An Introduction to Stability Theory for Nonlinear PDEs

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Abstract

These notes were prepared for the 2017 Participating School in Analysis of PDE: Stability of Solitons and Periodic Waves held in KAIST in Daejeon, Korea during August 21 - August 25, 2017. Their main focus is to introduce participants to some of the basic methodologies and techniques for obtaining spectral, linear and nonlinear stability results for nonlinear wave solutions to special classes of PDEs, with explicit examples being worked out whenever possible.

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1 Introduction

The purpose of this summer school is to introduce students and early career researchers to various aspects of the stability theory for special classes of solutions in some important nonlinear partial differential equations (PDEs). Many times, this theory mimics classical finite-dimensional ODE theory, while making appropriate modifications accounting for the fact that the state space for PDEs is inherently infinite dimensional. Consequently, we will begin with a very brief review of finite-dimensional ODE stability theory.

To begin, consider a nonlinear ODE of the form

$$\dot{u} = F(u), \quad u = u(t) \in \mathbb{R}^n \tag{1}$$

where here $F : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth and $\dot{u}$ refers to the derivative of u with respect to the dependent variable t. Basic results in ODE theory guarantee that for each initial condition $u_0 \in \mathbb{R}^n$, there will exist a unique solution $u(t; u_0)$ of (1) with $u(0; u_0) = u_0$ that exists at least locally, i.e. for at least $|t| \ll 1$. The basic question in stability theory is the following:

Question: *If $u_0 \in \mathbb{R}^n$ is a fixed point of F, so that $F(u_0) = 0$, and if $|u_1 - u_0| \ll 1$, will $u(t; u_1)$ remain near u_0 for all $t > 0$? If not, what happens?*

Observe that by continuous dependence, we know that since $|u_1 - u_0| \ll 1$ we will have $|u(t; u_1) - u_0| \ll 1$ for at least a short amount of time. A natural and often used method to determine if $u(t; u_1)$ stays close to u_0 for *all* time $t > 0$ is to first approximate the dynamics of (1) near u_0 by studying a suitable approximating system. This leads to the process of linearization.

To this end, note that we may write, for so long as it exists,

$$u(t; u_1) = u_0 + v(t)$$

so that now $u(t; u_1)$ is considered as a perturbation of u_0. From (1), the function $v(t)$ solves the initial value problem (IVP)

$$\dot{v} = F(u_0 + v), \quad v(0) = u_1 - u_0 \tag{2}$$

Noting that we can write

$$F(u_0 + v) = F(u_0) + DF(u_0)v + \underbrace{(F(u_0 + v) - F(u_0) - DF(u_0))}_{\mathcal{O}(|v|^2)},$$

where here DF is the $n \times n$ matrix valued derivative of F at u_0, it follows v satisfies an IVP of the form

$$\dot{v} = DF(u_0)v + \mathcal{O}(|v|^2), \quad v(0) = u_1 - u_0, \tag{3}$$

where we have used here the fact that $F(u_0) = 0$ by construction. Using Duhamel's formula or, equivalently, variation of parameters, we can rewrite the above evolution equation for the perturbation v as the implicit integral equation

$$v(t) = e^{DF(u_0)t}v(0) + \int_0^t e^{DF(u_0)(t-s)}\mathcal{O}(|v(s)|^2)ds$$

where here $e^{DF(u_0)t}$ is the $n \times n$ matrix exponential of the linear operator $DF(u_0)$. In particular, for every $v(0) \in \mathbb{R}^n$ the vector $e^{DF(u_0)t}v(0)$ is the unique solution to the linear system

$$\dot{v} = D_uF(u_0)v, \quad v(0) = u_1 - u_0. \tag{4}$$

Since the $\mathcal{O}(|v|^2)$ are small compared to the $D_uF(u_0)v$ terms when v is small, we expect that so long as $|v(t)|$ remains small that solutions of (2) will be well approximated by solutions of (4), the solutions of which are governed *completely* by the eigenvalues of the matrix $DF(u_0)$. In particular, if λ is an eigenvalue of $DF(u_0)$ with eigenfunction $w \in \mathbb{R}^n$ then the function $v(t) = e^{\lambda t}w$ is a solution of (4). It follows that if all the eigenvalues of $DF(u_0)$ have strictly negative real part, then all the solutions of (4) decay exponentially as $t \to \infty$, while if there is any eigenvalue with positive real part then there exists some solution of (4) that blows up as $t \to \infty$. Finally, if $DF(u_0)$ has an eigenvalue on the imaginary axis $i\mathbb{R}$, then (4) has a solution that remains bounded for all $t \in \mathbb{R}$, being oscillatory and not decaying for large time.

A natural question, and one of the key problems in classical ODE stability theory, is when the predictions from the linearized system (4) carry over to the nonlinear system (3). In classical ODE theory, the simplest result in this direction is the Stable Manifold Theorem. To motivate this result, suppose for a moment that all eigenvalues of $DF(u_0)$ are semisimple, i.e. their algebraic multiplicity agrees with their geometric multiplicity (so no Jordan blocks). Then we can express

$$e^{Lt} = \sum_{j=1}^{k} e^{\lambda_j t}\Pi_j$$

where the Π_j are the spectral projections onto the eigenspaces associated with the k distinct eigenvalues of $DF(u_0)$. It follows that if $\omega = \max_j \Re(\lambda_j)$ then there exists a constant $C > 0$ such that

$$\left|e^{Lt}\right| \leq Ce^{\omega t}$$

for all $t > 0$. In particular if $\Re(\lambda_j) < -\gamma$ for some constant $\gamma > 0$ and all j then every solution of (4) will satisfy $|v(t)| \leq Ce^{-\gamma t}|v(0)|$ for all $t > 0$. The fact that this observation carries over for the nonlinear equation (3), even when $DF(u_0)$ may admit Jordan blocks, follows by the following fundamental result.

Theorem 1 (Stable Manifold Theorem)**.** *Suppose there exists a constant $\gamma > 0$ such that every eigenvalue $\lambda \in \mathbb{C}$ of the matrix $DF(u_0)$ satisfies $\Re(\lambda) < -\gamma$. Then there exists* *constants $\delta > 0$ and $C \geq 1$ such that if $|v_0| < \delta$ then the associated unique solution $v(t)$ of* (3) *exists for all $t > 0$ and satisfies the exponential decay estimate*

$$|v(t)| \leq Ce^{-\gamma t}|v(0)|$$

for all $t \geq 0$. In particular, u_0 is an asymptotically stable *solution of* (1).

Remark 1. *By essentially taking $t \to -t$ in the proof of the Stable Manifold Theorem, one can show that if $DF(u_0)$ has an eigenvalue with strictly positive real part, then u_0 is an unstable solution of* (1)*: there exists an $\epsilon > 0$ such that for every $\delta \in (0, \epsilon)$ there exists initial data $v(0)$ with $|v(0)| < \delta$ such that the associated solution $v(t)$ satisfies $|v(T)| > \epsilon$ for some finite $T > 0$. This result is known as the Unstable Manifold Theorem*

Remark 2. *We note that if $DF(u_0)$ has an eigenvalue on the imaginary axis, one must work harder in order to determine the stability of u_0: it may be stable or unstable depending on the particular nonlinearity. In this case, one usually either uses a center manifold reduction, or tries the study the stability of u_0 through energy (i.e. Lyapunov) methods instead.*

In these notes, we are interested in extending the above results to be suitable for PDE applications. Some of the initial challenges include:

(1) In the PDE case, establishing that the PDE can be solved, even locally in time, for initial data "near" the background wave u_0 is a much more delicate matter. One thing that complicates this is evolutionary PDE's of the form $u_t = F(u)$, where here F may be a nonlinear differential operator with possibly non-constant coefficients, describe the evolution of functions in *infinite dimensional* vector spaces. Consequently, there are many non-equivalent topologies one may use to define what "close" means, and identifying the appropriate topology in which to work is not always easy.

(2) The linearized operator $DF(u_0)$ is now generally a differential operator with coefficients depending on the function u_0. Describing the spectral properties of $DF(u_0)$ then takes significantly more care, since linear operators on infinite dimensional vector spaces may fail to be invertible in more ways than losing injectivity, i.e. there maybe more than eigenvalues we have to worry about. Such issues are discussed in Section 2 below. Furthermore, turning spectral properties into decay bounds on the linear solution operator[2] e^{Lt} also becomes a much more delicate matter, and is discussed in Section 3.3.

(3) In PDE's modeling extended systems, i.e. defined on the infinite line $\mathbb{R}$, the linearized operator often does *not* have a spectral gap, i.e. the spectrum of $DF(u_0)$ includes points on the imaginary axis. This is often due to the presence of continuous symmetries in the governing PDE, such as the PDE being translationally invariant, which causes the linearized operator $DF(u_0)$ to fail to be invertible: see Remark 6 in Section

[2] In fact, since differential operators are unbounded from spaces into themselves, it is not a-priori clear how to even define e^{Lt}.

3.2.2. Dealing with this lack of spectral gap is an important matter, and something that is fundamental to the stability analysis of PDE's. We will discuss some methods in this direction in Section 4.2 and Section 4.3 below.

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Disclaimer: These notes were extracted from a class I taught during the Fall 2015 semester at the University of Kansas. The primary text for that course was the book [KP] by Todd Kapitula and Keith Promislow. As such, while these notes do not follow [KP] verbatim, they were *heavily* influenced it. Most everything in these notes can be found in [KP], and, due to time constraints when writing these notes, I make no attempt to flush out precise page or theorem number references. Also, there are hundreds of additional references that should be cited throughout these notes, but for the same reasons they are not listed here. My apologies to all those who should be cited, and to anyone reading these notes.

2 Spectral Theory: Survey of Results

In this section, we briefly discuss the relevant definitions from spectral theory. We will then discuss the nature of the spectrum for classes of differential operators that arise naturally in stability theory for nonlinear waves.

Consider an n^{th}-order, scalar linear differential operator of the form

$$L = \partial_x^n + a_{n-1}(x)\partial_x^{n-1} + a_{n-2}(x)\partial_x^{n-2} + \ldots + a_1(x)\partial_x + a_0(x) \tag{5}$$

where here the functions a_j are sufficiently smooth. In PDE applications, such operators arise naturally as the linearizations of a PDE about a some nonlinear wave solution, where the coefficient functions a_j depend on the background wave. Since for this workshop we are interested in the stability of solitary and periodic waves, we will briefly discuss the spectral properties of operators of the form L in when the a_j are either exponentially localized or spatially periodic, corresponding to the cases when L is the linearization about a solitary or periodic wave, respectively.

We begin with some basic definitions.

Definition 1. *Suppose L acts on a complex Banach space X. The resolvent set of L on X, denoted by $\rho(L)$ is the set of all $\lambda \in \mathbb{C}$ where $L - \lambda I$ is invertible with bounded (i.e. continuous) inverse. Here, I denotes the identity operator. Further, the spectrum of L on X, denoted $\sigma(L)$ is defined as $\sigma(L) := \mathbb{C} \setminus \rho(L)$. Finally, the $\lambda \in \sigma(L)$ is an eigenvalue of L if $\mathrm{Ker}(L - \lambda I) \neq \{0\}$, i.e. if there exists a $v \in X \setminus \{0\}$ such that $Lv = \lambda v$..*

Remark 3. *Note the choice of the Banach space X on which L acts is crucial in our understanding of the the spectrum of L. As we will see below, the spectral properties of a given operator can change dramatically if one changes the space X.*

In many applications, it is useful to further decompose the spectrum of L. There are *several* non-equivalent ways of doing this, all of which appear in the literature. For the purpose of this lecture, we will use the following definition.

Definition 2. *Given a linear operator L acting on a complex Banach space X as above, the point spectrum of L, denoted $\sigma_p(L)$ (on X) is the set of all isolated eigenvalues of L (on X) with finite multiplicities. Further, the essential spectrum is $\sigma_{\text{ess}}(L) := \sigma(L) \setminus \sigma_p(L)$.*

As we will see, in stability theory the point and essential spectrum give drastically different information concerning the dynamics near a given nonlinear wave. In what follows, we discuss spectral properties for classes of differential operators often occurring in applications, as well as how the spectrum may be calculated.

2.1 BVP with Separated Boundary Conditions

First, let's consider a differential operator of the form

$$L = \partial_x^2 + a_1(x)\partial_x + a_0(x) \tag{6}$$

where a_0, a_1 are some given smooth, real-valued functions and let $T > 0$ be finite. Consider the spectral problem

$$Lv = \lambda v, \quad x \in (-T, T) \tag{7}$$

equipped with homogeneous Dirichlet boundary conditions

$$v(-T) = v(T) = 0. \tag{8}$$

Precisely, we are considering L as a densely defined operator on $L^2(-T,T)$ with densely defined domain $D(L) := H_0^1(T,T) \cap H^2(-T,T)$. Our goal here is to characterize $\sigma(L)$.

Our first observation is that the operator L is self-adjoint with respect to the *weighted* inner product

$$\langle f, g\rangle := \int_{-T}^{T} \overline{f(x)} g(x)\rho(x)dx,$$

where the weight function is $\rho(x) := \exp\left(\int_0^x a_1(z)dz\right)$. Consequently, $\sigma(L) \subset \mathbb{R}$.

Next, we search for eigenvalues of L, i.e. we search for $\lambda \in \mathbb{C}$ such that there exists a non-trivial solution of (7) satisfying the B.C.'s at $\pm T$. To this end, let $\phi_\pm$ be the unique solutions of the IVP

$$\begin{cases} Lv = \lambda v, \quad x \in (-T, T) \\ v(\pm T) = 0 \\ v'(\pm T) = 1 \end{cases}$$

and note that for each $\lambda \in \mathbb{C}$ the function $\phi_-(\cdot;\lambda)$ satisfies the B.C. at $x = -T$, while the function $\phi_+(\cdot;\lambda)$ satisfies the B.C. at $x = T$. Observe that if $\phi_\pm$ are linearly dependent for some $\lambda \in \mathbb{C}$, then there exists a constant $C > 0$ such that

$$\phi_-(x;\lambda) = C\phi_+(x;\lambda) \quad \forall x \in [-T, T]$$

so that, in particular, the function ϕ_+ provides a non-trivial solution of (7) satisfying *both* boundary conditions at $x = \pm T$, and hence $\lambda \in \sigma_p(L)$. Note if we define the Wronskian[3]

$$E(\lambda) := \det \begin{pmatrix} \phi_+(x;\lambda) & \phi_-(x;\lambda) \\ \phi_+'(x;\lambda) & \phi_-'(x;\lambda) \end{pmatrix}$$

then the functions $\phi_\pm(\cdot;\lambda)$ are linearly dependent precisely on the zero set of E. It can be easily checked that $E(\lambda)$ is an entire function of λ, and that the multiplicity of λ as a root of E agrees with the algebraic multiplicity of λ as an eigenvalue of L. By basic results in complex analysis, it follows that the eigenvalues of L are isolated with no finite accumulation point, and all eigenvalues necessarily have finite algebraic (and hence geometric) multiplicity.

Remark 4. *Note that if L has constant coefficients $a_1, a_0 \in \mathbb{R}$ then $\sigma_p(L)$ can be found through elementary ODE techniques. Indeed, for every $\lambda \in \mathbb{C}$ one can write the general solution to the second-order ODE $Lv = \lambda v$ and determine for which λ this solution satisfies the appropriate boundary conditions.*

Next, I claim that if $\lambda \notin \sigma_p(L)$ then $\lambda \in \rho(L)$. To see this, note that if $E(\lambda) \neq 0$ then $\phi_\pm(\cdot;\lambda)$ provides two linearly independent solutions of the 2nd order homogeneous ODE $Lv + \lambda v = 0$. Given any $f \in L^2(-T,T)$, one can now show that the unique solution v of the equation

$$Lv - \lambda v = f$$

that satisfies the B.C.'s (8) at $\pm T$ is given explicitly by

$$\begin{aligned} v(x;\lambda) &= \frac{\phi_+(x;\lambda)}{E(\lambda)} \int_{-T}^{x} f(z)\phi_-(z;\lambda)dz + \frac{\phi_-(x;\lambda)}{E(\lambda)} \int_{x}^{T} f(z)\phi_+(z;\lambda)dz \\ &= \int_{-T}^{T} G(x,z;\lambda)f(z)dz, \end{aligned}$$

where G here is the Green's function, given explicitly as

$$G(x,z;\lambda) = \frac{\phi_+(x;\lambda)\phi_-(z;\lambda)\chi_{(-T,x)}(z) + \phi_-(x;\lambda)\phi_+(z;\lambda)\chi_{(x,T)}(z)}{E(\lambda)}$$

where $\chi_{(a,b)}$ denotes the characteristic function of the set (a,b). Since a simple application of Holder's inequality implies that

$$\|v(x;\lambda)\|_{L^2(-T,T)} \leq C(\lambda)\|f\|_{L^2(-T,T)},$$

[3]It is easily checked that the determinant defining $E(\lambda)$ is independent of x. However, in applications it is sometimes helpful to have the flexibility in choosing at what $x \in [-T,T]$ to do the evaluation. This function $E(\lambda)$ can also be identified as the so-called "Evans function" for the given boundary value problem.

for all $\lambda \in \mathbb{C}$ with $E(\lambda) \neq 0$, i.e. $\lambda \notin \sigma_p(L)$, it follows that $(L - \lambda I)^{-1}$ is a bounded linear operator on $L^2(-T, T)$, and hence that $\lambda \in \rho(L)$, as claimed.

It is important to note that the fact that $\sigma(L) = \sigma_p(L)$ in the above example holds in much more generality. Indeed, it holds for general n^{th}-order linear differential operators defined bounded intervals $[a, b] \subset \mathbb{R}$, when equipped with n separated boundary conditions at $x = a$ and $x = b$. It can even be seen to hold in higher dimensions through the use of the Rellich-Kondrachov compactness theorem. The fact that $\sigma(L) \subset \mathbb{R}$ does *not* generally extend to such operators, but holds here due to the fact that the second-order operator L is a Sturm-Liouville operator. Furthermore, using the fact that L is second order, we have the following version of Sturm's Oscillation Theorem that is often helpful in studying $\sigma_p(L)$ in such cases.

Theorem 2 (Sturm's Oscillation Theorem for BVP's with Separated Boundary Conditions)**.** *Let L be a second-order, linear differential operator of the form* (6)*, considered as an operator on $L^2(a, b)$ with separated boundary conditions at $x = a$ and $x = b$. Then $\sigma_p(L)$ consists of infinitely many simple eigenvalues which can be numerated in strictly decreasing order as*

$$\lambda_0 > \lambda_1 > \lambda_2 > \ldots, \quad \lim_{n\to\infty} \lambda_n = -\infty.$$

Further, an eigenfunction associated with λ_j has exactly j zeroes on (a, b), all of which are simple.

2.2 Exponentially Localized Coefficients

Next, we consider the case when the functions a_j are exponentially localized in space, i.e. there exists an $r > 0$ and constants $a_0, a_1 \ldots, a_{n-1} \in \mathbb{R}$ such that

$$\lim_{x\to\infty} e^{r|x|}|a_j(x) - a_j| = 0$$

for all $j = 0, 1, 2, \ldots, n-1$. Here, we consider L as acting on the Lebesgue space $L^2(\mathbb{R})$ with densely defined domain $H^n(\mathbb{R})$ (a Sobolev space). Such operators arise naturally when linearizing a PDE defined on all of $\mathbb{R}$ about a wave that is asymptotically constant, such as a solitary wave or a front. We first discuss the essential spectrum of L. The key result here is known as the Weyl Essential Spectrum Theorem which, in this context, roughly states that the essential spectrum of L is controlled entirely by the behavior of the operator at spatial infinity. To make this precise, define the constant-coefficient asymptotic operator

$$L_\infty := \partial_x^n + a_{n-1}\partial_x^{n-1} + \ldots + a_1\partial_x + a_0$$

We then have the following key result.

Theorem 3 (Weyl Essential Spectrum Theorem)**.** *With L and L_∞ as above, we have*

$$\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(L_\infty).$$

The proof of this theorem relies on the fact that since the coefficient functions $a_j(x)$ of L are exponentially localized, the operator L is a *relatively compact perturbation* of the operator L_∞. Since functional analysis tells us the essential spectrum of an operator is invariant under relatively compact perturbations, the result follows.

Remark 5. *In the case where the functions a_j are exponentially localized to* different *limiting values $a_1^\pm, a_0^\pm$ at $\pm\infty$, such as what appears when linearizing a PDE about a front solution, the essential spectrum of L may then be a subset of $\mathbb{C}$ with non-zero two-dimensional measure. Consequently, Weyl's Essential Spectrum Theorem becomes more complicated in that scenario. Nevertheless, if we define asymptotic operators $L_{\pm\infty}$ at $x=\pm\infty$, respectively, one can show that the one-dimensional curves $\sigma_{\text{ess}}(L_{-\infty})$ and $\sigma_{\text{ess}}(L_{+\infty})$, known as the "Fredholm boundaries" of L, form the* boundary *of the essential spectrum of L. Most importantly for stability purposes, if the essential spectrum for both asymptotic operators $L_{\pm\infty}$ lies in the left half plane, then the essential spectrum of L also lies in the left half plane. Consequently, for stability purposes it is enough to just calculate the Fredholm boundaries.*

It remains to determine how to calculate the essential spectrum for the constant coefficient operator L_∞ on $L^2(\mathbb{R})$. Thankfully, this is actually very simple! Indeed, define the polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \ldots a_0$ so that $L_\infty = p(\partial_x)$: the polynomial p is called the *symbol* of L. For a given $\lambda \in \mathbb{C}$, we now try to solve the equation $(L - \lambda I)v = w$ for a given $w \in L^2(\mathbb{R})$. Since this equation has constant coefficients, this is easily solved using the Fourier transform. Indeed, denoting the Fourier transform of a function $v \in L^2(\mathbb{R})$ as

$$\mathcal{F}(v)(\xi) = \hat{v}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} v(x)dx$$

and noting that $\widehat{\partial_x^m v}(\xi) = (-i\xi)^m \hat{v}(\xi)$ for all positive integers m, it follows that

$$(L - \lambda I)v = w \quad \Rightarrow \quad (p(i\xi) - \lambda)\, \hat{v}(\xi) = \hat{w}(\xi) \quad \Rightarrow \quad v(x) = \mathcal{F}^{-1}\left(\frac{\hat{w}(\cdot)}{p(\cdot) - \lambda}\right)(x)$$

Clearly, since $\xi \in \mathbb{R}$, if $\lambda \in p(i\mathbb{R})$ one can find a $w \in L^2(\mathbb{R})$ such that the function v defined above is *not* in $L^2(\mathbb{R})$. Consequently, $(L - \lambda I)^{-1}$ is not well defined on all of $L^2(\mathbb{R})$ for any $\lambda \in p(i\mathbb{R})$. Furthermore, if $\lambda \notin p(i\mathbb{R})$ then the above defines a unique solution $v \in H^n(\mathbb{R})$ for each $w \in L^2(\mathbb{R})$. Indeed, the Parseval inequality implies that for $\lambda \notin p(i\mathbb{R})$ we have the bound

$$\left\| (L - \lambda I)^{-1}(w) \right\|_{H^n(\mathbb{R})} \leq \sum_{j=0}^{n} \left(\left\| \frac{(i\cdot)^j}{p(i\cdot) - \lambda} \right\|_{L^\infty(\mathbb{R})} \right) \|w\|_{L^2(\mathbb{R})},$$

which implies that $\lambda \in \rho(L)$. Combining these insights with the Weyl Essential Spectrum Theorem, it follows that for a differential operator L of the form (5), one has

$$\sigma_{\text{ess}}(L) = p(i\mathbb{R})$$

where $p(z)$ is the symbol of the constant coefficient, asymptotic operator L_∞.

While the essential spectrum in this case is rather easy to describe, the point spectrum is in general not. There is an analytical tool known as the Evans function, which is a sort of infinite dimensional characteristic polynomial for differential operators, that has proven to be very useful in both numerical and theoretical investigations. We won't develop the Evans function here, although interested readers are encouraged to read [KP] for details. Here, we observe that classical ODE theory give us the following result which applies at least for 2nd order linear differential operators.

Theorem 4 (Sturm-Liouville Theory on $\mathbb{R}$). *Consider a linear differential operator*

$$L = \partial_x^2 + a_1(x)\partial_x + a_0(x)$$

acting on $L^2(\mathbb{R})$, where the coefficient functions a_1, a_0 are exponentially constant, i.e.

$$\lim_{x\to\pm\infty} e^{r|x|}|a_1(x) - a_1^{\pm}| = \lim_{x\to\pm\infty} e^{r|x|}|a_0(x) - a_0^{\pm}| = 0$$

for some $r > 0$ and constants $a_1^{\pm}, a_0^{\pm} \in \mathbb{R}$. Then $\sigma_p(L)$ consists of a finite number, possibly zero, of real simple eigenvalues which can be enumerated in strictly decreasing order

$$\lambda_0 > \lambda_1 > \ldots > \lambda_n > \max\{a_0^-, a_0^+\} = \max \sigma_{\text{ess}}(L).$$

Further, for each $j = 0, 1, \ldots, N$ any eigenfunction $v_j(x)$ associated to the eigenvalue λ_j has exactly j zeroes on $\mathbb{R}$, all of which are simple.

The above is sometimes referred to as Sturm's oscillation theorem, and is a fundamental result that is used heavily in many classical stability results. We emphasize that this result is sensitive to the boundary conditions, as we will see below in the study of periodic boundary conditions.

2.3 Periodic Coefficients

Finally, we consider the case where the operator (5) has smooth periodic coefficients, i.e. there exists a finite $T > 0$ such that $a_j(x+T) = a_j(x)$ for ll $x \in \mathbb{R}$ and $j = 0, \ldots n-1$. Such an operator arises naturally when linearizing a PDE about an equilibrium solution that is spatially T-periodic. Observe that while L here is defined on a bounded domain, the fact that the boundary conditions are *not separated* implies different analysis is needed from that of Section 2.1.

For such operators, there are (at least) two natural classes of boundary conditions one can enforce on L.

1. One can consider L as a linear operator on

   $$L^2_{\text{per}}(0, mT) := \left\{f \in L^2_{\text{loc}}(\mathbb{R}) : f(x+mT) = f(x) \quad \forall x \in \mathbb{R}\right\}$$

   for some integer $m \geq 1$ with densely defined domain $H^n_{\text{per}}(0, mT)$. This is most natural when considering the stability of a spatially periodic equilibrium solution of a PDE to perturbations with the same periodic structure as the underlying solution.

2. One can consider L as a linear operator on $L^2(\mathbb{R})$ with densely defined domain $H^n(\mathbb{R})$. This is most natural when considering the stability of a spatially periodic equilibrium solution of a PDE to "localized" perturbations, i.e. perturbations that are integrable on $\mathbb{R}$.

It turns out that the nature of $\sigma(L)$ depends sensitively on which class of perturbations are considered above. We consider these below separately.

2.3.1 Periodic Boundary Conditions

First, consider L as acting on $L^2_{\text{per}}(0, mT)$ for some integer $m \geq 1$. The eigenvalues of L are found by seeking nontrivial solutions in $L^2_{\text{per}}(0, mT)$ of the ODE $Lv = \lambda v$, which can be rewritten as a first order system of the form

$$Y' = A(x; \lambda)Y \tag{9}$$

where here $Y = (v, v', \ldots, v^{(n-1)})^t$ and $A(x; \lambda)$ is an $n \times n$ matrix valued function with $A(x + T; \lambda) = A(x; \lambda)$ for all $x \in \mathbb{R}$. It follows that $\lambda \in \sigma_p(L)$ exactly when the ODE (9) has a non-trivial nT-periodic solution. By Floquet theory, we know that any fundamental matrix solution of (9) has the form

$$\Phi(x; \lambda) = P(x; \lambda)e^{B(\lambda)x} \tag{10}$$

where here $P(x + T; \lambda) = P(x; \lambda)$ for all $x \in \mathbb{R}$ and $B(\lambda)$ is a generically complex $n \times n$ matrix. Defining the "monodromy operator"

$$M(\lambda) = \Phi(T; \lambda)\Phi(0; \lambda)^1 = P(T; \lambda)e^{B(\lambda)T}P(0; \lambda)^{-1}$$

it follows that any solution $Y(x; \lambda)$ of (9) will satisfy

$$Y(x + nT) = M(\lambda)^n Y(x)$$

for all $x \in \mathbb{R}$. In particular, we see $\lambda \in \sigma_p(L)$ precisely when $1 \in \sigma(M(\lambda)^n)$ or, equivalently, when

$$\det\left(M(\lambda) - e^{2\pi ji/n}I\right)$$

vanishes for some $j = 1, 2, \ldots n$. In particular, if we define the function

$$D(\lambda, \xi) = \det\left(M(\lambda) - e^{i\xi T}I\right)$$

then $\lambda \in \sigma_p(L)$ exactly when $D(\lambda, \frac{2\pi j}{nT}) = 0$ for some $j = 1, 2, \ldots n$.

The function $D(\lambda, \xi)$ is called the "periodic Evans function" and can be shown to be an entire function of λ and analytic in ξ. From basic results in complex analysis, it follows that the eigenvalues of L on $L^2_{\text{per}}(0, nT)$ are isolated with no finite accumulation point, and that all eigenvalues necessarily have finite algebraic (and hence geometric) multiplicities.

Finally, if $\lambda \notin \sigma_p(L)$ one can construct a Green's function[4] for the operator $L - \lambda I$ so that, in particular, one can uniquely and continuously solve the nonhomogeneous equation

$$Lv - \lambda v = f$$

for every $f \in L^2_{\text{per}}(0, nT)$. It follows then that $\sigma(L) = \sigma_p(L)$, and hence that there is no essential spectrum in this case. While determining the point spectrum of L is in general a difficult task, when L is a second-order operator we are aided by the following version of Sturm's oscillation theorem.

Theorem 5 (Sturm's Oscillation Theorem: Periodic Case)**.** *Consider a linear differential operator of the form*

$$L = \partial_x^2 + a_1(x)\partial_x + a_0(x)$$

where the coefficients a_1, a_0 are smooth and T-periodic. Considering L as acting on $L^2_{\text{per}}(0, T)$, it follows that $\sigma_p(L)$ consists of infinitely many eigenvalues which can be enumerated in non-increasing order

$$\lambda_0 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots, \quad \lim_{n\to\infty} \lambda_n = -\infty.$$

Furthermore, for each $j = 0, 1, 2, \ldots$ let v_j denote an eigenfunction for λ_j. Then v_0 has no zeroes on $[0, T)$, while for each $j \geq 1$ the eigenfunctions $\{v_{2n-1}, v_{2n}\}$ both have exactly $2n$ simple zeroes on $[0, T)$.

Observe that in the above theorem, the eigenvalues λ_j for $j \geq 1$ need not be simple. This is in stark contrast to the other versions of Sturm's theorem that we have seen, and is a reflection of the complications induced by the the non-separated boundary conditions present here.

2.3.2 Acting on Whole Line

Finally, consider the T-periodic coefficient differential operator (5) acting on $L^2(\mathbb{R})$. In this case, it is relatively easy to see that $\sigma_p(L) = \emptyset$. Indeed, from the Floquet decomposition (10) we know that every solution of the ODE $Lv = \lambda v$ will be of the form

$$v(x\lambda) = e^{\mu(\lambda)x} p(x; \lambda)$$

for some T-periodic function p, where here $\mu(\lambda)$ is an eigenvalue[5] of $B(\lambda)$. In particular, we observe that if $\Re(\mu(\lambda)) < 0$, then while the solution $v(\cdot; \lambda)$ decays to zero at an exponential rate as $x \to +\infty$, it necessarialy blows up exponentially fast as $x \to -\infty$. Similarly, solutions when $\Re(\mu(\lambda)) > 0$ necessarialy blow up as $x \to -\infty$. It follows that the equation $Lv = \lambda v$ can never have a non-trivial solution that decays at *both* spatial infinities, and hence for every $\lambda \in \mathbb{C}$ the equation $Lv = \lambda v$ will never have a solution in $L^p(\mathbb{R})$ for any finite $1 \leq p < \infty$.

[4]Fundamentally, this follows since for $\lambda \notin \sigma_p(L)$ the operator $L - \lambda I$ admits an *exponential dichotomy.*

[5]In particular $e^{\mu(\lambda)T}$ is an eigenvalue of the monodromy operator $M(\lambda)$.

From above, it follows the solutions of $Lv = \lambda v$ can at best be bounded on $\mathbb{R}$, happening precisely when the matrix $e^{B(\lambda)T}$, and hence the monodromy operator $M(\lambda)$ has an eigenvalue on the unit circle. In fact, it can be shown that $\lambda \in \sigma(L)$ precisely when the spectral problem $Lv = \lambda v$ has an $L^\infty(\mathbb{R})$-eigenfunction' of the form

$$v(x; \lambda, \xi) = e^{i\xi x} w(x; \lambda, \xi)$$

for some $\xi \in [-\pi/T, \pi/T)$ and $w \in L^2_{\text{per}}(0, T)$. Since $\sigma_p(L) = \emptyset$, it follows that this characterizes precisely the essential spectrum of L acting on $L^2(\mathbb{R})$.

In particular, $\lambda \in \sigma(L)$ if and only if there exists a $\xi \in [-\pi/T, \pi/T)$ such that there exists a non-trivial T-periodic solution of the equation

$$L_\xi w = \lambda w, \quad \text{where} \quad (L_\xi w)(x) := e^{i\xi x} L \left[e^{i\xi \cdot} w(\cdot) \right](x)$$

the T-periodic problem

$$\begin{cases} e^{-i\xi x} L e^{i\xi x} w = \lambda w \\ w(x+T) = w(x) \;\; \forall x \in \mathbb{R}, \end{cases}$$

i.e. when there exists a $\xi \in [-\pi/T, \pi/T)$ such that λ is a T-periodic eigenvalue of the operator

$$L_\xi := e^{-i\xi x} L e^{i\xi x}.$$

Here, the parameter ξ is referred to as the Bloch, or Floquet-Bloch, parameter and the one-parameter family of operators $\{L_\xi\}_{\xi \in [-\pi/T, \pi/T)}$, each acting on $L^2(0, T)$, are referred to as the Bloch operators. Observe that L_0 corresponds to considering the operator L with T-periodic, i.e. co-periodic, boundary conditions. Since the Bloch operators L_ξ act on the space of T-periodic functions $L^2_{\text{per}}(0, T)$, we know from the previous section that their spectrum consists entirely of isolated, discrete eigenvalues that depend continuously on the Bloch parameter ξ. Thus, the $L^2(\mathbb{R})$-spectrum of L consists entirely of $L^\infty(\mathbb{R})$-eigenvalues and may be decomposed into countably many curves $\lambda(\xi)$ such that $\lambda(\xi) \in \sigma_{L^2_{\text{per}}(0,T)}(L_\xi)$ for $\xi \in [-\pi/T, \pi/T)$. In other words, we have the decomposition

$$\sigma_{L^2(\mathbb{R})}(L) = \bigcup_{\xi \in [-\pi/T, \pi/T)} \sigma_{L^2_{\text{per}}(0,T)}(L_\xi),$$

Note that, as in our previous discussion, the T-periodic eigenvalues of L_ξ for a given ξ are given precisely by the zero set of the periodic Evans function $D(\cdot; \xi)$. The periodic Evans function has proven to be a very useful tool in both analytical and numerical studies of the stability of periodic waves.

3 Linear Dynamics and Stability

In this section, we begin to study the connection between the spectrum of a linearization about a wave and the local dynamics near the wave. To this end, consider a PDE of the abstract form

$$u_t = F(u) \tag{11}$$

posed on a Hilbert space X, and suppose there exists a dense subspace $Y \subset X$ such that (11) is locally well posed on Y, i.e. for very $u_0 \in Y$ there exists a time $T = T(u_0) > 0$ such that there exists a unique solution $u(t) \in Y$ of (11) with initial condition $u(0) = u_0$ for $t \in [0, T)$. Here, $F : Y \subset X \to X$ is a generically nonlinear operator. If $\phi \in Y$ is an equilibrium solution of (11), so that $F(\phi) = 0$, we note that if take initial data $u(0) = \phi + v_0$ with $\|v_0\|_Y$ small and let $u(t) = \phi + v(t)$ be the associated unique local solution, then so long as $\|v(t)\|_Y$ remains sufficiently small it is natural to expect the dynamics of (11) near ϕ are well approximated by the linear evolution equation

$$v_t = Lv \tag{12}$$

where here $L = DF(\phi)$ denotes the linear differential operator obtained by linearizing[6] F at ϕ. Often, understanding the dynamics of the linear problem (12) is the key to controlling the fully nonlinear dynamics generated by (11).

Definition 3. *An equilibrium solution ϕ of* (11) *is said to be linearly stable provided that $v = 0$ is a stable solution of the linearized system* (11)*, i.e. if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $v_0 \in Y$ with $\|v_0\|_Y < \delta$ the unique solution $v(t)$ of* (12) *with $v(0) = v_0$ satisfies $\|v(t)\|_Y < \epsilon$ for all $t \geq 0$.*

As in the case of ODE theory, it is often the case that the dynamics of the linearized system (12) are governed by the spectrum of the operator L. Consequently, our first goal is to understand how solutions of (12) are influenced by the spectrum of the linear differential operator L.

3.1 Dynamics Induced by the Spectrum

Suppose first that L has an eigenvalue $\lambda_0 \in \mathbb{C}$ with eigenfunction v_0. An easy calculation shows that the function $v(t) = e^{\lambda_0 t} v_0$ solves the ODE (12) with initial data $v(0) = v_0$ and, furthermore, we have the growth rate $|v(t)| = e^{\Re(\lambda_0)t}|v_0|$. This gives rise to an exponentially growing or decaying solution of (12) depending on the sign of $\Re(\lambda_0)$. Note also that if the algebraic and geometric multiplicities of λ_0 do not agree, indicating the existence of a Jordan block, one expects additional polynomial growth. For example, suppose there exists non-trivial v_0, v_1 such that

$$Lv_0 = \lambda_0 v_0, \quad (L - \lambda_0)v_1 = v_0.$$

In this case, $v(t) = e^{\lambda_0 t}(v_0 + tv_1)$ again solves (12). In particular, if $\Re(\lambda_0) = 0$, this construction yields a solution that grows polynomially in time, giving instability. It follows that a *necessary* condition for linear stability of $v = 0$ is that $\Re(\sigma(L)) \leq 0$ and that all $\lambda_0 \in \sigma(L)$ with $\Re(\lambda_0) = 0$ are "semi-simple", i.e. the algebraic and geometric multiplicities of λ_0 are the same.

As may be expected, the dynamics associated to $\sigma_{\text{ess}}(L)$ is more subtle. To motivate this, suppose for simplicity $L = p(\partial_x)$ for some polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with $a_j \in \mathbb{R}$

[6]Precisely, F is the Gautaux derivative of F at ϕ.

and $a_n \neq 0$, and suppose that L acts on $L^2(\mathbb{R})$ with densely defined domain $H^n(\mathbb{R})$. We recall from Weyl's Essential Spectrum Theorem that in this case

$$\sigma(L) = \sigma_{\text{ess}}(L) = p(i\mathbb{R}).$$

Mimicing the above arguments, we clearly see that for all $\xi \in \mathbb{R}$ that $p(i\xi) \in \sigma_{\text{ess}}(L)$ and that

$$v(t) = e^{i\xi x + p(i\xi)t}$$

formally solves the PDE (12). While it is clear that these functions do not lie in $L^2(\mathbb{R})$ for any $t \geq 0$, we do see that if $\Re(p(i\xi)) < 0$ for all ξ then all of these solutions exponentially decay as $t \to \infty$, which should indicate some sort of stability. To make this intuition rigorous, we must consider the effect of the essential spectrum on L^2 initial data, rather than $L^\infty(\mathbb{R})$. This can be achieved via the Fourier transform: let $v_0 \in L^2(\mathbb{R})$ and $v(t)$ be the solution of the IVP

$$v_t = p(\partial_x)v, \quad v(0) = v_0$$

considered as an evolution equation on $L^2(\mathbb{R})$. Taking the Fourier transform we find that

$$\hat{v}(\xi, t) = e^{p(i\xi)t}\hat{v_0}(\xi),$$

so that the Fourier transform of our solution grows or decays exponentially depending on the sign of $\Re(p(i\xi))$. If we assume there exists a $\sigma > 0$ such that $\Re(p(i\xi)) < -\sigma$ for all $\xi \in \mathbb{R}$ then

$$|\hat{v}(\xi, t)| \leq e^{-\sigma t}|\hat{v_0}(\xi)|,$$

from which Plancherl's theorem implies the solution v obeys the exponential decay estimate

$$\|v(t)\|_{L^2(\mathbb{R})} \leq e^{-\sigma t}\|v_0\|_{L^2(\mathbb{R})}$$

for all $t > 0$.

Conversely, unstable essential spectrum gives rise to growing solutions of (12). To see this, suppose there exists a $\xi^* \in \mathbb{R}$ such that $p(i\xi^*) > 0$ and $\Re(p(i\xi)) < p(i\xi^*)$ for all $\xi \neq \xi^*$. In other words, $p(i\xi^*)$ is real and is the *most unstable* part of the essential spectrum. Suppose furthermore that $\Re(p(i\xi))$ is locally quadratic near $\xi = \xi^*$, i.e. for $|\xi - \xi^*| \ll 1$ it satisfies

$$p(i\xi) = p(i\xi^*) + i\alpha(\xi - \xi^*) - \beta(\xi - \xi^*)^2 + \mathcal{O}(|\xi - \xi^*|^3) \tag{13}$$

for some constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$. From above, the function

$$v(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x + p(i\xi)t}\hat{v_0}(\xi)d\xi$$

solves (12) with initial data $v(x, 0) = v_0(x)$. Using (13), a simple stationary phase argument shows that for $t \gg 1$ we have

$$v(x, t) \approx \underbrace{e^{p(i\xi^*)t}}_{I} \underbrace{e^{i\xi^* x}e^{-(x+\alpha t)^2/4\beta t}}_{II} \underbrace{\frac{\hat{v_0}(\xi^*)}{\sqrt{2\beta t}}}_{III}.$$

t

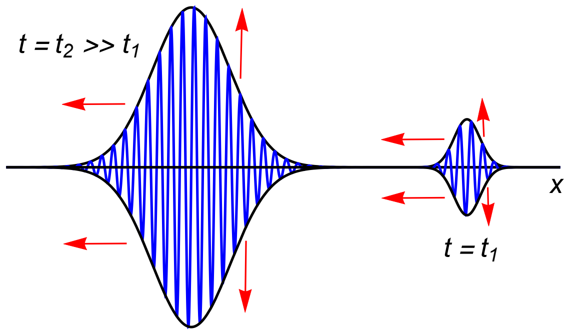


Figure 1: An illustrative picture showing the evolution of (12) when L has constant coefficients and has unstable essential spectrum.

Here, term I gives exponential growth, II gives a convecting, oscillatory wave packet with localized Gaussian envelope and speed α, and III is just a polynomially decaying scalar factor. See Figure 1 for an illustrative picture.

From the above considerations, is evident that a necessary condition that the equilibrium solution ϕ to be a linearly stable solution of (11) is that $\Re(\sigma(L)) \leq 0$.

Definition 4. *An equilibrium solution ϕ of* (11) *is said to be spectrally stable if its linearization $L = DF(\phi)$ satisfies*

$$\sigma(L) \cap \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} = \emptyset.$$

Else, ϕ is said to be spectrally unstable.

Equipped with the techniques from the previous section, we are can now (finally!) do some examples.

3.2 Examples

We now present a number of examples illustrating how one can use the above tools to determine the spectral stability of a given equilibrium solution.

3.2.1 Reaction Diffusion on Bounded Domain

Consider the PDE

$$u_t = u_{xx} + u^3, \quad t > 0, \quad x \in (0, \pi) \tag{14}$$

equipped with homogeneous Dirichlet boundary conditions $u(0,t) = u(\pi,t) = 0$ for all $t \geq 0$. We consider the above as an evolution equation on $L^2(0,\pi)$. Clearly $u = 0$ is an equilibrium solution of (14), and here we are interested in the stability of this so-called "trivial solution". To this end, we consider the above PDE with the initial condition

$$u(x,0) = 0 + v_0(x)$$

where $\|v_0\|_{H^1(0,\pi)} \ll 1$ and $v_0(0) = v_0(\pi) = 0$, and note there exists a unique solution $v(t)$ of (14) defined locally in time and v(t) satisfies $v(0,t) = v(\pi,t) = 0$ for so long as $v(t)$ is defined. Defining the operator $L := \partial_x^2$ and the nonlinear operator $N(v) = v^3$, we note $v(t)$ solves the evolution equation

$$v_t = Lv + N(v) \tag{15}$$

Here, we consider L as being densely defined on $L^2(0,\pi)$ with domain $D(L) := H_0^1(0,\pi) \cap H^2(0,\pi)$. It follows the linearization of (14) about $u = 0$ is

$$v_t = Lv$$

considered on $L^2(0,\pi)$ (with Dirichlet B.C.'s), and spectral stability is determined by finding $\sigma(L)$.

By the above work, we know that $\sigma(L) = \sigma_p(L)$. Since here L has constant coefficients, we can easily use ODE theory to find that

$$\sigma_p(L) = \left\{\lambda_n = -n^2 : n = 1, 2, 3, \dots\right\}$$

and that, for each n, λ_n has a one-dimensional eigenspace spanned by $\sin(nx)$. Since $\Re(\sigma(L)) \leq -1$, it follows that $u = 0$ is a spectrally stable solution of (14).

3.2.2 Stationary Pulse of Reaction Diffusion Equation

Consider the reaction diffusion equation

$$u_t = u_{xx} - u + u^3 \tag{16}$$

and note that equilibrium solution satisfy the ODE

$$u_{xx} - u + u^3 = 0 \tag{17}$$

which, being a Hamiltonian ODE, can be solved via quadrature as

$$\frac{u_x^2}{2} = E - \left(u^4 - u^2\right)$$

where $E \in \mathbb{R}$ is an integration constant. Elementary phase plane analysis shows that this ODE admits positive solution that is homoclinic to $u = 0$ (occuring at $E = 0$). This homoclinic orbit corresponds to a one-parameter family of smooth "pulse" like solutions of the form $\{\phi(\cdot + \gamma)\}_{\gamma \in \mathbb{R}}$ where $\phi(x) > 0$ for all x and can be chosen to be even with $\phi'(x) > 0$ for all $x < 0$. To investigate the stability of ϕ, we note the linearized operator about ϕ is

$$L := \partial_x^2 - 1 + 3\phi^2,$$

considered here as a densely defined operator on $L^2(\mathbb{R})$. By the Weyl essential spectrum theorem, we easily find that $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(\partial_x^2 - 1) = (-\infty, -1]$, which is stable. To investigate the point spectrum of L, begin by observing that by differentiating the profile ODE (17) (with $u = \phi$) with respect to x gives $L\phi' = 0$ and, since ϕ' decays to 0 at an exponential rate, $\phi' \in L^2(\mathbb{R})$. It follows that $0 \in \sigma_p(L)$, and we will have that ϕ is a spectrally stable solution of (16) if and only if $\lambda = 0$ is the *largest* eigenvalue of L. However, since ϕ necessarialy has a critical point at $x = 0$ it follows that ϕ' has exactly one root, which, by the Sturm Oscillation Theorem, implies that 0 is the *second* largest eigenvalue of L. In particular, there exists a $\lambda_0 > 0$ such that $\lambda_0 \in \sigma_p(L)$. Consequently, the pulse solution ϕ is a *spectrally unstable* solution of the scalar reaction diffusion equation (16).

3.2.3 Stationary Front in Reaction Diffusion Equation

Consider now the reaction diffusion equation

$$u_t = u_{xx} + u - u^3. \tag{18}$$

Following the above example, it is clear (18) admits a one-parameter family of smooth "front" like solutions $\{\phi(\cdot + \gamma)\}_{\gamma \in \mathbb{R}}$ that satisfy

$$\lim_{x \to -\infty} \phi(x) = -1, \quad \lim_{x \to \infty} \phi(x) = 1$$

and $\phi'(x) > 0$ for all $x \in \mathbb{R}$. Here, the linearized operator is $L := \partial_x^2 + 1 - 3\phi^2$, and the Weyl essential spectrum theorem implies $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(\partial_x^2 - 2) = (-\infty, -2]$, which is stable. Furthermore, as above we have $L\phi' = 0$ and hence, since $\phi' \in L^2(\mathbb{R})$, it follows that 0 is an eigenvalue of L. Since ϕ is strictly monotone on $\mathbb{R}$, Sturm's Oscillation Theorem implies that 0 is the largest eigenvalue of L, and hence that ϕ is a *spectrally stable* solution of the reaction diffusion equation (18).

In the coming sections, we will continue this investigation to show that the stationary front ϕ of (18) is in fact nonlinearly stable in a very particular sense.

Remark 6. *In both of the above examples, a crucial observation came from the fact that $L\phi' = 0$, hence that we could actually identify the kernel of L. While this can be verified directly by differentiating the profile equation in each case, this fact actually follows from the translation invariance of the PDE's in each case. To see this, note that if we write the PDE's above as*

$$u_t = F(u)$$

the fact that the coefficients of F are independent of x implies that $F(\phi(\cdot + \gamma)) = 0$ for every $\gamma \in \mathbb{R}$. Differentiating with respect to γ at $\gamma = 0$ gives

$$\frac{\partial}{\partial \gamma}\Big|_{\gamma=0} F(\phi(\cdot + \gamma)) = DF(\phi)\phi' = 0,$$

where here $DF(\phi)$ denotes the linearization of F about ϕ. This observation extends to equations that have additional symmetries as well, such as phase invariance $\phi \mapsto e^{i\beta}\phi$ and Galilean boosts, and can be seen as a sort of a linearized Noether's theorem.

3.2.4 The KdV Equation

For something a little different, consider the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0 \tag{19}$$

The KdV equation admits a one-parameter family of solitary wave solutions of the form $u(x,t) = u(x-ct)$ with wave speed $c > 0$ that are asymptotic (at an exponential rate) to 0 as $x \to \pm\infty$. Indeed, such solutions are equilibrium solutions of the KdV equation in traveling coordinates $\xi = x - ct$, which reads

$$u_t - cu_x + u_{xxx} + uu_x = 0. \tag{20}$$

By reducing the profile equation to quadrature form

$$\frac{u_x^2}{2} = E - \left(\frac{1}{3}u^3 - \frac{c}{2}u^2\right).$$

we see that such a solitary wave corresponds to the homoclinic orbit to $u = 0$ that exists when $E = 0$. The linearization of (20) about ϕ is given by

$$v_t = -v_{xxx} + cv_x - (\phi v)_x =: Lv.$$

By the Weyl Essential Spectrum Theorem, we see that

$$\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}\left(-\partial_x^3 + c\partial_x\right) = \mathbb{R}i.$$

Thus, the entire imaginary axis belongs to the essential spectrum of L. With more work, one can show that $\sigma_p(L) = \{0\}$ so that the solitary wave ϕ is indeed spectrally stable. However, due to the neutral nature of the stability (all spectrum lies on the imaginary axis) it is not at all clear how one could hope to obtain linear (or even nonlinear) stability from this information.

Notice, however, that if we consider perturbations of ϕ of the form $e^{ax}w(x,t)$ with $w(\cdot,t) \in L^2(\mathbb{R})$ and $a \in \mathbb{R}$ then w would solve the linear equation

$$w_t = e^{-ax}Le^{ax}w.$$

To such perturbations, spectral stability is determined by $\sigma(e^{-a\cdot}Le^{a\cdot})$. Note that for all $w \in L^2(\mathbb{R})$ we have

$$e^{-ax}Le^{ax}v = (\partial_x - a)\left(-(\partial_x - a)^2 + c - \phi\right)v$$

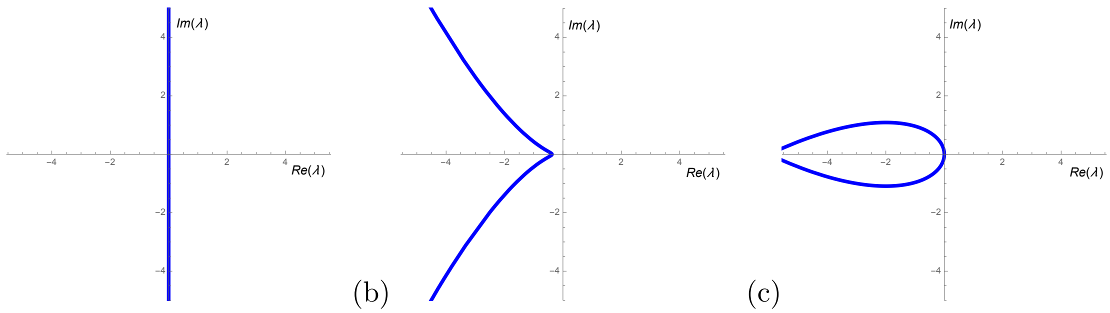
and hence, by the Weyl Essential Spectrum Theorem, we have $\sigma_{\text{ess}}(e^{-a\cdot}Le^{a\cdot}) = p_a(\mathbb{R}i)$ where

$$p_a(z) = (z-a)\left(-(z-a)^2 + c\right).$$

Setting $z = ik$ with $k \in \mathbb{R}$ we find that

$$\Re(p_a(ik)) = a^3 - ca - 3ak^2$$

which is strictly negative for all $k \in \mathbb{R}$ provided $a > 0$ and $0 < a < \sqrt{c}$; see Figure 2. Thus, in this case we can change the class of perturbations in order to ensure that at least the essential spectrum is moved into the left half plane, opening the door to the possibility of obtaining some sort of stability result.



(a) (b) (c)

Figure 2: For the KdV equation, plots of $\sigma_{\text{ess}}(e^{-a\cdot}Le^{a\cdot})$ when $c = 1$ for (a) $a = 0$, (b) $a = \frac{1}{2}$, and (c) $a = 1$. While the essential spectrum is stable for $a \in (0, 1)$, when $a > 1$ one finds that the essential spectrum crosses back into the unstable right half plane.

3.3 Linear Stability

We now aim at establishing linear stability from spectral stability information. To begin, we first consider a simpler case where we don't have essential spectrum to worry about.

To this end, recall that $u = 0$ is a spectrally stable equilibrium solution of

$$\begin{cases} u_t = u_{xx} + u^3, & t > 0, \quad x \in (0, \pi) \\ u(0,t) = u(\pi,t) = 0 & \forall t \geq 0. \end{cases} \tag{21}$$

In fact, the linearization in this case is given by $v_t = Lv$ with $L = \partial_x^2$ and, moreover, the eigenvalues are given explicitly by $\lambda_{-}n^2$, $n = 1, 2, 3, \ldots$ with corresponding eigenfunctions $v_n(x) = \sin(nx)$. Using the fact that $\{\sin(n\cdot)\}_{n=1}^{\infty}$ forms an orthogonal basis of $L^2(0, \pi)$, it follows from linearity that given any $v_0 \in L^2(0, \pi)$ the unique solution of the evolution equation $v_t = Lv$ with $v \in H_0^1(0, \pi) \cap H^2(0, \pi)$ and $v(x, 0) = v_0(x)$ is given by

$$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx)$$

where $a_n = \frac{2}{\pi}\int_0^{\pi} v_0(x)\sin(nx)dx$ are the Fourier sine coefficients of v_0. In particular, we have from Parseval's inequality that

$$\|v(\cdot, t)\|_{L^2(0,\pi)} \leq e^{-t}\|v_0\|_{L^2(0,\pi)},$$

implying that $u = 0$ is a *linearly stable* solution of (21).

In the above example, notice we an write $v(t) = T(t)v_0$ where the operator $T(t)$ is given by

$$T(t) = \sum_{n=1}^{\infty} \frac{2}{\pi} \langle \sin(nx), \cdot \rangle_{L^2(0,\pi)} e^{-n^2 t} \sin(nx).$$

It is elementary to verify that $T(t)$ is a bounded linear operator on $L^2(0, \pi)$ for all $t > 0$ and that, furthermore, it satisfies

(1) $T(0) = I$, the identify operator.

(2) $T(s)T(t) = T(s+t) = T(t)T(s)$ for all $s, t \geq 0$.

(3) For all $v_0 \in L^2(0,\pi)$, $T(t)v_0 \to v_0$ in $L^2(0,\pi)$ as $t \to 0^+$.

Any set of bounded linear operators $\{T(t)\}_{t\geq 0}$ that satisfies (1)-(3) above is called a *strongly continuous, or C^0, semigroup of operators*. Notice from the functional properties (1)-(3) it makes sense to denote

$$T(t) = e^{Lt} \quad t \geq 0.$$

Furthermore, in the above example we observed that the fact that $\Re(\sigma(L)) \leq -1$ implied that

$$\left\| e^{Lt} v_0 \right\|_{L^2(0,\pi)} \leq e^{-t} \| v_0 \|_{L^2(0,\pi)}$$

so that the decay rate on the linearized solution operator e^{Lt} is given exactly by the maximum real part of the spectrum of L.

Not surprisingly, everything said above becomes more difficult when the essential spectrum is present. For definiteness, let

$$L = \partial_x^n + a_{n-1}(x)\partial_x^{n-1} + \ldots + a_1(x)\partial_x + a_0(x)$$

be an n^{th}-order linear differential operator that is densely defined on $L^2(\mathbb{R})$ with the coefficients a_j being smooth and exponentially localized functions. In this case, we have the following lemma[7].

Lemma 1. *Suppose that the exponentially localized operator L above, acting on $L^2(\mathbb{R})$, is well-posed, i.e. there exists an $\alpha > 0$ such that $\Re\left(\sigma_{\text{ess}}(L)\right) < \alpha$. Then L generates a C^0 semigroup of operators $\{T(t)\}_{t\geq 0}$ on $H^k(\mathbb{R})$ for every $k \leq n$.*

The above lemma guarantees that so long as L is well-posed, then for every $v_0 \in L^2(\mathbb{R})$ the unique solution to the linear evolution equation

$$v_t = Lv$$

with $v(0) = v_0$ is given by $v(t) = T(t)v_0$. As before, due to the fact that $T(t)$ satisfies properties (1)-(3) above, we often denote $T(t) = e^{Lt}$. Now, a natural question remains regarding the connection between the maximum real part of the spectrum of L and the decay of the linearized semigroup $\{e^{Lt}\}_{t\geq 0}$. This is addressed in general by the Gearhart-Prüss Theorem.

Theorem 6 (Gearhart-Prüss)**.** *Let X be a Hilbert space and assume $L : X \to X$ is a linear operator with densely defined domain. Let Π be a finite dimensional spectral projection[8] associated with L. If there exists constants $M, \sigma > 0$ such that*

$$\left\| (L - \lambda I)^{-1}(I - \Pi) f \right\|_X \leq M \| f \|_X \quad \forall f \in X \tag{22}$$

[7] The forthcoming results come from [KP, Section 4.1].

[8] In particular, one could take Π to be the zero operator.

on the set $\Re(\lambda) \geq -\sigma$, then there exists a constant $C > 0$ such that the C^0 semigroup associated with $(I-\Pi)L$ satisfies the decay estimate

$$\left\| e^{(I-\Pi)Lt} f \right\|_X = \left\| e^{Lt}(I-\Pi)f \right\|_X \leq Ce^{-\sigma t}\|f\|_X$$

for every $f \in X$.

Admittedly, it is often a very difficult task to verify the resolvent operator of L is uniformly bounded on a half space $\Re(\lambda) > -\sigma$. However, we emphasize that an obvious necessary condition is $\Re(\sigma((I-\Pi)L)) < -\sigma$ so that, in particular, it is necessary that L to be spectrally stable with a spectral gap away from $\mathbb{R}i$ when acting on the invariant subspace $(I-\Pi)X$. Often times, in practice, Π corresponds to projecting out some finite dimensional eigenspace of L. Thankfully, however, there is a commonly occurring class of operators where the resolvent bound always holds.

Lemma 2. *Suppose the exponentially localized linear operator L above is 2nd-order, and let Π be a finite dimensional spectral projection for L. Then the uniform resolvent bound* (22) *holds on $\Re(\lambda) \geq -\sigma + \epsilon$ for every $\epsilon > 0$ provided that*

$$\Re(\sigma((I-\Pi)L)) < -\sigma.$$

In particular, in this case we are guaranteed that for every $\omega \in (-\sigma, 0)$ there exists a constant $C = C(\omega) > 0$ such that

$$\left\| e^{Lt}(I-\Pi)f \right\|_{H^1(\mathbb{R})} \leq Ce^{-\omega t}\|f\|_{H^1(\mathbb{R})}$$

for every $f \in H^1(\mathbb{R})$.

In other words, the above lemma guarantees us that if L is a second-order, exponentially localized differential opeartor, then one can obtain decay bounds on the semigroup $\{e^{Lt}\}_{t\geq 0}$ directly from the spectral information on L. We will use this observation heavily in our forthcoming examples. Note that this observation extends to a more general class of so-called "sectorial" operators, where the linearized solution operator can be defined via the holomorphic functional calculus as

$$e^{Lt} = \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda t}}{\lambda I - L}\, d\lambda,$$

where here Γ is some curve in $\mathbb{C}$ going from $\alpha - i\infty$ to $\alpha + i\infty$ for some $\alpha > \max \Re(\sigma(L))$ that is always to the right of the spectrum of L. While there is some serious fun one can have here trying to obtain properties of e^{Lt} through the use of complex analytic techniques, we leave this as an exercise for the interested reader.

3.4 Linear Stability of Monotone Front

Consider again the reaction diffusion equation

$$u_t = u_{xx} + u - u^3, \quad x \in \mathbb{R}, \quad t > 0 \tag{23}$$

and recall above we verified that (23) admits a stationary front solution ϕ that is strictly monotone and satisfies

$$\lim_{x\to-\infty} \phi(x) = -1, \qquad \lim_{x\to+\infty} \phi(x) = 1.$$

Furthermore, recall that we have already established the following spectral properties of $L = \partial_x^2 + 1 - 3\phi^2$, considered here as an operator on $L^2(\mathbb{R})$ with densely defined domain $H^1(\mathbb{R})$:

1. The essential spectrum is given explicitly by $\sigma_{\text{ess}}(L) = (-\infty, -2]$.
2. $0 \in \sigma_p(L)$ with eigenfunction ϕ'.
3. There exists a $\gamma > 0$ such that $\sigma_p(\mathbb{R}) \setminus \{0\} < -\gamma$.

Defining $\Pi : L^2(\mathbb{R}) \to \ker(L)$ to be the spectral projection

$$\Pi := \frac{\langle \phi', \cdot \rangle_{L^2}}{\|\phi'\|_{L^2}^2}\ \phi'$$

and noting that $\sigma(\Pi L) < -\gamma$, it follows from the work in the previous section that L generates a C^0 semigroup $\{e^{Lt}\}_{t\geq 0}$ and, furthermore, that for every $\omega \in (0, \gamma)$ there exists a constant $C = C(\omega) > 0$ such that

$$\left\| e^{(I-\Pi)Lt} v(0) \right\|_{L^2} \lesssim e^{-\omega t} \|v(0)\|_{L^2}$$

for all $t \geq 0$.

Since we had to project out the kernel of L to establish the above linear decay result, we can not conclude linear stability of ϕ from here. Indeed, it is a-priori possible that initially nearby solution $u(t)$ simply "drifts" along the center manifold associated with $\mathrm{Ker}(L)$. Nevertheless, in this case we can show the following *linear, asymptotic orbital stability* result:

$$\|v(t) - \Pi(v(0))\|_{L^2} \leq C e^{-\omega t} \|v(0)\|_{L^2}.$$

In particular, this motivates that $v(t)$ will converge to some multiple, depending on $v(0)$, of ϕ' as $t \to \infty$, i.e. for given initial data, to a fixed element of the "center subspace". In terms of the original solution $u(x,t)$, this suggests that an initially nearby solution $u(x,0)$ will satisfy

$$\begin{aligned} u(x,t) &\approx \phi(x) + v(x,t) \\ &\approx \phi(x) + \gamma_\infty \phi'(x) \quad \text{for} \quad t \gg 1 \\ &\approx \phi\left(x + \gamma_\infty\right) \end{aligned}$$

if $\|v(0)\|_{H^1} \ll 1$, where here $\gamma_\infty = \frac{\langle \psi', v(0)\rangle_{L^2}}{\|\phi'\|^2_{L^2}}$ and the last approximation follows by Taylor's theorem. This implies that one should expect that a solution that starts near the stationary front ϕ will evolve into a slight spatial translate of ϕ. In a later section, we will verify this behavior at the nonlinear level.

3.5 The Periodic Case

Recall that if L is a linear operator of the form (5) with T-periodic coefficients, considered here as acting on $L^2(\mathbb{R})$, then

$$\sigma(L) = \sigma_{\text{ess}}(L) = \bigcup_{\xi \in [-\pi/T, \pi/T)} \sigma_{L^2_{\text{per}}(0,T)}(L_\xi)$$

where here $L_\xi := e^{-i\xi x} L e^{i\xi x}$ are the Bloch operators. A natural question arises in how T-periodic spectral information about the Bloch operators L_ξ influence the behavior of the linearized smigroup e^{Lt}. This is most easily seen through the introduction of the Bloch, or Floquet-Bloch, transform.

To motivate this, notice that given any $v \in L^2(\mathbb{R})$ we can express v in terms of its inverse Bloch representation as

$$v(x) = \int_{-T/2}^{T/2} e^{i\xi T x} \check{v}(\xi, x) d\xi$$

where here $\check{v}(\xi, x) := \sum_{k \in \mathbb{Z}} e^{ikTx} \hat{v}(\xi + kT)$ are T-periodic functions of x, where here $\check{v}$ denotes the Fourier transform of v. Indeed, the above formulas may be easily checked on the Schwartz class by grouping frequencies that differ by T in the standard Fourier transform representation of v:

$$v(x) = \sum_{k \in \mathbb{Z}} \int_{-T/2}^{T/2} e^{i(\xi + kT)x} \hat{v}(\xi + kT) d\xi = \int_{-T/2}^{T/2} e^{i\xi x} \check{v}(\xi, x) d\xi.$$

The Bloch transform

$$\mathcal{B} : L^2(\mathbb{R}) \to L^2([-T/2, T/2); L^2_{\text{per}}(0, T))$$

given by $\mathcal{B}(v)(\xi, x) := \check{v}(\xi, x)$ is then well-defined, bijective and continuous. In fact, for a given $v \in L^2(\mathbb{R})$ one can show that $\mathcal{B}(Lv)(\xi, x) = L_\xi[\check{v}(\xi, \cdot)](x)$, and hence that the Bloch operators L_ξ may be viewed as operator-valued symbols under $\mathcal{B}$, acting on $L^2_{\text{per}}(0, T)$. Similarly, from the identity $\mathcal{B}\left(e^{Lt} v\right) = \left(e^{L_\xi t} \check{v}(\xi, \cdot)\right)(x)$, we find the *Bloch solution formula* for the periodic-coefficient operator L:

$$\left(e^{Lt} v\right)(x) = \int_{-\pi/T}^{\pi/T} e^{i\xi x} \left(e^{L_\xi t} \check{v}(\xi, \cdot)\right)(x) d\xi. \tag{24}$$

It follows that the Bloch transform $\mathcal{B}$ diagonalizes the periodic coefficient operators L in the same way that the Fourier transform diagonalizes constant-coefficient operators.

Using the representation formula (24), bounds on the Bloch solution operators $e^{L_\xi t}$, which are governed by the the T-periodic eigenvalues of the Bloch operators L_ξ, can be converted to bounds on the linearized solution operator e^{Lt}. Indeed, using the classical Parseval theorem we note that for all $v \in L^2(\mathbb{R})$ that

$$\|v\|_{L^2(\mathbb{R})}^2 = 2\pi \int_{-T/2}^{T/2} \int_0^T |\mathcal{B}(v)(\xi, x)|^2 \, dz \; d\xi$$

so that the rescaled Bloch transform $\sqrt{2\pi}\mathcal{B}$ is an isometry from $L^2(\mathbb{R})$ into the space $L^2([-T/2, T/2); L^2_{\text{per}}(0,T))$. More generally, by interpolating with the triangle inequality, corresponding to $(p,q) = (\infty, 1)$ below, we obtain the generalized Hausdorff-Young inequality

$$\|v\|_{L^p(\mathbb{R})} \le C_{p,q} \|\mathcal{B}(v)\|_{L^q([-\pi/T,\pi/T);L^p_{\text{per}}(0,T)}$$

for $q \le 2 \le p$ and $\frac{1}{p} + \frac{1}{q} = 1$. This immediately gives us the result

$$\left\|e^{Lt} v\right\|_{L^p(\mathbb{R})} \le C_{p,q} \left\|e^{L_\xi t} \check{v}(\xi, \cdot)\right\|_{L^q_\xi([-\pi/T,\pi/T);L^p_{\text{per}}(0,T))} \tag{25}$$

which allows one to obtain estimates on the linearized semigroup e^{Lt} from estimates on the Bloch semigroups $e^{L_\xi t}$.

4 Nonlinear Stability: Examples

In this section, we present a few examples where the above theories can be applied to yield nonlinear stability of solutions of PDE's. In the first example, we will see an example where the linearized operator about the trivial solution admits a spectral gap, and will establish that this trivial solution is in fact asymptotically stable In the second example, we study the case when the spectrum of the linearized operator is stable with a spectral gap, except for the existence of a simple eigenvalue at the origin coming from the translation invariance of the PDE. Finally, we will discuss complications that arise when considering the stability of periodic patterns to localized (i.e. integrable on $\mathbb{R}$) perturbations, and how the ideas presented in these notes can be used to study the nonlinear stability of such patterns.

4.1 Reaction Diffusion on Bounded Domain

Consider the PDE

$$u_t = u_{xx} + u^3, \quad t > 0, \quad x \in (0, \pi) \tag{26}$$

equipped with the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi, t) = 0 \quad \forall t \ge 0.$$

We consider the above as an evolution equation on $L^2(0,\pi)$. Clearly $u=0$ is an equilibrium solution of (26), and here we are interested in the stability of this so-called "trivial solution". To this end, we consider the above PDE with the initial condition

$$u(x,0)=0+v_0(x)$$

where $\|v_0\|_{H^1(0,\pi)} \ll 1$ and $v_0(0)=v_0(\pi)=0$, and note there exists a unique solution $v(t)$ of (26) defined locally in time and $v(t)$ satisfies $v(0,t)=v(\pi,t)=0$ for so long as $v(t)$ is defined. Defining the operator $L := \partial_x^2$ and the nonlinear operator $N(v)=v^3$, we note $v(t)$ solves the evolution equation

$$v_t = Lv + N(v) \tag{27}$$

Here, we consider L as being densely defined on $L^2(0,\pi)$ with domain $D(L) := H_0^1(0,\pi) \cap H^2(0,\pi)$. Recall from previous work that we have $\Re(\sigma(L)) \leq -1$ and that the associated linearized semigroup $\{e^{Lt}\}_{t\geq 0}$ obeys the exponential decay bound

$$\left\|e^{Lt}f\right\|_{H^1(0,\pi)} \leq Ce^{-t}\left\|f\right\|_{H^1(0,\pi)} \quad \forall f \in D(L)$$

valid for some constant $C>0$.

To establish the nonlinear stability of $u=0$, we note that we can rewrite the nonlinear perturbation equation (27) as the equivalent integral equation

$$v(t) = e^{Lt}v(0) + \int_0^t e^{L(t-s)}N(v(s))ds,$$

which, as it is for ODE, is sometimes called "Duhamel's formula". Our goal is to prove that $u=0$ is nonlinearly stable in the following sense.

Theorem 7. *For each $\omega \in (-1,0)$, there exists a δ, $C = C(\omega,\delta) > 0$ such that if $u_0 \in D(L)$ with*

$$\|u_0\|_{H^1(0,\pi)} < \delta$$

then the unique solution $u(t)$ of (26) with $u(0)=u_0$ satisfies

$$\|u(t)\|_{H^1(0,\pi)} \leq Ce^{-\omega t}\|u_0\|_{H^1(0,\pi)}.$$

for all $t \geq 0$.

Remark 7. *This is purely a* local *stability result. Indeed, one can show that if $\|u_0\|_{H^1(0,\pi)}$ is too large, then the solution $u(t)$ of the IVBVP (26) blows up in finite time. Specifically, one can show that if*

$$S(t) := \frac{1}{\pi}\int_0^\pi u(x,t)\sin(x)dx$$

denotes the first Fourier sine coefficient of the solution $u(t)$, then S satisfies the differential inequality

$$S'(t) \geq -S(t) + \frac{\pi^2}{4}S(t)^3$$

which, thanks to Gronwall's inequality, implies there exists a finite $t_* > 0$ such that $S(t) \to +\infty$ as $t \to t_*^-$. It follows that $\|u(t)\|_{H^1(0,\pi)}$ must blow up at least by time t_*. Note it could blow up before time t_* if a higher order Fourier coefficient blows up first. Regardless, this emphasizes the fact that stability is a purely local *theory.*

To prove this theorem, we start by noting that, thanks to the Sobolev embedding $L^\infty(0,\pi) \subset H^1(0,\pi)$, the nonlinear term $N(v)$ is well defined in H^1 and satisfies

$$\|N(u)\|_{H^1(0,\pi)} \leq M\|u\|^3_{H^1(0,\pi)}$$

for some constant $M > 0$ independent of u; indeed, recall that $H^s(0,\pi)$ is an *algebra* when $s > \frac{1}{2}$. Furthermore, by the local well posedness, given any $R > 0$ sufficiently small and $u_0 \in H^1(0,\pi)$ with $\|u_0\|_{H^1(0,\pi)} < \frac{R}{2}$ then there exists a $T = T(u_0) > 0$ such that the (26) with initial condition $u(0) = u_0$ has a unique solution $u(t) \in H^1(0,\pi)$ defined for all $t \in [0,T)$ such that

$$\|u(t)\|_{H^1(0,\pi)} < R \quad \forall t \in [0,T).$$

By the exponential decay bound on e^{Lt} and the triangle inequality, we find from Duhamel's formula that

$$\begin{aligned}
\|u(t)\|_{H^1(0,\pi)} &\leq Ce^{-t}\|u(0)\|_{H^1(0,\pi)} + C\int_0^t e^{-(t-s)}\,\|N(u(s))\|_{H^1(0,\pi)}\,ds \\
&\leq Ce^{-t}\|u(0)\|_{H^1(0,\pi)} + CM\int_0^t e^{-(t-s)}\|u(s)\|^3_{H^1(0,\pi)}ds.
\end{aligned}$$

The above integral inequality implies the desired decay bound. We will establish this using two separate arguments: the first via a more classical ODE approach using Gronwall's inequality, and the second using a different approach that will be more suitable to our forthcoming PDE applications.

For our first proof of the above claim, observe that since $\|u(t)\|_{H^1(0,\pi)} < R$ for all $t \in [0,T)$ we have

$$e^t\|u(t)\|_{H^1(0,\pi)} \leq C\|u(0)\|_{H^1(0,\pi)} + CMR^2\int_0^t e^s\|u(s)\|_{H^1(0,\pi)}.$$

Gronwall's inequality now immediately gives that

$$e^t\|u(t)\|_{H^1(0,\pi)} \leq C\|u(0)\|_{H^1(0,\pi)}e^{CMR^2t}$$

for all $t \in [0,T)$. Choosing $R > 0$ sufficiently small that $\omega := 1 - CMR^2 > 0$ and choosing $\epsilon \in (0, \frac{R}{2})$ such that $C\epsilon < \frac{R}{4}$ it follows that if $\|u(0)\|_{H^1(0,\pi)} < \epsilon$ then

$$\|u(t)\|_{H^1(0,\pi)} \leq C\|u(0)\|_{H^1(0,\pi)}e^{-\omega t} \quad \forall t \in [0,T)$$

and, further, that $\|u(t)\|_{H^1(0,\pi)} < \frac{R}{2}$ for all $t \in [0,T)$. By extensibility, it follows that $T = +\infty$ and the above bounds hold for all $t \geq 0$, as claimed. We note that while the above "Gronwall" argument works very well in this case, it is not suitable for our intended applications to problems on the whole line.

As an alternative for the above Gronwall based argument, for all $t \in [0, T)$, define the quantity

$$Z(t) := \sup_{0 \le s \le t} e^s \|u(s)\|_{H^1(0,\pi)}$$

and note the expected decay rate follows from showing that Z is uniformly bounded. Now, for all $t \in [0, T)$ and $s \in [0, t]$ we have

$$\|u(s)\|_{H^1(0,\pi)} = e^{-s} \left(e^s \|u(s)\|_{H^1(0,\pi)} \right) \le e^{-s} Z(t)$$

so that, if we fix $t \in [0, T)$ we have for all $t' \in [0, t)$

$$\|u(t')\|_{H^1(0,\pi)} \le Ce^{-t'} \|u(0)\|_{H^1(0,\pi)} + M \int_0^{t'} e^{-(t'-s)} e^{-3s} Z(s)^3 ds$$

and hence

$$\begin{aligned} e^{t'} \|u(t')\|_{H^1(0,\pi)} &\le C\|u(0)\|_{H^1(0,\pi)} + CM \left(\int_0^{t'} e^{-2s} ds \right) Z(t)^3 \\ &\le C\|u(0)\|_{H^1(0,\pi)} + \tilde{M} Z(t)^3 \end{aligned}$$

where here $\widetilde{M} = CM$. Taking the supremium in $t' \in [0, t)$ it follows that

$$Z(t) \le C\|u(0)\|_{H^1(0,\pi)} + \widetilde{M} Z(t)^3.$$

for all $t \in [0, T)$. Now, note that if we define the polynomial

$$P(r) := \widetilde{M} r^3 - r + C\|u(0)\|_{H^1(0,\pi)}$$

then the above inequality simply states that $P(Z(t)) \ge 0$ for all $t \in [0, T)$. When $\|u(0)\|_{H^1(0,\pi)}$ is small, we find that $P(r)$ has roots given by

$$r_1 = C\|u(0)\|_{H^1(0,\pi)} + \mathcal{O}(\|u(0)\|^2_{H^1(0,\pi)}), \quad r_2 = \frac{1}{\widetilde{M}} + \mathcal{O}(\|u(0)\|_{H^1(0,\pi)});$$

see Figure 3. Taking $C > 1$ above and noting that $0 \le Z(0) = \|u(0)\|_{H^1(0,\pi)}$, it follows by continuity of $Z(t)$ as a function of t that if $\|u(0)\|_{H^1(0,\pi)}$ is sufficiently small then

$$Z(t) \le r_1 \le 2C\|u(0)\|_{H^1(0,\pi)}$$

for all $t \in [0, T)$ which, by definition, implies that

$$\|u(t)\|_{H^1(0,\pi)} \le 2C\|u(0)\|_{H^1(0,\pi)} e^{-t}$$

for all $t \in [0, T)$. Again by extensibility, it follows that $T = +\infty$ and hence that the above bounds hold for all $t \ge 0$, as claimed.

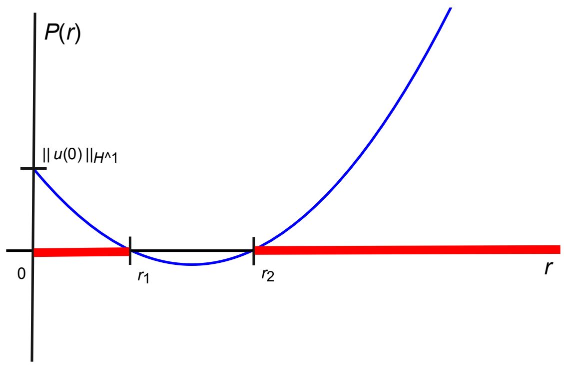


Figure 3: An illustration of the polynomial $P(r)$ when $\|u(0)\|_{H^1(0,\pi)}$ is sufficiently small. Since $P(Z(t)) \geq 0$, it follows by continuity that $Z(t)$ must lie in one of the red regions for so long as it is defined.

4.2 Reaction Diffusion: Stationary Front

Consider now the reaction diffusion equation

$$u_t = u_{xx} + u - u^3, \quad x \in \mathbb{R}, \quad t > 0 \tag{28}$$

Previously, we have seen that this equation admits a one-parameter family of stationary front solutions of the form $\phi(x + \alpha)$, $\alpha \in \mathbb{R}$, where ϕ is strictly monotone and satisfies

$$\lim_{x \to -\infty} \phi(x) = -1, \qquad \lim_{x \to +\infty} \phi(x) = 1.$$

To investigate the nonlinear stability of ϕ, we consider (28) with the initial data

$$u(0) = \phi + v_0, \quad \|v_0\|_{H^1} \ll 1$$

and note, since (28) is locally well posed on the space

$$X := \left\{ f : 1 - |f|^2 \in H^1(\mathbb{R}) \right\},$$

there exists a unique local solution $u(t) \in H^1(\mathbb{R})$ defined for $t \in [0, T)$ for some $T > 0$. A simple calculation shows the linearization of (28) about ϕ is given by

$$v_t = v_{xx} + v - 3\phi^2 v =: Lv \tag{29}$$

In this case, we have seen that $0 \in \sigma_p(L)$ with eigenfunction ϕ', owing to the spatial-translational invariance of the PDE (28), and that $\Re(\sigma(L) \setminus \{0\})$ is uniformly bounded away from the imaginary axis. Thus, the main difference between the current example and the previous is the presence of an eigenvalue at the origin coming from a continuous symmetry of the underlying equation. In this example, we will illustrate how to deal with this additional complication through the introduction of a "modulation function" that, in some sense, will ensure the nonlinear perturbation always lies in the stable subspace of the linearized operator L.

Recall, through a detailed study of $\sigma(L)$ and the resulting bounds on the linearized semigroup $\{e^{Lt}\}_{t\geq 0}$, we have seen that by defining $\Pi : L^2(\mathbb{R}) \to \ker(L)$ to be the spectral projection

$$\Pi := \frac{\langle \phi', \cdot \rangle_{L^2}}{\|\phi'\|_{L^2}^2}\ \phi'$$

the linearized semigroup $\{e^{Lt}\}_{t\geq 0}$ satisfies the decay estimate: for every $\omega \in (0, \gamma)$ there exists a constant $C = C(\omega) > 0$ such that

$$\left\|e^{Lt}(I-P)v(0)\right\|_{L^2} \lesssim e^{-\omega t}\|v(0)\|_{L^2}$$

for all $t \geq 0$. In particular, we demonstrated that this gives the following *linear, asymptotic orbital stability* result:

$$\|v(t) - \Pi(v(0))\|_{L^2} \leq Ce^{-\omega t}\|v(0)\|_{L^2}.$$

Of particular importance, this leads us to suspect that $v(t)$ will converge to a multiple, depending on $v(0)$, of ϕ' as $t \to \infty$, i.e. for given initial data, to a fixed element of the "center subspace". In terms of the original solution $u(x,t)$, this suggests that an initially nearby solution $u(x, 0)$ will satisfy

$$u(x,t) \approx \phi\left(x + \gamma_\infty\right) \quad \text{for} \quad t \gg 1,$$

where here $\gamma_\infty = \frac{\langle \psi', v(0)\rangle_{L^2}}{\|\phi'\|_{L^2}^2}$. This implies that one should expect that a solution that starts near the stationary front ϕ will evolve into a slight spatial translate of ϕ.

To verify the above prediction at a nonlinear level, first notice in the "classical" notion of asymptotic stability, one aims at showing $u(t) \to \phi$ as $t \to \infty$ and hence we want to control the distance from $u(t)$ to ϕ. This motivates introducing the perturbation variable

$$v(t) = u(t) - \phi$$

defined for $t \in [0, T)$. However, here we expect that $u(t) \to \phi(\cdot + \gamma_\infty)$ as $t \to \infty$ for some γ_∞ small. Consequently, in this case we want to control the distance from $u(t)$ to the 1-dimensional manifold

$$\mathcal{M} := \{\phi(\cdot + \gamma) : \gamma \in \mathbb{R}\} \subset H^1(\mathbb{R}). \tag{30}$$

This motivates the introduction of the new perturbation variable

$$v(t) = u(t) - \phi(\cdot - \gamma(t)), \quad t \in [0, T)$$

where $\gamma(t)$ is some function to be determined. The function $\gamma(t)$ is sometimes called a "modulation function". Note that regardless of how $\gamma(t)$ is chosen above, for all $t \in [0, T)$ we have

$$\operatorname{dist}(u(t), \mathcal{M}) := \inf_{\gamma\in\mathbb{R}} \|u(t) - \phi(\cdot + \gamma)\|_{H^1} \leq \|u(t) - \phi(\cdot + \gamma(t)\|_{H^1},$$

and so we are free to choose $\gamma(t)$ above to fit our needs.

Motivated by above, we begin by determining a suitable foliation for a small "tubular" neighborhood of the manifold $\mathcal{M}$ in $H^1(\mathbb{R})$, i.e. a local coordinate system defined in a neighborhood of the orbit of ϕ that is suitable for our needs. This can be done in a number of ways, and one will find many different such foliations in the literature. Since the linearized semigroup e^{Lt} is exponentially stable when restricted to $\operatorname{Ker}(L)^\perp$, here we would like to choose $\gamma(t)$ such that $v \in \operatorname{Ker}(L)^\perp$ for all $t \in [0, T)$.

Lemma 3. *There exists a $\delta > 0$ and smooth functions $(\gamma, v) : H^1(\mathbb{R}) \to \mathbb{R} \times H^1(\mathbb{R})$ with $\gamma(\phi) = 0$, $v(\mathcal{M}) = 0$ such that if $u \in H^1(\mathbb{R})$ with*

$$\operatorname{dist}(u, \mathcal{M}) := \inf_{z \in \mathbb{R}} \|u - \phi(\cdot + z)\|_{H^1} < \delta,$$

then

$$u = \phi(\cdot + \gamma(u)) + v(u)$$

where $v(u) \in \operatorname{Ker}(L)^\perp$.

Proof. Suppose, without loss of generality, $\|u - \phi\|_{H^1}$ is small. For each such u, want to show there exists a $\gamma \in \mathbb{R}$ with $\gamma(\phi) = 0$ such that

$$\underbrace{\left\langle \phi', u - \phi(\cdot + \gamma) \right\rangle_{H^1}}_{g(\gamma, u)} = 0.$$

Since g is smooth with $g(0, \phi) = 0$ and $\partial_\gamma(0, \phi) = -\|\phi'\|_{L^2}^2 \neq 0$, the result follows by the implicit function theorem. $\square$

Thus, if $\|u - \phi\|_{H^1} < \frac{\delta}{2}$ and

$$v(t) := u(t) - \phi(\cdot + \gamma(t))$$

then by possibly choosing $T > 0$ smaller above we can assume $\gamma(t)$ is such that

$$v(t) \in \operatorname{Ker}(L)^\perp \quad \forall\ t \in [0, T)$$

and, further, without loss of generality that $\gamma(0) = 0$ (else, can consider the stability of a translate of ϕ.

Now, since $u(t)$ solves (28) for all $t \in [0, T)$, it follows that the functions $v(t)$ and $\gamma(t)$ satisfy

$$\partial_t \left(v + \phi(\cdot + \gamma(t))\right) = \partial_x^2 \left(v(t) + \phi(\cdot + \gamma(t))\right) + \left(v + \phi(\cdot + \gamma(t))\right) - \left(v + \phi(\cdot + \gamma(t))\right)^3$$

for all $t \in [0, t)$ which, using that $\phi(\cdot + \gamma(t))$ is a stationary solution of (28), can be rewritten as

$$v_t + \phi'(\cdot + \gamma(t))\gamma'(t) = L_{\gamma(t)} v + N_{\gamma(t)}(v) \tag{31}$$

where here

$$L_{\gamma(t)} = \partial_x^2 + 1 - 3\phi(\cdot + \gamma(t))^2$$

denotes the linearization of (28) about the translate $\phi(\cdot + \gamma(t))$ and

$$N_{\gamma(t)}(v) = -3v^2\phi(\cdot + \gamma(t)) - v^3$$

is a nonlinear functional. Now, we can *not* apply our semigroup theory and estimates to (31) since the coefficients of the linear operator $L_{\gamma(t)}$ depend on the evolution variable t. To compensate for this, we simply observe that

$$L_{\gamma(t)} = L + \left[L_{\gamma(t)} - L\right]$$

and hence treat the operator $L_{\gamma(t)} - L$ as an additional nonlinearity. It follows that $v(t)$ and $\gamma(t)$ satisfy the equation

$$v_t + \phi'(\cdot + \gamma(t))\gamma'(t) = Lv + \underbrace{\left[N_{\gamma(t)}(v) + (L_{\gamma(t)} - L)v\right]}_{\mathcal{R}(\gamma(t),v)} \tag{32}$$

Next, we decompose (32) according to the orthogonal decomposition $H^1 = \mathrm{Ker}(L) \oplus \mathrm{Ker}(L)^\perp$. Applying the spectral projection Π to (32) gives

$$\Pi(v_t) + \Pi\left(\phi'(\cdot + \gamma(t))\phi'(t)\right) = \Pi L v + \Pi\mathcal{R}(\gamma(t), v).$$

Since $\Pi(v) = 0$ for all $t \in [0,T)$ we have $\Pi(v_t) = \partial_t\Pi(v) = 0$. Furthermore, since Π is a spectral projection for L, the operators L and Π commute and hence $\Pi \circ L = L \circ \Pi = 0$. We may thus rewrite the above equation as

$$\left\langle \phi', \phi'(\cdot + \gamma(t))\right\rangle \gamma'(t) = \left\langle \phi', \mathcal{R}(\gamma(t), v)\right\rangle$$

From here, we an derive estimates on γ' as follows. Note that

$$\left\langle \phi', \phi'(\cdot + \gamma(t))\right\rangle = \left\langle \phi', \phi'\right\rangle + \left\langle \phi', \underbrace{\phi'(\cdot + \gamma(t)) - \phi'}_{z(t)}\right\rangle$$

Since $\|z(t)\| \leq \|\phi''\||\gamma(t)|$, it follows that so long as $\gamma(t)$ is small, say for all $t \in [0,T)$ with $T > 0$ possibly smaller than before, the above says

$$|\gamma'(t)| \leq C(1 + \gamma(t))\left|\left\langle \phi', \mathcal{R}(\gamma(t), v)\right\rangle\right|$$

for all $t \in [0,T)$ for some constant $C > 0$. Similarly, noting that

$$\|\mathcal{R}(\gamma(t), v)\| \leq C\left(\|v\|^2 + |\gamma(t)|\|v\|\right)$$

it follows that, by possibly choosing $T > 0$ smaller yet again (so that $\|v\|_{H^1}$ is sufficiently small for all $t \in [0,T)$),

$$|\gamma'(t)| \leq C\left(\|v(t)\|_{H^1}^2 + |\gamma(t)|\|v(t)\|\right)$$

for all $t \in [0, T)$.

Next, we project (32) onto the stable subspace $\text{Ker}(L)^\perp$. Applying $(I - \Pi)$ to (32) we get

$$v_t + (I - \Pi)\phi'(\cdot + \gamma(t))\gamma'(t) = Lv + (I - \Pi)\mathcal{R}(\gamma(t), v).$$

As above, we can rewrite the second term as

$$(I - \Pi)\left[\left(\phi' + (\phi'(\cdot + \gamma(t)) - \phi')\right)\gamma'(t)\right] = (I - \Pi)z(t)\gamma'(t)$$

so that

$$v_t = Lv + (I - \Pi)\left[\mathcal{R}(\gamma(t), v) - z(t)\gamma'(t)\right].$$

Setting $\mathcal{R}_F(\gamma(t), v) := \mathcal{R}(\gamma(t), v) - z(t)\gamma'(t)$, it follows that for all $t \in (0, T)$ the functions $\gamma(t)$ and $v(t)$ satisfy the coupled system

$$\begin{cases} \gamma'(t) = \mathcal{O}\left(\|v(t)\|_{H^1}^2 + |\gamma(t)|\|v(t)\|_{H^1}\right) \\ v(t) = e^{Lt}v(0) + \displaystyle\int_0^t e^{L(t-s)}\left(I - \Pi\right)\mathcal{R}_F(\gamma(s), v(s))ds. \end{cases} \tag{33}$$

Now, we know that if $\omega \in (0, \gamma)$ then there exists a $C = C(\omega) > 0$ such that

$$\left\|e^{Lt}f\right\|_{H^1} \le Ce^{-\omega t}\|f\|_{H^1} \quad \forall\ f \in \text{Ker}(L)^\perp.$$

Fix such an ω and fix $\tilde{\omega} \in (\omega/2, \omega)$ and define for all $t \in [0, T)$ the functions

$$M_v(t) := \sup_{0 \le s \le t} e^{\tilde{\omega}s}\|v(s)\|_{H^1}, \quad M_\gamma(t) := \sup_{0 \le s \le t} |\gamma(s)|.$$

Note that for all $t \in [0, T)$ and $s \in [0, t]$ we have

$$\|v(s)\|_{H^1} \le e^{-\tilde{\omega}s}M_v(t), \quad |\gamma(s)| \le M_\gamma(t).$$

Our goal is to show that M_v and M_γ are uniformly bounded in time.

To this end, notice that if we fix $t \in [0, T)$ and integrate (33)(i) from $[0, t']$ for some $t' \in [0, t]$ we have

$$\gamma(t') - \gamma(0) = \int_0^{t'} \mathcal{O}\left(\|v(s)\|_{H^1}^2 + |\gamma(s)|\|v(s)\|_{H^1}\right)ds.$$

Since $\gamma(0) = 0$ by choice, it follows that

$$|\gamma(t')| \le C\left[\left(\int_0^{t'} e^{-2\tilde{\omega}s}ds\right)M_v(t)^2 + \left(\int_0^{t'} e^{-\tilde{\omega}s}\right)M_\gamma(t)M_v(t)\right]$$

for some constant $C > 0$. Noting that the scalar integrals above are uniformly bounded in t', taking the supremium over $t' \in [0, t]$ implies there exists a constant $C_1 > 0$ such that

$$M_\gamma(t) \le C_1\left(M_v(t)^2 + M_\gamma(t)M_v(t)\right)$$

for all $t \in [0, T)$. Similarly, for $0 \leq t' \leq t < T$ as above, from (33)(ii) we get

$$\|v(t')\|_{H^1} \leq Ce^{-\omega t'} \|v(0)\|_{H^1} + C \int_0^{t'} e^{-\omega(t'-s)} \|(I - \Pi)\mathcal{R}_F(\gamma(s), v(s))\|_{H^1} \, ds.$$

Since $(I - \Pi)$ is clearly a bounded linear operator on H^1, we have by the above work that

$$\begin{aligned}\|(I - \Pi)\mathcal{R}_F(\gamma(s), v(s))\|_{H^1} &\leq C \left(\|v(s)\|_{H^1}^2 + |\gamma(s)| \|v(s)\|_{H^1}\right) \\ &\leq C \left(e^{-2\tilde{\omega}s} M_v(t)^2 + e^{-\tilde{\omega}s} M_\gamma(t) M_v(t)\right)\end{aligned}$$

Consequently,

$$\|v(t')\|_{H^1} \leq Ce^{-\omega t'} \|v(0)\|_{H^1} + Ce^{-\omega t'} \left[\left(\int_0^{t'} e^{(\omega - 2\tilde{\omega})s} ds \right) M_v(t)^2 + \left(\int_0^{t'} e^{(\omega - \tilde{\omega})s} ds \right) M_\gamma(t) M_v(t) \right]$$

Recalling that $\tilde{\omega} \in (\omega/2, \omega)$, we find

$$\int_0^{t'} e^{(\omega - 2\tilde{\omega})s} ds \leq \frac{1}{2\tilde{\omega} - \omega} \qquad \int_0^{t'} e^{(\omega - \tilde{\omega})s} ds \leq \frac{1}{\omega - \tilde{\omega}} e^{(\omega - \tilde{\omega})t'}.$$

It follows that

$$\|v(t')\|_{H^1} \leq C \left(e^{-\omega t'} \|v(0)\|_{H^1} + e^{-\omega t'} M_v(t)^2 + e^{-\tilde{\omega} t'} M_\gamma(t) M_v(t) \right)$$

and hence that

$$e^{\tilde{\omega} t'} \|v(t')\|_{H^1} \leq C \left(e^{(\tilde{\omega} - \omega)t'} \|v(0)\|_{H^1} + e^{(\tilde{\omega} - \omega)t'} M_v(t)^2 + M_\gamma(t) M_v(t) \right)$$

Noting that the exponentials on the right hand side above are uniformly bounded above in t', we find by taking the supremum in $t' \in [0, t]$ that there exists a constant $C_2 > 0$ such that

$$M_v(t) \leq C_2 \left(\|v(0)\|_{H^1} + M_v(t)^2 + M_\gamma(t) M_v(t) \right).$$

Next, we claim that if $\|v(0)\|_{H^1}$ and $T > 0$ are such that

$$M_v(t) \leq \frac{1}{2C_1} \quad \forall t \in [0, T)$$

then $T = +\infty$ and, in particular, $M_v(t)$ is well defined and uniformly bounded for all $t \geq 0$. To see this, note that by the above condition on T we have

$$M_\gamma(t) \leq \frac{1}{2} M_\gamma(t) + C_1 M_v(t)^2$$

and hence that

$$M_\gamma(t) \leq 2C_1 M_v(t)^2 \quad \forall t \in [0, T).$$

Inserting this into the bound for $M_v(t)$ gives

$$M_v(t) \leq \tilde{C} \left(\|v(0)\|_{H^1} + M_v(t)^2 + M_v(t)^3 \right) \quad \forall t \in [0, T).$$

where $\tilde{C} > 0$ is some constant. Now, define the polynomial

$$P(r) = r^3 + r^2 - \frac{1}{\tilde{C}} r + \|v(0)\|_{H^1}$$

and note the above inequality states that $P(M_v(t)) \geq 0$ for all $t \in [0, T)$. One can easily check that for $\|v(0)\|_{H^1}$ sufficiently small, $P(r)$ has two consecutive positive roots r_1, r_2 satisfying

$$0 < r_1 = \tilde{C}\|v(0)\|_{H^1} + \mathcal{O}(\|v(0)\|_{H^1}^2) \ll r_2$$

with $P(r) \geq 0$ for $r \in [0, r_1] \cup [r_2, \infty)$ and $P(r) < 0$ for $r \in (r_1, r_2)$. Taking $\tilde{C} > 1$ above, it follows that

$$M_v(0) = \|v(0)\|_{H^1} < r_1$$

so that, by continuity of $M_v(t)$ on t, we have

$$M_v(t) \leq r_1 \leq 2\tilde{C}\|v(0)\|_{H^1}$$

for all $t \in [0, T)$, provided that $\|v(0)\|_{H^1}$ is sufficiently small. In particular, if we choose $\|v(0)\|_{H^1}$ sufficiently small that $r_1 < \frac{1}{2C_1}$ then the above argument can be continued to give $T = +\infty$. We conclude that if $\|v(0)\|_{H^1}$ is sufficiently small, then there exists a constant $C > 0$ such that $M_v(t) \leq C\|v(0)\|_{H^1}$ for all $t \geq 0$ so that, in particular,

$$\|v(t)\|_{H^1} \leq Ce^{-\tilde{\omega}t}\|v(0)\|_{H^1} \quad \forall t \geq 0.$$

This verifies that the perturbed solution $u(t)$ converges as $t \to \infty$ to the one-dimensional manifold $\mathcal{M}$. To show that it converges to a *particular element* of $\mathcal{M}$, recall that for all $t \geq 0$ we have

$$\gamma'(t) = \mathcal{O}\left(\|v(t)\|_{H^1}^2 + |\gamma(t)|\|v(t)\|_{H^1} \right)$$

and that $|\gamma(t)| \leq M_\gamma(t) \leq 2C_1 M_v(t)^2$. By the above bound on $M_v(t)$ it follows that

$$\gamma'(t) = \mathcal{O}\left(e^{-2\tilde{\omega}t}\|v(0)\|_{H^1}^2 + e^{-\tilde{\omega}t}\|v(0)\|_{H^1}^3 \right).$$

Fixing $0 \leq t_1 < t_2$ and integrating over $[t_1, t_2]$, it follows there exists a constant $C > 0$ such that

$$\begin{aligned}
|\gamma(t_2) - \gamma(t_1)| &\leq C \int_{t_1}^{t_2} \left(e^{-2\tilde{\omega}t}\|v(0)\|_{H^1}^2 + e^{-\tilde{\omega}t}\|v(0)\|_{H^1}^3 \right) dt \\
&\leq C \left(\frac{1}{2\tilde{\omega}} e^{-2\tilde{\omega}t_1}\|v(0)\|_{H^1}^2 + \frac{1}{\tilde{\omega}} e^{-\tilde{\omega}t_1}\|v(0)\|_{H^1}^3 \right) \\
&\leq \tilde{C} e^{-\tilde{\omega}t_1}\|v(0)\|_{H^1}^2
\end{aligned}$$

provided that $\|v(0)\|_{H^1}$ is sufficiently small. It follows that the sequence $\{\gamma(t)\}_{t\geq 0}$ is a Cauchy sequence in $\mathbb{R}$ and hence there exists a $\gamma_\infty \in \mathbb{R}$ such that

$$\gamma(t) \to \gamma_\infty \;\text{ as }\; t \to \infty.$$

In particular, the above bound gives

$$|\gamma(t) - \gamma_\infty| \leq \tilde{C} e^{-\tilde{\omega} t} \|v(0)\|_{H^1}^2$$

so that $\gamma(t)$ converges to γ_∞ at an exponential rate.

Putting everything together, we have shown that the perturbed solution $u(t)$ converges to $\phi(\cdot + \gamma_\infty) \in \mathcal{M}$ at an exponential rate as $t \to \infty$, as claimed.

4.3 Discussion of the Periodic Case

Finally, let's briefly discuss how the previous techniques might be applied in the periodic setting. To this end, consider a PDE of the form

$$u_t = F(u) \tag{34}$$

where we assume F is a constant coefficient (in both x and t) nonlinear operator, and suppose that (34) has a T-periodic equilibrium solution $\phi(x)$. For this discussion, we are interested in determining the stability of ϕ to so-called "localized" perturbations[9], i.e. to perturbations in $L^2(\mathbb{R})$. The linearized operator, obtained by linearizing the right hand side of (34) about ϕ is the operator

$$L := DF(\phi)$$

which will be a linear differential operator with T-periodic coefficients. From our previous work, we know the spectrum of L can be decomposed as

$$\sigma_{L^2(\mathbb{R})}(L) = \bigcup_{\xi \in [-\pi/T, \pi/T)} \sigma_{L^2_{\text{per}}(0,T)}(L_\xi),$$

where the Bloch operators $L_\xi := e^{-i\xi x} L e^{i\xi x}$ are considered to act on $L^2_{\text{per}}(0,T)$ for each $\xi \in [-\pi/T, \pi/T)$. By Remark 6 in Section 3.2.3 above, it follows that ϕ' satisfies the ODE

$$L\phi' = 0. \tag{35}$$

Unlined the analysis in Section 3.2.2 and Section 3.2.3, this does *not* imply that $\lambda = 0$ is an eigenvalue[10] of L since ϕ', being T-periodic, clearly does not belong to $L^2(\mathbb{R})$. Rather, (35) implies that $\lambda = 0$ is an eigenvalue for the co-periodic Bloch operator L_0.

For the sake of simplicity, assume that $\lambda = 0$ is a *simple* eigenvalue of L_0, and that $0 \notin \sigma_p(L_\xi)$ for any $\xi \neq 0$. Then as ξ is varied near $\xi = 0$, there exists a curve of essential

[9]Note if one wishes to study the stability of ϕ to periodic perturbations, as discussed in Section 2.3.1, then $\sigma_p(L)$ is purely discrete and stability may be studied in much the same way as in the above examples.

[10]And, recall that we already know that $\sigma_p(L) = \emptyset$ anyways.

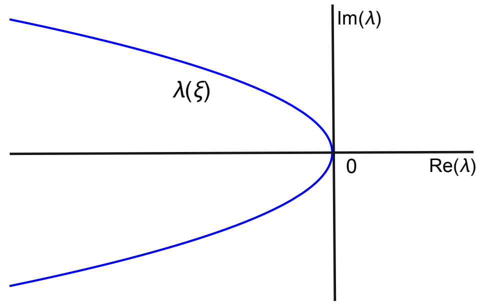


Figure 4: An illustration of the spectrum about a stable periodic equilibrium solution of (34) near the origin $\lambda = 0$.

spectrum $\lambda(\xi) \in \sigma_p(L_\xi)$ near the origin which, by basic results in spectral perturbation theory, is analytic in ξ and satisfies

$$\lambda(\xi) = \overline{\lambda(-\xi)}$$

for all $|\xi| \ll 1$. It follows that if ϕ is to be a stable equilibrium solution of (34), then $\lambda(\xi)$ must admit a Taylor expansion for $|\xi| \ll 1$ of the form

$$\lambda(\xi) = i\alpha\xi - \beta\xi^2 + \mathcal{O}(|\xi|^3)$$

for some constants $\alpha \in \mathbb{R}$ and $\beta \geq 0$. If we assume the non-degeneracy condition $\beta \neq 0$, it follows that[11]

$$\Re(\sigma_p(L_\xi)) \leq -\theta\xi^2 \quad \forall 0 < |\xi| \ll 1. \tag{36}$$

Considering now the linear stability of ϕ, let $\epsilon_0 > 0$ be small and recall the Bloch solution formula

$$\left(e^{Lt}v\right)(x) = \int_{|\xi|<\epsilon_0} e^{i\xi x}\left(e^{L_\xi t}\check{v}(\xi,\cdot)\right)(x)d\xi + \int_{\epsilon_0<|\xi|<\frac{\pi}{T}} e^{i\xi x}\left(e^{L_\xi t}\check{v}(\xi,\cdot)\right)(x)d\xi, \tag{37}$$

where here we have separated the low- and high-Bloch number components of the integral. If we assume that there exists a constant $\sigma > 0$ such that

$$\Re(\sigma_p(L_\xi)) < -\sigma \quad \text{for all} \;\; \epsilon_0 < |\xi| < \frac{\pi}{T},$$

it follows by the generalized Hausdorff-Young (25) inequality that

$$\left\|\int_{\epsilon_0<|\xi|<\frac{\pi}{T}} e^{i\xi x}\left(e^{L_\xi t}\check{v}(\xi,\cdot)\right)(x)d\xi\right\|_{L^2(\mathbb{R})} \leq Ce^{-\sigma t}\|\check{v}\|_{L^2([-\pi/T,\pi/T);L^2(0,T))} = Ce^{-\sigma t}\|v\|_{L^2(\mathbb{R})},$$

[11]Concerning modeling, such a condition may be expected to hold for stable waves in diffusive, either fully or partially, systems. In energy conserving Hamiltonian systems (such as the KdV equation (19)), rather, spectral stability is equivalent to $\sigma(L) \subset \mathbb{R}i$. In such a case, it is not yet known how to establish nonlinear stability through the method of linearization, and one typically relies on different methods, such as the study of an appropriate Lyapunov functional.

so that the second term in (37) decays exponentially fast in time. On the other hand, the first term in (37) satisfies

$$\begin{aligned}\left\|\int_{|\xi|<\epsilon_0} e^{i\xi x}\left(e^{L_\xi t}\check{v}(\xi,\cdot)\right)(x)d\xi\right\|_{L^2_x(\mathbb{R})} C &\leq \left\|e^{-\theta\xi^2 t}\check{v}(\xi,x)\right\|_{L^2([-\pi/T,\pi/T);L^2(0,T)} \\ &\leq \left\|e^{-\theta\xi^2 t}\right\|_{L^\infty_\xi}\|\check{v}\|_{L^2([-\pi/T,\pi/T);L^2(0,T)} \\ &\leq C\|v\|_{L^2(\mathbb{R})}\end{aligned}$$

Consequently, these calculations show the first term in (37) is only *bounded* in time, and hence does not decay.

Remark 8. *It is actually possible to sharpen the above estimate to yield a $t^{-1/4}$ decay rate, but this ends up being too slow to close any reasonable iteration scheme that might lead to a nonlinear stability result.*

So then, how might one go about establishing a nonlinear stability result in this case? In a sense, we should follow the ideas in Section 4.2 and introduce an appropriate "modulation" function to accommodate for the lack of decay. As expected, however, this is more complicated in this case. To see why, observe that in Section 4.2 that a spatially uniform translation of the front still satisfies the same boundary conditions as the front. In particular, the $L^p(\mathbb{R})$ norm between any two elements of the manifold $\mathcal{M}$ in (30) is finite. For our T-periodic equilibrium solution of (34), however, periodicity implies that for any $\gamma \neq \mathbb{Z}T$ that

$$\|\phi - \phi(\cdot+\gamma)\|_{L^p(\mathbb{R})} = +\infty \quad \text{for all} \quad 1 \leq p < \infty$$

In other words, a spatially uniform translation of a periodic wave is *not* a small, or even bounded, perturbation in $L^p(\mathbb{R})$ for any finite $1 \leq p < \infty$.

To proceed, let's think about the expected dynamics about a "stable" periodic wave ϕ. Suppose ϕ is as in Figure 5(a), and note that we can decompose x-axis into periodic "cells" corresponding to the periodicity of ϕ. If we perturb ϕ by a small localized perturbation as in Figure 5(a), it is natural to expect that it effect of this perturbation has a finite speed of propagation. In particular, at any given time each periodic cell may react to the initial perturbation differently. To further illustrate this point, it is known that there exists "spectrally stable" periodic traveling wave solutions to the Kuramoto-Sivashinsky (KS) equation

$$u_t + u_{xxx} + uu_x + \delta(u_{xx} + u_{xxxx}) = 0. \tag{38}$$

While we won't go into the details of the analysis or proof, which can be found in [BJNRZ], it is instructive to view the dynamics of such a stable solution: see Figure 5(b). In this figure, start at $t = 0$ with a stable periodic traveling wave solution of (38) and perturb it by some small, localized "bump" around $x = 130$. The effect of the perturbation on the solution tracked as time is increased: here, the green and blue curves denote the "peaks"

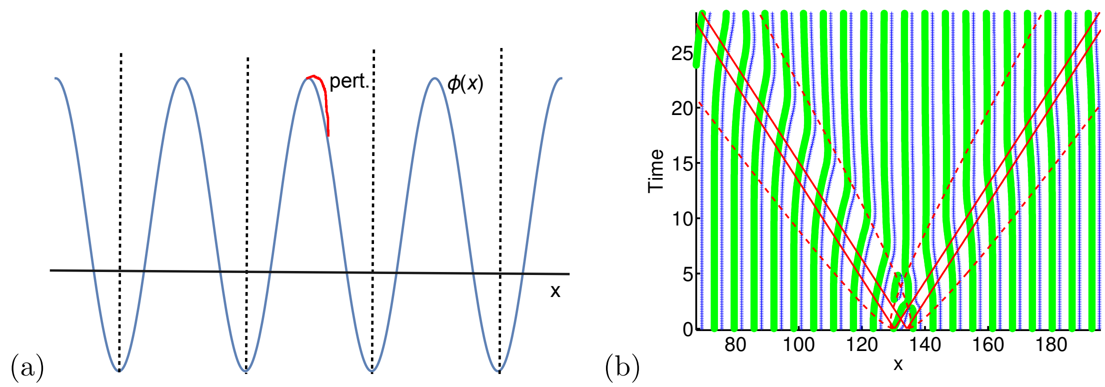


Figure 5: (a)An illustration of a periodic wave ϕ that is perturbed by a small localized perturbation. Here, the perturbation is in red, and the vertical dashed lines are there to simply decompose ϕ into disjoint periodic "cells". (b) The temporal evolution of a slightky perturbed stable periodic traveling wave solution (in traveling coordinates, so the background wave is stationary) of the KS equation (38). This picture was taken from the work [BJNRZ].

and "troughs" of the solution as it evolves over time[12]. What we observe is that the effect of the perturbation has a finite speed of propagation, signaling some sort of "hyperbolicity" in the system. Indeed, for a fixed time $t > 0$ the solution is essentially unchanged outside of the outermost dashed red lines, while between the innermost dashed red lines we essentially see a copy of the original wave. However, upon careful inspection one will observe that the periodic wave between the inner most dashed red curves is slightly out of phase with the original wave: indeed, between the "left" two dashed red lines and the "right" two dashed red lines, we can see that the solution is undergoing a spatially localized phase shift, and that this phase shift remains small and spreads out over time. For details, see [BJNRZ]. If we are interested in proving the stability[13] of the periodic wave depicted in Figure 5(b), the above considerations motivate introducing a modulation function that would allow ϕ to change *differently* on each periodic cell, i.e. our modulation function should be spatially dependent!

In summary, if u_0 is near ϕ in $L^2(\mathbb{R})$, the above discussion motivates decomposing the associated solution $u(t)$ with initial data u_0 as

$$u(x,t) = \phi(x + \psi(x,t)) + v(x,t)$$

where here ψ is our spatially dependent modulation function. The fact that ψ depends on x suggests that it will solve a system of $PDE's$ that are coupled to the evolution

[12] We are thus recording the evolution of the periodic structure of the wave, and not its shape or magnitude between consecutive local max and min's.

[13] Here, the sense of stability has to be defined appropriately. In this context, the notion of "nonlinear space-modulated stability" has been shown to be powerful. See [JNRZ].

equation for v. Compare this to the analysis in Section 4.2, where the evolution for the nonlinear perturbation v couples to an ODE satisfied by the x-independent modulation $\gamma(t)$. Consequently, establishing nonlinear stability of periodic patterns in this context is considerably more complicated than other examples we have considered in these notes. Thankfully, this is the subject of some of the presentations at this workshop, so I refer students to the presentation notes prepared by your peers.

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