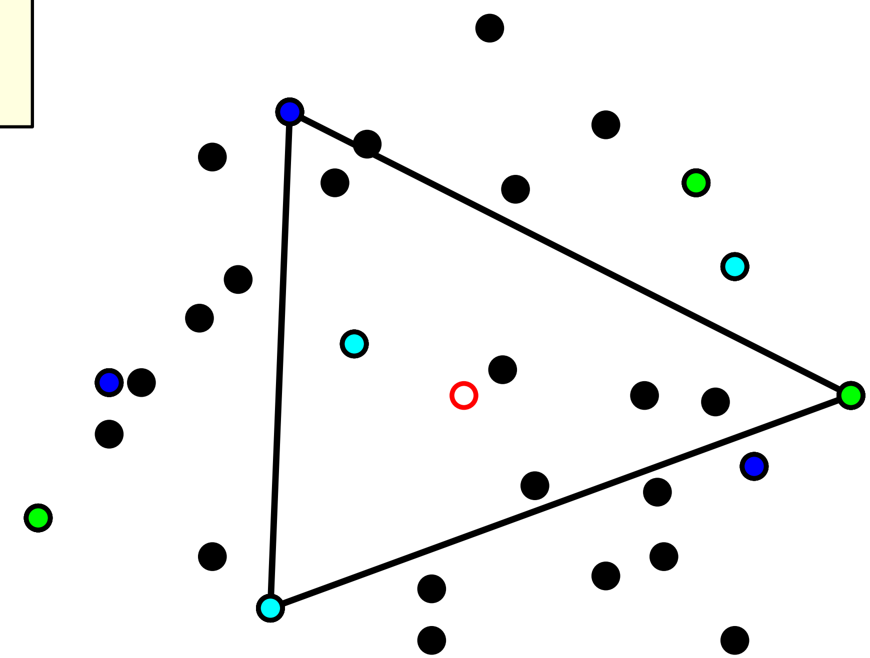
$$\Rightarrow \binom{\frac{n}{d+1}}{d+1} \approx \frac{1}{(d+1)!(d+1)^{d+1}} n^{d+1}$$

distinct simplices contain $p$.



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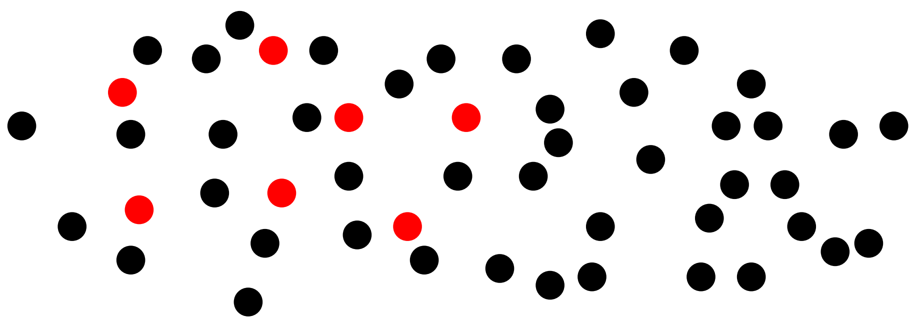
**(2)** We choose a weak $\epsilon$-net greedily: Let $N_i$ be defined. If there exists a convex set $S$ containing more than $\epsilon n$ of the points of $\mu$, where $S \cap N_i = \emptyset$, let $N_{i+1} = N_i \cup p$ where $p$ is chosen using **(1)**.



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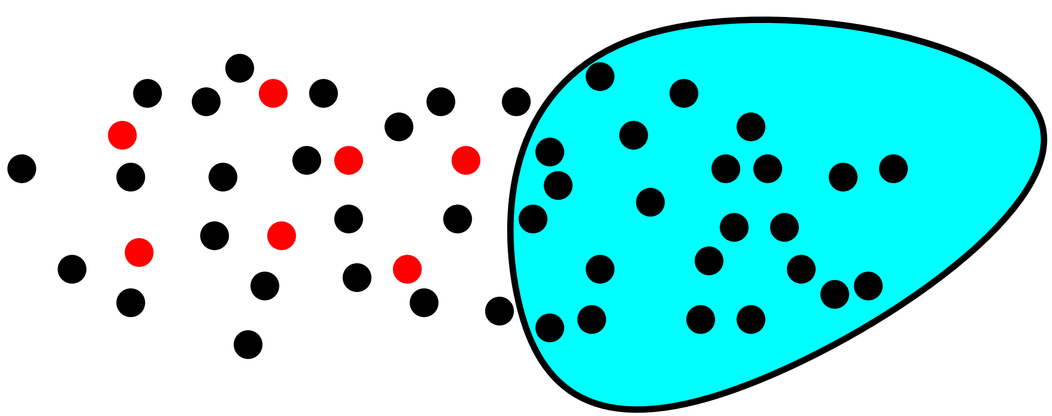


The process ends in a finite number of steps depending only on $\epsilon$ and $d$: Each step kills at least $c_d(\epsilon n)^{d+1}$ simplices.

$$\Rightarrow n(\epsilon, d) \leq O(\frac{1}{\epsilon^{d+1}})$$

# Line transversals: Definitions

$F = \{S_1, \ldots, S_n\}$ : Family of convex sets in the plane.

*Common transversal* : A straight line that intersects every member of $F$.

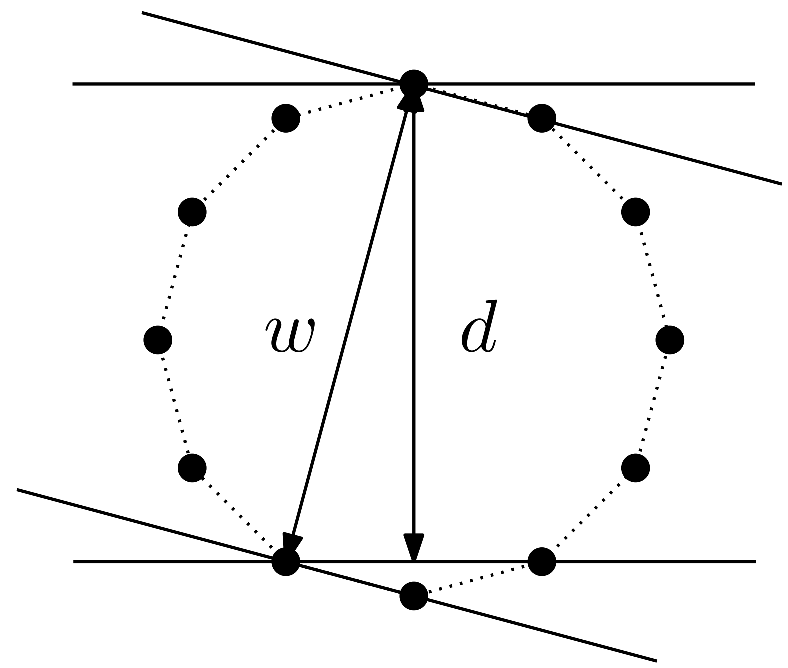
$T(k)$ - *family* : Every subfamily of size at most $k$ has a common transversal.

$\alpha$ - *transversal* : A straight line that intersects at least $\alpha n$ members of $F$ $(0 \leq \alpha \leq 1)$.

# No Helly type theorem for line transversals

For every positive integer $k$ there exists a $T(k)$-family that does not have a common transversal.

Regular $(k+1)$ - gon.

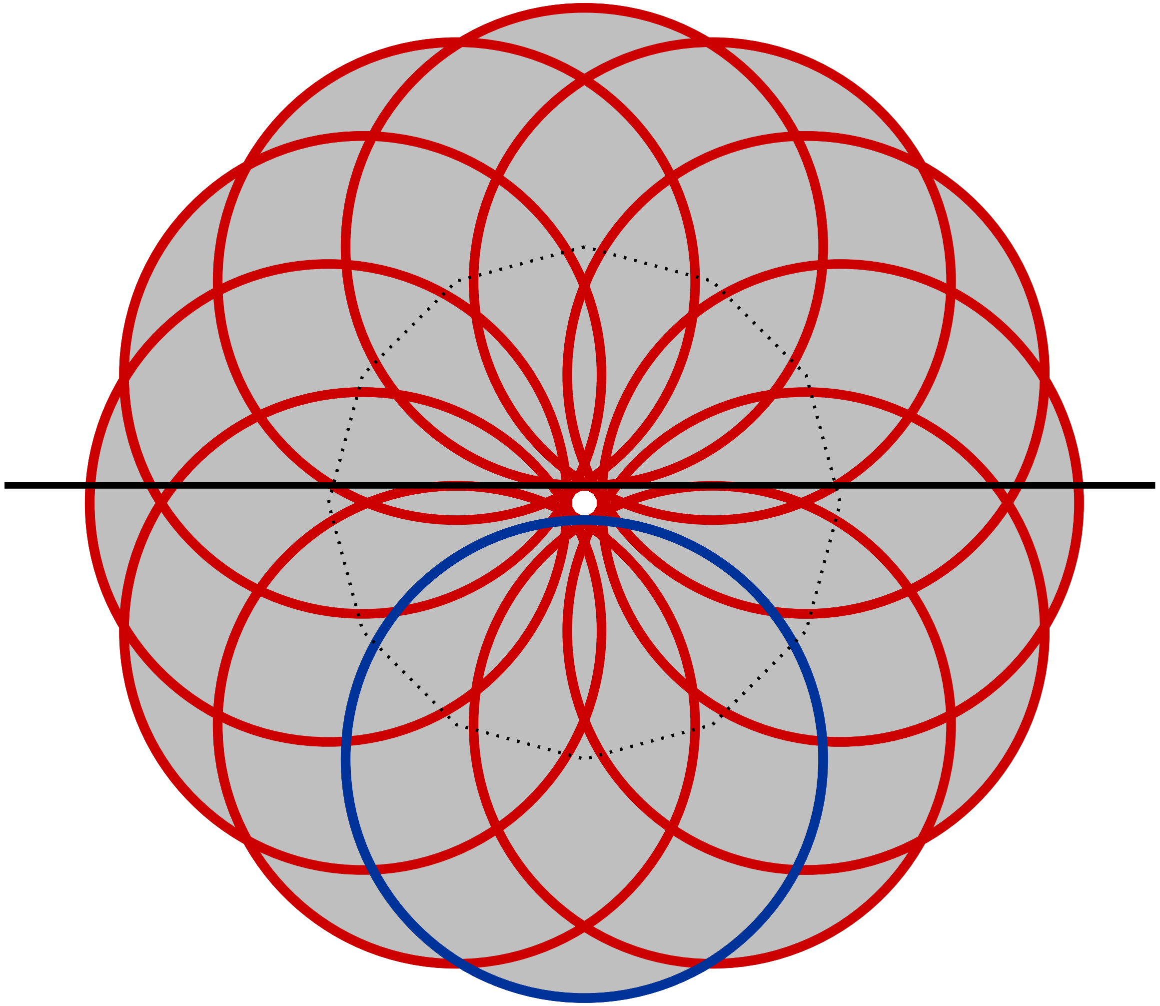



$d < w$

# No Helly type theorem for line transversals

For every positive integer $k$ there exists a $T(k)$-family that does not have a common transversal.

$F$ has a $\frac{k}{k+1}$ - transversal.



## A basic result

**Theorem.** (Katchalski-Liu, 1980)

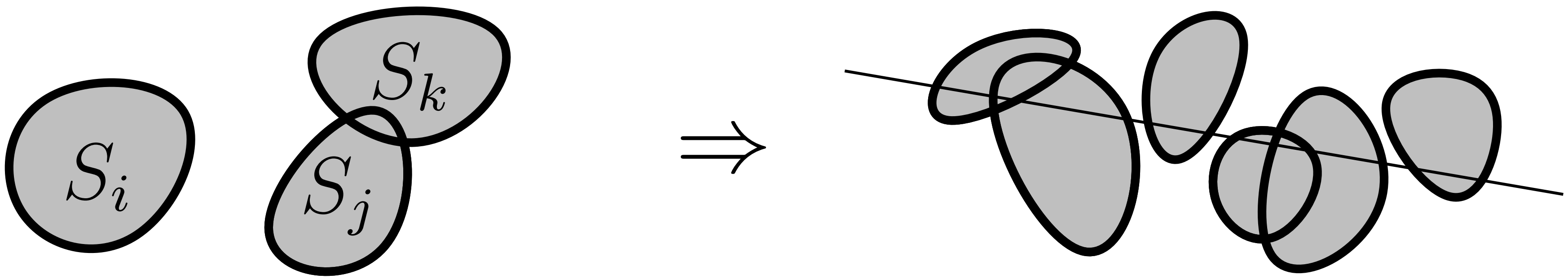
For every $k \geq 3$ there exists a maximal number $\alpha(k) \in (0,1)$ such that every $T(k)$-family has an $\alpha(k)$-transversal. Moreover,

$$\lim_{k \to \infty} \alpha(k) = 1$$

**Problem.** Determine the function $\alpha(k)$.

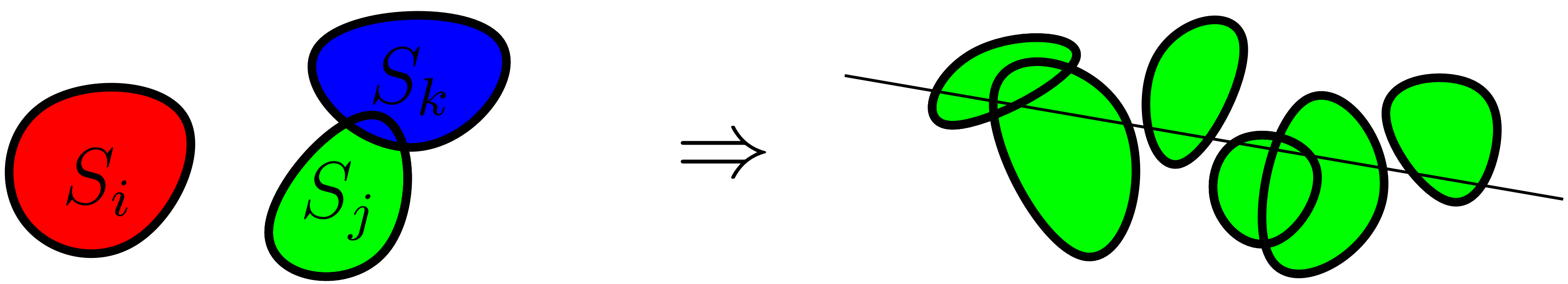
# Hadwiger's transversal theorem

**Theorem.** (Wenger, 1990)
Let $F = \{S_1, S_2, \ldots, S_n\}$. If for every $1 \leq i < j < k \leq n$ we have $S_j \cap \mathrm{conv}(S_i \cup S_k) \neq \emptyset$, then $F$ has a transversal.

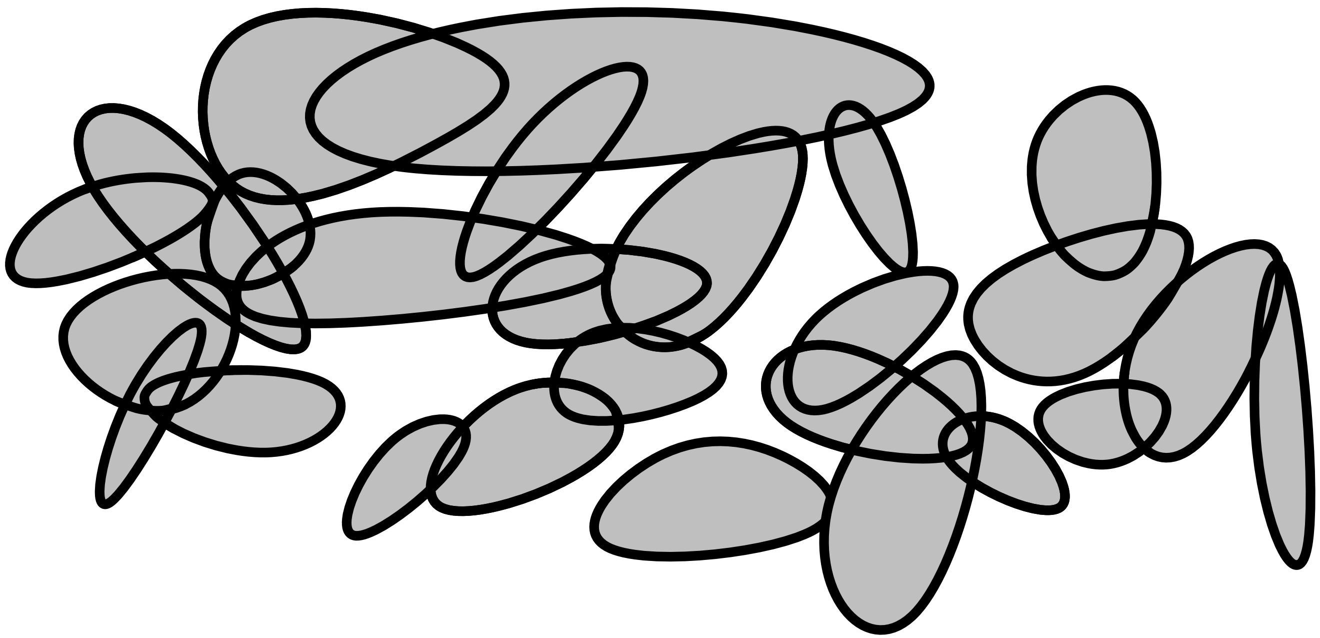


# Hadwiger's transversal theorem

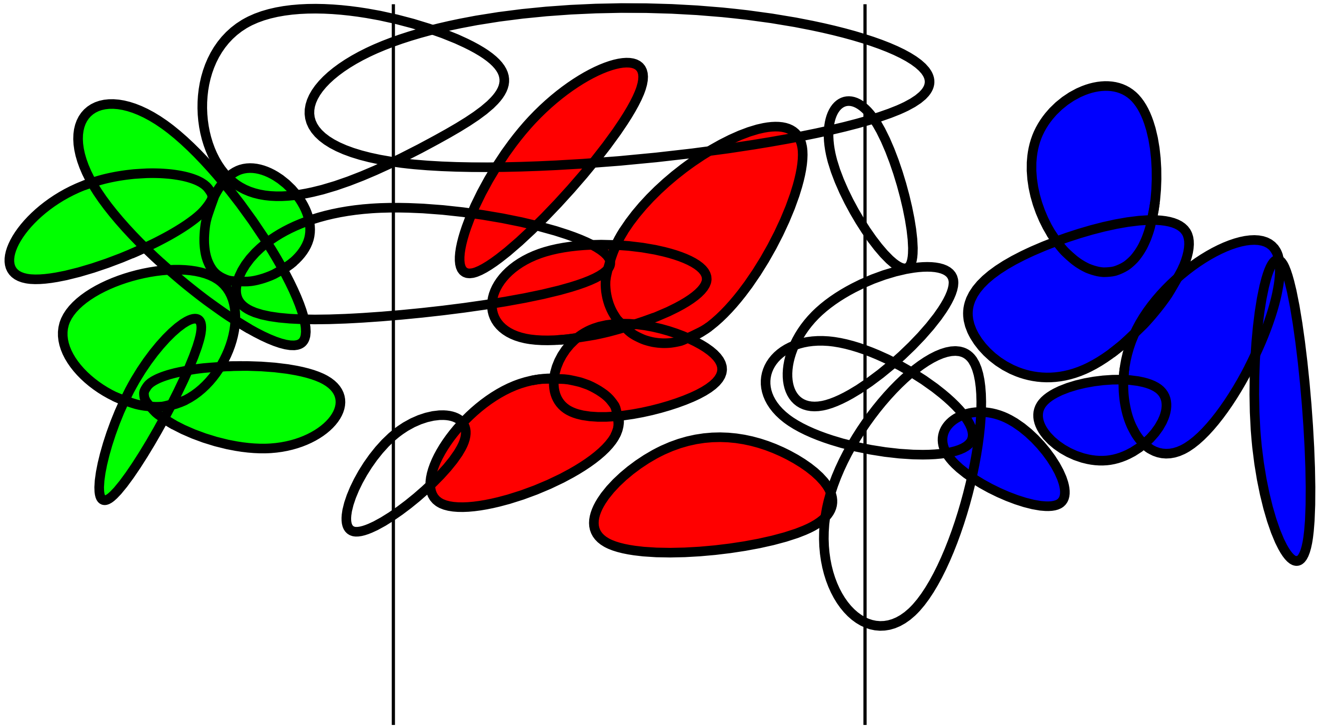
**Theorem.** (Arocha-Bracho-Montejano, 2008)
Let $F = F_1 \cup F_2 \cup F_3 = \{S_1, S_2, \ldots, S_n\}$. If for every $1 \leq i < j < k \leq n$ where $S_i$, $S_j$, $S_k$ belong to distinct parts ($F_p$'s) we have $S_j \cap \mathrm{conv}(S_i \cup S_k) \neq \emptyset$, then one of the $F_p$ has a transversal.



# Application to general $T(3)$-families



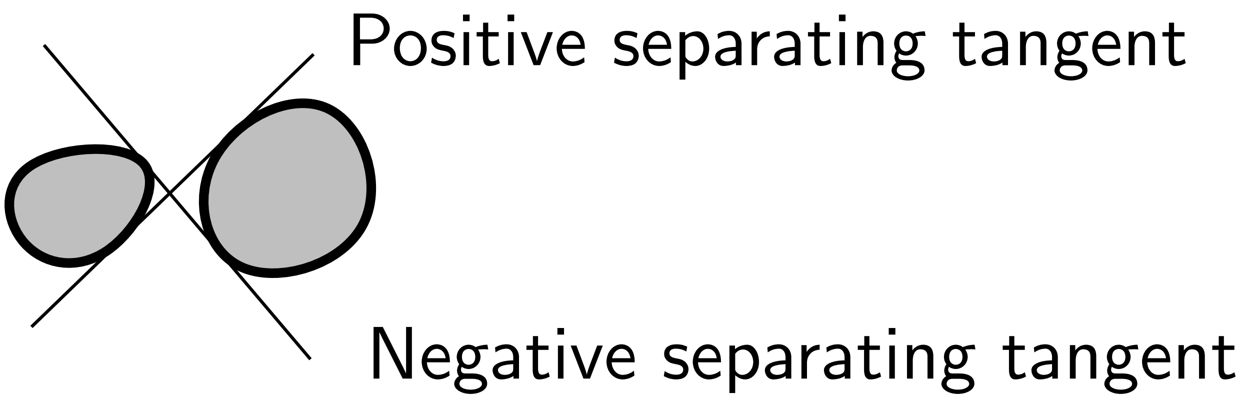
# Application to general $T(3)$-families



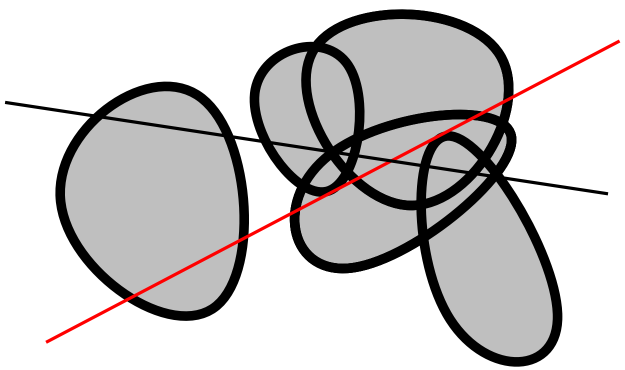
$\Rightarrow \frac{1}{5}$ - transversal.

# The space of transversals

Disjoint pairs:




**Observation.** Suppose $F$ contains at least one disjoint pair. Then $F$ has a transversal if and only if a positive separating tangent of some disjoint pair of $F$ is transversal to $F$.



# Lower bound for $\alpha(k)$

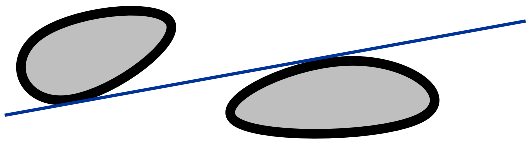
Suppose $F$ contains $d\binom{n}{2}$ intersecting pairs, $0 \le d < 1$.

$X$ : $k$-tuples containing at least one disjoint pair.

$Y$ : $k$-tuples containing only intersecting pairs.

$$|X| \ge (1 - d^{k/2})\binom{n}{k}$$

$$\frac{|X|}{(1-d)\binom{n}{2}} \ge \binom{n}{k} / \binom{n}{2}$$

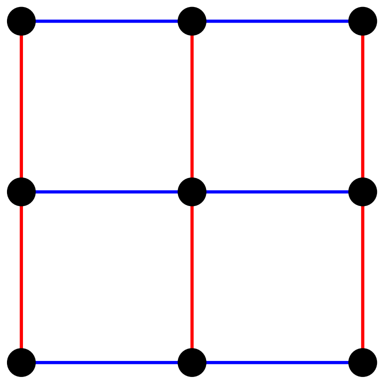


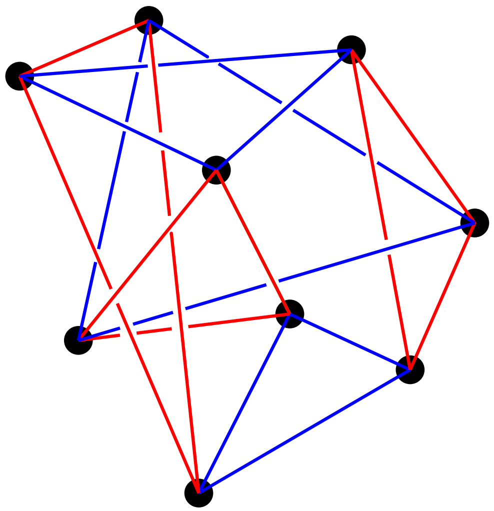
$$\Rightarrow \alpha(k) \ge \left(\frac{2}{k(k-1)}\right)^{\frac{1}{k-2}}$$

$\alpha(3) \ge \frac{1}{3}$, $\alpha(4) \ge$ ~~$0.408\cdots$~~ $\frac{1}{2}$, $\alpha(5) \ge$ ~~$0.464\cdots$~~ $\frac{1}{2}$, $\alpha(6) \ge 0.508\cdots, \ldots$

Better bounds due to Eckhoff (1973)

# A problem (or perhaps an exercise?)





Show that for any embedding of the combinatorial configuration (on the left) into $\mathbb{R}^3$ (on the right) there is a line that intersects all the red triangles or all the blue triangles. (Due to Luis Montejano)