Presentation on *Multichannel Nonlinear Scattering for Nonintegrable equations* by A. Soffer and M. I. Weinstein

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Part I

Nonlinear Schrödinger equation

Consider the nonlinear Schrödinger equation with a potential V

$$\begin{cases} i\Phi_t = [-\Delta + V(x) + |\Phi|^2]\Phi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3 \\ \Phi(0) = \Phi_0 \in H^1(\mathbb{R}^3). \end{cases} \tag{1}$$

Aim : Asymptotic stability

Given initial conditions which lie in a neighborhood of a solitary wave $e^{i\gamma_0}\psi_{E_0}$, the solution

$$\Phi(t) = e^{-i\left(\int_0^t E(s)\,ds - \gamma(t)\right)}(\psi_{E(t)} + \phi(t))$$

converges asymptotically to a solitary wave of nearby energy $E_\pm$ and phase $\gamma_\pm$ in L^4, as $t \to \pm\infty$, i.e.,

$$\Phi(t) \sim e^{-i\int_0^t E(s)\,ds} e^{i\gamma_\pm}\psi_{E_\pm}, \qquad t \to \pm\infty.$$

Hypotheses for a potential V

Hypotheses

Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function satisfying

(V1) $V \in \mathcal{S}(\mathbb{R}^3)$.

(V2) $-\Delta + V$ has exactly one negative eigenvalue E_* on $L^2(\mathbb{R}^3)$ with corresponding L^2 normalized eigenfunction ψ_*.

(V3) $V(x) = V(|x|)$.

Nonresonance Condition (NR)

V satisfies (NR) condition if 0 is neither an eigenvalue nor a resonance of $-\Delta + V$.

Nonlinear bound state

Consider a time periodic and spatially localized solution to (1) of the form

$$\Phi(t,x) = e^{-iEt}\psi_E(x).$$

ψ_E satisfies

$$\begin{aligned} H(E)\psi_E \equiv [-\Delta + V(x) + |\psi_E|^2]\psi_E = E\psi_E \\ \psi_E \in H^2, \qquad \psi_E > 0 \end{aligned} \tag{2}$$

An H^2-solution ψ_E is called a *nonlinear bound state* or *solitary wave profile.*

Note that the solution $e^{-iEt}\psi_E$ does not converge to $e^{-iE_0 t}\psi_{E_0}$, since there is a family of solitary waves.

Nonlinear bound state

Theorem-Existence of ψ_E

Let $E \in (E_*, 0)$. Then, there exists a solution $\psi_E > 0$ to (2) such that

(a) $\psi_E \in H^2$.

(b) The function $E \mapsto \|\psi_E\|_{H^2}$ is smooth for $E \neq E_*$, and

$$\lim_{E \to E_*} \|\psi_E\|_{H^2} = 0,$$

i.e. (E, ψ_E) bifurcates from the zero solution at $(E_*, 0)$ in H^2 (and therefore in L^p, $2 \leq p \leq \infty$ thanks to Sobolev embedding).

(c) For all $\varepsilon > 0$,

$$|\psi_E(x)| \leq C_{E,\varepsilon} e^{-(|E|-\varepsilon)|x|}.$$

(d) As $E \to E_*$,

$$\psi_E = \left(\frac{E - E_*}{\int \psi_*^4} \right)^{\frac{1}{2}} [\psi_* + O(E - E_*)]$$

in H^2. Here ψ_* is the normalized ground state of $-\Delta + V$ with corresponding eigenvalue E_*.

Nonlinear bound state

Corollary

For all $E \in \Omega$, any compact subinterval of $(E_*, 0)$, we have $\|\psi_E\|_{H^2} \leq C_\Omega \|\psi_E\|_{L^2}$.

Theorem-Weighted estimates

Let $E \in (E_*, 0)$. Also, E lie in a sufficiently small neighborhood of E_*. Then, for $k \in \mathbb{Z}_+$ and $s \geq 0$,

$$\|\langle x \rangle^k \psi_E\|_{H^s} \leq C_{k,s} \|\psi_E\|_{H^s}$$

and

$$\|\langle x \rangle^k \partial_E \psi_E\|_{H^s} \leq C'_{k,s} |E - E_*|^{-1} \|\psi_E\|_{H^s}$$

Remark : By above theorems and corollary, we can regard any weighted L^p norm of ψ_E and $\partial_E \psi_E$ as a constant, which tends to 0 as $E \to E_*$, in various estimates appearing in the analysis.

Decay estimates

Decay estimate

Let $K = -\Delta + V$ acting on $L^2(\mathbb{R}^3)$, and assume **Hypotheses** on V. Also, V satisfies (NR). Let $P_c(K)$ denote the projection onto the continuous spectral part of K. If $1/p + 1/q = 1$, $2 \le q \le \infty$, then

$$\|e^{itK} P_c(K)\psi\|_{L^q} \le C_q |t|^{-(3/2-3/q)} \|\psi\|_{L^p}.$$

If ψ is more regular ($\psi \in H^1$), then

$$\|e^{itK} P_c(K)\psi\|_{L^q} \le C_q \langle t \rangle^{-(3/2-3/q)} (\|\psi\|_{L^p} + \|\psi\|_{H^1}).$$

A simple consequence is the following local decay estimate

Local decay estimate

Under the same assumption as in the above theorem, let $\sigma > 3/2 - 3/q$. Then

$$\|\langle x \rangle^{-\sigma} e^{itK} P_c(K)\psi\|_{L^2} \le C_q |t|^{-(3/2-3/q)} \|\psi\|_{L^p}.$$

Decomposition of the solution Φ

We decompose the solution to (1) as

$$\Phi(t) = e^{-i\Theta}(\psi_{E(t)} + \phi(t))$$

where

$$\Phi(0) = \Phi_0 = e^{i\gamma_0}(\psi_{E_0} + \phi_0)$$

$$\Theta = \int_0^t E(s)\,ds - \gamma(t)$$

$$E(0) = E_0, \qquad \gamma(0) = \gamma_0$$

Orthogonality Condition

$$\langle \psi_{E_0}, \phi_0 \rangle = 0 \quad \text{and} \quad \frac{d}{dt}\langle \psi_{E_0}, \phi(t) \rangle = 0$$

The orthogonality condition ensures that $\phi(t)$ lies in the Range of $P_c(H(E_0))$.

Decomposition of the solution Φ

$$\begin{cases} i\phi_t = [H(E_0) - E_0]\phi + [E_0 - E(t) + \dot{\gamma}(t)]\phi + \mathbf{F}, \\ \phi(0) = \phi_0 \end{cases} \tag{3}$$

where

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2},$$
$$\mathbf{F_1} = \dot{\gamma}\psi_E - i\dot{E}\partial_E\psi_E,$$
$$\mathbf{F_2} = \mathbf{F_{2,lin}} + \mathbf{F_{2,nl}}.$$

Here $\mathbf{F_{2,lin}}$ is a linear term in ϕ of the form

$$\mathbf{F_{2,lin}} = (2\psi_E^2 - \psi_{E_0}^2)\phi + \psi_E^2\overline{\phi}$$

and $\mathbf{F_{2,nl}}$ is a nonlinear term in ϕ of the form

$$\mathbf{F_{2,nl}} = 2\psi_E|\phi|^2 + \psi_E\phi^2 + |\phi|^2\phi.$$

Decomposition of the solution Φ

The Orthogonality condition says

$$\phi(0) = \phi_0 = P_c(H(E_0))\phi_0,$$

which implies

$$\mathbf{F} = P_c(H(E_0))\mathbf{F}.$$

Moreover, we know

$$\dot{E}(t) = \langle \partial_E \psi_E, \psi_{E_0} \rangle^{-1} \mathrm{Im} \langle \mathbf{F_2}, \psi_{E_0} \rangle$$

and

$$\dot{\gamma}(t) = -\langle \psi_E, \psi_{E_0} \rangle^{-1} \mathrm{Re} \langle \mathbf{F_2}, \psi_{E_0} \rangle.$$

Linear propagator of dispersive part ϕ

Consider the homogeneous linear equation

$$\begin{cases} iu_t = (H(E_0) - E_0)u + (E_0 - E(t) + \dot{\gamma}(t))u, \\ u(s) = f. \end{cases} \tag{4}$$

Let $U(t,s)$ be the propagator associated to (4), i.e.

$$u(t) = U(t,s)f, \qquad U(s,s) = Id.$$

Using the gauge transform

$$u(t) = e^{-i\int_s^t [E_0 - E(\tau)]\, d\tau - i(\gamma(t) - \gamma(s))} v(t),$$

(4) is equivalent to the equation $iv_t = (H(E_0) - E_0)v$ with the initial data $v(s) = f$. The solution v is of the form

$$v(t) = e^{-i(H(E_0) - E_0)(t-s)} f.$$

Hence

$$U(t,s) = e^{-i\int_s^t [E_0 - E(\tau)]\, d\tau - i(\gamma(t) - \gamma(s))} e^{-i(H(E_0) - E_0)(t-s)}. \tag{5}$$

Linear propagator of dispersive part ϕ

Now (3) can be rewritten as the integral equation, in addition to the Orthogonality condition,

$$\phi(t) = U(t,0)P_c(H(E_0))\phi_0 - i\int_0^t U(t,s)P_c(H(E_0))\mathbf{F}(s)\,ds.$$

We remark that the gauge transform (5) preserves L^p or weighted L^2 norms, i.e.,

$$\|U(t,s)g\|_X = \|e^{-i(H(E_0)-E_0)(t-s)}g\|_X$$

where $X = L^p$ or a weighted L^2.

Well-posedness theory

- Contraction mapping principle $\Rightarrow$ Local well-posedness
- The equation (1) admits the following mass and energy conservation laws:

$$\mathcal{N}[\Phi(t)] \equiv \int_{\mathbb{R}^3} |\Phi(x)|^2 \, dx = \mathcal{N}[\Phi_0]$$

$$\begin{aligned}\mathcal{H}[\Phi(t)] \equiv \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\Phi(x)|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|\Phi(x)|^2 \, dx + \frac{1}{4}\int_{\mathbb{R}^3} |\Phi(x)|^4 \, dx \\ = \mathcal{H}[\Phi_0]\end{aligned}$$

- For $C_0 > 0$ such that $|V(x)| \le C_0$,

$$\|\Phi(t)\|_{H^1}^2 \le 2\mathcal{H}[\Phi_0] + (C_0+1)\mathcal{N}[\Phi_0] \le C(\|\Phi_0\|_{H^1}^2 + \|\Phi_0\|_{H^1}^4)$$

- Local well-posedness implies Global well-posedness

Main Theorem

Theorem-Asymptotic stability

Let $\Omega_\eta = (E_*, E_* + \eta)$, where η is positive and sufficiently small. Then for all $E_0 \in \Omega_\eta$ and $\gamma_0 \in [0, 2\pi)$, there exists a positive number $\epsilon = \epsilon(E_0, \eta)$ such that if

$$\Phi(0) = e^{i\gamma_0}(\psi_{E_0} + \phi_0)$$

where

$$\|\phi_0\|_{L^1(\mathbb{R}^3_x)} + \|\phi_0\|_{H^1(\mathbb{R}^3_x)} < \epsilon$$

then

$$\Phi(t) = e^{-i\int_0^t E(s)\,ds + i\gamma(t)}(\psi_{E(t)+\phi(t)})$$

with

$$\dot{E}(t), \dot{\gamma}(t) \in L^1(\mathbb{R}_t) \quad (\Rightarrow \exists \lim_{t\to\pm\infty}(E(t), \gamma(t)) = (E_\pm, \gamma_\pm))$$

Main Theorem

Theorem A. - Asymptotic stability

and $\phi(t)$ is purely dispersive in the sense that

$$\|\langle x\rangle^{-\sigma}\phi(t)\|_{L^2(\mathbb{R}^3)} = O(\langle t\rangle^{-\frac{3}{2}})$$

for $\sigma > 2$, and

$$\|\phi(t)\|_{L^4(\mathbb{R}^3)} = O(\langle t\rangle^{-\frac{3}{4}})$$

as $|t| \to \infty$.

Decomposition of initial data

Let $\widetilde{E} \in (E_*, 0)$ and $\widetilde{\gamma} \in [0, 2\pi)$ be given. Consider the initial data Φ_0, which is nearby a nonlinear bound state:

$$\Phi_0 = e^{i\widetilde{\gamma}}\psi_{\widetilde{E}} + \delta\Phi.$$

In general, $\langle \psi_{\widetilde{E}}, \delta\Phi \rangle \neq 0$, so we can find E_0 and γ_0 such that

$$\langle e^{-i\gamma_0}\Phi_0 - \psi_{E_0}, \psi_{E_0} \rangle = 0,$$

i.e.

$$\begin{aligned} \Phi_0 &:= e^{i\gamma_0}(\psi_{E_0} + \phi_0) \\ &= e^{i\gamma_0}\psi_{E_0} + [e^{i\widetilde{\gamma}}\psi_{\widetilde{E}} - e^{i\gamma_0}\psi_{E_0} + \delta\Phi]. \end{aligned}$$

Indeed, let

$$F[E, \gamma, \delta\Phi] := \langle \psi_E, \phi_0 \rangle = \langle e^{i\gamma}\psi_E, e^{i\widetilde{\gamma}}\psi_{\widetilde{E}} - e^{i\gamma}\psi_E + \delta\Phi \rangle.$$

Then $F[\widetilde{E}, \widetilde{\gamma}, 0] = 0$.

Decomposition of initial data

We write

$$F[E,\gamma,\delta\Phi] = F_1[E,\gamma,\delta\Phi] + iF_2[E,\gamma,\delta\Phi].$$

The Jacobian matrix of $(E,\gamma,\delta\Phi) \mapsto (F_1,F_2)$ is given by

$$\begin{bmatrix} -\frac{1}{2}\frac{d}{dE}\int |\psi_E|^2\Big|_{E=\widetilde{E}} & 0 \\ 0 & \int |\psi_E|^2\Big|_{E=\widetilde{E}} \end{bmatrix}$$

at $(\widetilde{E},\widetilde{\gamma},0)$. Since the curve $E \mapsto \|\psi_E\|_{L^2}^2$ has no critical point for $E \in (E_*,0)$, the determinant of the Jacobian matrix at $(\widetilde{E},\widetilde{\gamma},0)$ is nonzero. By the implicit function theorem, for any $\delta\Phi$ near 0, there uniquely exists (E_0,γ_0) near $(\widetilde{E},\widetilde{\gamma})$ such that $F[E_0,\gamma_0,\delta\Phi] = 0$, i.e. the decomposition

$$\Phi_0 = e^{i\gamma_0}(\psi_{E_0} + \phi_0)$$

with $\langle \psi_{E_0}, \phi_0 \rangle = 0$ holds.

Thank You
for Your Attention!!