Rational Homogeneous Manifold and its Rigidity

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Abstract

An important theorem in complex geometry is the Riemann’s uniformization theorem which says that three basic Riemann surfaces, namely the Riemann sphere (the projective space) $\mathbb{CP}^1$, the complex plane $\mathbb{C}$ and the unit disc $\Delta$, exhaust all simply connected Riemann surfaces (the complex manifolds of dimension 1). The natural higher-dimensional analogs of $\mathbb{CP}^1$, $\mathbb{C}$ and $\Delta$ are Hermitian symmetric spaces.

In the talk, we introduce the rational homogeneous manifolds which are the larger class of the complex manifolds including Hermitian symmetric spaces of compact type. These are just homogeneous Fano manifolds, where Fano manifolds are complex manifolds $X$ with positive anti-canonical bundles $K_X^{-1} = \det(T_X) = \bigwedge^n T_X$. We will give the related definitions, examples and basic properties of the rational homogeneous manifold $S$ and consider mainly when $S$ is of Picard number 1.

A rational homogeneous manifold $S$ of Picard number 1 is a homogeneous space $G/P$ for a complex simple Lie group $G$ and a maximal parabolic subgroup $P \subset G$. In particular, an irreducible Hermitian symmetric space $S = G/P$ of compact type is a rational homogeneous manifold of Picard number 1 such that the isotropy representation of $P$ on $T_x(S)$ at a base point $x \in S$ is irreducible.

By studying the deformation of minimal rational curves and their associated varieties of minimal rational tangents (VMRT for short), we will consider the rigidity of rational homogeneous manifolds of Picard number 1 under Kähler deformation and the characterization of standard embeddings of rational homogeneous manifolds of Picard number 1.

1 Introduction : complex manifold and its rigidity

My research objects are ‘complex manifolds’, which are topological (differentiable) manifolds that are locally modeled on open polydiscs $B \subset \mathbb{C}^n$ with holomorphic transition maps. For details on complex manifolds, see [7] and [4].

**Definition 1.1.** 1. A topological manifold is a locally Euclidean paracompact Hausdorff topological space.

2. An $m$-dimensional differentiable manifold is a topological manifold $M$ together with an open covering $M = \bigcup U_i$ and homeomorphisms $\varphi_i : U_i \cong V_i$ onto open subsets
Figure 1: Georg Friedrich Bernhard Riemann (1826 – 1866) was an influential German mathematician who opened up major parts of the theories of Riemannian geometry, algebraic geometry, and complex manifold theory.

Let $V_i \subset \mathbb{R}^m$ such that the transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are $C^\infty$-maps (smooth maps). Here the datum $\{(U_i, \varphi_i)\}$ is called an atlas (or differentiable structure) and each tuple $(U_i, \varphi_i)$ is a chart.

3. A complex manifold of dimension $n$ is a real differentiable manifold of (real) dimension $2n$ endowed with a holomorphic atlas $\{(U_i, \varphi_i)\}$, that is, an atlas of the form $\varphi_i : U_i \cong \varphi_i(U_i) \subset \mathbb{C}^n$ such that the transition maps $\varphi_j \circ \varphi_i^{-1}$ are holomorphic.

**Definition 1.2.** Let $X$ and $Y$ be two complex manifolds. A map $f : X \to Y$ is a holomorphic map (or, a morphism) if for any holomorphic charts $(U, \varphi)$ and $(U', \varphi')$ of $X$ and $Y$, respectively, the map $\varphi' \circ f \circ \varphi^{-1} : \varphi(f^{-1}(U') \cap U) \to \varphi'(U')$ is holomorphic.

**Example 1.3.**

1. Affine space. The most basic complex manifold is the $n$-dimensional complex vector space $\mathbb{C}^n$ and an open subset of $\mathbb{C}^n$.

  ◊ **Cautions! (a difference between differentiable and complex manifolds) :** $\mathbb{C}$ is not biholomorphic to a bounded open set by Liouville’s theorem. In particular, $\mathbb{C}^n$ and the unit disc $D^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ are not biholomorphic. Furthermore, the theorem of Poincaré states that the polydisc $B_{(1,1)}(0) = \{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < 1\}$ and the unit disc $D^2$ are not biholomorphic.

2. Projective space. The complex projective space $\mathbb{P}^n := \mathbb{CP}^n$ is the most important compact complex manifold. By definition, $\mathbb{P}^n$ is the set of lines in $\mathbb{C}^{n+1}$ or, equivalently, $\mathbb{P}^n = (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by multiplication on $\mathbb{C}^{n+1}$.
Remark. Holomorphic functions have good properties as followings.

1. Maximum Modulus Principle: If $f$ is a holomorphic function defined on an open connected subset of $\mathbb{C}^n$, then $|f|$ has no maximum, unless $f$ is constant.

2. Riemann Extension Theorem [One complex variable]: Let $U \subset \mathbb{C}$ be an open subset. If $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and bounded on $B_r(z_0) = \{z : 0 < |z - z_0| < r\}$, then $f$ can be extended to a holomorphic function $f : U \to \mathbb{C}$.

3. Hartogs Extension Theorem [Several complex variables]: Let $n \geq 2$ and $U \subset \mathbb{C}^n$ be an open subset. Suppose that $K$ is a compact subset of $U$ such that $U \setminus K$ is connected. Then any holomorphic function $f$ on $U \setminus K$ extends to a holomorphic function on $U$.

4. Complex torus. The complex torus $\mathbb{C}^n / \mathbb{Z}^{2n}$, where $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n} = \mathbb{C}^n$. They are diffeomorphic to $(S^1)^{2n}$. But, from the complex view point the situation is very difficult.

3. Grassmannian manifold. Let $V$ be a complex vector space of dimension $n + 1$. The Grassmannian $\text{Gr}(k, V)$ is defined as the set of all $k$-dimensional subspaces of $V$, that is, $\text{Gr}(k, V) := \{W \subset V : \dim(W) = k\}$. In particular, $\text{Gr}(1, V) = \mathbb{P}(V)$ and $\text{Gr}(n, V) = \mathbb{P}(V^*)$. The dimension of Grassmannian manifold $\text{Gr}(k, V)$ is $\dim(\text{Gr}(k, V)) = k(n + 1 - k)$.

4. Complex torus. The complex torus is the quotient $\mathbb{C}^n / \mathbb{Z}^{2n}$, where $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n} = \mathbb{C}^n$. They are diffeomorphic to $(S^1)^{2n}$. But, from the complex view point the situation is very difficult.

Let $\Gamma_1, \Gamma_2 \subset \mathbb{C}^n$ be the two lattices, that is, free abelian, discrete subgroups of order $2n$. If we pick $\Gamma_1, \Gamma_2 \subset \mathbb{C}^n$ randomly, then $\mathbb{C}^n / \Gamma_1$ and $\mathbb{C}^n / \Gamma_2$ will not be isomorphic as complex manifolds.

Note that any connected compact complex Lie group is abelian, hence a complex torus.

Remark. Holomorphic functions have good properties as followings.

1. Maximum Modulus Principle: If $f$ is a holomorphic function defined on an open connected subset of $\mathbb{C}^n$, then $|f|$ has no maximum, unless $f$ is constant.

2. Riemann Extension Theorem [One complex variable]: Let $U \subset \mathbb{C}$ be an open subset. If $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and bounded on $B_r(z_0) = \{z : 0 < |z - z_0| < r\}$, then $f$ can be extended to a holomorphic function $f : U \to \mathbb{C}$.

3. Hartogs Extension Theorem [Several complex variables]: Let $n \geq 2$ and $U \subset \mathbb{C}^n$ be an open subset. Suppose that $K$ is a compact subset of $U$ such that $U \setminus K$ is connected. Then any holomorphic function $f$ on $U \setminus K$ extends to a holomorphic function on $U$.
4. Osgood Theorem: A bijective holomorphic map between complex manifolds is a biholomorphism.

Thus complex manifolds are “more rigid” than differentiable manifolds.

**Proposition 1.4.** A holomorphic function $f$ defined on a compact connected complex manifold $X$ is constant.

**Proof.** Since $X$ is compact, $f$ attain a maximum modulus. By Maximum Modulus Principle, $f$ must be constant. \qed

Using Hartogs Extension Theorem, we know the following.

**Proposition 1.5.** Let $X$ be a connected complex manifold of dimension at least 2 and $x \in X$. Then every holomorphic function on $X\setminus\{x\}$ extends uniquely to a holomorphic function on $X$.

## 2 Riemann’s uniformization theorem

An important theorem in complex geometry is the *Riemann’s uniformization theorem* on the Riemann surface.

**Definition 2.1.** A complex manifold of dimension one is called a *Riemann surface* or a *complex curve*.

For details on Riemann surfaces, see [11].

**Theorem 2.2** (Riemann’s uniformization theorem, Theorem 4.4.1 of [11]). Let $\Sigma_1$ be a compact Riemann surface of genus $g$. Then there exists a biholomorphism (conformal diffeomorphism) $f : \Sigma_1 \to \Sigma_2$, where $\Sigma_2$ is
1. a Riemann sphere $\mathbb{C}P^1 = S^2$ in the case $g = 0$;

2. a compact Riemann surface of the form $\mathbb{C}/L$ in the case $g = 1$, where $L$ is a lattice of $\mathbb{C}$, that is, $L = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}, \frac{\omega_2}{\omega_1} \notin \mathbb{R}$;

3. a compact Riemann surface of the form $H/\Gamma$ in the case $g \geq 2$, where $H$ is the hyperbolic space (the Lobachevski upper-half plane) and $\Gamma$ is a discrete subgroup of the hyperbolic isometry group $PSL(2, \mathbb{R})$ and without fixed points on $H$.

**Corollary 2.3.** The universal cover of a compact Riemann surface is biholomorphic to the Riemann sphere (the projective space) $\mathbb{C}P^1 = S^2$, the complex plane $\mathbb{C}$ or the unit disc (the Poincaré disc) $\Delta$.

A compact connected Riemann surface of genus 0 is called a rational curve.

**Remark.** We know from the topological classification of 2-dimensional manifold that $\Sigma_1$ is always homeomorphic (in fact, diffeomorphic) to one of the three types occurring in the statement of Theorem 2.1. But “homeomorphic” or “diffeomorphic” does not imply “biholomorphic”, hence we should show that this map can be deformed to a biholomorphism.

**Proof of Theorem 2.1.** The proofs of Case 2 (Euclidean type) and Case 3 (Noncompact type) use the existence of a harmonic map, the implicit function theorem and the regularity theory. Because this case is out of our interest, we will omit this proof.

The proof of Case 1 is a consequence of the Riemann-Roch Theorem and use the method in algebraic geometry.

Let $\Sigma$ be a compact Riemann surface of genus 0. We need to construct a holomorphic diffeomorphism $h : \Sigma \rightarrow S^2$. For that purpose, we show that there exists a meromorphic function $h$ on $\Sigma$ with a single and simple pole by using the Riemann-Roch Theorem and then interpret $h$ as such a holomorphic diffeomorphism.

**Theorem 2.4 (Riemann-Roch Theorem).** Let $\Sigma$ be a compact Riemann surface of genus $g$.

(a) If $E$ is a rank $r$ holomorphic vector bundle over $\Sigma$, then

$$\chi(\Sigma, E) := h^0(\Sigma, E) - h^1(\Sigma, E) = (1 - g)r + \deg(E),$$

where $\chi(\Sigma, E)$ is the Euler-Poincaré characteristic and $h^k(\Sigma, E)$ denotes the dimension of the cohomology space $H^k(\Sigma, E)$.

(b) Let $D$ be a divisor of $\Sigma$. Then

$$h^0(D) = 1 - g + \deg(D) + h^0(K - D),$$

where $h^0(D) := \dim_{\mathbb{C}}\{\text{meromorphic functions } f \text{ on } \Sigma : f \equiv 0 \text{ or } D + (f) \geq 0\}$ and $K$ is the canonical line bundle or canonical divisor of $\Sigma$. And recall that $\deg(D) := \sum_{\nu=1}^{n} a_{\nu}$ for a divisor $D = \sum_{\nu=1}^{n} a_{\nu}z_{\nu}, \ z_{\nu} \in \Sigma.$
We choose any point \( z_0 \in \Sigma \) and consider this as divisor \( D = z_0 \). Thus the Riemann-Roch Theorem implies that

\[
h^0(D) = 1 - g + \deg(D) + h^0(K - D) = 2 + h^0(K - D) \geq 2.
\]

In fact, since \( \deg(K) = -2 \) and so \( \deg(K - D) = -3 \), \( h^0(K - D) = 0 \) and \( h^0(D) = 2 \). Therefore, we can find a nonconstant meromorphic function \( h \) with \( D + (h) \geq 0 \), that is, with a simple pole at \( z_0 \). Such a meromorphic function can be considered as a holomorphic map \( h : \Sigma \to S^2 \) which is bijective, and hence is a biholomorphism.

\[\square\]

### 3 Hermitian symmetric spaces of compact type

Hermitian symmetric spaces are the higher-dimensional analogs of three basic Riemann surfaces: \( \mathbb{P}^1 \), \( \mathbb{C} \) and \( \Delta \). Recall that these have constant curvatures: (1) the curvature of \( \mathbb{P}^1 \) endowed with the Fubini-Study metric is +1; (2) the curvature of \( \mathbb{C} \) endowed with the standard metric is 0; (3) the curvature of \( \Delta \) endowed with the Poincaré metric is -1.

You might regard the generalizations of three basic Riemann surfaces as manifolds having the constant sectional curvature which are called the space forms. But these are too much simple and so uninterested.

**Theorem 3.1** (Theorem 4.1 in Chapter 8 of [2]). Let \( M^n \) be a complete Riemannian manifold with constant sectional curvature \( K \). Then the universal covering \( \tilde{M} \) of \( M \), with the covering metric, is isometric to
1. \( S^n \) if \( K = 1 \),
2. \( \mathbb{R}^n \) if \( K = 0 \),
3. \( H^n \) if \( K = -1 \).

Because there is no newness, we expect less restrictive geometric condition. To obtain “good” geometric model in higher dimensional case, we ask that its curvature tensor is parallel, that is, \( \nabla R = 0 \), where \( \nabla \) is the Riemannian connection(or the Levi-Civita connection) and \( R \) is the curvature tensor of \( M \).

**Theorem 3.2.** Let \( (M, g) \) be a Riemannian manifold. Then the followings are equivalent.

1. A Riemannian manifold \( (M, g) \) is said to be Riemannian locally symmetric.
2. \( \nabla R = 0 \), where \( \nabla \) is the Riemannian connection and \( R \) is the curvature tensor of \( M \), that is, \( R(X, Y, Z, W) = \langle R^\nabla(X, Y)Z, W \rangle = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, W \rangle \) for the vector fields \( X, Y, Z, W \) on \( M \).
3. For each point \( x \in M \), there exist a neighborhood \( U \) of \( x \) and an involutive local symmetry \( \sigma_x \) such that \( \sigma^2_x = \text{id} \) and \( x \) is an isolated fixed point of \( \sigma_x \) in \( U \).
4. The universal covering \( \tilde{M} \) of \( M \) is a Riemannian symmetric space, that is, there exist a global isometry given by the involution \( \sigma_x \) for every \( x \in M \).
5. \( \tilde{M} = G/K \) is a homogeneous space with a Killing metric \( g \), where \( G = \text{Aut}_0(M, g) \) is the identity component of the group of isometries of \( (M, g) \) and \( K = G_x \) is the isotropy subgroup at a point \( x \).
   In fact, the isometry group \( \text{Aut}(M, g) \) is a real Lie group by Myers-Steenrod Theorem and \( K < G \) is a compact Lie subgroup.

For details on symmetric spaces, see [14] and Chapter 5 of [12].

**Definition 3.3.** Let \( X \) be a complex manifold. The complex structure of \( X \) gives rise to a \( J \)-operator which is defined by \( J(\frac{\partial}{\partial z_i}) = \frac{\partial}{\partial y_i} \), \( J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial z_i} \), when a system of local holomorphic coordinates on \( X \) is written as \( z_i = x_i + \sqrt{-1}y_i \).

An Hermitian metric \( g \) on \( X \) is a Riemannian metric compatible with the almost complex structure \( J \), i.e., \( g(Ju, Jv) = g(u, v) \) for tangent vectors \( u \) and \( v \). The complex manifold \( X \) endowed with an Hermitian metric is called an Hermitian manifold.

**Definition 3.4.** A Riemannian symmetric space \( (X, g) \) is said to be an Hermitian symmetric space if \( (X, g) \) an Hermitian manifold and the involution \( \sigma_x \) at each point \( x \in X \) is a holomorphic isometry.

**Example 3.5.** 1. \( \mathbb{C}^n \) equipped with the Euclidean metric : the involution at \( p \in \mathbb{C}^n \) is the map \( \sigma_p(x) = 2p - x \).
Élie Joseph Cartan (1869 - 1951) was an influential French mathematician, who did fundamental work in the theory of Lie groups and their geometric applications. He was the father of another influential mathematician, Henri Paul Cartan (1904 – 2008).

2. $\mathbb{P}^n$ equipped with the Fubini-Study metric: for a line $L$ in $\mathbb{C}^{n+1}$ we may consider the reflection at $L$, i.e., $s|_L = \text{id}$ and $s|_{L^\perp} = -\text{id}$. Then this reflection induces an isometry $\sigma_p$ of $\mathbb{P}^n$ with fixed point $p = \pi(L) = [L]$ and $d\sigma = -\text{id}: T_{[L]}\mathbb{P}^n \to T_{[L]}\mathbb{P}^n$. The isometry group $G = \text{Aut}(\mathbb{P}^n, g_{FS})$ is $SU(n+1)$ and the isotropy subgroup $K = SU(n)$. Thus the projective space can be realized as the homogeneous space $\mathbb{P}^n = SU(n+1)/SU(n) = SL(n+1, \mathbb{C})/P_1$, where $P_1$ is the parabolic subgroup associated with the simple root $\alpha_1$ of the Lie algebra $\mathfrak{sl}(n+1, \mathbb{C})$.

4 Rational homogeneous manifolds

The rational homogeneous manifolds are the larger class of the complex manifolds including Hermitian symmetric spaces of compact type.

Let $G$ be a complex simple Lie group and $\mathfrak{g}$ its complex simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h}$ and a root system $\Phi \subset \mathfrak{h}^*$ of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$. For readers unfamiliar with the Lie theory, we collect some definitions of terminology. See [3] and [15].

**Definition 4.1.** A complex Lie algebra $\mathfrak{g}$ is a complex vector space together with a skew-symmetric bilinear map $[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ for all vectors $X,Y,Z \in \mathfrak{g}$.

We say that $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ has no nonzero solvable ideals. A Lie algebra $\mathfrak{g}$ is semisimple if and only if it is a direct sum of simple Lie algebras.

**Definition 4.2.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

1. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an abelian subalgebra all of whose elements are semisimple, and that is not contained in any larger such subalgebra.
2. For \( \lambda \in \mathfrak{h}^* \), we set \( g_\lambda := \{ x \in \mathfrak{g} : [h,x] = \lambda(h)x \text{ for all } h \in \mathfrak{h} \} \). The nonzero weight \( \lambda \in \mathfrak{h}^* \) with \( g_\lambda \neq \{0\} \) is called the root and its weight space \( g_\lambda \) is called the root space.

3. A subsystem \( \Delta \) of a root system \( \Phi \) is called a system of simple roots (or a base) if the elements of \( \Delta \) are linearly independent and any \( \beta \in \Phi \) presents in the form \( \beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \), where \( k_\alpha \) are simultaneously either nonnegative or nonpositive integers.

Let \( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \) be a simple root system of \( \mathfrak{g} \). Choose a distinguished simple root \( \alpha_i \). Given an integer \( k, -m \leq k \leq m \), \( \Phi_k \) is the set of all roots \( \alpha = \sum_{q=1}^{\ell} m_q \alpha_q \) with \( m_i = k \). Here \( m \) is the largest integer such that \( \Phi_m \neq 0 \). For \( \alpha \in \Phi \), let \( g_\alpha \) be the corresponding root space. Define

\[
g_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha, \quad g_k = \bigoplus_{\alpha \in \Phi_k} \mathfrak{g}_\alpha, \quad k \neq 0.
\]

**Definition 4.3. (Maximal parabolic subgroup)** We say that \( p = g_0 \oplus g_1 \oplus \cdots \oplus g_m \) is the maximal parabolic subalgebra associated to the simple root \( \alpha_i \) and a corresponding Lie subgroup is a maximal parabolic subgroup \( P \) of \( G \).

**Remark.** (1) In general, a parabolic subalgebra is a Lie subalgebra containing a Borel subalgebra \( \mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \) which is a maximal solvable subalgebra of \( \mathfrak{g} \), where \( \Phi^+ \) is the set of the positive roots of \( \mathfrak{g} \).

(2) A Lie subgroup \( P \) of \( G \) is parabolic if and only if the quotient \( G/P \) is a compact manifold (in fact, a projective algebraic variety). This homogeneous space \( G/P \) is called the rational homogeneous manifold (or the generalized flag variety).

(3) The rational homogeneous manifold \( S = G/P \) is simply connected and has the positive Ricci curvature. As a complex manifold, \( S \) is Kähler manifold. Conversely, Wang has shown that all compact simply connected homogeneous Kähler manifold are of this form. For details, see [1].

(4) The rational homogeneous manifolds are just **homogeneous Fano manifolds**, where **Fano manifolds** are complex manifolds \( X \) with positive anti-canonical bundles \( K_X^{-1} = \det(T_X) = \wedge^n T_X \).

**Definition 4.4.** A rational homogeneous manifold \( S \) of Picard number 1 is a homogeneous space \( G/P \) for a complex simple Lie group \( G \) and a maximal parabolic subgroup \( P \subset G \).

In particular, an irreducible Hermitian symmetric space \( S = G/P \) of compact type is a rational homogeneous manifold of Picard number 1 such that the isotropy representation of \( P \) on \( T_x(S) \) at a base point \( x \in S \) is irreducible.

**Definition 4.5.** (Marked Dynkin diagram) The **marked Dynkin diagram** \( (\mathcal{D}(G), \alpha_i) \) of the rational homogeneous manifold \( G/P \) is the Dynkin diagram marked nodes corresponding to the simple roots \( \alpha_i \).
Figure 5: Dynkin diagrams. Every semisimple Lie algebra is classified by its Dynkin diagram. The semisimple Lie algebras over the complex numbers were first classified by Wilhelm Killing (1888-90), though his proof lacked rigor. His proof was made rigorous by Élie Cartan (1894) in his Ph.D. thesis, who also classified semisimple real Lie algebras. This was subsequently refined, and the present classification by Dynkin diagrams was given by then 22-year old Eugene Dynkin in 1947.

5 Deformation rigidity of rational homogeneous manifolds of Picard number 1

We consider the deformation rigidity problem of rational homogeneous manifolds.

Definition 5.1. Let $\mathcal{X}$ and $B$ be connected complex manifolds and let $\pi : \mathcal{X} \to B$ be a smooth proper morphism. Then the fibers $\mathcal{X}_t := \pi^{-1}(t), \ t \in B$ are compact complex submanifolds of $\mathcal{X}$.

(a) We say that $\pi : \mathcal{X} \to B$ is a smooth family of complex manifolds parametrized by $B$.

(b) Let $\pi : \mathcal{X} \to B$ be a smooth family and $0 \in B$. Then $\pi : \mathcal{X} \to B$ is called a deformation of the (central) fiber $X = \mathcal{X}_0$.

Theorem 5.2 (Ehresmann). Let $\pi : \mathcal{X} \to B$ be a smooth proper family of differentiable manifolds. Then all fibers are diffeomorphic. So every proper family of differentiable manifolds is locally diffeomorphic to a product.

Now, we have a natural question about deformation as a complex manifold: “when the general fibers are all isomorphic(biholomorphic), does the central fiber still converge to the model manifold?” But the answer is “No”.

Example 5.3. Let us consider the manifold $\mathcal{X}$ of all tuples $(x, y, t) \in \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{C}$ satisfying $x_0^2y_0 - x_1^2y_1 - tx_0x_1y_2 = 0$. Then the projection $\pi : \mathcal{X} \to \mathbb{C}$ defines a family of smooth surfaces. For $t \neq 0$, the fiber $\mathcal{X}_t$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ from the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathcal{X}_t$, $(x, y) = ([x_0 : x_1], [y_0 : y_1]) \mapsto (x, [(x_1y_0 + x_0y_1)^2 : (x_0y_0 - x_0y_1)^2 : 4t^{-1}x_0x_1y_0y_1], t)$. However, the central fiber $\mathcal{X}_0$ is not at all isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, it is the Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ over $\mathbb{P}^1$. 

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So a product of two Riemann spheres can be holomorphically deformed to a Hirzebruch surface, hence $\mathbb{P}^1 \times \mathbb{P}^1$ is not rigid under deformation. We can regard the reason of this as the fact that $\mathbb{P}^1 \times \mathbb{P}^1$ has two different hyperplane, more precisely, Picard number 2. In the case that the general fiber is a rational homogeneous manifold $S$ of Picard number 1, $S$ is rigid under deformation.

**Theorem 5.4** (Main Theorem of [10] and Theorem 1.2 of [8]). Let $\pi : \mathcal{X} \to \Delta$ be a smooth and projective morphism from a complex manifold $\mathcal{X}$ to the unit disc $\Delta$. Suppose for any $t \in \Delta \setminus \{0\}$, the fiber $\mathcal{X}_t = \pi^{-1}(t)$ is biholomorphic to a rational homogeneous manifold $S$ of Picard number 1 except $F$ and projective morphism from a complex manifold $S$ is rigid under deformation.

**Remark.** (1) In the above statement, $\mathbb{F}^5 = SO(7, \mathbb{C})/P_2 = (B_3, \alpha_2)$ means the space of lines on the 5-dimensional hyperquadric $Q^5 \subset \mathbb{P}^6$. Then $\mathbb{F}^5$ is a 7-dimensional uniruled projective manifold of Picard number 1 and homogeneous under the Lie group $PSO(7, \mathbb{C}) = SO(7, \mathbb{C})/\{\pm I\}$ of projective motions of $\mathbb{P}^6$ preserving the quadric $Q^5$. This exception exists because of the fact that the quadric $Q^5$ is homogeneous under the smaller Lie subgroup of $PSO(7, \mathbb{C})$, the group of type $G_2$.

(2) The method of proof consists of studying the deformation of minimal rational curves and their associated varieties of minimal rational tangents (VMRT for short), where VMRT is the subvariety of $\mathbb{P}(TX)$ consisting of the tangent directions of minimal rational curves immersed in $X$.

For a general reference on the theory of rational curves, see [13] and [9].

**Definition 5.5.** Let $X$ be a polarized uniruled projective manifold of dimension $n$.

(a) We say that a (parameterized) rational curve $h : \mathbb{P}^1 \to X$ is free if $h^*(TX)$ is nonnegative in the sense that $h^*(TX)$ splits into a direct sum $O(a_1) \oplus \cdots \oplus O(a_n)$ of line bundles of degree $a_i \geq 0$ for all $i = 1, \ldots, n$.

(b) A minimal rational curve is a free rational curve of minimal degree among all free rational curves on $X$. Let $\mathcal{J}$ be a connected component of minimal rational curves and let $\mathcal{K} := \mathcal{J}/\text{Aut}(\mathbb{P}^1)$ be the quotient space of (unparameterized) minimal rational curves. We call $\mathcal{K}$ a minimal rational component.

(c) A rational curve $h : \mathbb{P}^1 \to X$ is said to be standard if $h^*(TX) \cong O(2) \oplus [O(1)]^p \oplus O^q$ for some nonnegative integers $p$ and $q$. There is a subvariety $E$ of $X$ such that for any $x \in X \setminus E$, any minimal rational curve passing through $x$ is free and a general minimal rational curve passing through $x$ is standard. The smallest subvariety $E$ of $X$ with this property is called the bad locus of $\mathcal{K}$.

(d) For a general point $x \in X$ consider the subvariety $\mathcal{K}_x$ of $\mathcal{K}$ consisting of minimal rational curves belonging to $\mathcal{K}$ marked at $x$. Define the tangent map $\Phi_x : \mathcal{K}_x \to \mathbb{P}(T_x X)$ by $\Phi_x([h(\mathbb{P}^1)]) = [dh(T_o \mathbb{P}^1)]$, where $h : \mathbb{P}^1 \to X$ is a minimal rational curve with $h(o) = x$. The image $C_x(X) := \Phi_x(\mathcal{K}_x)$ is called the variety of minimal rational tangents of $X$ at $x$.

**Example 5.6.** Examples of Varieties of minimal rational tangents of rational homogeneous manifolds.
1. If \( X = \mathbb{P}^n \), then VMRT at any point \( x \in \mathbb{P}^n \) is \( C_x(X) = \mathbb{P}^{n-1} = \mathbb{P}(T_xX) \).

2. If \( X = \mathbb{Q}^n \) is a quadric in \( \mathbb{P}^{n+1} \), then VMRT at a general point \( x \in \mathbb{Q}^n \) is \( C_x(X) = \mathbb{Q}^{n-2} \subset \mathbb{P}(T_xX) = \mathbb{P}^{n-1} \).

3. If \( X = G/K = SU(p+q)/S(U(p) \times U(q)) = Gr(p,p+q) \) is the Grassmannian of \( p \)-planes in \( \mathbb{C}^{p+q} \), then VMRT at a base point \( o \) of \( X \) is \( C_o(X) = \mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \subset \mathbb{P}(T_oX) \).

4. Let \( X = G/P = \left( A_4, \alpha_2 \right) = Gr(2,5) \) be the Grassmannian of 2-planes in \( \mathbb{C}^5 \) and \( Z = G_0/P_0 = \left( A_3, \alpha_2 \right) = Gr(2,4) \) be the Grassmannian of 2-planes in \( \mathbb{C}^4 \). Note that \( Z = G_0/P_0 \) is a rational homogeneous manifold associated to a subdiagram of the marked Dynkin diagram of \( X = G/P \).

Then VMRT at a base point \( o \) of \( X \) is \( C_o(X) = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}(T_oX) \) and VMRT at a base point \( o \) of \( Z \) is \( C_o(Z) = \mathbb{P}^1 \times \mathbb{P}^1 \).

6 Characterization of standard embeddings of rational homogeneous manifolds of Picard number 1

Let \( X = G/P \) be a rational homogeneous manifold of Picard number one. A subdiagram of the marked Dynkin diagram of \( G/P \) induces naturally an embedding \( \varphi : G_0/P_0 \to G/P \) of a rational homogeneous manifold \( G_0/P_0 \) into \( G/P \). By a standard embedding of \( G_0/P_0 \) into \( G/P \) we will mean the composite \( g \circ \varphi \) for any \( g \in G = \text{Aut}(X) \). When \( G/P \) is associated to a long root and \( G_0/P_0 \) is not linear, we have a characterization of standard embeddings of \( G_0/P_0 \) into \( G/P \) by means of varieties of minimal rational tangents as follows.

**Theorem 6.1** (Hong-Mok, 2010, Theorem 1.2 of [5]). Let \( X = G/P \) be a rational homogeneous manifold associated to a long simple root and let \( Z = G_0/P_0 \) be a rational homogeneous manifold associated to a subdiagram of the marked Dynkin diagram of \( X \). Assume that \( Z \) is not linear. If \( f : U \to X \) is a holomorphic embedding from a connected open subset \( U \) of \( Z \) into \( X \) which respects varieties of minimal rational tangents for a general point \( z \in U \), then \( f \) is the restriction of a standard embedding of \( Z \) into \( X \).

**Remark.** Recently, we prove an analog of Theorem 6.1 in the case that \( G/P \) is associated to a short root. When \( X \) is associated to a short root, varieties of minimal rational tangents are not homogeneous any more so that the proof uses projective geometry of varieties of minimal rational tangents, while the proof is more representation theoretic in the case that \( X \) is associated to a long root.
**Theorem 6.2** (Hong-Park, 2011, Theorem 1.2 of [6]). Let $X = G/P$ be a rational homogeneous manifold associated to a short simple root and let $Z = G_0/P_0$ be a rational homogeneous manifold associated to a subdiagram of the marked Dynkin diagram of $X$. Assume that $Z$ is not linear. If $f : U \to X$ is a holomorphic embedding from a connected open subset $U$ of $Z$ into $X$ which respects varieties of minimal rational tangents for a general point $z \in U$, then $f$ is the restriction of a standard embedding of $Z$ into $X$.

If $Z$ is linear, then there is a non-standard embedding of an open subset of $Z$ into $Z$. In this case we will consider only the image $f(U)$. We also have a characterization of maximal linear spaces in $X$.

**Theorem 6.3** (Hong-Park, 2011, Theorem 1.3 of [6]). Let $X = G/P$ be a rational homogeneous manifold associated to a simple root and let $Z$ be a linear space. Let $f : U \to X$ be a holomorphic embedding from a connected open subset $U$ of $Z$ into $X$ such that $df(P(T_z Z))$ is contained in $C_f(z)(X)$ for any $z \in U$. If there is a maximal linear space $Z_{\text{max}}$ of $X$ of dimension $\dim U$ which is tangent to $f(U)$ at some point $f(z) \in f(U)$, then $f(U)$ is contained in $Z_{\text{max}}$, excepting when $(X, Z_{\text{max}})$ is given by

1. $X$ is associated to $(B_\ell, \alpha_i) (i \leq \ell - 1)$ and $Z_{\text{max}}$ is $\mathbb{P}^{d-i}$;

2. $X$ is associated to $(C_\ell, \alpha_\ell)$ and $Z_{\text{max}}$ is $\mathbb{P}^1$;

3. $X$ is associated to $(F_4, \alpha_1)$ and $Z_{\text{max}}$ is $\mathbb{P}^2$.

**References**


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