# Nonlocalized Modulation of Periodic Reaction Diffusion Waves: Nonlinear Stability

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# Periodic traveling reaction-diffusion waves

Consider a periodic (period one) traveling wave solution of a system of reaction diffusion

$$u_t = u_{xx} + f(u), \quad u \in \mathbb{R}^n,\ x \in \mathbb{R},\ t \geq 0,$$

or, equivalently, a stationary 1-periodic solution $u(x,t) = \bar{u}(x)$ of

$$k_* u_t = k_*^2 u_{xx} + f(u) + k_* c u_x, \tag{RD}$$

where $c$ is the speed of the original traveling wave, and the wave number $k_*$ is chosen so that $\bar{u}(x+1) = \bar{u}(x)$.

**Goal:** Spectral stability implies nonlinear mudulational stability of $\bar{u}(x)$ of (RD) under small perturbations consisting of a **nonlocalized modulation** plus a localized perturbation.

# Previous result - Localized modulational stability

- Spectral stability implies nonlinear stability of $\bar{u}(x)$ of (RD) under small localized ($L^1$) perturbation. (Assume $\bar{u}$ is spectrally stable)

**Theorem (Mathew A. Johnson, Kevin Zumbrun, 2011)**

*Let $\bar{u}$ be a 1-periodic stationary solution of $k_* u_t = k_*^2 u_{xx} + f(u) + k_* c u_x$, and let $E_0 := \|\tilde{u}_0(\cdot) - \bar{u}(\cdot)\|_{L^1(\mathbb{R}) \cap H^K(\mathbb{R})}$. Then, there exists a solution $\tilde{u}(x,t)$ of (RD) with initial data $\tilde{u}_0$ and a phase function $\psi(x,t)$ with $\psi(x,0) = 0$ such that for $t > 0$ and $2 \le p \le \infty$,*

$$\|\tilde{u}(\cdot - \psi(\cdot,t), t) - \bar{u}(\cdot)\|_{L^p(\mathbb{R})} \lesssim E_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}},$$

$$\|\tilde{u}(\cdot - \psi(\cdot,t), t) - \bar{u}(\cdot)\|_{H^k(\mathbb{R})} \lesssim E_0 (1+t)^{-\frac{3}{4}},$$

$$\|(\psi_t, \psi_x)(\cdot,t)\|_{H^k(\mathbb{R})} \lesssim E_0 (1+t)^{-\frac{3}{4}},$$

$$\|\tilde{u}(\cdot,t) - \bar{u}(\cdot)\|_{L^p(\mathbb{R})}, \quad \|\psi(\cdot,t)\|_{L^p(\mathbb{R})} \lesssim E_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})}.$$

# Nonlocalized modulational stability

We study nonlinear stability of $\bar{u}(x)$ under nonlocalized initial perturbation

$$\tilde{u}_0(x) - \bar{u}(x) \notin L^1(\mathbb{R}).$$

• By setting $h_0 := \psi(x,0)$ and $d_0(x) = \tilde{u}_0(x - h_0(x)) - \bar{u}(x)$, we replace the nonlocalized perturbation $\tilde{u}_0(x) - \bar{u}(x)$ by a different nonlocalized perturbation $d_0 + h_0\bar{u}'$. (for the localized case, we set $h_0(x) = 0$.)

• We estimate nonlocalized modulational perturbation

$$\tilde{u}(x - \psi(x,t), t) - \bar{u}(x)$$

for some $\psi(x,0) = h_0(x) \notin L^1(\mathbb{R})$ under small perturbations

$$E_0 := \|\tilde{u}_0(\cdot - h_0) - \bar{u}(\cdot)\|_{L^1(\mathbb{R}) \cap H^K(\mathbb{R})} + \|\partial_x h_0\|_{L^1(\mathbb{R}) \cap H^K(\mathbb{R})}.$$

• Roughly speaking, what if we shift $\bar{u}$ slightly out of phase at $\pm\infty$, i.e. $\tilde{u}_0(x) \approx \bar{u}(x + h_\pm)$? Will it still relax back to its original form?

## Theorem (Johnson, Noble, Rodrigues, Zumbrun, 2013)

*Let* $K \geq 3$. *Assuming* $(H1)-(H2)$ *and* $(D1)-(D3)$, *let*

$$E_0 := \|\tilde{u}_0(\cdot - h_0) - \bar{u}(\cdot)\|_{L^1(\mathbb{R})\cap H^K(\mathbb{R})} + \|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^K(\mathbb{R})}$$

*be sufficiently small. Then, there exists a solution* $\tilde{u}(x,t)$ *of (RD) with initial data* $\tilde{u}_0$ *and a phase function* $\psi(x,t)$ *with initial data* $\psi(x,0) = h_0$ *such that for* $t > 0$ *and* $2 \leq p \leq \infty$,

$$\begin{aligned}
\|\tilde{u}(\cdot - \psi(\cdot,t),t) - \bar{u}(\cdot)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})},\\
\|\tilde{u}(\cdot - \psi(\cdot,t),t) - \bar{u}(\cdot)\|_{H^k(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{4}},\\
\|(\psi_t,\psi_x)(\cdot,t)\|_{W^{k+1,p}(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})},\\
\|\tilde{u}(\cdot,t) - \bar{u}(\cdot)\|_{L^\infty(\mathbb{R})}, \quad \|\psi(\cdot,t)\|_{L^\infty(\mathbb{R})} &\lesssim E_0.
\end{aligned}$$

$(D1)-(D3)$ : spectral stability assumption.

- Slower decay by factor $(1+t)^{1/2}$ than in the localized data case.
- No longer asymptotic stability, only bounded stability in $L^\infty$.

## Linearization and Bloch operators

By linearization of (RD) about $\bar{u}$,

$$k_* v_t = k_* L v := (k_*^2 \partial_x^2 + k_* c \partial_x + b) v, \quad b = df(\bar{u}(x)) \tag{0.1}$$

operating on $L^2(\mathbb{R})$ with densely defined domain $H^2(\mathbb{R})$. Since all coefficients of $L$ are 1-periodic, by the Floquet theory,

$$v(x) = e^{i\xi x} w(\xi, x) \quad \text{with} \quad w(\xi, x+1) = w(\xi, x).$$

Inserting into (0.1), define a family of Bloch operators, for $\xi \in [-\pi, \pi)$,

$$k_* L_\xi := e^{-i\xi x} k_* L e^{i\xi x} = k_*^2 (\partial_x + i\xi)^2 + k_* c(\partial_x + i\xi) + b,$$

acting on $L^2_{\text{per}}([0,1])$ with densely defined domain $H^2_{\text{per}}([0,1])$, that is,

$$L(e^{i\xi x} f) = e^{i\xi x}(L_\xi f) \quad \text{for 1-periodic functions } f.$$

Moreover, $\sigma_{L^2(\mathbb{R})}(L) = \bigcup_{\xi \in [-\pi,\pi)} \sigma_{L^2_{per}([0,1])}(L_\xi)$.

From the inverse Fourier transform, introduce the inverse Bloch transform:

$$g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} \check{g}(\xi, x) d\xi,$$

where $\check{g}(\xi, x) := \sum_{j \in \mathbb{Z}} e^{i2\pi jx} \hat{g}(\xi + 2\pi j) \in L^2_{\text{per}}([0,1])$ : Bloch transform.

Since $L(e^{i\xi x} g) = e^{i\xi x}(L_\xi g)$ for 1-periodic $g$, Bloch solution formula for periodic-coefficients operators $L$ is

$$(S(t)g)(x) := e^{Lt} g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t} \check{g}(\xi, x) d\xi,$$

that is, $(e^{t\check{L}} g)(\xi, x) = (e^{tL_\xi} \check{g}(\xi, \cdot))(x)$.

For $q \le 2 \le p$ with $1 = \frac{1}{p} + \frac{1}{q}$, denote

$$\|g\|_{L^q([-\pi,\pi],L^p([0,1]))} := \Big( \int_{\pi}^{\pi} \|g(\xi,\cdot)\|^q_{L^p([0,1])} d\xi \Big)^{1/q}.$$

Then For any 1-periodic $g(\xi,\cdot) \in L^q([-\pi,\pi], L^p([0,1]))$, there is $f \in L^p(\mathbb{R})$ such that $\check{f}(\xi,x) = g(\xi,x)$. In fact,

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} g(\xi,x) d\xi.$$

By the generalized Hausdorff-Young inequality

$$\|u\|_{L^p(\mathbb{R})} \le \|\check{u}\|_{L^q([-\pi,\pi)),L^p([0,1]))},$$

for 1-periodic functions $g(\xi,\cdot)$, $\xi \in [-\pi,\pi)$,

$$\Big\| \int_{-\pi}^{\pi} e^{i\xi\cdot} g(\xi,\cdot) d\xi \Big\|_{L^p(\mathbb{R})} \le (2\pi)^{1/p} \|g\|_{L^q([-\pi,\pi)),L^p([0,1]))}.$$

: this is a key formulation for the linear estimates $e^{Lt}g$.

# Diffusive spectral stability conditions

$L\bar{u}'(x) = 0$ and $\bar{u}'(x)$ is 1-periodic on $[0, 1]$ ; 0 is an eigenvalue of $L_0$.

(D1) $\sigma(L) \subset \{\lambda \in \mathbb{C} : \mathcal{R}(\lambda) < 0\} \cup \{0\}$.
(D2) There exists a constant $\theta > 0$ such that $\sigma(L_\xi) \subset \{\lambda \mid \mathcal{R}\lambda \leq -\theta|\xi|^2\}$ for each $\xi \in [-\pi, \pi)$.
(D3) $\lambda = 0$ is a simple eigenvalue of $L_0$.

**Remark.** By (D3), there exists a simple zero eigenfunction $\bar{u}'$ of $L_0$, thus bifurcates to an eigenfunction $\phi(\xi, \cdot)$, with associated left eigenfunction $\tilde{\phi}(\xi, \cdot)$ and eigenvalue

$$\lambda(\xi) = ai\xi - d\xi^2 + O(|\xi|^3), \quad |\xi| \leq \xi_0 \text{ sufficiently small,}$$

where $a$ and $d$ are real with $d > 0$.

# Nonlinear Perturbation Equations

For $\tilde{u}(x,t)$ satisfying $k_*\tilde{u}_t = k_*^2\tilde{u}_{xx} + f(\tilde{u}) + k_* c\tilde{u}_x$ and $\psi(x,t)$ to be determined, set

$$u(x,t) = \tilde{u}(x-\psi(x,t),t) \quad \text{and} \quad v(x,t) = u(x,t) - \bar{u}(x).$$

## Lemma (Nonlinear perturbation equation)

*The modulated perturbations $v(x,t)$ satisfies*

$$k_*(\partial_t - L)(v + \psi\bar{u}') = \mathcal{Q} + \mathcal{R}_x + (k_*\partial_t + k_*^2\partial_x^2)\mathcal{S} + \mathcal{T},$$

*where*

$$\mathcal{Q} := f(v+\bar{u}) - f(\bar{u}) - df(\bar{u})v,$$

$$\mathcal{R} := -k_* v\psi_t - k_*^2 v\psi_{xx} + k_*^2(\bar{u}_x + v_x)\frac{\psi_x^2}{1-\psi_x},$$

$$\mathcal{S} := v\psi_x, \quad \textit{and} \quad \mathcal{T} := -(f(v+\bar{u}) - f(\bar{u}))\psi_x.$$

Setting $\mathcal{N} := \mathcal{Q} + \mathcal{R}_x + (k_* \partial_t + k_*^2 \partial_x^2)\mathcal{S} + \mathcal{T}$, by Duhamel's principle,

$$v(x,t) = -\bar{u}'(x)\psi(x,t) + S(t)(d_0 + \bar{u}' h_0) + \int_0^t S(t-s)\mathcal{N}(s)ds,$$

where

$$(S(t)g)(x) := e^{Lt}g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t}\check{g}(\xi, x)d\xi.$$

**TODO:**

- Estimate linear behavior $S(t)$ in terms of the localized data $d_0$ and nonlocalized data $\bar{u}' h_0$.
- Define an appropriate modulation $\psi(x,t)$ with $\psi(x,0) = h_0$. (Idea : Decompose S(t) into a bad part "$\bar{u}s^p(t)$" and a good part "$\tilde{S}(t)$" and define $\psi(x,t) \approx s^p(t)(d_0 + \bar{u}' h_0)$; $v \sim \psi_x$ )
- Estimate $v$ for nonlinear stability.

# Decomposition of the solution operator

Recalling the solution operator of $L$

$$(S(t)g)(x) = e^{Lt}g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t}\check{g}(\xi, x)d\xi,$$

decompose this as

$$S(t) = S^p(t) + \tilde{S}(t), \quad S^p(t) = \bar{u}' s^p(t),$$

$$(s^p(t)g)(x) := \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}\langle\tilde{\phi}(\xi,\cdot), \check{g}(\xi,\cdot)\rangle_{L^2[0,1]}d\xi,$$

$$\begin{aligned}(\tilde{S}(t)g)(x) :=& \int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))(e^{L_\xi t}\check{g}(\xi))(x)d\xi \\ &+ \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)(e^{L_\xi t}\tilde{\Pi}(\xi)\check{g}(\xi))(x)d\xi \\ &+ \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}(\phi(\xi,x)-\phi(0,x))\langle\tilde{\phi}(\xi,\cdot), \check{g}(\xi,\cdot)\rangle_{L^2[0,1]}d\xi\end{aligned}$$

- $\alpha(\xi)$ is a smooth cut off function such that $\alpha(\xi) = 1$ for sufficiently small $|\xi| \leq \frac{1}{2}\xi_0$ and $\alpha(\xi) = 0$ for $|\xi| \geq \xi_0$.

  For sufficiently small $|\xi| \leq \xi_0$,

- $\phi(\xi, x)$ : right eigenfunction of $L_\xi$ associated to $\lambda(\xi)$ bifurcating from $\bar{u}'(x)$ at $\xi = 0$.

- $\Pi^p(\xi) := \phi(\xi)\langle\tilde{\phi}(\xi), \cdot\rangle_{L^2([0,1])}$ : the eigenprojection onto the eigenspace Range$\{\phi(\xi)\}$ bifurcating from Range$\{\bar{u}'\}$ at $\xi = 0$. In particular,

$$(e^{L_\xi t}\Pi^p(\xi)\check{g}(\xi))(x) = e^{\lambda(\xi)t}\phi(\xi, x)\langle\tilde{\phi}(\xi, \cdot), \check{g}(\xi, \cdot)\rangle_{L^2[0,1]}$$

- $\tilde{\phi}(\xi, x)$ : the associated left eigenfunction of $L_\xi$ and $\tilde{\Pi} := Id - \Pi^p$.

# Basic linear estimates

GOAL : Estimate the linear contribution of the localized data.

**Proposition.** Under assumptions $(D1)-(D3)$, for all $t>0$ and $2\le p\le\infty$,

$$\left\|\partial_x^l\partial_t^m s^p(t)\partial_x^r g\right\|_{L^p(\mathbb{R})}\lesssim \min\begin{cases}(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{l+m}{2}}\|g\|_{L^1(\mathbb{R})},\\(1+t)^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{p})-\frac{l+m}{2}}\|g\|_{L^2(\mathbb{R})},\end{cases}$$

for $0\le r\le K+1$, and for some $\eta>0$ and $0\le l+2m,\ r\le K+1$,

$$\begin{aligned}&\left\|\partial_x^l\partial_t^m\tilde{S}(t)\partial_x^r g\right\|_{L^p(\mathbb{R})}\\&\quad\lesssim\min\begin{cases}(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|g\|_{L^1(\mathbb{R})\cap H^{l+2m+1}(\mathbb{R})},\\e^{-\eta t}\|\partial_x^r g\|_{H^{l+2m+1}(\mathbb{R})}+(1+t)^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}\|g\|_{L^2(\mathbb{R})}.\end{cases}\end{aligned}$$

## Proof : $s^p$ bounds

(i) The first estimate on $s^p$: treat the case $l = m = r = 0$.

$$
\begin{aligned}
&(s^p(t)(g))(x) \\
&= \int_{\pi}^{\pi} \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \langle \tilde{\phi}, \check{g} \rangle_{L^2([0,1])}(\xi) d\xi \\
&= \sum_{j \in \mathbb{Z}} \int_{\pi}^{\pi} \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \langle \tilde{\phi}(\xi, \cdot), e^{i2\pi j \cdot} \rangle_{L^2([0,1])} \hat{g}(\xi + 2\pi j) d\xi \\
&= \sum_{j \in \mathbb{Z}} \int_{\pi}^{\pi} \alpha(\xi) e^{\lambda(\xi)t} e^{i\xi x} \hat{\tilde{\phi}}_j(\xi)^* \hat{g}(\xi + 2\pi j) d\xi
\end{aligned}
$$

where $\hat{\tilde{\phi}}_j(\xi)$ denotes the $j$th Fourier coefficient in the Fourier expansion of periodic function $\tilde{\phi}(\xi)$, and $z^*$ denotes complex conjugate.

By Hausdorff-Young's inequality, $\|\hat{g}\|_{L^\infty(\mathbb{R})} \leq \|g\|_{L^1(\mathbb{R})}$ and $|e^{\lambda(\xi)t}\alpha^{1/2}(\xi)| \leq e^{-\eta\xi^2 t}$, $\eta > 0$, we thus get

$$\begin{aligned}
\|s^p(t)(g)\|_{L^p(\mathbb{R})} &\lesssim \|e^{-\eta\xi^2 t}\|_{L^q(\xi;[-\pi,\pi])}\|g\|_{L^1(\mathbb{R})} \sup_{|\xi|\leq\xi_0} \sum_j |\hat{\tilde{\phi}}_j(\xi)| \\
&\lesssim (1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\|g\|_{L^1(\mathbb{R})} \sup_{|\xi|\leq\xi_0} \|\tilde{\phi}(\xi,\cdot)\|_{H^1([0,1])} \\
&\lesssim (1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\|g\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Here, we used by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\alpha(\xi)^{1/2}\sum_j |\hat{\tilde{\phi}}_j(\xi)| &\leq \alpha(\xi)^{1/2}\sqrt{\sum_j (1+|j|^2)|\hat{\tilde{\phi}}_j(\xi)|^2 \sum_j (1+|j|^{-2})} \\
&\lesssim \alpha(\xi)^{1/2}\|\tilde{\phi}(\xi)\|_{H^1([0,1])}.
\end{aligned}$$

Examining $s^p(t)g = \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}\langle\tilde{\phi}(\xi,\cdot),\check{g}(\xi,\cdot)\rangle_{L^2[0,1]}d\xi$, we see that $\partial_t$ yields additional factor of $\lambda(\xi)\sim|\xi|$ in the integrand, while $\partial_x$ yields factor exactly $i\xi$. This yields an additional factor of $t^{-\frac{1}{2}}$ decay in the estimates, verifying the bounds claimed for $\partial_{x,t}s^p$, $r=0$. For $r>0$, pass $\partial_x^r$ derivatives onto $\tilde{\phi}$ in the inner product using integration by parts.

**Remark:** $S^p = \bar{u}'s^p$ includes $\bar{u}'$, hence $\partial_x$ does not improve $L^p$ estimates for the full operator $S^p$. This is a crucial point, and the reason we need to separate out $s^p$ in our analysis.

(ii) The second estimate on $s^p$: for $\frac{1}{r}+\frac{1}{s}=1$,

$$\begin{aligned}
&\|(s^p(t)(g))(x)\|_{L^p(\mathbb{R})} \\
&= \| \int_{\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}\langle\tilde{\phi},\check{g}\rangle_{L^2([0,1])}(\xi)d\xi\|_{L^p(\mathbb{R})} \\
&\lesssim \|\alpha(\xi)e^{\lambda(\xi)t}\langle\tilde{\phi},\check{g}\rangle_{L^2([0,1])}\|_{L^q([-\pi,\pi],L^p([0,1]))} \\
&\lesssim \|e^{-\eta\xi^2 t}\|\check{g}\|_{L^2([0,1])}\|_{L^q([-\pi,\pi])} \\
&\lesssim \|e^{-\eta\xi^2 t}\|_{L^{rq}([0,1])}\|\check{g}\|_{L^{sq}([-\pi,\pi],L^2([0,1]))} \\
&\lesssim (1+t)^{-\frac{1}{2rq}}\|\check{g}\|_{L^{sq}([-\pi,\pi],L^2([0,1]))}
\end{aligned}$$

Choose $s$ such that $\frac{1}{sq}=\frac{1}{2}$; then

$$\|(s^p(t)(g))(x)\|_{L^p(\mathbb{R})} \lesssim (1+t)^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}\|g\|_{L^2(\mathbb{R})}.$$

# Proof : $\tilde{S}$ bounds

Recall that $(\tilde{S}(t)g)(x)$ consists of three terms:

$$\int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))(e^{L_\xi t}\check{g}(\xi))(x)d\xi, \quad \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)(e^{L_\xi t}\tilde{\Pi}(\xi)\check{g}(\xi))(x)d\xi$$

$$\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}(\phi(\xi,x)-\phi(0,x))\langle\tilde{\phi}(\xi,\cdot),\check{g}(\xi,\cdot)\rangle_{L^2[0,1]}d\xi.$$

The last term has an additional factor $O(|\xi|)$ in the integrand by Taylor expansion, yielding the claimed bounds by the same proof as $s^p$ bounds.

The first and second terms satisfy time-exponential $H^s \to H^s$ decay bounds due to spectral gap of $L_\xi$ on $\xi \in supp t\alpha$ or restricted to Range($\tilde{\Pi}(\xi)$).

# Linear behavior for modulational data

GOAL : Estimate the linear contribution of data $h_0\bar{u}'$ (nonlocalized data) in terms of $\partial_x h_0$ (localized data).

**Proposition.** Under assumptions $(D1)-(D3)$, for all $t>0$ and $2\leq p\leq\infty$,

$$\left\|\partial_x^l\partial_t^m s^p(t)(h_0\bar{u}')\right\|_{L^p(\mathbb{R})}\lesssim(1+t)^{-\frac{1}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{l+m}{2}}\|\partial_x h_0\|_{L^1(\mathbb{R})}$$

for $l+m\leq 1$ or else $l=m=0$ and $p=\infty$, and for $0\leq l+2m\leq K+1$,

$$\left\|\partial_x^l\partial_t^m \tilde{S}(t)(h_0\bar{u}')\right\|_{L^p(\mathbb{R})}\lesssim(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^{l+2m+1}},$$

and when $t\leq 1$

$$\left\|\partial_x^l\partial_t^m (S^p(t)-Id)(h_0\bar{u}')\right\|_{L^p(\mathbb{R})}\lesssim\|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^{l+2m+1}},$$

$$\left\|\partial_x^l\partial_t^m (s^p(t)(h_0\bar{u}')-h_0)\right\|_{L^p(\mathbb{R})}\lesssim\|\partial_x h_0\|_{L^1(\mathbb{R})\cap H^{l+2m+1}}.$$

## Proof : $s^p$ bounds

(i) treat case $l = 1$ and $m = 0$; other similar. Re-express

$$
\begin{aligned}
&\partial_x(s^p(t)(h_0\bar{u}'))(x) \\
&= \int_{\pi}^{\pi} i\xi\alpha(\xi)e^{\lambda(\xi)t}e^{i\xi x}\langle\tilde{\phi}\bar{u}', \check{h}_0\rangle_{L^2([0,1])}(\xi)d\xi \\
&= \sum_{j\in\mathbb{Z}} \int_{\pi}^{\pi} i\xi\alpha(\xi)e^{\lambda(\xi)t}e^{i\xi x}\langle\tilde{\phi}(\xi,\cdot)\bar{u}'(\cdot), e^{i2\pi j\cdot}\rangle_{L^2([0,1])}\hat{h}_0(\xi+2\pi j)d\xi \\
&= \sum_{j\in\mathbb{Z}} \int_{\pi}^{\pi} i\xi\alpha(\xi)e^{\lambda(\xi)t}e^{i\xi x}\widehat{\tilde{\phi}(\xi)\bar{u}'_j}^{\,*}\hat{h}_0(\xi+2\pi j)d\xi \\
&= \sum_{j\in\mathbb{Z}} \int_{\pi}^{\pi} \frac{\xi}{\xi+2\pi j}\alpha(\xi)e^{\lambda(\xi)t}e^{i\xi x}\widehat{\tilde{\phi}(\xi)\bar{u}'_j}^{\,*}\widehat{\partial_x h}_0(\xi+2\pi j)d\xi,
\end{aligned}
$$

where $\widehat{\tilde{\phi}(\xi)\bar{u}'_j}$ denotes the $j$th Fourier coefficient in the Fourier expansion of periodic function $\tilde{\phi}(\xi)\bar{u}'$.

By H-Y and $|e^{\lambda(\xi)t}\alpha^{1/2}(\xi)| \le e^{-\eta\xi^2 t}$, $\eta > 0$, we thus get

$$\|\partial_x s^p(t)(h_0\bar{u}')\|_{L^p(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\|\partial_x h_0\|_{L^1(\mathbb{R})} \sup_{|\xi|\le\xi_0} \sum_j \Big|\frac{\widehat{\tilde{\phi}(\xi)\bar{u}'_j}}{1+|j|}\Big|$$

yielding the result together with the Cauchy-Schwarz inequality

$$\sum_j \Big|\frac{\widehat{\tilde{\phi}(\xi)\bar{u}'_j}}{1+|j|}\Big| \le \sqrt{\sum_j (1+|j|)^{-2} \sum_j |\widehat{\tilde{\phi}(\xi)\bar{u}'_j}|^2} \lesssim \|\tilde{\phi}(\xi)\bar{u}'\|_{L^2([0,1])}.$$

# Proof : $\tilde{S}$ bounds

(i) Re-express $\tilde{S}(t)(h_0\bar{u}')(x)$ as

$$\int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))(e^{L_\xi t}\check{h}_0(\xi)\bar{u}')(x)d\xi$$
$$+\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)(e^{L_\xi t}\tilde{\Pi}(\xi)\check{h}_0\bar{u}'(\xi))(x)d\xi$$
$$+\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}(\phi(\xi,x)-\phi(0,x))\langle\tilde{\phi}(\xi,\cdot),\hat{h}_0(\xi)\bar{u}'\rangle_{L^2[0,1]}d\xi$$
$$=\sum_{j\in\mathbb{Z}}\int_{-\pi}^{\pi} e^{i\xi x}(1-\alpha(\xi))\frac{(e^{L_\xi t}(\bar{u}'e^{2i\pi j\cdot}))(x)}{i(\xi+2\pi j)}\widehat{\partial_x h_0}(\xi+2\pi j)d\xi$$
$$+\sum_{j\in\mathbb{Z}}\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)\frac{(e^{L_\xi t}\tilde{\Pi}(\xi)(\bar{u}'e^{2i\pi j\cdot}))(x)}{i(\xi+2\pi j)}\widehat{\partial_x h_0}(\xi+2\pi j)d\xi$$
$$+\sum_{j\in\mathbb{Z}}\int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi)e^{\lambda(\xi)t}\frac{(\phi(\xi,x)-\phi(0,x))\widehat{\tilde{\phi}(\xi)\bar{u}'_j}^{*}}{i(\xi+2\pi j)}\widehat{\partial_x h_0}(\xi+2\pi j)d\xi.$$

Use $1-\alpha(\xi)=O(|\xi|)$, $\phi(\xi,x)-\phi(0,x)=O(|\xi|)$, and

$$\tilde{\Pi}(\xi)\bar{u}'=\tilde{\Pi}[\tilde{\Pi}(\xi)-\tilde{\Pi}(0))\bar{u}'], \quad |\tilde{\Pi}(\xi)-\tilde{\Pi}(0))|\lesssim|\xi|$$

to bound key $j=0$ terms, plus C-S inequality for $j\neq 0$.

(ii) For $t\leq 1$, expand

$$S^p(t)-Id=S^p(t)-S(0)=(S^p(t)-S^p(0))-\tilde{S}(0)=t\partial_t S^p(s(t))-\tilde{S}(0)$$

for some $s(t)\in(0,t)$ and use previous t-derivative bounds of $S^p$.

Similarly, expand

$$s^p(t)(h_0\bar{u}')-h_0=(s^p(t)-s^p(0))(h_0\bar{u}')+(s^p(0)(h_0\bar{u}')-h_0).$$

# Nonlinear Damping Estimate

**Proposition.** Let $v(\cdot,0) \in H^K(\mathbb{R})$ and suppose that for some $T > 0$, $\|(v,\psi_t)(t)\|_{H^K(\mathbb{R})}$ and $\|\psi_x(t)\|_{H^{K+1}(\mathbb{R})}$ remain bounded by a sufficiently small constant for all $0 \le t \le T$. Then there are positive constants $\theta$ and $C$, independent of $T$, such that, for all $0 \le t \le T$,

$$\|v(t)\|^2_{H^K(\mathbb{R})} \le Ce^{-\theta t}\|v(0)\|^2_{H^K} + C\int_0^t e^{-\theta(t-s)}\Big(\|v(s)\|^2_{L^2(\mathbb{R})} + \|(\psi_t,\psi_x)(s)\|_{H^K(\mathbb{R})}\Big)ds.$$

**Proof.** From the nonlinear perturbation equation

$$(1-\psi_x)v_t - v_{xx} - cv_x = df(\bar{u})v + \mathcal{Q} - (\bar{u}+v_x)\psi_t + ((\bar{u}_x+v_x)\psi_x)_x + \left((\bar{u}_x+v_x)\frac{\psi_x^2}{1-\psi_x}\right)_x - f(\bar{u}+v)\psi_x,$$

taking $L^2(\mathbb{R})$ inner product against $\sum_{j=0}^{K} \frac{(-1)^j \partial_x^{2j} v}{1-\psi_x}$, integrating by parts,

$$\frac{d}{dt}\|v(t)\|^2_{H^K(\mathbb{R})} \le -\tilde{\theta}\|\partial_x^{K+1} v(t)\|^2_{L^2(\mathbb{R})} + C\Big(\|v(t)\|^2_{H^K(\mathbb{R})} + \|(\psi_t, \psi_x)(t)\|^2_{H^K(\mathbb{R})}\Big),$$

for some $\tilde{\theta} > 0$ so long as $\|(v, \psi_t, \psi_x, \psi_{xx})(t)\|_{H^K(\mathbb{R})}$ remains small. By Sobolev interpolation $\|g\|^2_{H^K(\mathbb{R})} \le C^{-1}\|\partial_x^{K+1} g\|^2_{L^2(\mathbb{R})} + C\|g\|^2_{L^2(\mathbb{R})}$,

$$\frac{d}{dt}\|v(t)\|^2_{H^K(\mathbb{R})} \le \theta\|v(t)\|^2_{H^K(\mathbb{R})} + C\Big(\|v(t)\|^2_{L^2(\mathbb{R})} + \|(\psi_t, \psi_x)(t)\|^2_{H^K(\mathbb{R})}\Big),$$

from which the desired estimate follows by Gronwall's inequality.

## Nonlinear Iteration Scheme

Recall a integral representation $v$ from the nonlinear perturbation equation

$$v(x,t) = -\bar{u}'(x)\psi(x,t) + S(t)(d_0 + \bar{u}'h_0) + \int_0^t S(t-s)\mathcal{N}(s)ds,$$

where $d_0(x) = v(x.0) \in L^1(\mathbb{R}) \cap H^K(\mathbb{R})$, $h_0(x) = \psi(x,0)$ and $\partial_x h_0 \in L^1(\mathbb{R}) \cap H^K(\mathbb{R})$.

Recalling $S(t) = \bar{u}'s^p(t) + \tilde{S}(t)$, define $\psi(t)$ to cancel $s^p$ contributions

$$\begin{aligned}\psi(t) &= s^p(t)(d_0 + h_0\bar{u}') + \int_0^t s^p(t-s)\mathcal{N}(s)ds \\ &\quad - (1-\chi(t))\Big(s^p(t)(d_0 + h_0\bar{u}') - h_0 + \int_0^t s^p(t-s)\mathcal{N}(s)ds\Big),\end{aligned}$$

where $\chi(t)$ is a smooth cutoff that is zero for $t \leq 1/2$ and one for $t \geq 1$.

Then we have

$$v(t) = \tilde{S}(t)(d_0 + h_0\bar{u}') + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)ds$$
$$+ (1-\chi(t))\Big(S^p(t)d_0 + (S^p(t) - Id)h_0\bar{u}' + \int_0^t S^p(t-s)\mathcal{N}(s)ds\Big).$$

- We impose a smooth cutoff function to have $\psi(x,0) = h_0(x)$.
- $\mathcal{N}$ consists of $v$ and $\partial_x$-derivatives of $(\psi_x, \psi_t)$; so consider the system $\|(v, \psi_x, \psi_t)\|_{H^K(\mathbb{R})}$. From the linear estimate, we easily guess $v \sim \psi_x$.

## Nonlinear Stability

Define $\eta(t) := \sup_{0\le s\le t} \|(v, \psi_t, \psi_x)(s)\|_{H^K(\mathbb{R})}(1+s)^{\frac{1}{4}}$.

**Lemma**

*For $E_0 := \|(d_0, \partial_x h_0)\|_{L^1(\mathbb{R})\cap H^3(\mathbb{R})}$ sufficiently small and $t \ge 0$,*

$$\eta(t) \le C(E_0 + \eta^2(t))$$

*so long as $\eta \le C$.*

**Proof:** Bounding $x$-derivatives by definition and $t$-derivative using equation, we have

$$\|\mathcal{N}(t)\|_{L^1(\mathbb{R})\cap H^1(\mathbb{R})} \lesssim \|(v, \psi_t, \psi_x)(t)\|^2_{H^3(\mathbb{R})} \le C\eta(t)^2(1+t)^{-\frac{1}{2}},$$

so long as $\eta(t)$ bounded.

Applying the linearized (basic and modulational bounds), get

$$\begin{aligned}\|v(t)\|_{L^p(\mathbb{R})} &\le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}E_0 \\ &\quad + C\eta(t)^2\int_0^t (1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+s)^{-\frac{1}{2}}ds \\ &\le C_p(E_0+\eta^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\end{aligned}$$

($C_p$ depends on $p$, i.e., $C_p \to \infty$ as $p \to \infty$) and

$$\begin{aligned}\|(\psi_t,\psi_x)(t)\|_{W^{K+1,p}(\mathbb{R})} &\le C(1+t)^{-\frac{1}{2}}E_0 \\ &\quad + C\eta(t)^2\int_0^t (1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+s)^{-\frac{1}{2}}ds \\ &\le C_p(E_0+\eta^2(t))(1+t)^{-\frac{1}{2}(1-\frac{1}{p})};\end{aligned}$$

so gives $\|(\psi_t,\psi_x)(t)\|_{H^{K+1}(\mathbb{R})}$ small, verifying the hypotheses of nonlinear damping estimates bounding $\|v\|_{L^2}$ and $\|(\psi_t,\psi_x)(t)\|_{H^K(\mathbb{R})}$. $\square$

# Proof of the main theorem

Recalling $\eta(t) := \sup_{0\le s\le t} \|(v,\psi_t,\psi_x)(s)\|_{H^K}(\mathbb{R})(1+s)^{\frac{1}{4}}$.

If $E_0 := \|(d_0, \partial_x h_0)\|_{L^1(\mathbb{R})\cap H^3(\mathbb{R})}$ sufficiently small, say $E_0 < \frac{1}{4C^2}$, then $\eta(t) \le 2CE_0$ for all $t$; thus we obtain immediately, for all $t > 0$,

$$\|(v,\psi_t,\psi_x)(s)\|_{H^K}(\mathbb{R}) \lesssim E_0(1+t)^{\frac{1}{4}}.$$

Cycling these back through our estimates gives the $L^p$ bounds for $p \le$ any fixed $p_*$, finite (set $p_* > 4$):

$$\|v(t)\|_{L^p(\mathbb{R})}, \quad \|(\psi_t,\psi_x)(t)\|_{W^{K+1,p}(\mathbb{R})} \le CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}.$$

We extend to $p \ge p_*$ by using different linear estimates and

$$\|(\mathcal{Q},\mathcal{R},\mathcal{S},\mathcal{T})(t)\|_{L^2(\mathbb{R})} \le \|(v,\psi_t,\psi_x,\psi_{xx})(t)\|^2_{L^4(\mathbb{R})} \le CE_0(1+t)^{-\frac{3}{4}}.$$

**Thank you for your attention!**