Finite topological spaces

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Introduction

1. Spaces with finite number of open sets studied by Alexandroff;
2. Fundamental papers of McChord and Stong in 1966;
3. Almost nothing in the 20^{th} century;
4. Dozens of articles since 2000.

Topological spaces

Let X be a non empty set. A topology on X is a collection $\mathcal{O}$ of subsets of X such that

- $\emptyset$ and X belong to $\mathcal{O}$;
- if $(O_j)_j$ is a collection of elements of $\mathcal{O}$, then $\cup_j O_j$ is also an element of $\mathcal{O}$;
- if $(O_j)_j$ is a **finite** collection of elements of $\mathcal{O}$, then $\cap_j O_j$ is also an element of $\mathcal{O}$.

Elements of a topology $\mathcal{O}$ are called *open sets*, their complements are called *closed sets*. $(X, \mathcal{O})$ is called *topological space* (or just *space*).

Examples

Let X be a set, then $\mathcal{O} = \{\emptyset, X\}$ is a topology on X called *coarse topology* or *undiscrete topology* on X.

Let X be a set, then $\mathcal{O} = \mathcal{P}(X)$ (the set of all subsets of X) is a topology on X called *discrete topology* on X.

$X = \{0, 1\}$, with $\mathcal{O} = \{\{0\}, \{0, 1\}\}$ is a topological space called *Serpinski space*.

Remark on finite spaces

Let X be a finite topological space. Then every union of closed sets is closed (since it is a finite union!). So the closed sets for a topology $\mathcal{O}$ also define a topology on X called *opposite topology*.

Minimal basis of a finite space

Let X be a finite topological space. Then every intersection of open sets is also open.

Let x be an element of X. We denote U_x the intersection of all open sets of $\mathcal{O}$ containing x. And $\mathcal{B} = \{ \ U_x / x \in X \ \}$

Theorem

U_x is the smallest open set of $\mathcal{O}$ containing x. Moreover, $\mathcal{B}$ is a minimal basis of $\mathcal{O}$. Here, minimal means that if $\mathcal{C}$ is another basis of $\mathcal{O}$, then $\mathcal{B}$ in contained in $\mathcal{C}$.

Examples

Let X be a finite topological space. Then:

- If X is endowed with the discrete topology, then $U_x = \{x\}$ for every $x \in X$.
- Conversely, if X is endowed with the coarse topology, then $U_x = X$ for every $x \in X$.

Let $X = \{0, 1\}$ be the Serpinski space. Then $U_0 = \{0\}$ and $U_1 = \{0, 1\}$

Characterization of minimal basis

Theorem

Let X be a non empty finite set and $\mathcal{B}$ a family of subsets of X. Then $\mathcal{B}$ is a minimal basis for some topology on X if and only if the three following properties are satisfied:

- The elements of $\mathcal{B}$ cover X;
- For every A and B in $\mathcal{B}$, $A \cap B$ is an union of elements of $\mathcal{B}$;
- If $(A_j)_{j\in J}$ is a family of elements of $\mathcal{B}$ such that $\bigcup_{j\in J} A_j$ is also in $\mathcal{B}$, then there $j_0 \in J$ such that $A_{j_0} = \bigcup_{j\in J} A_j$.

Preorder of a finite space

Let $(X, \mathcal{O})$ be a finite topological space and $\mathcal{B} = \{ U_x \ / \ x \in X \}$ its minimal basis. We can define a relation $\leq$ on X by

$$x \leq y \Leftrightarrow U_x \subset U_y$$

Alternatively, $x \leq y$ if and only if $x \in U_y$.

The relation $\leq$ is a preorder, i.e. it is reflexive and transitive.

Examples

Let X be a finite topological space. Then:

- If X is endowed with the discrete topology, then an element of X is only comparable to itself. So $\leq$ is the equality.
- Conversely, if X is endowed with the coarse topology, then every element is related to every element of X.

Let $X = \{0, 1\}$ be the Serpinski space. Then we have $0 < 1$.

Axioms of separation

Let X be a (non necesarly finite) topological space. Then we say that:

- X is T_0 if for every pair (a, b) of distinct points of X, there exists an open set O which contains a but not b, or there exists an open set O which contains b but not a.
- X is T_1 if for every pair (a, b) of distinct points of X, there exists an open set O which contains a but not b, and there exists an open set O which contains b but not a.
- X is T_2 (or Haussdorff) if for every pair (a, b) of distinct points of X, there exists an open set O and an open set U such that O contains a, U contains b and O and U are disjoint.

T_2 implies T_1 and T_1 implies T_0.

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T_2 implies T_1 and T_1 implies T_0.

Examples

Examples:

A coarse space is not T_0.

A discrete space is T_2.

The Serpinski space is T_0 but not T_1.

Two important theorems (1)

Let $(X, \mathcal{O})$ be a finite topological space. Then the following properties are equivalent:

- the topology $\mathcal{O}$ is T_1;
- the topology $\mathcal{O}$ is Haussdorff;
- $\mathcal{O}$ is the discrete topology;
- $\mathcal{O}$ is metrizable.

Two important theorems (2)

Let $(X, \mathcal{O})$ be a finite topological space. Then the following properties are equivalent:

- the topology $\mathcal{O}$ is T_0;
- the application $x \mapsto U_x$ is injective;
- the preorder $\leq$ is an order.

Connectedness of finite spaces

Theorem:
Let X be a finite topological space. Then U_x is arcwise connected for every x in X.

Corollary:
A finite topological space is locally arcwise connected.

Corollary:
A finite topological space is connected if and only if it is arcwise connected.

Total order

Proposition:

Let X be a finite topological space and $\leq$ its associated preorder. If $\leq$ is a total order, then X is arcwise connected.

Remark: The converse statement is false, even if one assumes $\leq$ to be an order. For, consider $X = \{a, b, c\}$ with minimal basis $U_a = \{a\}$, $U_b = \{b\}$ and $U_c = \{a, b, c\}$. Then, X is arcwise connected but $\leq$ is not a total order.

Let $X = \{1, \ldots, n\}$ be a set with n element. The generalized Serpinski topology on X is the topology whose minimal basis consists of $U_k = \{1, \ldots, k\}$, for every $k \in X$. Notice that the preorder of the generalized Serpinski topology is the natural order on X.

Proposition:

Let $(X, \mathcal{O})$ be a finite topological space. Then, if the associated preorder $\leq$ is a total order, then $(X, \mathcal{O})$ is homeomorphic to a generalized Serpinski space.

Continuity

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two finite topological spaces and $f : X \to Y$ be an application. Then f is continuous if and only if f is order-preserving.

$$\forall x, y \in X, \quad x \leq_X y \;\Rightarrow\; f(x) \leq_Y f(y)$$

Homeomorphisms

Let $(X, \mathcal{O})$ be a finite topological space and f a continuous map from X to itself. Then, f is an homeomorphism if and only if f is injective or surjective.

Let $(X, \mathcal{O})$ be a finite topological space and f a bijective map from X to itself. Then, f is an homeomorphism if and only if f is continuous.

Theorem of Kuratowski

Let X be a (non necesarly finite) topological space and A a subset of X. Applying the two operators $^{-}$ (closure) and ^{o} (interior), how many different sets can we obtain?

Kuratowski theorem

We can obtain at most 7 different sets. Moreover, there exists a topological space X and a subset A of X such that, applying $^{-}$ and ^{o} to A, we can obtain exactly 7 different sets.

Finite version of Kuratowski theorem

Definition: Let X be a topological space and A a subset of X. The Kuratowski number of A is the number of different sets one can obtain applying $\overline{}$ and ^{o} to A.

Question 1: Does there exist a finite topological space X such that there exists a subset A of X whose Kuratowski number is 7?

Question 2: What is the minimal cardinal for such a space X?

Let X be the space $X = \{a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4\}$ with topology with basis

$$\begin{array}{llll} U_{a_1} = \{a_1, a_2\}, & U_{a_2} = \{a_2\}, & & \\ U_{b_1} = \{b_1, b_2\}, & U_{b_2} = \{b_1, b_2\}, & & \\ U_{c_1} = \{c_1\}, & U_{c_2} = \{c_1, c_2\}, & U_{c_3} = \{c_1, c_2, c_3, c_4\}, & U_{c_4} = \{c_4\}, \end{array}$$

Finally, we put $A = \{a_1\} \sqcup \{b_1\} \sqcup \{c_1, c_3\}$.

The Kuratowski number of A is 7.

For the second question, the answer is between 5 and 8.

Topogeneous matrix of a finite space

Let $(X, \mathcal{O})$ be a finite topological space with n elements.

We denote $x_1, x_2, \ldots, x_n$ the elements of X and U_i the minimal open set containing x_i. Then we define a topogeneous matrix of X to be the square matrix of order n whose $(i, j)^{th}$ entry is 1 if x_j belong to U_i and 0 otherwise.

For example, a topogeneous of a discrete space is the identity matrix and the topogeneous matrix of a coarse space is the matrix whose entries are all 1.

Topogeneous matrix(2)

Theorem:

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two finite topological spaces and M and N be topogeneous matrices associated to X and Y respectively. Then X and Y are homeomorphic if and only if there exists a permutation matrix P such that $N = P'MP$ (where P' is the transpose matrix of P).

Spaces with few points

Up to homeomorphisms:

1. There exists only 1 topological space with one point: the discrete space;
2. there exists 3 topological spaces with two points: the discrete space, the coarse space and the Serpinski space;
3. there exists 9 topological spaces with three points.
4. there exists 33 topological spaces with four points.

Problem: find names for those spaces.

Problem: is there a way to simply describe all the finite space?

Non-Haussdorff cones and suspensions

Let X be a finite topological space. We put $X = \{x_1, \dots, x_n\}$ and denote $\mathcal{B} = \{U_1, \dots, U_n\}$ its minimal basis.

Definition

The non-Haussdorff cone over X is the space $\mathbb{C}X = X \sqcup \{+\}$ with minimal basis $\mathcal{B}' = \{U_1, \dots, U_n, U_+\}$ where $U_+ = X \sqcup \{+\}$.

Definition

The non-Haussdorff suspension over X is the space $\S X = X \sqcup \{-, +\}$ with minimal basis $\mathcal{B}' = \{U'_-, U'_1, \dots, U'_n, U'_+\}$ where $U'_- = X \sqcup \{-\}$, $U'_j = U_j \sqcup \{j\}$ and $U'_+ = X \sqcup \{+\}$.

The "product problem"

Let X and Y be two finite topological spaces. Then $X \times Y$, endowed with the product topology, is also a finite space (!)

Problem: Given a finite topological space Z, how one can know if Z is homeomorphic to a non trivial product of topological spaces?

Case of topological groups

Definition:
A topological group is a topological space G with a group structure such that the maps $g \mapsto g^{-1}$ and $(g, h) \mapsto gh$ are continuous.

Theorem:
A finite topological is a disjoint union of coarse spaces with the same cardinal. Therefore, a finite topological group is a product of a discrete space and a coarse space. Moreover, the coarse space containing the identity is a normal closed subgroup.

Group of homeomorphisms

Theorem:
Let X be a finite topological space. Then, for the compact-open topology, the group $(H = Homeo(X), \circ)$ of homeomorphisms of X is a topological group. Moreover, the preorder on H is

$$f \leq_H g \Leftrightarrow \forall x \in X, f(x) \leq g(x)$$

Theorem:
If X is T_0, then $Homeo(X)$ is discrete.

Homotopy

Let X and Y be two topological spaces and f and g two continuous maps from X to Y. We say that f and g are homotopy equivalent if there exists a continuous map F from $X \times [0, 1]$ to Y such that $f(x, 0) = f(x)$ and $f(x, 1) = g(x)$ for every x in X.

If (X, x_0) and (Y, y_0) are ponited spaces and f and g satisfy $f(x_0) = g(x_0) = y_0$, then we also require that $F(x_0, t) = y_0$ for every t.

Homotopy equivalence of maps is an equivalence relation.

Homotopy groups of a topological space

Let X be a topological space and a base point x_0 in X. A loop is a continuous map γ from $[0,1]$ to X such that $\gamma(0) = \gamma(1) = x_0$.

$\pi_1(X, x_0)$ is the set of the equivalence classes of loops of X at x_0 for the homotopy equivalence relation. One can define multiplication and inverse of theses classes and $\pi_1(X, x_0)$ becomes a group.

Taking maps from $(S^n, *)$ to (X, x_0), one can also define homotopy groups $\pi_n(X, x_0)$ for every $n \geq 0$.

Moreover, every map from (X, x_0) to (Y, y_0) induces group homomorphisms f_* from $\pi_i(X, x_0)$ to $\pi_i(Y, y_0)$. X and Y are said weak homotopy equivalent if all these maps are isomorphisms.

Finite circle

Let $X = \{a, b, c, d\}$ with minimal basis $U_a = \{a\}$, $U_b = \{b\}$, $U_c = \{a, b, c\}$ and $U_d = \{a, b, d\}$. Then X is weakly homotopy equivalent to the circle S^1.

From finite posets to finite T_0-spaces

We know that to a finite T_0-space, we can associate a finite poset.

Conversely, if $(X, \leq)$ is a finite poset, then we can define a topology on X whose associated preorder is $\leq$:

$$\forall x \in X, U_x = \{ \ y \in X \ / \ y \leq x \ \}$$

This clearly defines a minimal basis for a topology on X.

Simplicial complexes

Definition:
A simplicial complex on a finite set V is a collection K of subsets of V such that:

1. $\emptyset$ is element of K;
2. for every σ in K, we have: $\tau \subset \sigma \Rightarrow \tau \in K$.

Geometric realization of a simplicial complex

Definition

A geometric complex is a finite collection $\mathcal{C}$ of simplexes of $\mathbb{R}^n$ such that

- If C belongs to $\mathcal{C}$, then every face of C belongs to $\mathcal{C}$;
- If C and C' belong to $\mathcal{C}$, then $C \cap C'$ is a facet of C and a facet of C'.

The support of $\mathcal{C}$ is the union $|\mathcal{C}|$ of all its elements. We say that $\mathcal{C}$ is a realization of a simplicial complex K (on a set V) if there is a bijection ϕ from V to the set of vertices of $\mathcal{C}$ such that for every σ in K $\phi(\sigma)$ is the set of vertices of some element of $\mathcal{C}$.

Order complex of a finite topological space

Let X be a finite T_0 topological space and $\leq$ its associated order. A chain of X is a sequence $(x_1, \ldots, x_p)$ of elements of X such that $x_j < x_{j+1}$ for every j.

We define the order complex $\mathcal{K}(X)$ of X as follows:

1. $V = X$;
2. $\emptyset$ belongs to $\mathcal{K}(X)$;
3. $\{x_1, \ldots, x_p\}$ belongs to $\mathcal{K}(X)$ if and only if up to order $(x_1, \ldots, x_p)$ is a chain of X.

Examples

Let $X = \{a, b, c, d\}$ the space with $U_a = \{a\}$, $U_b = \{a, b\}$, $U_c = \{a, b, c\}$ and $U_d = \{a, b, d\}$ as minimal basis. Then $\mathcal{K}(X)$ is the complex whose vertices are a, b, c and d and with two triangles $\{a, b, c\}$ and $\{a, b, d\}$.

But, if consider the minimal basis $U_a = \{a, b, c, d\}$, $U_b = \{b, c, d\}$, $U_c = \{c\}$ and $U_d = \{d\}$, we obtain the same complex.

Face poset of a simplicial complex

Let K be a simplicial poset on a set V. Then the face poset of K is the topological space $\mathcal{F}(K)$ whose poset is $(K, \subset)$.

Proposition

For every simplicial complex K, $K' = \mathcal{K}(\mathcal{F}(K))$ is the barycentric subdivision of K. As a consequence, $|K'|$ and $|K|$ are homeomorphic.

Definition

Let X and Y be two topological spaces. A map f from X into Y is said weak homotopy equivalence if it induces isomorphisms from $\pi_i(X)$ onto $\pi_i(Y)$ for every $i \geq 0$.

Theorem

$|\mathcal{K}(X)|$ is weak homotopy equivalent to X.

Corollary

For every simplicial complex K, $|K|$ and $\mathcal{F}(K)$ are weakly homotopy equivalent.

The weak homotopy equivalence

If X is a finite T_0 space, then, for every element σ of $\mathcal{K}(X)$, then $\sigma = \{x_{j_1}, \ldots, x_{j_r}\}$ where $x_{j_1} < \cdots < x_{j_r}$. Then the map f from $\mathcal{K}(X)$ to X defined by $f(\sigma) = x_{j_1}$ is a weak homotopy equivalence.

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