Modulational instability in equations of KdV type

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Abstract

The authors discuss some resent advances in the mathematical understanding of the dynamics, in particular, the instability, of slowly modulated waves for equations of KdV type. They provide a rigorous proof of Whitham's formal theory.

4.1 Introduction

4.2 Preiodic traveling waves of generalized KdV equations

We study the stability of the periodic traveling wave solution of the KdV type equation.

$$u_t = u_{xxx} + f(u)_x. \tag{4.1}$$

A traveling solution is given by $u(x,t) = u(z)$, $z = x - ct$ for some $c \in \mathbb{R}$, which satisfies

$$u_{zzz} + cu_z + f(u)_z = 0. \tag{4.3}$$

For the KdV equation case, $f(u)$ is given by $f(u) = u^2$. Integrating both sides, we obtain

$$u_{zz} + cu + f(u) = a, \tag{4.4}$$

$$\tfrac{1}{2}u_z^2 = E - V, \qquad V = F(u) + \tfrac{c}{2}u^2 - au, \tag{4.5}$$

where $a, E \in \mathbb{R}$ and $F(u) = \int_0^u f(s)ds$.

4.2.1 Some explicit solutions

4.2.2 General existence theory

For simplicity, we assume the KdV equation $f(u) = u^2$. In general, the third order polynomial has three distinct real root if its discriminant is positive. Three real roots of $g(u) = E - \frac{u^3}{3} - \frac{c}{2}u^2 + au = 0$ are denoted by $u_{--} < u_- < u_+$. For any $(a, E, c) \in \mathbb{R}^3$ satisfying $\mathrm{disc}(E - \frac{u^3}{3} - \frac{c}{2}u^2 + au) > 0$, their exists a unique periodic traveling wave solution $U(z; a, E, c)$. Its period is calculated from (4.5).

$$T = 2\int_{u_-}^{u_+} \frac{dx}{du}du = 2\int_{u_-}^{u_+} \frac{du}{(\frac{du}{dx})} = 2\int_{u_-}^{u_+} \frac{du}{\sqrt{2(E-V)}}. \tag{4.18}$$

Equation (4.1) admits three conserved quantities. For periodic traveling wave solution $U(x; a, E, c)$, they are defined by

$$\begin{aligned} M &= \int_0^T U dx = 2\int_{u_-}^{u_+} u\frac{dx}{du}du = 2\int_{u_-}^{u_+} \frac{u}{(\frac{du}{dx})}du = 2\int_{u_-}^{u_+} \frac{u}{\sqrt{2(E-V)}}du, \\ P &= \int_0^T U^2 dx = 2\int_{u_-}^{u_+} \frac{u^2}{\sqrt{2(E-V)}}du, \\ H &= \int_0^T (\tfrac{1}{2}U_x^2 - F(U))dx = 2\int_{u_-}^{u_+} \frac{E-V-F(u)}{\sqrt{2(E-V)}}du. \end{aligned} \tag{4.19}$$

Let $K = 2\int_{u_-}^{u_+} \sqrt{2(E-V)}du$. Then it holds that

$$T = \frac{\partial K}{\partial E}, \qquad M = \frac{\partial K}{\partial a}, \qquad P = -2(\frac{\partial K}{\partial c}). \tag{4.21}$$

Form this relation, $\frac{\partial(T,M,P)}{\partial(a,E,c)}$ coincides with (up to a constant factor) the Hessian of the $K(a, E, c)$. In Whitham's theory, we change variables from (a, E, c) to (T, M, P) to characterize the periodic traveling wave solution. This change of variables is possible if $\frac{\partial(T,M,P)}{\partial(a,E,c)}$ is non-singular.

4.3 Formal asymptotics and Whitham's modulation theory

4.3.1 Linear dispersive waves

4.3.2 Nonlinaer disipersive waves

This section is summary of Section 4 in [3]. Throughout this section, we consider (4.1) in the moving coordinate frame. Let $u(z,t) = u(x,t)$, $z = x - c_0 t$ ($c_0 \in \mathbb{R}$). Then it satisfies

$$u_t = u_{zzz} + f(u)_z + c_0 u_z. \tag{1}$$

We introduce slow variables $(X, S) = (\epsilon z, \epsilon t)$. Let $u(X,S) = u(x,t)$. Then it satisfies

$$u_S = \epsilon^2 u_{XXX} + f(u)_X + c_0 u_X. \tag{4.32}$$

Following the Whitham's theory, we seek the following form of solution.

$$u(X,S) = u^{(0)}(y,X,S) + \epsilon u^{(1)}(y,X,S) + O(\epsilon^2), \qquad y = \tfrac{\phi(X,S)}{\epsilon}, \tag{4.33}$$

where $u^{(j)}(y,X,S)$ $(j = 0, 1)$ are 1-periodic functions in y. We substitute (4.33) to (4.32). Since $\frac{d}{dS} = \partial_S + \phi_S \partial_y$, $\frac{d}{dX} = \partial_X + \phi_X \partial_y$, we see that

$$\epsilon^{-1}(\epsilon\partial_S + \phi_S\partial_y)u = \epsilon^{-1}(\epsilon\partial_X + \phi_X\partial_y)^3 u + \epsilon^{-1}(\epsilon\partial_X + \phi_X\partial_y)f(u) + c_0\epsilon^{-1}(\epsilon\partial_X + \phi_X\partial_y)u \tag{2}$$

At the order of ϵ^{-1} in (2), we find that

$$\phi_S\partial_y u^{(0)} = \partial_z^3 u^{(0)} + \partial_z f(u^{(0)}) + c_0\partial_z u^{(0)}, \qquad \partial_z = \phi_X\partial_y \tag{4.34}$$

Let $k(X,S) = \phi_X(X,S)$, $\omega(X,S) = \phi_S(X,S)$ and $\delta(X,S) = -\omega(X,S)/k(X,S)$. Since $u^{(0)}(y,X,S)$ is 1-periodic solution of (4.34), we may choose $u^{(0)}(y,X,S)$ as

$$u^{(0)}(y,X,S) = U(z; a(X,S), E(X,S), c_0 + \delta(X,S)), \qquad z = \tfrac{y}{k(X,S)}, \tag{4.35}$$

where $U(z;a,E,c)$ is a function defined in Section 4.2.2 and $T(X,S) = \frac{1}{k(X,S)}$ represents the period of $U(z; a(X,S), E(X,S), c_0 + \delta(X,S))$. For example, we consider the following constant case.

$$\epsilon = 0, \qquad a(X,S) = a_0, \qquad E(X,S) = E_0, \qquad \delta(X,S) = 0$$

Let T_0 be the period of $U(z; a_0, E_0, c_0)$. Then $\phi(X,S)$ is determined by

$$\phi_X = k_0 = \tfrac{1}{T_0}, \qquad \phi_S = -c_0 k_0.$$

Solving this equation, we get $\phi(X,S) = k_0(X - c_0 S)$. Therefore it follows that

$$u(X,S) = u^{(0)}(y,X,S) = U(\tfrac{\phi(X,S)}{\epsilon}; a_0, E_0, c_0) = U(x - c_0 t; a_0, E_0, c_0).$$

From definition of $k(X,S)$ and $\omega(X,S)$, it follows that

$$k_S = \omega_X = -(\delta k)_X \tag{4.36}$$

We next collect terms of the order of ϵ^0 in (2).

$$\begin{aligned}\partial_S u^{(0)} + \phi_S\partial_y u^{(1)} &= \partial_X(\phi_X^2\partial_y^2 u^{(0)}) + \phi_X\partial_y(\partial_X\phi_X\partial_y u^{(0)} + \phi_X\partial_y\partial_X u^{(0)} + (\phi_X\partial_y)^2 u^{(1)}) \\ &\quad + c_0\partial_X u^{(0)} + c_0\phi_x\partial_y u^{(1)} + \partial_X f(u^{(0)}) + \phi_X\partial_y(f'(u^{(0)})u^{(1)}).\end{aligned}$$

This equation is rewritten as

$$\begin{aligned}\partial_z\{(\partial_z^2 + c_0 + \delta + f'(u^{(0)}))u^{(1)}\} &= \partial_S u^{(0)} - \partial_X(\partial_z^2 u^{(0)}) - c_0\partial_X u^{(0)} \\ &\quad - \partial_X f(u^{(0)}) - \partial_z(\partial_X\phi_X\partial_y u^{(0)} + \phi_X\partial_y\partial_X u^{(0)}), \qquad \partial_z = \phi_X\partial_y.\end{aligned} \tag{3}$$

We note that $\partial_X\phi_X\partial_y u^{(0)} + \phi_X\partial_y\partial_X u^{(0)}$ is 1-periodic in y. Therefore integrating (3) in y and changing variables from y to $z = \frac{y}{k}$, we get

$$0 = \partial_S\{\tfrac{1}{T}\textstyle\int_0^T \tilde{u}^{(0)} dz\} - \partial_X\{\tfrac{1}{T}\textstyle\int_0^T (\partial_z^2\tilde{u}^{(0)} + c_0 u^{(0)} + f(\tilde{u}^{(0)}))dz\}, \qquad T = \tfrac{1}{k},$$

where $\tilde{u}^{(0)}(z,X,S)=u^{(0)}(y,X,s)$ with $y=kz$. Let

$$M(X,S)=\int_0^T \tilde{u}^{(0)}(z,X,S)dz, \qquad P(X,S)=\int_0^T (\tilde{u}^{(0)})^2(z,X,S)dz. \tag{4}$$

Since $\tilde{u}^{(0)}$ satisfies (4.4) with $c=c_0+\delta$, it follows that

$$\begin{aligned}0&=\partial_S\{\tfrac{1}{T}\int_0^T \tilde{u}^{(0)}dz\}-\partial_X\{\tfrac{1}{T}\int_0^T(a(X,S)-\delta(X,S)\tilde{u}^{(0)})dz\}\\&=\partial_S(kM)-\partial_X(a-k\delta M).\end{aligned} \tag{4.40-1}$$

Furthermore by a direct computation, we see that

$$\begin{aligned}&\partial_X\{u^{(0)}(\phi_X\partial_y)^2u^{(0)}-\tfrac{(\phi_X\partial_y u^{(0)})^2}{2}\}+\phi_X\partial_y\{u^{(0)}(\partial_X\phi_X\partial_y+\phi_X\partial_y\partial_X)u^{(0)}-\partial_X u^{(0)}\cdot\phi_X\partial_y u^{(0)}\}\\&\quad=u^{(0)}\cdot\partial_X(\phi_X^2\partial_y^2u^{(0)})+u^{(0)}\cdot\phi_X\partial_y(\partial_X\phi_X\partial_y u^{(0)}+\phi_X\partial_y\partial_X u^{(0)})+O(\epsilon).\end{aligned} \tag{5}$$

$$\partial_X\{u^{(0)}f(u^{(0)})-F(u^{(0)})\}=u^{(0)}f(u^{(0)}). \tag{6}$$

We multiply (3) by $u^{(0)}$ and integrate both sides in y. Since $u^{(0)}\in\ker(\{\partial_z(\partial_z^2+c+f'(u^{(0)})\}^{\dagger})$, using (5)-(6), we compute in the same way as the above.

$$0=\partial_S\{\tfrac{1}{T}\int_0^T\tfrac{(\tilde{u}^{(0)})^2}{2}dz\}-\partial_X\{\tfrac{1}{T}\int_0^T\{u^{(0)}\partial_z^2u^{(0)}-\tfrac{(\partial_z u^{(0)})^2}{2}+\tfrac{c_0}{2}(u^{(0)})^2+\tilde{u}^{(0)}f(\tilde{u}^{(0)})-F(\tilde{u}^{(0)})\}dz\}.$$

Since $\tilde{u}^{(0)}$ satisfies (4.4)-(4.5), it follows from (3) that

$$\begin{aligned}0&=\partial_S\{\tfrac{1}{T}\int_0^T\tfrac{(\tilde{u}^{(0)})^2}{2}dz\}-\partial_X\{\tfrac{1}{T}\int_0^T\{u^{(0)}(a-\delta\tilde{u}^{(0)})-(E-\tfrac{c}{2}(\tilde{u}^{(0)})^2+a\tilde{u}^{(0)})\}dz\}\\&=\partial_S(\tfrac{kP}{2})-\partial_X(kaM-k\delta P-E+\tfrac{kcP}{2}-kaM)\\&=\partial_S(\tfrac{kP}{2})+\partial_X(\tfrac{k\delta P}{2}+E).\end{aligned} \tag{4.40-2}$$

Therefore from (4.36), (4.40-1) and (4.40-2), we obtain

$$\begin{cases}\partial_S k-\partial_X(-\delta k)=0,\\ \partial_S(kM)-\partial_X(a-k\delta M)=0,\\ \partial_S(kP)-\partial_X(-k\delta P-2E)=0.\end{cases}$$

We now assume that

$$\frac{\partial(k,M,p)}{\partial(a,E,c)}=-\frac{1}{T^2}\frac{\partial(T,M,P)}{\partial(a,E,c)}\neq 0, \qquad k=\frac{1}{T}.$$

This is the condition mentioned in Section 4.2.2. From this relation, (a,E,c) is uniquely determined by (k,M,P). Therefore (6) is closed system for (k,M,P).

Here we consider (k,M,p) as a function of (a,E,c), $c=c_0+\delta$.

$$(k,P,M)=(k(a,E,\delta),P(a,E,\delta),M(a,E,\delta)).$$

In the variable (a,E,c), (6) becomes closed system for (a,E,c). We now consider the stability of the periodic traveling wave solution $U(z;a_0,E_0,c_0)$. Clearly $(a_0,E_0,0)$ is a stationary solution of (6), which corresponds to the periodic traveling wave solution $U(z;a_0,E_0,c_0)$. We linearize (6) around $(a_0,E_0,0)$.

$$\begin{cases}(k_a,k_E,k_\delta)|_{(a,E,\delta)=(a_0,E_0,0)}\cdot\partial_S(\tilde{a},\tilde{E},\tilde{\delta})\\ \qquad-((-\delta k)_a,(-\delta k)_E,(-\delta k)_\delta)|_{(a,E,\delta)=(a_0,E_0,0)}\cdot\partial_X(\tilde{a},\tilde{E},\tilde{\delta})=0,\\ ((kM)_a,(kM)_E,(kM)_\delta)|_{(a,E,\delta)=(a_0,E_0,0)}\cdot\partial_S(\tilde{a},\tilde{E},\tilde{\delta})\\ \qquad-((a-k\delta M)_a,(a-k\delta M)_E,(a-k\delta M)_\delta)|_{(a,E,\delta)=(a_0,E_0,0)}\cdot\partial_X(\tilde{a},\tilde{E},\tilde{\delta})=0,\\ ((kP)_a,(kP)_E,(kP)_\delta)|_{(a,E,\delta)=(a_0,E_0,0)}\cdot\partial_S(\tilde{a},\tilde{E},\tilde{\delta})\\ \qquad-((-k\delta P-2E)_a,(-k\delta P-2E)_E,(-k\delta P-2E)_\delta)|_{(a,E,\delta)=(a_0,E_0,0)}\cdot\partial_X(\tilde{a},\tilde{E},\tilde{\delta})=0.\end{cases}$$

This is equivalent to

$$\begin{aligned}&\frac{\partial(k,kM,kP)}{\partial(a,E,\delta)}|_{(a,E,\delta)=(a_0,E_0,0)}\partial_S(\tilde{a},\tilde{E},\tilde{\delta})\\&\qquad-\frac{\partial(-\delta k,a-k\delta M,-k\delta P-2E)}{\partial(a,E,\delta)}|_{(a,E,\delta)=(a_0,E_0,0)}\partial_X(\tilde{a},\tilde{E},\tilde{\delta})=0.\end{aligned} \tag{4.45}$$

We seek the following form of solutions.

$$(\tilde{a},\tilde{E},\tilde{\delta}) = e^{-\lambda S - i\kappa X}(\tilde{a}_0,\tilde{E}_0,\tilde{\delta}_0), \qquad \lambda\in\mathbb{C},\ \kappa\in\mathbb{R}. \tag{4.46}$$

If (4.45) admits a solution with a from of (4.46), (λ,κ) must satisfy

$$\mathcal{D}(\lambda,\kappa) = \det\{(\lambda\tfrac{\partial(k,kM,kP)}{\partial(a,E,\delta)} - i\kappa\tfrac{\partial(-\delta k,a-k\delta M,-k\delta P-2E)}{\partial(a,E,\delta)})|_{(a,E,\delta)=(a_0,E_0,0)}\}. \tag{4.47}$$

Since a solution of linearized problem (4.45) is given by (4.46), for the case $\lambda\notin i\mathbb{R}$, we conclude that the stationary solution $(a_0,E_0,0)$ corresponding the periodic traveling wave solution $U(z;a_0,E_0,c_0)$ is unstable. Let $\mu=\frac{i\kappa}{\lambda}$. In the following computation, the value of (k,M,P) is estimated at $(a,E,c)=(a_0,E_0,0)$.

$$\begin{aligned}
\lambda^{-3}\mathcal{D}(\lambda,\kappa) &= \mathcal{D}(1,\tfrac{\kappa}{\lambda})\\
&= \det\left\{\begin{pmatrix} k_a & k_E & k_\delta\\ (kM)_a & (kM)_E & (kM)_\delta\\ (kP)_a & (kP)_E & (kP)_\delta\end{pmatrix}\right.\\
&\qquad\left. + \mu\begin{pmatrix} (\delta k)_a & (\delta k)_E & (\delta k)_\delta\\ (-a+k\delta M)_a & (-a+k\delta M)_E & (-a+k\delta M)_\delta\\ (k\delta P+2E)_a & (k\delta P+2E)_E & (k\delta P+2E)_\delta\end{pmatrix}\right\}\\
&= \det\left\{\begin{pmatrix} k_a & k_E & k_\delta\\ (kM)_a & (kM)_E & (kM)_\delta\\ (kP)_a & (kP)_E & (kP)_\delta\end{pmatrix} + \mu\begin{pmatrix} 0 & 0 & k\\ -1 & 0 & kM\\ 0 & 2 & kP\end{pmatrix}\right\}\\
&= \det\begin{pmatrix} k_a & k_E & k_\delta+\mu k\\ (kM)_a-\mu & (kM)_E & (kM)_\delta+\mu kM\\ (kP)_a & (kP)_E+2\mu & (kP)_\delta+\mu kP\end{pmatrix}\\
&= \det\begin{pmatrix} k_a & k_E & k_\delta\\ (kM)_a-\mu & (kM)_E & (kM)_\delta\\ (kP)_a & (kP)_E+2\mu & (kP)_\delta\end{pmatrix} + k\mu\det\begin{pmatrix} k_a & k_E & 1\\ (kM)_a-\mu & (kM)_E & M\\ (kP)_a & (kP)_E+2\mu & P\end{pmatrix}\\
&= \det\begin{pmatrix} k_a & k_E & k_\delta\\ kM_a-\mu & kM_E & kM_\delta\\ kP_a & kP_E+2\mu & kP_\delta\end{pmatrix} + k\mu\det\begin{pmatrix} 0 & 0 & 1\\ kM_a-\mu & kM_E & M\\ kP_a & kP_E+2\mu & P\end{pmatrix}\\
&= \det\begin{pmatrix} k_a & k_E & k_\delta\\ kM_a & kM_E & kM_\delta\\ kP_a & kP_E+2\mu & kP_\delta\end{pmatrix} + \det\begin{pmatrix} 0 & k_E & k_\delta\\ -\mu & kM_E & kM_\delta\\ 0 & kP_E+2\mu & kP_\delta\end{pmatrix}\\
&\quad + k\mu\det\begin{pmatrix} kM_a-\mu & kM_E\\ kP_a & kP_E+2\mu\end{pmatrix}\\
&= \det\begin{pmatrix} k_a & k_E & k_\delta\\ kM_a & kM_E & kM_\delta\\ kP_a & kP_E & kP_\delta\end{pmatrix} + \det\begin{pmatrix} k_a & 0 & k_\delta\\ kM_a & 0 & kM_\delta\\ kP_a & 2\mu & kP_\delta\end{pmatrix} + \mu\det\begin{pmatrix} k_E & k_\delta\\ kP_E+2\mu & kP_\delta\end{pmatrix}\\
&\quad + k\mu\det\begin{pmatrix} kM_a & kM_E\\ kP_a & kP_E+2\mu\end{pmatrix} + k\mu\det\begin{pmatrix} -\mu & kM_E\\ 0 & kP_E+2\mu\end{pmatrix}\\
&= k^2\det\begin{pmatrix} k_a & k_E & k_\delta\\ M_a & M_E & M_\delta\\ P_a & P_E & P_\delta\end{pmatrix} + 2\mu\det\begin{pmatrix} k_\delta & k_a\\ kM_\delta & kM_a\end{pmatrix} + \mu\det\begin{pmatrix} k_E & k_\delta\\ kP_E & kP_\delta\end{pmatrix}\\
&\quad + \mu\det\begin{pmatrix} 0 & k_\delta\\ 2\mu & kP_\delta\end{pmatrix} + k\mu\det\begin{pmatrix} kM_a & kM_E\\ kP_a & kP_E\end{pmatrix} + k\mu\det\begin{pmatrix} kM_a & 0\\ kP_a & 2\mu\end{pmatrix}\\
&\quad + k\mu\det\begin{pmatrix} -\mu & kM_E\\ 0 & kP_E+2\mu\end{pmatrix}\\
&= k^2\det\tfrac{\partial(k,M,P)}{\partial(a,E,\delta)} + 2k\mu\det\tfrac{\partial(k,M)}{\partial(\delta,a)} + k\mu\det\tfrac{\partial(k,P)}{\partial(E,\delta)} - 2k_\delta\mu^2 + k^3\mu\tfrac{\partial(M,P)}{\partial(a,E)}\\
&\quad + 2k^2M_a\mu^2 - k\mu^2(kP_E+2\mu)\\
&= k^2\det\tfrac{\partial(k,M,P)}{\partial(a,E,\delta)} + k\mu(2\det\tfrac{\partial(k,M)}{\partial(\delta,a)} + \det\tfrac{\partial(k,P)}{\partial(E,\delta)} + k^2\tfrac{\partial(M,P)}{\partial(a,E)})\\
&\quad + \mu^2(-2k_\delta + 2k^2M_a - k^2P_E) - 2k\mu^3.
\end{aligned}$$

Since $T = \frac{1}{k}$, it holds that

$$\lambda^{-3}\mathcal{D}(\lambda,\kappa) = \tfrac{-1}{T^4}\det\tfrac{\partial(T,M,P)}{\partial(a,E,\delta)} + \tfrac{\mu}{T^3}(-2\det\tfrac{\partial(T,M)}{\partial(\delta,a)} - \det\tfrac{\partial(T,P)}{\partial(E,\delta)} + \tfrac{\partial(M,P)}{\partial(a,E)}) + \tfrac{\mu^2}{T^2}(2T_\delta + 2M_a - P_E) - \tfrac{2\mu^3}{T}.$$

From (4.19)-(4.20), we easily see that

$$M_a = P_E = -2T_\delta.$$

Furthermore from (4.21) (see Section 6.2 in [1]), it holds that

$$\det\tfrac{\partial(T,M)}{\partial(\delta,a)} = \det\begin{pmatrix} T_\delta & T_a \\ M_\delta & M_a \end{pmatrix} = \det\begin{pmatrix} K_{E\delta} & K_{Ea} \\ K_{a\delta} & K_{aa} \end{pmatrix} = \det\begin{pmatrix} K_{E\delta} & K_{a\delta} \\ K_{Ea} & K_{aa} \end{pmatrix}$$
$$= \det\begin{pmatrix} \frac{-1}{2}P_E & \frac{-1}{2}P_a \\ M_E & M_a \end{pmatrix} = \tfrac{-1}{2}\det\tfrac{\partial(P,M)}{\partial(E,a)} = \tfrac{-1}{2}\det\tfrac{\partial(M,P)}{\partial(a,E)}$$

Therefore we obtain

$$\mathcal{D}(\lambda,\kappa) = \tfrac{-\lambda^3}{T^4}\det\tfrac{\partial(T,M,P)}{\partial(a,E,\delta)} + \tfrac{\lambda^3\mu}{T^3}(-\det\tfrac{\partial(T,P)}{\partial(E,\delta)} + 2\det\tfrac{\partial(M,P)}{\partial(a,E)}) - \tfrac{2\lambda^3\mu^3}{T}$$
$$= \tfrac{\lambda^3}{T^4}\{-\det\tfrac{\partial(T,M,P)}{\partial(a,E,\delta)} + T\mu(-\det\tfrac{\partial(T,P)}{\partial(E,\delta)} + 2\det\tfrac{\partial(M,P)}{\partial(a,E)}) - 2T^3\mu^3\}.$$

Let $\xi = T\mu$. Then it follows that

$$\mathcal{D}(\lambda,\kappa) = \tfrac{2\lambda^3}{T^4}\{-\xi^3 + \tfrac{\xi}{2}(-\det\tfrac{\partial(T,P)}{\partial(E,\delta)} + 2\det\tfrac{\partial(M,P)}{\partial(a,E)}) - \tfrac{1}{2}\det\tfrac{\partial(T,M,P)}{\partial(a,E,\delta)}\}. \tag{7}$$

Since $\xi = T\mu$, $\mu = \frac{i\kappa}{\lambda}$, if the polynomial on the right-hand side has a non-real root, $(a_0, E_0, 0)$ is unstable. In general, the discriminant of the the third order polynomial $f(\xi) = a\xi^3 + b\xi^2 + c\xi + d$ is given by

$$\mathrm{disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

From this formula, it follows that

$$\mathrm{disc}(-\xi^3 + \tfrac{\xi}{2}(-\det\tfrac{\partial(T,P)}{\partial(E,\delta)} + 2\det\tfrac{\partial(M,P)}{\partial(a,E)}) - \tfrac{1}{2}\det\tfrac{\partial(T,M,P)}{\partial(a,E,\delta)})$$
$$= \tfrac{1}{2}(-\det\tfrac{\partial(T,P)}{\partial(E,\delta)} + 2\det\tfrac{\partial(M,P)}{\partial(a,E)})^3 - \tfrac{27}{4}(\det\tfrac{\partial(T,M,P)}{\partial(a,E,\delta)})^2.$$

Therefore if the right-hand side is negative, $(a_0, E_0, 0)$ is unstable.

4.4 Rigorous theory of modulational instability

4.4.1 Analytic setup

In this section, we consider (4.1) in the moving coordinate as in Section 4.3.2.

$$u_t = u_{xxx} + f(u)_x + cu_x. \tag{4.5.1}$$

Let u_0 be the periodic traveling wave solution of (4.1) with speed c, which is stationary solution of (4.5.1). We linearize (4.5.1) around u_0.

$$v_t = Lv, \tag{4.52}$$

where the operator is defined as

$$Lv = \partial_x\{(\partial_x^2 + f'(u_0) + c)v\}, \qquad L : D(L) = H^3(\mathbb{R}) \to L_2(\mathbb{R}). \tag{4.53}$$

A goal of this section is to study the $L^2(\mathbb{R})$-spectrum of L. We recall definition of the spectrum of linear operators. The spectral $\sigma(L)$ is a set of $\lambda \in \mathbb{C}$ such that $L - c : H^3(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$ does not have a continuous inverse from $L^2(\mathbb{R};\mathbb{C}) \to H^3(\mathbb{R};\mathbb{C})$. Since (4.52) has invariant the following two transformation

$$v \to \bar{v} \qquad \text{and} \qquad \lambda \to \bar{\lambda}, \tag{4.57}$$
$$x \to -x \qquad \text{and} \qquad \lambda \to -\lambda, \tag{4.58}$$

the spectrum of L is symmetric with respect to reflections about both the real and imaginary axis. Consequently u_0 is spectrally stable if and only if the $L^2(\mathbb{R})$-spectrum of L is only on the imaginary axis. Let

$$L_\xi = e^{-i\xi x}Le^{i\xi x}, \qquad \xi \in [-\tfrac{\pi}{T}, \tfrac{\pi}{T}). \tag{4.61}$$

We apply the decomposition property of a linear operator with T-periodic coefficients (see Theorem A.4 in [5]).

$$\mathrm{spec}_{L^2(\mathbb{R})}(L) = \overline{\bigcup_{\xi\in[-\frac{\pi}{T},\frac{\pi}{T})} \mathrm{spec}_{L^2_{\mathrm{per}}(0,T)}(L_\xi)}$$

Since the inverse of $L_\xi + \kappa$ becomes a compact operator for some $\kappa \in \mathbb{R}$, the spectrum of L_ξ coincides with the eigenvalues of L_ξ. From now, we study the spectrum of L_ξ for $|\xi| \ll 1$. For simplicity, we set

$$\{f,g\}_{x,y} = \det \tfrac{\partial(f,g)}{\partial(x,y)}, \qquad \{f,g,h\}_{x,y,z} = \det \tfrac{\partial(f,g,h)}{\partial(x,y),z}$$

and let

$$\begin{aligned} \phi_0 &= \{T,u\}_{a,E}, \qquad \phi_1 = \{T,M\}_{a,E}u_x, \qquad \phi_2 = \{u,T,M\}_{a,E,c}, \\ \psi_0 &= 1, \qquad \psi_1 = \textstyle\int_0^x \phi_2(s)ds, \qquad \psi_2 = -\{T,M\}_{E,c} + \{T,M\}_{a,E}u. \end{aligned} \tag{4.65}$$

Furthermore we denote $(\cdot,\cdot)_{L^2(0,T)}$ by $(\cdot,\cdot)$. These functions are all T-periodic. We recall the following result.

Lemma 4.1 (Propostion 1 in [5], Proposition 4 in [1])**.** *Suppose that T_E, $\{T,M\}$ and $\{T,M,P\}_{a,E,c}$ are not zero at (a_0,E_0,c_0). Then it holds that*

$$\begin{aligned} L_0\phi_0 = L_0\phi_1 = 0, \qquad L_0\phi_2 = -\phi_1, \\ L_0^\dagger\psi_0 = L_0\psi_2 = 0, \qquad L_0^\dagger\psi_1 = -\psi_2. \end{aligned} \tag{4.66}$$

In particular, ϕ_j $(j = 0,1,2)$ for a bases for the generalized null space of L_0 and ψ_j $(j = 0,1,2)$ for a bases for the generalized null space of $L_0^\dagger$. Furthermore $(\psi_j,\phi_i) = 0$ if $i \neq j$.

Let X_ξ be the subspace in $L^2_{\mathrm{per}}(\mathbb{R})$ spanned by generalized eigenfunctions of L_ξ with $|\lambda| < \epsilon$. We denote by Π_ξ a total eigen-projection from $L^2_{\mathrm{per}}(\mathbb{R})$ to X_ξ. This is represented as

$$\Pi_\xi = \tfrac{-1}{2\pi i}\textstyle\int_\Gamma (L_\xi - \lambda)^{-1} d\lambda. \tag{8}$$

Then it should hold that

$$\Pi_\xi(L^2_{\mathrm{per}}(\mathbb{R})) = X_\xi, \qquad L_\xi X_\xi = X_\xi. \tag{9}$$

The presenter does not know whether (9) is correct or not. For the case $\xi = 0$, it holds that $\Pi_\xi|_{\xi=0}\phi_i = \phi_i$ $(i = 0,1)$, $\tilde{\Pi}_0|_{\xi=0}\psi_i = \psi_i$ $(i = 0,2)$ and

$$\begin{aligned} \Pi_\xi|_{\xi=0}\phi_2 &= \tfrac{-1}{2\pi}\textstyle\int_\Gamma (L_0 - \xi)^{-1}\phi_2 d\xi = \tfrac{-1}{2\pi}\int_\Gamma (\tfrac{-\phi_2}{\xi} + \tfrac{\phi_1}{\xi^2}) d\xi = \phi_2, \\ \Pi_\xi^\dagger|_{\xi=0}\psi_2 &= \tfrac{-1}{2\pi}\textstyle\int_\Gamma (L_0^\dagger - \xi)^{-1}\psi_2 d\xi = \tfrac{-1}{2\pi}\int_\Gamma (\tfrac{-\psi_2}{\xi} + \tfrac{\psi_1}{\xi^2}) d\xi = \psi_2. \end{aligned}$$

If we assume (9), our problem is reduced to finite dimensional eigenvalue problem on X_ξ.

$$L_\xi|_{X_\xi}\phi = \lambda\phi. \tag{10}$$

Let $\{v_j(\xi)\}_{j=0,1,2}$ and $\{\tilde{v}_j(\xi)\}_{j=0,1,2}$ be bases of X_ξ and $X_\xi^\dagger$ satisfying

$$v_j(\xi) = \phi_j + (i\xi)v_j^{(1)} + o(\xi), \qquad \tilde{v}_j(\xi) = \psi_j + (i\xi)\tilde{v}_j^{(1)} + o(\xi). \tag{4.84}$$

Define 3×3 matrix as

$$[M_\xi]_{ij} = (\tilde{v}_i(\xi), L_\xi v_j(\xi))_{L^2_{\mathrm{per}}}.$$

Let $v = \alpha_1 v_1(\xi) + \alpha_2 v_2(\xi) + \alpha_3 v_3(\xi) \in X_\xi$ be an eigenfunction of $L_\xi|_{X_\xi}$ with eigenvalue λ. Then it holds that

$$(\tilde{v}_i(\xi), L_\xi(v_j(\xi)))\alpha_j = (\tilde{v}_i(\xi), L_\xi v) = (\tilde{v}_i(\xi), \lambda v) = \lambda(\tilde{v}_i(\xi), v_j(\xi))\alpha_j.$$

Therefore for the case where (10) has distinct three eigenvalues, eigenvalues of (10) coincide with eigenvalues of

$$M_\xi = \lambda I_\xi, \tag{11}$$

where $[I_\xi]_{ij} = (\tilde{v}_i(\xi), v_j(\xi))$. For the case $\xi = 0$, this is written as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (\psi_1, L_0\phi_2) \\ 0 & 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} (\psi_0,\phi_0) & 0 & 0 \\ 0 & (\psi_1,\phi_1) & 0 \\ 0 & 0 & (\psi_2,\phi_2) \end{pmatrix}. \tag{12}$$

Let

$$\hat{M}_\xi = \Sigma(\xi)^{-1} M_\xi \Sigma(\xi), \qquad \hat{I}_\xi = \Sigma(\xi)^{-1} I_\xi \Sigma(\xi),$$

where

$$\Sigma(\xi) = \begin{pmatrix} i\xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i\xi \end{pmatrix}$$

We now consider

$$\hat{M}_\xi = \mu \hat{I}_\xi. \tag{13}$$

Eigenvalues of (11) and (13) satisfy

$$\lambda = i\mu\xi. \tag{14}$$

Our goal is to investigate of (13) for $\xi = 0$. To do that, we expand L_ξ as (see (4.61))

$$L_\xi = L_0 + i\xi L_1 + \frac{(i\xi)^2}{2} L_2 + O(\xi^3), \tag{4.70}$$

where

$$L_0 = \partial_x\{\partial_x^2 + f'(u_0) + c\}, \qquad L_1 = \partial_x^2 + f'(u_0) + c - 2\partial_x^2, \qquad L_2 = -3\partial_x. \tag{4.71}$$

By using (4.84), (4.70) and Lemma 4.1, we obtain

$$\hat{M}_\xi = \begin{pmatrix} (\psi_0, L_1\phi_0) & m_{12} & (\psi_0, L_1\phi_2) + (\tilde{v}_0^{(1)}, L_0\phi_2) \\ 0 & (\psi_1, L_1\phi_1) + (\psi_1, L_0 v_1^{(1)}) & (\psi_1, L_0\phi_2) \\ (\psi_2, L_1\phi_0) & m_{32} & (\psi_2, L_1\phi_2) + (\tilde{v}_2^{(1)}, L_0\phi_2) \end{pmatrix} + o(1), \tag{4.87}$$

where

$$m_{j2} = (\psi_j, L_2\phi_1 + L_1 v_1^{(1)}) + (\tilde{v}_j^{(1)}, L_1\phi_1 + L_0 v_1^{(1)}), \qquad j = 1, 3 \tag{4.88}$$

and

$$\hat{I}_\xi = \begin{pmatrix} (\psi_0, \phi_0) & (\psi_0, v_1^{(1)}) + (\tilde{v}_0^{(1)}, \phi_1) & 0 \\ 0 & (\psi_1, \phi_1) & 0 \\ 0 & (\psi_2, v_1^{(1)}) + (\tilde{v}_2^{(1)}, \phi_1) & (\psi_2, \phi_2) \end{pmatrix} + o(1). \tag{4.89}$$

To compute components of $\hat{M}_\xi$, $\hat{I}_\xi$, we need to determine $v_1^{(1)}$, $\tilde{v}_0^{(1)}$ and $\tilde{v}_2^{(1)}$. To simplify computations, we choose $\{v_j(\xi)\}_{j=0,1,2}$ and $\{\tilde{v}_j(\xi)\}_{j=0,1,2}$ suitably. The presenter does not understand this part. After long computations (see p16 in [4]), we obtain

$$\det(\hat{M}_\xi|_{\xi=0} - \mu\hat{I}_\xi|_{\xi=0}) = c(-\mu^3 + \tfrac{\mu}{2}(-\{T,P\}_{E,c} + 2\{M,P\}_{a,E}) - \tfrac{1}{2}\{T,M,P\}_{a,E,c}) \tag{4.97}$$

for some $c \neq 0$. Since $\lambda = i\mu\xi$ ($\xi \in \mathbb{R}$), if (4.97) has a non-real root, u_0 is unstable. This condition is equivalent to the condition given in the previous section (see (7)).

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