[2017 Participating School] Nonlinear Klein-Gordon Equation with a Potential

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1 Normal Form Transformation of NLKG

This section mainly covers [2, Section 1-4]. We consider a real-valued nonlinear Klein-Gordon equation with potential

$$u_{tt} - \Delta u + Vu + u + u^3 = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \tag{1.1}$$

where $V \in \mathcal{S}(\mathbb{R}^3; \mathbb{R})$. Mass is normalized to be 1, and nonlinearity is taken as u^3. For convenience, we restrict ourselves to the case that the Schrödinger operator $-\Delta + V$ has only one eigenvalue, which is in addition simple. This restriction makes the computation easier without losing the essence of the paper. We note that the original paper deals with more general cases; $-\Delta + V$ can have finitely many eigenvalues with finite multiplicities, and the nonlinearity can be replaced by some suitable $\beta'(u)$.

1.1 Assumptions and the Goal

Let us state our assumptions precisely.

H1. V is a Schwartz function.
H2. 0 is neither an eigenvalue nor a resonance for $-\Delta + V$.
H3. We assume that $\sigma_d(-\Delta + V) = \{-\lambda^2\}$ and $-\lambda^2$ is simple.
H4. Let $w := \sqrt{1 - \lambda^2}$ and assume $Nw < 1 < (N + 1)w$ for some $N \in \mathbb{N}$.
H5. (Nondegeneracy condition) See (2.2).

Let $K_0(t) = \frac{\sin(t\sqrt{-\Delta+1})}{\sqrt{-\Delta+1}}$ be the free Klein-Gordon propagator. With the above assumptions, the paper proves small data scattering of NLKG.

Theorem 1.1 (Small data scattering in $H^1 \times L^2$). *Assume the hypotheses (H1-H5). Then, there exist $\epsilon_0 > 0$ and $C > 0$ such that for any $\|(u_0, v_0)\|_{H^1\times L^2} \le \epsilon < \epsilon_0$, then it admits a unique global solution and there are $(u_\pm, v_\pm)$ with $\|(u_\pm, v_\pm)\|_{H^1\times L^2} \le C\epsilon$ and*

$$\lim_{t\to\pm\infty} \|u(t) - K_0'(t)u_\pm - K_0(t)v_\pm\|_{H^1} = 0.$$

Moreover, it is possible to write $u(t,x) = A(t,x) + \widetilde{u}(t,x)$ with $|A(t,x)| \le C_N(t)\langle x\rangle^{-N}$ for any N, with $\lim_{|t|\to\infty} C_N(t) = 0$ and

$$\|\widetilde{u}\|_{L_t^r W_x^{\frac{1}{p}-\frac{1}{r}+\frac{1}{2},p}} \le C\epsilon$$

for any admissible pair (r,p).

Remark. When $V \equiv 0$, one can take $A \equiv 0$ so that the Strichartz estimate is valid with $u = \widetilde{u}$.

As in the case of $V \equiv 0$, we know that the small data scattering occurs on the continuous mode. However, in our situation, there is an eigenvector $\varphi \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$ satisfying $(-\Delta + V)\varphi = -\lambda^2\varphi$. If there were no nonlinearity, the dynamics of φ is merely a rotation so we do not have scattering. Therefore, presence of the nonlinearity should account for the scattering, especially the leakage of energy from the discrete mode to the continuous mode. We will not prove Theorem 1.1, but we will present how this leakage can happen.

1.2 Original Hamiltonian Structure

Let $H^1(\mathbb{R}^3, \mathbb{R}) \times L^2(\mathbb{R}^3, \mathbb{R})$ be the phase space equipped with the symplectic form

$$\Omega\big((u_1, v_1); (u_2, v_2)\big) \coloneqq \langle u_1, v_2\rangle_{L^2} - \langle u_2, v_1\rangle_{L^2}.$$

The Hamiltonian

$$H = \underbrace{\int_{\mathbb{R}^3} \frac{1}{2}(v^2 + |\nabla u|^2 + Vu^2 + u^2)dx}_{H_L} + \underbrace{\int_{\mathbb{R}^3} \frac{1}{4}u^4 dx}_{H_P}$$

gives rise to NLKG. Indeed, the Hamilton equation reads

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ \Delta u - Vu - u - u^3 \end{bmatrix}.$$

The Hamiltonians H_L and H_P correspond to the linear and nonlinear evolution, respectively. Note that H_P can be regarded as a polynomial in u of order 4.

1.3 Spectral Decomposition of $-\Delta + V$

At first, in a spirit of spectral analysis, we decompose our phase space $H^1 \times L^2$ into the discrete mode and continuous mode. To do this, let $\varphi \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$ be the eigenvector associated to $-\lambda^2$. Express

$$u = q\varphi + P_c u, \qquad v = p\varphi + P_c v,$$

where P_c is the orthogonal projection onto the continuous spectrum part. We then introduce the operator

$$B := P_c(-\Delta + V + 1)^{1/2} P_c$$

and the complex variables

$$\xi := \frac{q\sqrt{w} + i\frac{p}{\sqrt{w}}}{\sqrt{2}}, \qquad f := \frac{B^{1/2} P_c u + i B^{-1/2} P_c v}{\sqrt{2}}.$$

This defines an isomorphism between phase spaces $H^1(\mathbb{R}^3, \mathbb{R}) \times L^2(\mathbb{R}^3, \mathbb{R})$ and $\mathcal{P}^{1/2,0} := \mathbb{C} \oplus P_c H^{1/2,0}(\mathbb{R}^3, \mathbb{C})$. Note that every function of u, v can be expressed by our variables $\xi, \bar{\xi}, f, \bar{f}$. With this coordinate,

$$\Omega\big((\xi^{(1)}, f^{(1)}); (\xi^{(2)}, f^{(2)})\big) = -2\mathrm{Im}[\xi^{(1)}\overline{\xi^{(2)}} + \langle f^{(1)}, \overline{f^{(2)}}\rangle].$$

The Poisson bracket becomes

$$\{H, K\} := i(\partial_\xi H \partial_{\bar{\xi}} K - \partial_{\bar{\xi}} H \partial_\xi K) + i\langle \nabla_f H, \nabla_{\bar{f}} K\rangle - i\langle \nabla_{\bar{f}} H, \nabla_f K\rangle.$$

One may compare with various expressions of symplectic forms: $\sum_j dx_j \wedge dy_j$, $\frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$, and the imaginary part of a Hermitian form. The Hamilton equations become

$$\dot{\xi} = -i\partial_{\bar{\xi}} H, \qquad \dot{f} = -i\nabla_{\bar{f}} H.$$

where the original Hamiltonian H in this coordinate reads

$$H = \underbrace{w|\xi|^2 + \langle \bar{f}, Bf\rangle}_{H_L} + \underbrace{\int_{\mathbb{R}^3} \frac{1}{4}\Big(\frac{\xi + \bar{\xi}}{\sqrt{2w}}\varphi + U\Big)^4 dx}_{H_P},$$

with $U := P_c u = B^{-\frac{1}{2}}(\frac{f + \bar{f}}{\sqrt{2}})$.

Remark. There are some estimates and regularity concerns of the Hamiltonian, but we will not discuss them.

1.4 Normal Form Transformation

We start with the nonlinearity H_P, which is of order 4. We will use the Birkhoff normal form theory. It canonically transforms the Hamiltonian equation by leaving the linear part and eliminating nonresonant nonlinearities. This enables us to examine the dynamics almost accurately for long time.

In our situation, as f is a continuous variable, we cannot perform the usual normal form transformation directly. In this subsection, we eliminate some nonresonant continuous part of the equation in some sense, up to order 1.

We first define the notion of normal form. A polynomial Z is in *normal form* if $Z = Z_0 + Z_1$ where Z_0 is a polynomial of $\xi, \bar{\xi}$ satisfying $\{H_L, Z_0\} = 0$ and Z_1 has monomials of the form $\xi^\mu \bar{\xi}^\nu \langle \Phi, f\rangle$ and $\xi^{\mu'} \bar{\xi}^{\nu'} \langle \Phi, \overline{f}\rangle$ with indices satisfying $w(\mu - \nu) < -1$ and $w(\mu' - \nu') > 1$.

As in the usual Birkhoff normal form theory, we will use Lie transformations generated by polynomials. Those transformations keep the symplectic structure. To find an appropriate polynomial, let us recall the Homological equation. Let K be a homogeneous polynomial given by

$$K = \sum_{\mu+\nu=M_1} K_{\mu\nu} \xi^\mu \bar{\xi}^\nu + \sum_{\mu'+\nu'=M_1-1} \xi^{\mu'} \bar{\xi}^{\nu'} \int \Phi_{\mu'\nu'} f + (\text{conjugate part}).$$

We want to find a polynomial χ and Z in normal form satisfying

$$\{H_L, \chi\} + Z = K.$$

To do this, let us examine the spectral property of the *homological operator* $\chi \mapsto \{H_L, \chi\}$.

Lemma 1.2 (Spectral property of the homological operator)**.** *We have*

$$\begin{aligned}
\{H_L, \xi^\mu \overline{\xi^\nu}\} &= -iw(\mu - \nu) \xi^\mu \overline{\xi^\nu}, \\
\{H_L, \xi^\mu \overline{\xi^\nu} \langle \Phi, f\rangle\} &= -i \xi^\mu \overline{\xi^\nu} \langle (B - w(\nu - \mu))\Phi, f\rangle, \\
\{H_L, \xi^\mu \overline{\xi^\nu} \langle \Phi, \overline{f}\rangle\} &= i \xi^\mu \overline{\xi^\nu} \langle (B - w(\mu - \nu))\Phi, \overline{f}\rangle.
\end{aligned}$$

The above lemma follows by an easy computation. The above lemma also says that when $w(\mu - \nu)$ is nonzero, we can invert $w(\mu - \nu)$ to delete the corresponding term in K. When $w(\nu - \mu) < 1$, we can invert the operator $B - w(\nu - \mu)$. Similarly, we can invert $B - w(\mu - \nu)$ when $w(\mu - \nu) < 1$. Therefore, we are led to the following lemma.

Lemma 1.3 (Solution to the homological equation). *Let K be given. Define*

$$\chi := \frac{iK_{\alpha\beta}\xi^\alpha\bar{\xi}^\beta}{w(\alpha-\beta)} + i\sum_{\mu,\nu}\xi^\mu\overline{\xi^\nu}\langle R_{\nu\mu}\Phi_{\mu\nu}, f\rangle - i\sum_{\mu',\nu'}\xi^{\mu'}\bar{\xi}^{\nu'}\langle R_{\mu'\nu'}\Psi_{\mu'\nu'}, \overline{f}\rangle,$$

$$Z := \text{(resonant part of } K)$$

where the indices μ, ν in χ ranges over

$$w(\alpha-\beta) \neq 0, \qquad w(\nu-\mu) < 1, \qquad w(\mu'-\nu') < 1$$

Then, Z and χ solve the homological equation.

As in the usual ODE case, one can prove the following (with some more technically delicate issues).

Theorem 1.4 (Normal Form). *For any integer r with $r \geq 0$, there exist an analytic canonical transformation $\mathcal{T}_r$ defined locally at the origin such that*

$$H^{(r)} := H \circ \mathcal{T}_r = H_L + Z^{(r)} + \mathcal{R}^{(r)}$$

where
1. $Z^{(r)}$ is a polynomial of degree $r+3$ which is in normal form,
2. $\mathcal{R}^{(r)}$ is of order $\geq r+4$ at the origin,
3. Near the origin, $\mathcal{T}_r$ is close to the identity map of order 3. More precisely,

$$\|z - \mathcal{T}_r(z)\| \leq C\|z\|^3.$$

Remark. There are more regularity concerns and error estimates in the paper.

2 Dynamics of the Normal form

This section mainly covers [2, Section 5]. We apply the normal form theorem with $r = 2N$. If we denote ξ, f by the transformed variable, we have

$$H = H_L(\xi, f) + Z_0(\xi) + Z_1(\xi, f) + R^{(2N)}(\xi, f)$$

where

$$Z_1(\xi, f) = \langle G, f\rangle + \langle \overline{G}, \overline{f}\rangle,$$
$$G = \sum_{\substack{2\leq\mu+\nu\leq 2N+2 \\ w(\mu-v)<-1}} \xi^\mu\overline{\xi^\nu}\Phi_{\mu\nu}.$$

This is our starting point.

2.1 Decoupling Variables ξ and f

We neglect the remainder term $R^{(2N)}$ from now on. So we have

$$\dot{f} = -i(Bf + \overline{G})$$
$$\dot{\xi} = -iw\xi - i\partial_{\overline{\xi}} Z_0 - i\langle \partial_{\overline{\xi}} G, f\rangle - i\langle \partial_{\overline{\xi}} \overline{G}, \overline{f}\rangle.$$

It is still hard to see why the energy leaks from discrete modes to continuous modes. Hence we make another coordinate change, which decouples the variables ξ and f. This will enable us to examine the equation in detail. Express

$$g := f + \overline{Y}(\xi, \overline{\xi}).$$

Then, the dynamics of g is given by

$$\dot{g} = -iBg - i\big\{\overline{G} + B\overline{Y}\big\} - \dot{\overline{Y}}.$$

Note that we will take Y of the form

$$\overline{Y} := \sum_{\substack{2\leq \mu+\nu \leq 2N+2 \\ w(\mu-v)>1}} \overline{Y}_{\mu\nu}\xi^{\mu}\overline{\xi}^{\nu}.$$

By the chain rule,

$$\dot{\overline{Y}} = [\dot{\xi}\partial_{\xi} + \dot{\overline{\xi}}\partial_{\overline{\xi}}]\overline{Y}$$
$$= [-iw\xi\partial_{\xi} + iw\overline{\xi}\partial_{\overline{\xi}}]\overline{Y} + (\text{remaining terms}),$$

where the remaining terms consist of $(\partial_{\overline{\xi}} Z_0)\partial_{\xi}\overline{Y}$, $\langle \partial_{\overline{\xi}} G, f\rangle\partial_{\xi}\overline{Y}$, and so on. Substituting this, we get

$$\dot{g} = -iBg - i\big\{\overline{G} - [B - w(\xi\partial_{\xi} - \overline{\xi}\partial_{\overline{\xi}})]\overline{Y}\big\} + (\text{remaining terms}),$$

where the remaining terms consist of high order polynomial in ξ (obtained from terms such as $(\partial_{\overline{\xi}} Z_0)\partial_{\xi}\overline{Y}$) and linear terms in f (obtained from terms such as $\langle \partial_{\overline{\xi}} G, f\rangle\partial_{\xi}\overline{Y}$). As the remaining terms can be considered as error terms, a detailed analysis of g will be possible if the curly bracket vanishes. To do this, using the formula of $\overline{Y}$, it suffices to find $Y_{\mu\nu}$ such that

$$\big(B - w(\mu - \nu)\big)\overline{Y}_{\mu\nu} = \overline{\Phi}_{\nu\mu}.$$

As different as the situation in the homological equation, we cannot invert the operator $B - w(\mu - \nu)$ in a trivial way because $w(\mu - \nu) \in \sigma(B)$. So we regularize the resolvent as follows.

$$R^{\pm}_{\mu\nu} = \lim_{\epsilon\to 0^+} \big(B - w(\mu - \nu) \mp i\epsilon\big)^{-1}$$
$$= R^{\pm}_{-\Delta+V}(k^2)(B + w(\mu - \nu)).$$

where $k^2 = w^2(\mu-\nu)^2 - 1$. This can be defined in the norm topology of $B(H^{2,s}, L^{2,-s})$ with $s > \frac{1}{2}$. We finally set

$$Y_{\mu\nu} \coloneqq R^-_{\mu\nu}\Phi_{\nu\mu}.$$

2.2 Leakage of Energy and Nondegeneracy Condition

By the coordinate change made in the previous subsection, we have

$$\dot{\xi} = -iw\xi - i\partial_{\bar{\xi}} Z_0 + i\langle \partial_{\bar{\xi}} G, \overline{Y}\rangle + i\langle \partial_{\bar{\xi}} \overline{G}, Y\rangle + (g\text{-term}).$$

We can neglect g-term because its spacetime norm can be estimated by some small quantity. (Actually, this combines spacetime norm estimates of f and ξ in the bootstrap argument, but we will not get into this.) We then normalize the equation as follows. Substituting the formula of Y and G (and neglecting g-term), we have

$$\dot{\xi} = -iw\xi - i\partial_{\bar{\xi}} Z_0 + i \sum_{\substack{\nu-\mu\in M\\ \mu'-\nu'\in M}} \left(\nu \xi^{\mu+\mu'}\bar{\xi}^{\nu'+\nu-1} c_{\mu\nu\mu'\nu'} + \mu \bar{\xi}^{\mu+\mu'-1}\xi^{\nu'+\nu}\overline{c}_{\mu\nu\mu'\nu'}\right),$$

where

$$\begin{aligned} M &\coloneqq \{\mu \in \mathbb{N}_0 : w\mu > 1, 2 \le \mu \le 2N+3\},\\ c_{\mu\nu\mu'\nu'} &\coloneqq \langle \Phi_{\mu\nu}, R^+_{\mu'\nu'}\overline{\Phi}_{\nu'\mu'}\rangle, \end{aligned}$$

From the formula of $c_{\mu\nu\mu'\nu'}$, one see that the summation is indeed finite. In the above summation, we may assume that $\mu = \nu' = 0$. Roughly, this is because any terms having nonzero μ or ν' correspond to some lower order term, which is typically larger. Therefore, we assume $\mu = \nu' = 0$ and further simplify the equation as follows.

$$\begin{aligned} \dot{\xi} &= -iw\xi - i\partial_{\bar{\xi}} Z_0 + \mathcal{G}_0(\xi),\\ \mathcal{G}_0(\xi) &\coloneqq i\sum_{\mu,\nu\in M} \nu\xi^\mu\bar{\xi}^{\nu-1} c_{0\nu\mu0}. \end{aligned}$$

We already know that $\partial_{\bar{\xi}} Z_0$ is resonant (with respect to $w|\xi|^2$), so we apply the normal form transformation to eliminate nonresonant terms in $\mathcal{G}_0(\xi)$. We let

$$\begin{aligned} \eta &\coloneqq \xi + \sum_{\substack{\mu,\nu\in M\\ w(\mu-\nu)\neq 0}} \frac{1}{iw(\mu-\nu)}\nu\xi^\mu\bar{\xi}^{\nu-1}c_{0\nu\mu0},\\ \mathcal{N}(\eta) &\coloneqq i\sum_{\substack{\mu,\nu\in M\\ w(\mu-\nu)=0}} \nu\eta^\mu\bar{\eta}^{\nu-1}c_{0\nu\mu0}, \end{aligned}$$

and obtain

$$\dot{\eta} = \Xi(\eta, \overline{\eta}) := -iw\eta - i\partial_{\overline{\eta}} Z_0 + \mathcal{N}(\eta).$$

To observe energy leaking, it is natural to compute the Lie derivative $\mathcal{L}_\Xi(w|\eta|^2)$. To this end, we will use the Plamelji formula

$$R^{\pm}_{\mu 0} = PV(B - w\mu)^{-1} \pm i\pi\delta(B - w\mu).$$

The distribution can be understood as follows. When $V \equiv 0$, we have

$$\begin{aligned}
\delta(B - w\mu)\Psi(x) &= c \int_{\lambda>0} \int_{|\theta|=1} \delta(\sqrt{\lambda^2+1} - w\mu)\widehat{\Psi}(\lambda\theta) e^{i\lambda\theta\cdot x} \lambda^2 d\sigma(\theta) d\lambda \\
&= c(w\mu)k \int_{|\theta|=1} \widehat{\Psi}(k\theta) e^{ik\theta\cdot x} d\sigma(\theta),
\end{aligned}$$

and

$$\begin{aligned}
\langle \overline{\Psi}, \delta(B - w\mu)\Psi \rangle &= c(w\mu)k \int_{|\theta|=1} \widehat{\Psi}(k\theta) \int_{\mathbb{R}^3} \overline{\Psi}(x) e^{ik\theta\cdot x} dx d\sigma(\theta) \\
&= c(w\mu)k \int_{|\theta|=1} |\Psi(k\theta)|^2 d\sigma(\theta), \qquad (2.1)
\end{aligned}$$

where c is the constant for the Fourier inversion and $k^2 = (w\mu)^2 - 1$. When V is nontrivial, the same formula holds where $\widehat{\Psi}$ should be understood by means of the distorted Fourier transform and $e^{ik\theta\cdot x}$ should be changed appropriately. Keeping this in mind, let us introduce the following notions:

$$\begin{aligned}
\Lambda &:= \bigcup_{\mu \in M} \{w \cdot \mu\} \\
M_\lambda &:= \{\mu \in M : w \cdot \mu = \lambda\} \text{ for } \lambda \in \Lambda \\
F_\lambda &:= \sum_{\mu \in M_\lambda} \overline{\eta}^\mu \Phi_{0\mu}, \\
B_\lambda &:= \pi\delta(B - \lambda).
\end{aligned}$$

With this notation, we now see that the leakage of energy occurs in our normalized system.

Lemma 2.1 (Leakage of Energy)**.** *The following formula holds*

$$\mathcal{L}_\Xi(w|\eta|^2) = -\sum_{\lambda \in \Lambda} \lambda \langle F_\lambda; B_\lambda \overline{F}_\lambda \rangle.$$

Moreover, the RHS is negative-semidefinite.

Proof. As $-iw\eta$ and $i\partial_{\overline{\eta}} Z_0$ are resonant terms, it suffices to compute $\mathcal{L}_{\mathcal{N}}(w|\eta|^2)$. Observe that

$$\begin{aligned}\mathcal{L}_{\mathcal{N}}(w|\eta|^2) &= w\overline{\eta}\mathcal{N}(\eta) + w\eta\overline{\mathcal{N}}(\eta) \\ &= -\mathrm{Im}\Big(\sum_{\substack{\mu,\nu\in M \\ w(\mu-\nu)=0}} w\nu\eta^\mu\overline{\eta}^\nu \langle \Phi_{0\nu}, (B - w\mu - i0)^{-1}\overline{\Phi}_{0\mu}\rangle \Big) \\ &= -\sum_{\lambda\in\Lambda} \lambda \mathrm{Im}\langle F_\lambda, (B-\lambda-i0)^{-1}\overline{F}_\lambda\rangle.\end{aligned}$$

By the Plamelji formula, we obtain

$$\mathcal{L}_{\Xi}(w|\eta|^2) = -\sum_{\lambda\in\Lambda} \lambda\langle F_\lambda; B_\lambda \overline{F}_\lambda\rangle.$$

This quantity is negative-semidefinite because

$$\lambda\langle F_\lambda; B_\lambda\overline{F}_\lambda\rangle = c\lambda k \int_{|\theta|=1} |\widehat{\Psi}(k\theta)|^2 d\sigma(\theta),$$

with $k^2 = \lambda^2 - 1$ by the equation (2.1). □

The nondegeneracy condition: there exists a positive constant C and a sufficiently small $\delta_0 > 0$ such that for all $|\eta| < \delta_0$,

$$\sum_{\lambda\in\Lambda} \lambda\langle F_\lambda; B_\lambda\overline{F}_\lambda\rangle \geq C\sum_{\mu\in M} |\eta^\mu|^2. \tag{2.2}$$

A Distorted Fourier Transform

Most of the results in this section are brought from [1]. Suppose that $V \in \mathcal{S}(\mathbb{R}^3;\mathbb{R})$ be our potential.

Theorem A.1 (Limiting Absorption Principle)**.** *Let $k^2 > 0$ and $s > \frac{1}{2}$ be given. Then,*

$$R^{\pm}_{-\Delta+V}(k^2) := \lim_{\epsilon\to 0^+} R_{-\Delta+V}(k^2 \pm i\epsilon)$$

is well-defined in the norm topology of $B(L^{2,s}, H^{2,-s})$. Moreover, if $f \in L^{2,s}$, then $u_\pm := R^{\pm}_{-\Delta+V}(k^2)f$ solve

$$(-\Delta + V - k^2)u_\pm = f.$$

Using the limiting absorption principle, we can naturally choose a substitute $\phi(\xi, x)$ of plane waves $e^{i\xi\cdot x}$. We express $\phi_{\pm}(\xi, x) = e^{i\xi\cdot x} + v_{\pm}(\xi, x)$ and require ϕ to satisfy

$$(-\Delta + V - |\xi|^2)\phi_{\pm}(\xi, x) = 0.$$

This is equivalent to

$$(-\Delta + V - |\xi|^2)v_{\pm}(\xi, x) = -V(x)e^{i\xi\cdot x}.$$

Hence we define

$$v_{\pm}(\xi, x) \coloneqq R^{\pm}_{-\Delta+V}(|\xi|^2)[V(x)e^{i\xi\cdot x}].$$

This allows us to define the distorted Fourier transform with respect to V.

Theorem A.2 (Distorted Fourier Transform)**.** *There exist two unitary maps $\mathcal{F}_{\pm} : L^2_{ac}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ such that the following diagram commutes.*

$$\begin{array}{ccc} H^2 \cap L^2_{ac} & \xrightarrow{\mathcal{F}_{\pm}} & H^2 \\ \Big\downarrow{\scriptstyle -\Delta+V} & & \Big\downarrow{\scriptstyle |\xi|^2 \ \text{mult.}} \\ L^2_{ac} & \xrightarrow{\mathcal{F}_{\pm}} & L^2 \end{array}$$

Remark. The scattering map $\mathcal{W}_{\pm}$ satisfies the formula

$$W_{\pm} = \mathcal{F}^*_{\pm}\mathcal{F}$$

where $\mathcal{F}$ is the usual Fourier transform.

References

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[2] D. Bambusi and S. Cuccagna, *On dispersion of small energy solutions to the nonlinear Klein Gordon equation with a potential*, Amer. J. Math. **133** (2011), no. 5, 1421–1468. MR2843104