Instruction for "Multichannel Nonliear Scattering for Nonintegrable Equations" by Soffer and Weinstein

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1 Organization and aim of the paper

The paper by Soffer and Weinstein [2] has 6 sections:

1. Introduction
2. The initial value problem, solitary waves and linear propagator estimates.
3. The equation for the localized and dispersive parts
4. Scattering and asymptotic stability theorem
5. The coupled channel equation
6. Scattering theory

and two appendices devoted to some technical estimates for the nonlinear term and solitary waves. However, due to the weak estimate in Theorem 2.5 of [2], the estimates in section 5 is quite complicated. In 1991, the estimate in Theorem 2.5 [2] was improved by Journé Soffer and Sogge [1] and using this improved estimate (which is the dispersive estimate), Soffer and Weinstein [3] gave a more simple proof of bootstrap argument than the one done in section 5 of [2]. Therefore, I think the first presenter should read sections 1-4 and 5.4 of [2] replacing Theorem 2.5 of [2] by Theorem 2.1 and Corollary 2.2 of [3], and the second presenter should read section 4 of [3]. Section 6 can be omitted.

The purpose of these papers [2] and [3] are to study the asymptotic stability of small solitary waves of NLS

$$\mathrm{i}\Phi_t = -\Delta\Phi + V\Phi + |\Phi|^2\Phi, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^3, \tag{NLS}$$

where $V\in\mathcal{S}$ (Schwartz function).

Remark 1.1. I think for simplicity of the presentation, it is better to reduce the number of parameters. So, I have fixed $n=3$, $m=3$, $\lambda=1$ and V to be a good function. Of course Soffer and Weinstein treats more general case.

Solitary waves are the solution of (NLS) with the form $e^{-\mathrm{i}Et}\psi_E(x)$. In [2], they consider the family of solitary waves which bifurcate from the ground state of $-\Delta+V$. Notice that since there is a family of solitary waves ψ_E, we cannot say something like

$$\text{if } \|u(0)-\psi_{E_0}\|_X \ll 1, \text{ then } \inf_{s\in\mathbb{R}} \|u(t)-e^{\mathrm{i}s}\psi_{E_0}\|_Y \to 0 \text{ as } t\to\infty,$$

for any suitable norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. This is because if we take $|E-E_0| \ll 1$, then $\|\psi_E-\psi_{E_0}\|_X \ll 1$. However, taking $u(0) = \psi_E$, the solution will be $e^{-\mathrm{i}Et}\psi_E$ and this will not converge to $e^{-\mathrm{i}E_0 t}\psi_{E_0}$ even though we adjust the gauge.

To overcome this difficulty, the authors take a simple strategy which is now standard.

One first show that solution $\Phi(t)$ near ψ_{E_0} can be decomposed as

$$\Phi(t) = e^{-\mathrm{i}\int_0^t E(s)\,ds+\gamma(t)}\left(\psi_{E(t)} + \phi(t)\right),$$

with an orthogonality condition $\langle \psi_{E_0}, \phi(t)\rangle \equiv 0$ (see (3.4) in [2]). By (NLS), and the orthogonality condition, we get a system consisting of one PDE for ϕ and two ODEs for γ and E (see, (3.6) and (3.9) in [2]).

Next, one solves the system and prove that the limits

$$\lim_{t\to\infty} E(t) = E_+, \quad \lim_{t\to\infty} \gamma(t) = \gamma_+,$$

exist and the dispersive part $\phi(t)$ converges to 0 in some norm. This gives us the asymptotic stability.

2 Detailed instruction

2.1 For presenter 1

- Introduce NLS (2.1) in [2] (but in the restricted form (NLS) in this note) and explain the aim of this paper based on section 1.
- Make a comment on the Global wellposedness and energy and mass conservation ($\mathcal{H}$, $\mathcal{N}$).
- Introduce Hypothesis V. However, $(V1)$ can be $V \in \mathcal{S}(\mathbb{R}^3;\mathbb{R})$ (Schwartz function). This is for only saving time.
- Introduce the solitary wave ψ_E and comment that ψ_E satisfies (2.3) in [2].
- Introduce Theorem 2.1-2.3 of [2] without proof. You do not have to state Corollary 2.2 and Theorem 2.3 so precisely. These simply mean that ψ_E are in $\mathcal{S}$ and their Sobolev norms with weight are all comparable to L^2 norm which is small when $E \to E_*$.
- In section 2.3 of [2], you do not have to state the precise definition of the nonresonance condition (it is not because the condition is unimportant, it is because we do not have time to go into detailed theory of linear estimates). Just say one word that for Theorem 2.4 of [2] and Theorem 2.1 and Corollary 2.2 of [3], you assume nonresonance condition.
- Introduce Theorem 2.4 of [2] without proof.
- Introduce Theorem 2.1 and Corollary 2.2 of [3] without proof. Further, introduce the estimate

  $$\|e^{\mathrm{i}tH}P_c(H)\psi\|_{L^q} \lesssim \langle t\rangle^{n/q-n/2}\left(\|\psi\|_{L^p} + \|\psi\|_{H^1}\right),$$

  which corresponds to (2.11) of [2].
- Introduce the decomposition (α) and the orthogonality condition (β). Substitute (3.2) of [2] into (3.1) of [2] and obtain (3.6) of [2]. Further, obtain (3.9ab) of [2] from the orthogonality condition (β).

- Now, move to section 5.4 of [2] and explain that for Φ_0 nearby $\psi_{\tilde{E}}$, one can decompose Φ_0 as decomposition (α) which satisfies the orthogonality condition (β).
- Coming back to section 3 of [2] again, introduce the propagator $U(t,s)$. Comment that because of the gauge transform (3.11), $U(t,s)$ has the same estimate as $e^{-\mathrm{i}(H(E_0)-E_0)(t-s)}$.
- Introduce Theorem 4.1 of [2]. However, the smallness conditions and decay should be change to the one based on Theorem 4.1 of [3]! Further, I think the smallness of $\|\phi_0\|_{L^2} + \|\phi_0\|_{L^1} + \|\phi_0\|_{L^{m+1=4}}$ should be replaced to $\|\phi_0\|_{H^1} + \|\phi_0\|_{L^1} + \|\phi_0\|_{L^{m+1(=4)}}$ (and of course the smallness of $\|\phi_0\|_{L^4}$ is unnecessary because of Sobolev embedding). Say one word about Theorem 4.2 of [2]. Here, in [2], it is assumed that $m < 3$. However, this will be no problem because the bootstrapping will be based on [3].

2.2 For presenter 2

- Presenter 2 will do the proof of Theorem 4.1 of [2], which all that written in section 4 of [3]. Since the set up is done by the presenter 1, I think the two presenters should communicate to adjust the time and the notations they use (basically you can follow Soffer and Weinstein's notation).
- First, introduce M_j ($j = 1,2,3$) and show that if these quantities are bounded and small, you have the proof.
- Claim (4.11), (4.18) and (4.19) of [3] as a lemma and show that if these estimates are true. You have the uniform bound of M_j under the smallness condition of the initial data.
- Prove each of the claim (4.11), (4.18) and (4.19).
- Of course, the presenter can rearrange the proof in his own way which feels comfortable for him. Some parts in the estimate can be omitted due to the time restriction.
- Some remarks:
  - To get the estimate in the second line of (4.4) of [3], I think you need to replace $\|\phi_0\|_2$ by $\|\phi_0\|_{H^1}$.

If you have some questions, please do not hesitate to ask me.
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References

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[3] A. Soffer and M. I. Weinstein, *Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data*, J. Differential Equations **98** (1992), no. 2, 376–390.

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