Presentation on *Multichannel Nonlinear Scattering for Nonintegrable equations* by A. Soffer and M. I. Weinstein

Presenter : Chulkwang Kwak

Facultad de Matemáticas
Pontificia Universidad Católica de Chile

2017 Participating School, KAIST
August 21–25, 2017

Part II

Goal

Theorem-Asymptotic stability

Let $\Omega_\eta = (E_*, E_* + \eta)$, where η is positive and sufficiently small. Then for all $E_0 \in \Omega_\eta$ and $\gamma_0 \in [0, 2\pi)$, there exists a positive number $\epsilon = \epsilon(E_0, \eta)$ such that if

$$\Phi(0) = e^{i\gamma_0}(\psi_{E_0} + \phi_0)$$

where

$$\|\phi_0\|_{L^1(\mathbb{R}^3_x)} + \|\phi_0\|_{H^1(\mathbb{R}^3_x)} < \epsilon$$

then

$$\Phi(t) = e^{-i\int_0^t E(s)\, ds + i\gamma(t)}(\psi_{E(t)+\phi(t)})$$

with

$$\dot{E}(t), \dot{\gamma}(t) \in L^1(\mathbb{R}_t) \quad (\Rightarrow \exists \lim_{t\to\pm\infty}(E(t), \gamma(t)) = (E^\pm, \gamma^\pm))$$

Goal

Theorem A. - Asymptotic stability

and $\phi(t)$ is purely dispersive in the sense that

$$\|\langle x\rangle^{-\sigma}\phi(t)\|_{L^2(\mathbb{R}^3)} = O(\langle t\rangle^{-\frac{3}{2}})$$

for $\sigma > 2$, and

$$\|\phi(t)\|_{L^4(\mathbb{R}^3)} = O(\langle t\rangle^{-\frac{3}{4}})$$

as $|t| \to \infty$.

Goal

Asymptotic stability theorem is reduced to

Proposition A

Assume

$$|E_0 - E_*| < \eta \quad \text{and} \quad \|\phi_0\|_{L^1} + \|\phi_0\|_{H^1} < \epsilon,$$

for sufficiently small $\eta > 0$ and $\epsilon = \epsilon(\eta) > 0$. Then, we have

$$\sup_{t\in\mathbb{R}} \langle t\rangle^{\frac{3}{2}} \|\langle x\rangle^{-\sigma}\phi(t)\|_{L^2} \lesssim \|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}, \tag{1}$$

$$\sup_{t\in\mathbb{R}} \langle t\rangle^{\frac{3}{4}} \|\phi(t)\|_{L^4} \lesssim \|\phi_0\|_{L^1} + \|\phi_0\|_{H^1} \tag{2}$$

and

$$\sup_{t\in\mathbb{R}} \langle t\rangle^{\frac{3}{2}} (|\dot{\gamma}(t)| + |\dot{E}(t)|) \lesssim \|\phi_0\|_{L^1} + \|\phi_0\|_{H^1} \tag{3}$$

Decay estimates

Decay estimate

Let $K = -\Delta + V$ acting on $L^2(\mathbb{R}^3)$, and assume **Hypotheses** on V. Also, V satisfies (NR). Let $P_c(K)$ denote the projection onto the continuous spectral part of K. If $1/p + 1/q = 1$, $2 \le q \le \infty$, then

$$\|e^{itK} P_c(K)\psi\|_{L^q} \le C_q |t|^{-(3/2-3/q)} \|\psi\|_{L^p}.$$

If ψ is more regular ($\psi \in H^1$), then

$$\|e^{itK} P_c(K)\psi\|_{L^q} \le C_q \langle t\rangle^{-(3/2-3/q)} (\|\psi\|_{L^p} + \|\psi\|_{H^1}).$$

A simple consequence is the following local decay estimate

Local decay estimate

Under the same assumption as in the above theorem, let $\sigma > 3/2 - 3/q$. Then

$$\|\langle x\rangle^{-\sigma} e^{itK} P_c(K)\psi\|_{L^2} \le C_q |t|^{-(3/2-3/q)} \|\psi\|_{L^p}.$$

Introduction of quantities $M_j(T)$, $j = 1, 2, 3$

We introduce quantities $M_j(T)$, $j = 1, 2, 3$ corresponding to (1), (2) and (3). Let $0 < T < \infty$. Define

$$M_1(T) = \sup_{|t| \le T} \langle t \rangle^{\frac{3}{2}} \| \langle x \rangle^{-\sigma} \phi(t) \|_{L^2},$$

$$M_2(T) = \sup_{|t| \le T} \langle t \rangle^{\frac{3}{4}} \| \phi(t) \|_{L^4}$$

and

$$M_3(T) = \sup_{|t| \le T} \| \phi(t) \|_{L^2}.$$

Once we have the uniform bound of $M_1(T)$ and $M_2(T)$ in T, by taking $T \to \infty$, we can prove (1) and (2). We note that $M_3(T)$ appears in the estimation of $M_1(T)$, and hence we will additionally control $M_3(T)$ by $M_1(T)$ and $M_2(T)$.

Bounds of $M_j(T)$

Lemma A - $M_2(T)$ bound

Under the assumptions in Theorem A. and definitions of M_1 and M_2, we have

$$M_2(T) \le C_2(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}) \\ + C_2(\psi_E, \partial_E \psi_E)(M_1(T) + M_2(T)^2 + M_2(T)^3) + C_2' M_2(T)^3,$$

where $C_2(\psi_E, \partial_E \psi_E) \to 0$ as $E \to E_*$.

Lemma B - $M_1(T)$ bound

Under the assumptions in Theorem A. and definitions of M_1, M_2 and M_3, we have

$$M_1(T) \le C_1(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}) \\ + C_1'(M_2(T)^2 + M_2(T)^3 + M_3(T) M_2(T)^2),$$

whenever $0 < |E - E_*| \ll 1$.

Bounds of $M_j(T)$

Lemma C - $M_3(T)$ bound

Under the assumptions in Theorem A. and definitions of M_1, M_2 and M_3, we have

$$M_3(T)^2 \leq C_3(\psi_E, \partial_E \psi_E)(M_1(T)^2 + M_2(T)^2 + M_2(T)^4) + C_3' M_2(T)^4,$$

where $C_3(\psi_E, \partial_E \psi_E) \to 0$ as $E \to E_*$.

Control of $\dot{E}$ and $\dot{\gamma}$

We first control $|\dot{E}|$ and $|\dot{\gamma}|$. From

$$\dot{E}(t) = \langle \partial_E \psi_E, \psi_{E_0} \rangle^{-1} \mathrm{Im} \langle \mathbf{F_2}, \psi_{E_0} \rangle$$

and

$$\dot{\gamma}(t) = -\langle \psi_E, \psi_{E_0} \rangle^{-1} \mathrm{Re} \langle \mathbf{F_2}, \psi_{E_0} \rangle,$$

where

$$\begin{aligned}
\mathbf{F_2} &= \mathbf{F_{2,lin}} + \mathbf{F_{2,nl}} \\
\mathbf{F_{2,lin}} &= (2\psi_E^2 - \psi_{E_0}^2)\phi + \psi_E^2 \overline{\phi} \\
\mathbf{F_{2,nl}} &= 2\psi_E |\phi|^2 + \psi_E \phi^2 + |\phi|^2 \phi,
\end{aligned}$$

by Hölder inequality, we have

$$\begin{aligned}
|\dot{E}| &\le |\langle \partial_E \psi_E, \psi_{E_0} \rangle|^{-1} |\langle \mathbf{F_2}, \psi_{E_0} \rangle| \\
&\le |\langle \partial_E \psi_E, \psi_{E_0} \rangle|^{-1} \Big[\| \langle x \rangle^{\sigma} (3\psi_E^2 + \psi_{E_0}^2) \psi_{E_0} \|_{L^2} \| \langle x \rangle^{-\sigma} \phi \|_{L^2} \\
&\qquad + \| 3\psi_E \psi_{E_0} \|_{L^2} \| \phi \|_{L^4}^2 + \| \psi_{E_0} \|_{L^4} \| \phi \|_{L^4}^3 \Big]
\end{aligned}$$

Control of $\dot{E}$ and $\dot{\gamma}$

and

$$\begin{aligned}|\dot{\gamma}| &\le |\langle \psi_E, \psi_{E_0}\rangle|^{-1}|\langle \mathbf{F_2}, \psi_{E_0}\rangle| \\ &\le |\langle \psi_E, \psi_{E_0}\rangle|^{-1}\Big[\|\langle x\rangle^{\sigma}(3\psi_E^2 + \psi_{E_0}^2)\psi_{E_0}\|_{L^2}\|\langle x\rangle^{-\sigma}\phi\|_{L^2} \\ &\qquad + \|3\psi_E\psi_{E_0}\|_{L^2}\|\phi\|_{L^4}^2 + \|\psi_{E_0}\|_{L^4}\|\phi\|_{L^4}^3\Big].\end{aligned}$$

Using the definitions of $M_1(T)$ and $M_2(T)$,

$$|\dot{E}(t)| \le C_E(\psi_E, \psi_{E_0})\langle t\rangle^{-\frac{3}{2}}(M_1(T) + M_2(T)^2 + M_2(T)^3)$$

and

$$|\dot{\gamma}(t)| \le C_\gamma(\psi_E, \psi_{E_0})\langle t\rangle^{-\frac{3}{2}}(M_1(T) + M_2(T)^2 + M_2(T)^3)$$

Proof of Proposition A

We assume that Lemmas A, B and C hold true. We remove $M_3(T)$ in

$$\begin{aligned} M_1(T) &\leq C_1(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}) \\ &\quad + C_1'(M_2(T)^2 + M_2(T)^3 + M_3(T)M_2(T)^2), \end{aligned}$$

by using

$$M_3(T)^2 \leq C_3(\psi_E, \partial_E \psi_E)(M_1(T)^2 + M_2(T)^2 + M_2(T)^4) + C_3' M_2(T)^4.$$

Then we have

$$M_1(T) \leq C_1(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}) + C_1'(M_2(T)^2 + M_2(T)^4) \tag{4}$$

for $0 < |E - E_*| \ll 1$. Substitution of (4) into

$$\begin{aligned} M_2(T) &\leq C_2(\|\phi_0\|_{L^{\frac{4}{3}}} + \|\phi_0\|_{H^1}) \\ &\quad + C_2(\psi_E, \partial_E \psi_E)(M_1(T) + M_2(T)^2 + M_2(T)^3) + C_2' M_2(T)^3, \end{aligned}$$

yields

$$M_2(T) \leq C_1(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}) + C_2 M_2(T)^2 + C_3 M_2(T)^3 + C_4 M_2(T)^4.$$

Proof of Proposition A

This can be rewritten as

$$M_2(T)f(M_2(T)) \le L,$$

where $f(x) = 1 - C_2x - C_3x^2 - C_4x^3$ and $L = C_1(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1})$.
For positive constants C_2, C_3, C_4, we can know that there exists $x_0 > 0$ such that

$$x_0 f(x_0) = \sup_{x>0} x f(x)$$

and $xf(x)$ is increasing on $(0, x_0)$.
Let $|E_0 - E_*| = 2\eta$, where $\eta > 0$ will be chosen sufficiently small such that

$$C_E(\psi_E, \psi_{E_0}), C_\gamma(\psi_E, \psi_{E_0}) \le \eta^{\frac{1}{2}}$$

and

$$\eta f(\eta) \le \frac{x_0 f(x_0)}{2}.$$

We choose $C_1\epsilon \le \eta f(\eta)$.

Proof of Proposition A

If
$$\|\phi_0\|_{L^1(\mathbb{R}^3_x)} + \|\phi_0\|_{H^1(\mathbb{R}^3_x)} \le \epsilon,$$
we know
$$L \le \eta f(\eta) \le \frac{x_0 f(x_0)}{2}$$
and
$$M_2(0) = \|\phi_0\|_{L^4} < \epsilon \le \eta.$$
By the continuity of $M_2(T)$, we have $M_2(T) \le \eta$ and therefore $M_1(T) \le C\eta$ for some $C > 0$. Hence, we have from
$$|\dot{E}(t)| \le C_E(\psi_E, \psi_{E_0})\langle t\rangle^{-\frac{3}{2}}(M_1(T) + M_2(T)^2 + M_2(T)^3)$$
and
$$|\dot{\gamma}(t)| \le C_\gamma(\psi_E, \psi_{E_0})\langle t\rangle^{-\frac{3}{2}}(M_1(T) + M_2(T)^2 + M_2(T)^3)$$
that
$$|\dot{E}| \le C_E \eta^{\frac{3}{2}} \langle t\rangle^{-\frac{3}{2}} \tag{5}$$
and
$$|\dot{\gamma}| \le C_\gamma \eta^{\frac{3}{2}} \langle t\rangle^{-\frac{3}{2}}. \tag{6}$$

Proof of Proposition A

Integration of (5) and (6) yields

$$\int_{-T}^{T} |\dot{E}(t)| + |\dot{\gamma}(t)|\, dt \le \widetilde{C}\eta^{\frac{3}{2}}, \tag{7}$$

where $\widetilde{C}$ is independent of T and η.
By choosing η sufficiently small, (7) ensures that

$$|E(t) - E_0| \le \int_0^t |\dot{E}(s)|\, ds \le \widetilde{C}\eta^{\frac{3}{2}} < \eta, \quad |t| \le T,$$

and thus

$$\sup\{t : |E(t) - E_0| < \eta\} = \infty.$$

Taking $T \to \infty$, we have

$$M_1(\infty) \le \eta$$

and

$$M_2(\infty) \le C\eta.$$

Proof of Lemma A

We first consider $\|\phi\|_{L^4}$. Recall

$$\phi(t) = U(t,0)P_c(H(E_0))\phi_0 - i\int_0^t U(t,s)P_c(H(E_0))\mathbf{F}(s)\,ds,$$

where

$$\begin{aligned}
\mathbf{F} &= \mathbf{F_1} + \mathbf{F_2}, \\
\mathbf{F_1} &= \dot{\gamma}\psi_E - i\dot{E}\partial_E\psi_E, \\
\mathbf{F_2} &= \mathbf{F_{2,lin}} + \mathbf{F_{2,nl}}, \\
\mathbf{F_{2,lin}} &= (2\psi_E^2 - \psi_{E_0}^2)\phi + \psi_E^2\overline{\phi}, \\
\mathbf{F_{2,nl}} &= 2\psi_E|\phi|^2 + \psi_E\phi^2 + |\phi|^2\phi.
\end{aligned}$$

We use

$$\|e^{itK}P_c(K)\psi\|_{L^q} \le C_q\langle t\rangle^{-(3/2-3/q)}(\|\psi\|_{L^p} + \|\psi\|_{H^1})$$

for the linear part and

$$\|e^{itK}P_c(K)\psi\|_{L^q} \le C_q|t|^{-(3/2-3/q)}\|\psi\|_{L^p}$$

for the nonlinear part to obtain that

Proof of Lemma A

$$
\begin{aligned}
\|\phi(t)\|_{L^4} &\le \|U(t,0)\phi_0\|_{L^4} + \int_0^t \|U(t,s)P_c(H(E_0))\mathbf{F}\|_{L^4}\, ds \\
&\le C\langle t\rangle^{-\frac{3}{4}}(\|\phi_0\|_{L^{\frac{4}{3}}} + \|\phi_0\|_{H^1}) \\
&+ C'\int_0^t |t-s|^{-\frac{3}{4}}\Big[|\dot\gamma(t)|\|\psi_E\|_{L^{\frac{4}{3}}} + |\dot E(t)|\|\partial_E\psi_E\|_{L^{\frac{4}{3}}} \\
&\qquad + \|(3\psi_E^2+\psi_{E_0}^2)\phi\|_{L^{\frac{4}{3}}} + \|3\psi_E|\phi|^2\|_{L^{\frac{4}{3}}} + \||\phi|^2\phi\|_{L^{\frac{4}{3}}}\Big]\, ds \\
&\le C_1\langle t\rangle^{-\frac{3}{4}}(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}) \\
&+ C_2\int_0^t |t-s|^{-\frac{3}{4}}[I+II+III+IV+V]\, ds.
\end{aligned} \tag{8}
$$

Proof of Lemma A

$\diamond$ Estimation of I.

$$|\dot{\gamma}(t)| \leq C_\gamma(\psi_E, \psi_{E_0})\langle t\rangle^{-\frac{3}{2}}(M_1(T) + M_2(T)^2 + M_2(T)^3)$$

implies

$$I \leq \widetilde{C}_\gamma(\psi_E, \psi_{E_0})\langle s\rangle^{-\frac{3}{2}}(M_1(T) + M_2(T)^2 + M_2(T)^3). \tag{9}$$

$\diamond$ Estimation of II.

$$|\dot{E}(t)| \leq C_E(\psi_E, \psi_{E_0})\langle t\rangle^{-\frac{3}{2}}(M_1(T) + M_2(T)^2 + M_2(T)^3)$$

implies

$$II \leq \widetilde{C}_E(\psi_E, \psi_{E_0})\langle s\rangle^{-\frac{3}{2}}(M_1(T) + M_2(T)^2 + M_2(T)^3). \tag{10}$$

$\diamond$ Estimation of III.

By Hölder inequality, we have

$$III \leq \|\langle x\rangle^{\sigma}(3\psi_E^2 + \psi_{E_0}^2)\|_{L^4}\|\langle x\rangle^{-\sigma}\phi\|_{L^2}. \tag{11}$$

Proof of Lemma A

⋄ Estimation of IV.
By Hölder inequality, we have

$$IV \leq 3\|\psi_E\|_{L^4}\|\phi\|_{L^4}^2. \tag{12}$$

⋄ Estimation of V.
We have

$$V = \|\phi\|_{L^4}^3. \tag{13}$$

Substitution of (9)-(13) into (8) yields, with the definitions of $M_j(T)$, $j = 1, 2, 3$,

$$\begin{aligned}\|\phi(t)\|_{L^4} \leq C_1\langle t\rangle^{-\frac{3}{4}}(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1})\\ + C_2(\psi_E, \partial_E\psi_E)\int_0^t |t-s|^{-\frac{3}{4}}[\langle s\rangle^{-\frac{3}{2}} + \langle s\rangle^{-\frac{9}{4}}]\, ds\\ \times (M_1(T) + M_2(T)^2 + M_2(T)^3)\\ + C_3\int_0^t |t-s|^{-\frac{3}{4}}\langle s\rangle^{-\frac{9}{4}}\, ds M_2(T)^3.\end{aligned} \tag{14}$$

Proof of Lemma A

Lemma

For $\alpha < 1$,

$$\int_0^t |t-s|^{-\alpha} \langle s \rangle^{-\beta} \, ds \leq C(\alpha, \beta) \langle t \rangle^{-\min(\alpha, \alpha+\beta-1)}. \tag{15}$$

Multiplication of both sides of (14) by $\langle t \rangle^{\frac{3}{4}}$ and taking supremum over $|t| \leq T$, after applying (15) to the right-hand side of (14), yields

$$\begin{aligned} M_2(T) \leq\, & C_1(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}) \\ & + C_2'(\psi_E, \partial_E \psi_E)(M_1(T) + M_2(T)^2 + M_2(T)^3) \\ & + C_3' M_2(T)^3 \end{aligned}$$

where $C_2'(\psi_E, \partial_E \psi_E) \to 0$ as $E \to E_*$.

Proof of Lemma B

We consider $\|\langle x\rangle^{-\sigma}\phi(t)\|_{L^2}$ similarly as in the proof of Lemma A, but we take a trick near $s = t$ at the nonlinear part in order to handle the nonintegrable time singularity of the operator $\langle x\rangle^{-\sigma}U(t,s)P_c(H(E_0))$. From

$$\phi(t) = U(t,0)P_c(H(E_0))\phi_0 - i\int_0^t U(t,s)P_c(H(E_0))\mathbf{F}(s)\,ds,$$

we have

$$\begin{aligned}\|\langle x\rangle^{-\sigma}\phi(t)\|_{L^2} &\le \|\langle x\rangle^{-\sigma}U(t,0)\phi_0\|_{L^2}\\ &\quad+\int_0^{t-1}\|\langle x\rangle^{-\sigma}U(t,s)P_c(H(E_0))\mathbf{F}(s)\|_{L^2}\,ds\\ &\quad+\int_{t-1}^{t}\|\langle x\rangle^{-\sigma}U(t,s)P_c(H(E_0))\mathbf{F}(s)\|_{L^2}\,ds\\ &=: A+B+D.\end{aligned} \tag{16}$$

Proof of Lemma B

⋄ Estimation of A

By

$$\|e^{itK}P_c(K)\psi\|_{L^q} \leq C_q\langle t\rangle^{-(3/2-3/q)}(\|\psi\|_{L^p} + \|\psi\|_{H^1}),$$

we have

$$A \leq C\langle t\rangle^{-\frac{3}{2}}(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}). \tag{17}$$

⋄ Estimation of B

By

$$\|\langle x\rangle^{-\sigma}e^{itK}P_c(K)\psi\|_{L^2} \leq C_q|t|^{-(3/2-3/q)}\|\psi\|_{L^p},$$

we have

$$\begin{aligned} B &= \int_0^{t-1} \|\langle x\rangle^{-\sigma}U(t,s)P_c(H(E_0))\mathbf{F}(s)\|_{L^2}\, ds \\ &\leq C_1\int_0^{t-1} |t-s|^{-\frac{3}{2}}\|\mathbf{F}(s)\|_{L^1}\, ds. \end{aligned} \tag{18}$$

Proof of Lemma B

Similarly as the estimations of $I - V$ in the proof of Lemma A, we have

$$
\begin{aligned}
&\|\mathbf{F}(s)\|_{L^1} \\
&\quad \le \|\psi_E\|_{L^1} |\langle \psi_E, \psi_{E_0} \rangle|^{-1} \Big[\|\langle x \rangle^{\sigma} (3\psi_E^2 + \psi_{E_0}^2) \psi_{E_0}\|_{L^2} \|\langle x \rangle^{-\sigma} \phi\|_{L^2} \\
&\qquad\qquad + \|3\psi_E \psi_{E_0}\|_{L^2} \|\phi\|_{L^4}^2 + \|\psi_{E_0}\|_{L^4} \|\phi\|_{L^4}^3 \Big] \\
&\quad + \|\partial_E \psi_E\|_{L^1} |\langle \partial_E \psi_E, \psi_{E_0} \rangle|^{-1} \Big[\|\langle x \rangle^{\sigma} (3\psi_E^2 + \psi_{E_0}^2) \psi_{E_0}\|_{L^2} \|\langle x \rangle^{-\sigma} \phi\|_{L^2} \\
&\qquad\qquad + \|3\psi_E \psi_{E_0}\|_{L^2} \|\phi\|_{L^4}^2 + \|\psi_{E_0}\|_{L^4} \|\phi\|_{L^4}^3 \Big] \\
&\quad + \|\langle x \rangle^{\sigma} (3\psi_E^2 + \psi_{E_0}^2)\|_{L^2} \|\langle x \rangle^{-\sigma} \phi\|_{L^2} \\
&\quad + 3\|\psi_E\|_{L^2} \|\phi\|_{L^4}^2 \\
&\quad + \|\phi\|_{L^3}^3
\end{aligned}
\tag{19}
$$

Proof of Lemma B

The interpolation of the L^3 norm between L^2 and L^4 yields

$$\|\phi\|_{L^3} \le \|\phi\|_{L^2}^{\frac{1}{3}} \|\phi\|_{L^4}^{\frac{2}{3}}, \tag{20}$$

and hence, substitution of (19) and (20) into (18), with the definitions of $M_j(T)$, $j = 1, 2, 3$, yields

$$\begin{aligned} B \le C_1(\psi_E, \partial_E \psi_E) \int_0^{t-1} |t-s|^{-\frac{3}{2}} (\langle s \rangle^{-\frac{3}{2}} + \langle s \rangle^{-\frac{9}{4}})\, ds \\ \times (M_1(T) + M_2(T)^2 + M_2(T)^3) \\ + \int_0^{t-1} |t-s|^{-\frac{3}{2}} \langle s \rangle^{-\frac{3}{2}}\, ds M_3(T) M_2(T)^2 \\ \le C_1(\psi_E, \partial_E \psi_E) \langle t \rangle^{-\frac{3}{2}} (M_1(T) + M_2(T)^2 + M_2(T)^3) \\ + C_2 \langle t \rangle^{-\frac{3}{2}} M_3(T) M_2(T)^2. \end{aligned} \tag{21}$$

Proof of Lemma B

⋄ Estimation of D

For D, we divide $\mathbf{F}$ into two parts: $\mathbf{F_{lin}}$ denotes the linear part of $\mathbf{F}$ in terms of ϕ of the form

$$\begin{aligned}\mathbf{F_{lin}} &= -\langle \psi_E, \psi_{E_0}\rangle^{-1}\psi_E \mathrm{Re}\langle \mathbf{F_{2,lin}}, \psi_{E_0}\rangle \\ &\quad - i\langle \partial_E\psi_E, \psi_{E_0}\rangle^{-1}\partial_E\psi_E \mathrm{Im}\langle \mathbf{F_{2,lin}}, \psi_{E_0}\rangle \\ &\quad + \mathbf{F_{2,lin}}\end{aligned}$$

and $\mathbf{F_{nl}}$ is the nonlinear functional of $\mathbf{F}$ dependence on ϕ as

$$\mathbf{F_{nl}} = \mathbf{F} - \mathbf{F_{lin}}.$$

The Minkowski inequality yields

$$\begin{aligned}D &\le \int_{t-1}^{t} \|\langle x\rangle^{-\sigma}U(t,s)P_c(H(E_0))\mathbf{F_{lin}}(s)\|_{L^2}\, ds \\ &\quad + \int_{t-1}^{t} \|\langle x\rangle^{-\sigma}U(t,s)P_c(H(E_0))\mathbf{F_{nl}}(s)\|_{L^2}\, ds \\ &=: D_1 + D_2.\end{aligned} \tag{22}$$

Proof of Lemma B

For D_1,

$$\|\langle x\rangle^{-\sigma}e^{itK}P_c(K)\psi\|_{L^2} \le C_q|t|^{-(3/2-3/q)}\|\psi\|_{L^p},$$

and Hölder inequality yield, with the definition of $M_1(T)$,

$$\begin{aligned}
D_1 \le C\int_{t-1}^{t}|t-s|^{-\frac{3}{4}}\Big[&|\langle\psi_E,\psi_{E_0}\rangle|^{-1}\|\psi_E\|_{L^{\frac{4}{3}}}\|\langle x\rangle^{\sigma}(3\psi_E^2+\psi_{E_0}^2)\psi_{E_0}\|_{L^2}\\
&+|\langle\partial_E\psi_E,\psi_{E_0}\rangle|^{-1}\|\partial_E\psi_E\|_{L^{\frac{4}{3}}}\|\langle x\rangle^{\sigma}(3\psi_E^2+\psi_{E_0}^2)\psi_{E_0}\|_{L^2}\\
&+\|\langle x\rangle^{\sigma}(3\psi_E^2+\psi_{E_0}^2)\|_{L^4}\Big]\|\langle x\rangle^{-\sigma}\phi\|_{L^2}\,ds\\
\le C(\psi_E,\partial_E\psi_E)&\int_{t-1}^{t}|t-s|^{-\frac{3}{4}}\langle s\rangle^{-\frac{3}{2}}\,dsM_1(T).
\end{aligned}$$

Since

$$\int_{t-1}^{t}|t-s|^{-\frac{3}{4}}\langle s\rangle^{-\frac{3}{2}}\,ds \le C\langle t\rangle^{-\frac{3}{2}},$$

we obtain

$$D_1 \le C(\psi_E,\partial_E\psi_E)\langle t\rangle^{-\frac{3}{2}}M_1(T). \tag{23}$$

Proof of Lemma B

For D_2, similarly as the estimations I, II, IV and V, we have for $\sigma > 1/4$ that

$$D_2 \le \int_{t-1}^{t} \|\langle x\rangle^{-\sigma}\|_{L^4} \|U(t,s)P_c(H(E_0))\mathbf{F_{nl}}(s)\|_{L^4}\, ds$$
$$\le C_1(\psi_E, \partial_E\psi_E) \int_{t-1}^{t} |t-s|^{-\frac{3}{4}} \left(\|\phi(s)\|_{L^4}^2 + \|\phi(s)\|_{L^4}^3\right) ds$$
$$+ C_2 \int_{t-1}^{t} |t-s|^{-\frac{3}{4}} \|\phi(s)\|_{L^4}^3\, ds.$$

With the definition of $M_2(T)$, the time integration yields

$$D_2 \le C_1(\psi_E, \partial_E\psi_E)\langle t\rangle^{-\frac{3}{2}}(M_2(T)^2 + M_2(T)^3) + C_2\langle t\rangle^{-\frac{9}{4}} M_2(T)^3. \quad (24)$$

Hence, substitution of (23) and (24) into (22) yields

$$D \le C_1(\psi_E, \partial_E\psi_E)\langle t\rangle^{-\frac{3}{2}}(M_1(T) + M_2(T)^2 + M_2(T)^3) + C_2\langle t\rangle^{-\frac{3}{2}} M_2(T)^3. \quad (25)$$

Proof of Lemma B

We substitute (17), (21) and (25) into (16). Then multiplication of both sides of (16) by $\langle t\rangle^{\frac{3}{2}}$ and taking supremum over $|t| \le T$ yields

$$\begin{aligned} M_1(T) \le\ & C_1(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}) \\ & + C_2(\psi_E, \partial_E\psi_E)(M_1(T) + M_2(T)^2 + M_2(T)^3) \\ & + C_2' M_3(T) M_2(T)^2 \\ & + C_3(\psi_E, \partial_E\psi_E)(M_1(T) + M_2(T)^2 + M_2(T)^3) + C_3' M_2(T)^3. \end{aligned}$$

The choice of sufficiently small $0 < |E - E_*| \ll 1$ implies

$$\begin{aligned} M_1(T) \le\ & C_1(\|\phi_0\|_{L^1} + \|\phi_0\|_{H^1}) \\ & + C_1'(M_2(T)^2 + M_2(T)^3 + M_3(T)M_2(T)^2). \end{aligned}$$

Proof of Lemma C

From

$$i\phi_t = [H(E_0) - E_0]\phi + [E_0 - E(t) + \dot{\gamma}(t)]\phi + \mathbf{F},$$

a direct calculation gives us that

$$\begin{aligned}
\frac{d}{dt}\|\phi(t)\|_{L^2}^2 &= 2\mathrm{Im}\int_{\mathbb{R}^3} \mathbf{F}(x)\overline{\phi(x)}\,dx \\
&\le \int_{\mathbb{R}^3} |\mathbf{F}(x)||\phi(x)|\,dx \\
&\le \int_{\mathbb{R}^3} \Big[|\dot{\gamma}||\psi_E(x)| + |\dot{E}||\partial_E\psi_E(x)| + (3\psi_E(x)^2 + \psi_{E_0}(x)^2)|\phi(x)| \\
&\qquad\qquad + 3\psi_E(x)|\phi(x)|^2 + |\phi(x)|^2\phi(x)\Big]|\phi(x)|\,dx.
\end{aligned}$$

Proof of Lemma C

From

$$
\begin{aligned}
|\dot{E}| &\le |\langle \partial_E \psi_E, \psi_{E_0}\rangle|^{-1} |\langle \mathbf{F_2}, \psi_{E_0}\rangle| \\
&\le |\langle \partial_E \psi_E, \psi_{E_0}\rangle|^{-1} \Big[\| \langle x\rangle^{\sigma} (3\psi_E^2 + \psi_{E_0}^2)\psi_{E_0}\|_{L^2} \|\langle x\rangle^{-\sigma}\phi\|_{L^2} \\
&\qquad + \|3\psi_E\psi_{E_0}\|_{L^2}\|\phi\|_{L^4}^2 + \|\psi_{E_0}\|_{L^4}\|\phi\|_{L^4}^3 \Big]
\end{aligned}
$$

and

$$
\begin{aligned}
|\dot{\gamma}| &\le |\langle \psi_E, \psi_{E_0}\rangle|^{-1} |\langle \mathbf{F_2}, \psi_{E_0}\rangle| \\
&\le |\langle \psi_E, \psi_{E_0}\rangle|^{-1} \Big[\| \langle x\rangle^{\sigma} (3\psi_E^2 + \psi_{E_0}^2)\psi_{E_0}\|_{L^2} \|\langle x\rangle^{-\sigma}\phi\|_{L^2} \\
&\qquad + \|3\psi_E\psi_{E_0}\|_{L^2}\|\phi\|_{L^4}^2 + \|\psi_{E_0}\|_{L^4}\|\phi\|_{L^4}^3 \Big],
\end{aligned}
$$

a similar estimates in the proofs of Lemmas A. and B. can be seen to imply

$$
\begin{aligned}
\frac{d}{dt}\|\phi(t)\|_{L^2}^2 &\le C_1(\psi_E, \partial_E\psi_E)\langle t\rangle^{-3}(M_1(T)^2 + M_2(T)^2 + M_2(T)^4) \\
&\quad + C_2\langle t\rangle^{-3} M_2(T)^4.
\end{aligned} \tag{26}
$$

Integration of (26) implies

$$
M_3(T)^2 \le C_3(\psi_E, \partial_E\psi_E)(M_1(T)^2 + M_2(T)^2 + M_2(T)^4) + C_3' M_2(T)^4.
$$

Thank You
for Your Attention!!