D. Bambusi and S. Cuccagna, On dispersion of small energy solutions of the nonlinear Klein Gordon equation with a potential

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2017 Aug. 22 at Daejeon

We consider the nonlinear Klein Gordon equation (NLKG):

$$\begin{cases} u_{tt} - \Delta u + Vu + u + u^3 = 0, \\ (u(0,x), u_t(0,x)) = (u_0(x), v_0(x)) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where

- $u = u(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$; unknown function,
- $V = V(x) \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$; given function.

Aim

To show scattering if energy is sufficiently small, by using Birkhoff normal form method.

Assumptions:

- 0 is neither an eigenvalue nor a resonant for $-\Delta + V$.
  i.e. there is no $u \in L^2(\mathbb{R}^3)$ s.t. $\Delta u = Vu$,
  and there is no $u \in C^\infty(\mathbb{R}^3)$ s.t. $\Delta u = Vu$ with $|u(x)| \lesssim \langle x \rangle^{-1}$
- $\sigma(-\Delta + V) = \{-\lambda^2\} \cup [0, \infty)$, $\lambda \in (0, 1)$, and
  $\sigma_d(-\Delta + V) = \{-\lambda^2\}$.
  Furthermore, we also assume that the multiplicity of λ is one,
  and let φ be its eigenfunction with $\|\varphi\|_{L^2} = 1$.
  (We can show that $\varphi \in \mathcal{S}$.)
- Let $\omega = \sqrt{1-\lambda}$. Then, we assume $\frac{1}{\omega} \notin \mathbb{Z}$.
  By this assumption, we take $N \in \mathbb{Z}$ s.t. $N\omega < 1 < (N+1)\omega$.
- Furthermore, we assume the condition (5.31), which is introduced by 2nd presenter.

Denote $K_0(t) = \frac{\sin(t\sqrt{-\Delta+1})}{\sqrt{-\Delta+1}}$ for $t > 0$.

Theorem ([Bambusi-Cuccagna '11])

There exists $\epsilon_0 > 0$ and $C > 0$ such that the following holds: If $\|(u_0, v_0)\|_{H^1\times L^2} \le \epsilon < \epsilon_0$, then the solution to (1.1) is global, and there exists $(u_\pm, v_\pm)$ with $\|(u_\pm, v_\pm)\|_{H^1\times L^2} \le C\epsilon$ such that

$$\lim_{t\to\pm\infty} \|u(t) - K_0'(t)u_\pm - K_0(t)v_\pm\|_{H^1} = 0.$$

Remark If the nonlinear term u^3 does NOT exist, then this theorem does NOT hold.

Proof Assume that the claim holds. Using the eigenfunction φ and $0 < \epsilon \ll 1$,

$$u(t) \equiv \epsilon \cos(\omega t)\varphi$$

is a solution to (1.1). However, this does NOT scatter, since

$$u(t) \not\to 0 \quad \text{in} \quad H^1.$$

Notations, function spaces and norms:

- $\mathbb{N} := \{1, 2, \cdots\}, \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- $\langle f, g \rangle := \int_{\mathbb{R}^3} f(x)g(x)dx$.
- For $k \in \mathbb{R}$, $1 < p < \infty$, $K = \mathbb{R}, \mathbb{C}$,
$$W^{k,p}(\mathbb{R}^3, K) := \{f : \mathbb{R}^3 \to K \ : \ \|f\|_{W^{k,p}} := \|(-\Delta + 1)^{k/2} f\|_{L^p} < \infty\}.$$
- $H^k(\mathbb{R}^3, K) = W^{k,2}(\mathbb{R}^3, K)$.
- For $p = 1, \infty$ and $k \in \mathbb{N}$,
$$W^{k,p}(\mathbb{R}^3, K) := \{f : \mathbb{R}^3 \to K \ : \ \partial_x^\alpha f \in L^p(\mathbb{R}^3, K) \text{ for all } |\alpha| \le k\}.$$
- For $s \in \mathbb{R}$,
$$H^{k,s}(\mathbb{R}^3, K) := \{f : \mathbb{R}^3 \to K \ : \ \|f\|_{H^{k,s}} := \|\langle x \rangle^s (-\Delta + 1)^{k/2} f\|_{L^2} < \infty\}.$$
- $L^{2,s} := H^{0,s}(\mathbb{R}^3, K)$.
- For an operator A, we denote $(A - z)^{-1}$ by $R_A(z)$.

Hamiltonian structure

It is known that various kinds of equations in physics have Hamiltonian structure, and (1.1) is one of these equations.

Consider $H^1(\mathbb{R}^3, \mathbb{R}) \times L^2(\mathbb{R}^3, \mathbb{R})$ with the standard symplectic form:

$$\Omega((u_1, v_1); (u_2, v_2)) := \langle u_1, v_2 \rangle_{L^2} - \langle u_2, v_1 \rangle_{L^2}.$$

The Hamiltonian:

$$\begin{aligned} H &= H_L + H_P, \\ H_L &:= \int_{\mathbb{R}^3} \frac{1}{2}(v^2 + |\nabla u| + Vu^2 + u^2)dx, \\ H_P &:= \int_{\mathbb{R}^3} \frac{1}{4} u^4 dx. \end{aligned}$$

(H_L: linear part, H_P: nonlinear part.)

$\longrightarrow$ We can rewrite the equation (1.1) into

$$\begin{cases} \dot{u} = \nabla_v H, \\ \dot{v} = -\nabla_u H, \end{cases}$$

where $\nabla_v H \in L^2(\mathbb{R}^3)$ is the L^2-gradient w.r.t. v, i.e.

$$\langle \nabla_v H(u,v), h \rangle = d_v H(u,v)h, \quad \forall h \in L^2(\mathbb{R}^3),$$

where $d_v H(u,v)$ is the Frechét derivative of $H(u,\cdot) : L^2 \to \mathbb{R}$.

And $\nabla_u H \in H^{-1}(\mathbb{R}^3)$ is the H^1-gradient w.r.t. u, i.e.

$$\langle \nabla_u H(u,v), h \rangle = d_u H(u,v)h, \quad \forall h \in H^1(\mathbb{R}^3),$$

where $d_u H(u,v)$ is the Frechét derivative of $H(\cdot, v) : H^1 \to \mathbb{R}$.

(Note that $(H^1)^* \simeq H^{-1}$ by the above representation.)

Small energy GWP of NLKG

We recall that NLKG (1.1) is GWP for small initial data:

Theorem (Small energy GWP of NLKG)

(i) There exists $\epsilon_0 > 0$ *and* $C > 0$ *such that the following holds:*
If $\|(u_0, v_0)\|_{H^1 \times L^2} \le \epsilon < \epsilon_0$*, then (1.1) has a unique solution* $u \in C^0(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, L^2)$.

(ii) For any bounded interval $I \subset \mathbb{R}$*, the map*

$$(u_0, v_0) \mapsto (u, v)$$

is continuous.

(iii) The Hamiltonian $H(u(t), v(t))$ *is conserved.*

(iv) $\|(u(t), v(t))\|_{H^1 \times L^2} \le C \|(u_0, v_0)\|_{H^1 \times L^2}$ *for all* $t \in \mathbb{R}$.

New variables

For our analysis, we introduce new variables.

Let $P_d := \langle \cdot, \varphi \rangle \varphi$ and $P_c := 1 - P_d$ (the orthogonal projection in L^2.)
Then, we can write

$$u = q\varphi + P_c u, \quad v = p\varphi + P_c u,$$

where $q, p \in \mathbb{R}$.

We introduce the operator B in $P_c L^2$:

$$B := P_c(-\Delta + V + 1)^{1/2} P_c$$

Remark Since $-\Delta + V + 1$ is positive operator, we can consider the fractional order.

Remark2 Note that $\sigma(B) = [1, \infty)$.

We set new variables (ξ, f) by

$$\xi := \frac{q\omega^{1/2} + ip\omega^{-1/2}}{\sqrt{2}}, \quad f := \frac{B^{1/2}P_c u + iB^{-1/2}P_c v}{\sqrt{2}}.$$

Then, we have the following fact:

Fact (Theorem 6.2)

$$\begin{array}{ccc} H^1(\mathbb{R}^3,\mathbb{R}) \times L^2(\mathbb{R}^3,\mathbb{R}) & \to & \mathcal{P}^{1/2,0} := \mathbb{C} \oplus P_c H^{1/2,0}(\mathbb{R}^3,\mathbb{C}) \\ \Cup & & \Cup \\ (u,v) & \mapsto & (\xi, f) \end{array}$$

is an isomorphism.

Thus, for these variables, our new phase space is $\mathcal{P}^{1/2,0}$.

New representation of Hamiltonian structure 1

By calculation, the Hamiltonian can be written by:

$$H_L = \omega|\xi|^2 + \langle \overline{f}, Bf \rangle$$

$$H_P = \int_{\mathbb{R}^3} \frac{1}{4} \left(\frac{\xi+\overline{\xi}}{\sqrt{2\omega}}\varphi + U(x) \right)^4 dx, \text{ where } U(x) := B^{-1/2}(f+\overline{f})/\sqrt{2}.$$

And the symplectic form becomes:

$$\Omega((\xi^{(1)}, f^{(1)}), (\xi^{(2)}, f^{(2)})) = -i(\overline{\xi}^{(1)}\xi^{(2)} - \xi^{(1)}\overline{\xi}^{(2)}) - i\left(\langle f^{(2)}, \overline{f}^{(1)} \rangle - \langle f^{(1)}, \overline{f}^{(2)} \rangle \right).$$

New representation of Hamiltonian structure 2

The Hamilton equations take the form:

$$\dot{\xi} = -i\frac{\partial H}{\partial \bar{\xi}}, \quad \dot{f} = -i\nabla_{\bar{f}}H.$$

We consider the Poisson bracket given by

$$\{H, K\} := i\left(\frac{\partial H}{\partial \xi}\frac{\partial K}{\partial \bar{\xi}} - \frac{\partial H}{\partial \bar{\xi}}\frac{\partial K}{\partial \xi}\right) + i\langle \nabla_f H, \nabla_{\bar{f}} K\rangle - i\langle \nabla_{\bar{f}} H, \nabla_f K\rangle.$$

Remark The intention of new variables:

$$H_L = \omega|\xi|^2 + \langle \bar{f}, Bf\rangle$$

discrete part (action) — $\omega|\xi|^2$

continuous part (sesqui-linear form of f) — $\langle \bar{f}, Bf\rangle$

Expand H_P in the order of U

By pointwise calculation, we have

$$\left(\frac{\xi+\overline{\xi}}{\sqrt{2\omega}}\varphi+U(x)\right)^4=\sum_{l=0}^{3}F_l(x,\xi)U^l+U^4$$

with

$$F_l(x,\xi)=\frac{4!}{l!(4-l)!}\left(\frac{\xi+\overline{\xi}}{\sqrt{2\omega}}\varphi\right)^4,\quad l=0,1,2,3.$$

The following lemma concerns with the regularity of F_l.

Lemma 3.2

(i) For any $k,s\in\mathbb{R}$ and $l=0,1,2,3$, the functions $\xi\mapsto F_l(\cdot,\xi)$ are in $C^\infty(\mathbb{C},H^{k,s})$, and $H_l(\xi,U):=\int_{\mathbb{R}^3}F_l(x,\xi)U^l dx$ are in $C^\infty(\mathbb{C}\times H^1,\mathbb{R})$.

(ii) There exists a constant C s.t. for $l=0,1,2,3$, $\|F_l(\cdot,\xi)\|_{H^{k,s}}\le C|\xi|^{4-l}$.

Strategy of Birkhoff normal form 1

First, we introduce the notion of canonical transformation.

Definition

Let $\mathcal{U} \subset \mathcal{P}^{1/2,0}$ be an open set. The map $\mathcal{T} : \mathcal{U} \to \mathcal{P}^{1/2,0}$ is a **canonical transformation** if
(i) $\mathcal{T} : \mathcal{U} \to \mathcal{T}(\mathcal{U})$ is C^∞-diffeomorphism,
(ii) $\mathcal{T}$ conserves the Hamiltonian structure, i.e. let $(\xi', f') := \mathcal{T}(\xi, f)$ be new variables, then for any smooth function $H : \mathcal{P}^{1/2,0} \to \mathbb{R}$,

$$\dot{\xi} = -i\frac{\partial H}{\partial \overline{\xi}}, \quad \dot{f} = -i\nabla_{\overline{f}} H$$

$$\iff \quad \dot{\xi}' = -i\frac{\partial (H \circ \mathcal{T}^{-1})}{\partial \overline{\xi'}}, \quad \dot{f}' = -i\nabla_{\overline{f'}}(H \circ \mathcal{T}^{-1}).$$

Strategy of Birkhoff normal form 2

Now, our Hamiltonian is $H = H_L + H_R$.
Given canonical transformation $\mathcal{T}$, let $H' := H \circ \mathcal{T}$. Then,

$$(\xi(t), f(t)) \text{ satisfies } \dot{\xi} = -i\frac{\partial H}{\partial \bar{\xi}}, \quad \dot{f} = -i\nabla_{\overline{f}} H \text{ (i.e. (1.1))}$$

$$\iff \quad (\xi'(t), f'(t)) := \mathcal{T}^{-1}(\xi(t), f(t)) \text{ satisfies}$$

$$\dot{\xi}' = -i\frac{\partial(H' \circ \mathcal{T}^{-1})}{\partial \overline{\xi'}}, \quad \dot{f}' = -i\nabla_{\overline{f'}}(H' \circ \mathcal{T}^{-1}).$$

Our wish is to find $\mathcal{T}$ which makes new Hamiltonian simpler.
The theory of Birkhoff normal form says, we can choose $\mathcal{T}$ so that

$$H' = H_L + Z + R$$

where Z is lower order polynomial of ξ, f, and R is higher order term of ξ, f.

Strategy of Birkhoff normal form 2

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Our wish is to find $\mathcal{T}$ which makes new Hamiltonian simpler.
The theory of Birkhoff normal form says, we can choose $\mathcal{T}$ so that

main terms (= $H_L + Z$) — negligible term (= R)

$$H' = H_L + Z + R$$

where Z is lower order polynomial of ξ, f, and R is higher order term of ξ, f.

Lie transform

One of the convenient way of constructing canonical transformations is Lie transform.

Set $z = (\xi, f)$.
Consider the function χ of the form

$$\chi(z) := \chi_0(\xi, \overline{\xi}) + \sum_{\mu+\nu=M_0+1} \xi^\mu \overline{\xi}^\nu (\langle \Phi_{\mu\nu}, f \rangle + \langle \Psi_{\mu\nu}, \overline{f} \rangle), \tag{4.1}$$

where

- $M_0 \in \mathbb{N}_0$,
- $\Phi_{\mu\nu}, \Psi_{\mu\nu} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$ with $\Phi_{\mu\nu} = \overline{\Psi_{\nu\mu}}$,
- χ_0 is a homogeneous polynomial of degree $M_0 + 2$ with $\chi_0(\xi, \overline{\xi}) = \overline{\chi_0(\xi, \overline{\xi})}$.

For $k, s \in \mathbb{R}$, let $\mathcal{P}^{k,s} := \mathbb{C} \oplus P_c H^{k,s}$.

Proposition 4.3 (Part I)

For any $\kappa, \tau \in \mathbb{R}$, there exists $\mathcal{U}^{\kappa,\tau} \subset \mathcal{P}^{\kappa,\tau}$; a neighborhood of $(0,0)$, s.t. for $z = (\xi, f) \in \mathcal{U}^{\kappa,\tau}$, there exists a unique solution $\phi^s(z) = (\xi(s), f(s))$ to the following ODE up to time $s = 1$:

$$\begin{cases} \dfrac{d\xi}{ds} = -i\dfrac{\partial \chi}{\partial \overline{\xi}}, \quad \dfrac{df}{ds} = -i\nabla_{\overline{f}}\chi \\ (\xi(0), f(0)) = (\xi, f), \end{cases}$$

and $\phi^s \in C^1([0,1], \mathcal{P}^{\kappa,\tau})$.

Set $\phi(z) := \phi^1(z)$. ϕ is called the **Lie transform** generated by χ.

Remark: The definition of ϕ is consistent with the choice of κ, τ. Namely, if $z \in \mathcal{P}^{\kappa,\tau} \cap \mathcal{P}^{\kappa',\tau'}$, then $\phi(z)$ constructed in $\mathcal{P}^{\kappa,\tau}$ and in $\mathcal{P}^{\kappa',\tau'}$ coincide with each other.

Then, we have the following property.

Proposition 4.3 (Part II)

(i) ϕ is a canonical transformation.
(ii) For any $k, \kappa, s, \tau \geq 0$, if $z \in \mathcal{U}^{-k,-s}$, then $z - \phi(z) \in \mathcal{P}^{\kappa,\tau}$. Moreover, there exists a constant $C_{k,\kappa,s,\tau} > 0$ s.t.

$$\|z - \phi(z)\|_{\mathcal{P}^{\kappa,\tau}} \leq C_{k,\kappa,s,\tau} |\xi|^{M_0} (|\xi| + \|f\|_{H^{-k,-s}})$$

Proposition 4.4

Let $K \in C^\infty(\mathcal{U}^{1/2,0}, \mathbb{C})$ satisfying $|K(z)| \leq C\|z\|_{\mathcal{P}^{1/2,0}}^{M_1}$ and $\|dK(z)\|_{\mathcal{P}^{-1/2,0}} \leq C\|z\|_{\mathcal{P}^{1/2,0}}^{M_1-1}$ with $M_1 \geq 2$. Then $K \circ \phi \in C^\infty(\mathcal{U}^{1/2,0}, \mathbb{C})$ and

$$|K(\phi(z))| \leq C_1 \|z\|_{\mathcal{P}^{1/2,0}}^{M_1}$$

$$|K(\phi(z)) - K(z)| \leq C_1 \|z\|_{\mathcal{P}^{1/2,0}}^{M_0+M_1}$$

for some constant $C_1 > 0$.

Normal form

We introduce the notion of normal form.

Definition

A polynomial $Z = Z(\xi, \overline{\xi}, f, \overline{f})$ is in **normal form** if we have

$$Z = Z_0 + Z_1$$

where

(i) $Z_0 = Z_0(\xi, \overline{\xi})$ (independent of $f, \overline{f}$), and Z_0 is a linear combination of monomials $\xi^\mu \overline{\xi}^\nu$ satisfying

$$\{H_L, \xi^\mu \overline{\xi}^\nu\} = 0$$

(ii) Z_1 is the form

$$\sum_{\omega(\mu-\nu)<-1} \xi^\mu \overline{\xi}^\nu \langle \Phi_{\mu\nu}, f\rangle + \sum_{\omega(\mu'-\nu')>1} \xi^{\mu'} \overline{\xi}^{\nu'} \langle \Psi_{\mu'\nu'}, \overline{f}\rangle$$

where $\Phi_{\mu\nu}, \Psi_{\mu'\nu'} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$.

This is the main theorem of 1st part:

Theorem (Birkhoff normal form (Theorem 4.9) 1/3)

For any $k, s > 0$ and $r \in \mathbb{N}_0$, there exists a neighborhood of the origin $\mathcal{U}_{r,k,s} \subset \mathcal{P}^{1/2,0}$ and $\mathcal{T}_r : \mathcal{U}_{r,k,s} \to \mathcal{P}^{1/2,0}$ with the following properties.
(1) $\mathcal{T}_r$ is a canonical transformation.
(2) $\mathcal{T}_r$ is consistent with the choice of k, s.
(3) Let $H^{(r)} := H \circ \mathcal{T}_r$. Then, we have

$$H^{(r)} = H_L + Z^{(r)} + R^{(r)},$$

where $Z^{(r)}$ and $R^{(r)}$ has the following properties.
(3-i) $Z^{(r)}$ is a polynomial of degree $r+3$ which is in normal form.
(The form $\langle \Phi, f \rangle$ is considered to be the monomial of f of 1 degree.)

Theorem (Birkhoff normal form (Theorem 4.9) 2/3)

(3-ii) $\mathcal{T}_r$ is close to identity. Namely, there is an open set $\mathcal{U}_r^{-k,-s} \subset \mathcal{P}^{-k,-s}$ s.t. $1 - \mathcal{T}_r$ can be extended into a C^∞-map from $\mathcal{U}_r^{-k,-s}$ to $\mathcal{P}^{k,s}$, and

$$\|z - \mathcal{T}_r(z)\|_{\mathcal{P}^{k,s}} \leq C_{r,k,s}\|z\|^3_{\mathcal{P}^{-k,-s}}.$$

(3-iii) $R^{(r)}$ is higher order term than $Z^{(r)}$. More precisely, we have $\mathcal{R}^{(r)} = \sum_{d=0}^4 R_d^{(r)}$ where $R_d^{(r)}$ is d-th order terms of f of the form

$$R_0^{(r)} = \sum_{\mu+\nu=r+4} \xi^\mu \overline{\xi}^\nu \int_{\mathbb{R}^3} a_{\mu\nu}^{(r)}(x, z, \mathrm{Re}B^{1/2}f(x))dx,$$

Theorem (Birkhoff normal form (Theorem 4.9) 3/3)

$$R_1^{(r)} = \sum_{\mu+\nu=r+3} \xi^\mu \overline{\xi}^\nu \int_{\mathbb{R}^3} \mathbf{\Lambda}_{\mu\nu}^{(r)}(x, z, \mathrm{Re} B^{1/2} f(x)) \cdot B^{-1/2} \mathbf{f} \, dx,$$

$$R_d^{(r)} = \int_{\mathbb{R}^3} F_d(x, z, \mathrm{Re} B^{-1/2} f(x)) \, [U(x)]^d dx, \ \textit{for } d = 2, 3,$$

$$R_4^{(r)} = \int_{\mathbb{R}^3} \frac{1}{4} [U(x)]^4 dx,$$

where $a_{\mu\nu}^{(r)}$, $\mathbf{\Lambda}_{\mu\nu}^{(r)}$ *are smooth functions, and* $\mathbf{f} = (f, \overline{f})$, $U = B^{-1/2}(f + \overline{f})$. *Moreover,*

$$\| F_2^{(r)}(\cdot, z, w) \|_{H^{k,s}} \leq C_{r,k,s} |\xi|.$$

Key lemma: Homological equation

Before moving on to the proof, we observe one lemma.

Consider a homogeneous polynomial

$$K = \sum_{\mu+\nu=M_1} K_{\mu\nu}\xi^{\mu}\overline{\xi}^{\nu} + \sum_{\mu'+\nu'=M_1-1} \xi^{\mu'}\overline{\xi}^{\nu'}\langle \Phi_{\mu'\nu'}, f\rangle + \sum_{\mu''+\nu''=M_1-1} \xi^{\mu''}\overline{\xi}^{\nu''}\langle \Psi_{\mu''\nu''}, \overline{f}\rangle,$$

where $\Phi_{\mu'\nu'}, \Psi_{\mu''\nu''} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$.

And let Z be the normal part of K. (Namely, taking the summation of the terms in normal form.)

Lemma 4.11.

There exists a polynomial χ s.t.

$$\{H_L, \chi\} + Z = K.$$

Moreover, this χ satisfies (4.1). (Namely, χ generates Lie transform.)

The role of this lemma

If we consider the Lie transform ϕ generated by χ.

$\longrightarrow$ The terms not in normal form become higher order terms in new Hamiltonian $K \circ \phi$.

(In other words, the terms in normal form are the remainder terms which we cannot remove by Lie transform.)

Iterating this procedure, we will have the result.

Sketch of proof of Theorem 4.9.

Induction on $r \in \mathbb{N}_0$

$\underline{r=0} \Longrightarrow T_0 := 1, Z^{(0)} := 0, R^{(0)} := H_P$ goes well.

Assume Theorem 4.9 holds for some $r \in \mathbb{N}_0$.
Define $\mathcal{R}_{02}^{(r)}, \mathcal{R}_{12}^{(r)}$ by

$$\mathcal{R}_0^{(r)} = \sum_{\mu+\nu=r+4} \xi^\mu \overline{\xi}^\nu \int_{\mathbb{R}^3} a_{\mu\nu}^{(r)}(x,0,0)dx + \mathcal{R}_{02}^{(r)}$$

$$\mathcal{R}_1^{(r)} = \sum_{\mu+\nu=r+3} \xi^\mu \overline{\xi}^\nu \int_{\mathbb{R}^3} \mathbf{\Phi}_{\mu\nu}^{(r)}(x) \cdot \mathbf{f} dx + \mathcal{R}_{12}^{(r)}$$

with $\mathbf{\Phi}_{\mu\nu}^{(r)}(x) := \mathbf{\Lambda}_{\mu\nu}^{(r)}(x,0,0)$.
The meaning of this decomposition:
1st term: principal term of $\mathcal{R}_0^{(r)}, \mathcal{R}_1^{(r)}$
2nd term: higher order term of $\mathcal{R}_0^{(r)}, \mathcal{R}_1^{(r)}$

Set

$$K_{r+1} := \sum_{\mu+\nu=r+4} \xi^{\mu}\overline{\xi}^{\nu} \int_{\mathbb{R}^3} a_{\mu\nu}^{(r)}(x,0,0)dx + \sum_{\mu+\nu=r+3} \xi^{\mu}\overline{\xi}^{\nu} \int_{\mathbb{R}^3} \mathbf{\Phi}_{\mu\nu}^{(r)}(x) \cdot \mathbf{f} dx.$$

(Summation of principal part)

We apply Lemma 4.11 to K_{r+1}. $\longrightarrow$ We take χ_{r+1}, Z_{r+1} s.t.

$$\{H_L, \chi_{r+1}\} + Z_{r+1} = K_{r+1}.$$

Let ϕ_{r+1} be the Lie transform generated by χ_{r+1}.

Define

$$\mathcal{T}_{r+1} := \mathcal{T}_r \circ \phi_{r+1}, \quad Z^{(r+1)} := Z^{(r)} + Z_{r+1}.$$

Then,

- $\mathcal{T}_{r+1}$ is s canonical transformation, and
- Z_{r+1} is in normal form.

Moreover, $H^{(r+1)} = H \circ \mathcal{T}_{r+1} = H^{(r)} \circ \phi_{r+1}$, and

$$
\begin{aligned}
H^{(r)} \circ \phi_{r+1} = H_L + \; & Z^{(r)} + Z_{r+1} \\
+ \; & Z^{(r)} \circ \phi_{r+1} - Z^{(r)} \\
+ \; & K_{r+1} \circ \phi_{r+1} - K_{r+1} \\
+ \; & H_L \circ \phi_{r+1} - (H_L + \{\chi_{r+1}, H_L\}) \\
+ \; & (\mathcal{R}_{02}^{(r)} + \mathcal{R}_{12}^{(r)}) \circ \phi_{r+1} \\
+ \; & \mathcal{R}_2^{(r)} \circ \phi_{r+1} \\
+ \; & \mathcal{R}_3^{(r)} \circ \phi_{r+1} \\
+ \; & \mathcal{R}_4^{(r)} \circ \phi_{r+1}
\end{aligned}
$$

Moreover, $H^{(r+1)} = H \circ \mathcal{T}_{r+1} = H^{(r)} \circ \phi_{r+1}$, and

$$H^{(r)} \circ \phi_{r+1} = H_L + Z^{(r)} + Z_{r+1}$$

$Z^{(r+1)}$ (new normal form)

$$+ Z^{(r)} \circ \phi_{r+1} - Z^{(r)}$$
$$+ K_{r+1} \circ \phi_{r+1} - K_{r+1}$$
$$+ H_L \circ \phi_{r+1} - (H_L + \{\chi_{r+1}, H_L\})$$
$$+ (\mathcal{R}_{02}^{(r)} + \mathcal{R}_{12}^{(r)}) \circ \phi_{r+1}$$
$$+ \mathcal{R}_2^{(r)} \circ \phi_{r+1}$$
$$+ \mathcal{R}_3^{(r)} \circ \phi_{r+1}$$
$$+ \mathcal{R}_4^{(r)} \circ \phi_{r+1}$$

higher order terms (absorbed in $\mathcal{R}^{(r+1)}$)

Hence, the induction works and we finish the proof.

We finish the 1st part.

Thank you for your attention.