Y. Martel, F. Merle, and T.-P. Tsai, Stability and Asymptotic Stability in the Energy Space of the Sum of N Solitons for Subcritical gKdV Equations, Comm. Math. Phys. 231 (2002), 347-373.

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1 Introduction

In this talk, we study the paper "Y. Martel, F. Merle, and T.-P. Tsai, *Stability and Asymptotic Stability in the Energy Space of the Sum of N Solitons for Subcritical gKdV Equations*, Comm. Math. Phys. **231** (2002), 347-373."

In this talk, we consider the generalized Korteweg–de Vries equations

$$\begin{cases} u_t + (u_{xx} + u^p)_x = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{gKdV}$$

for $p = 2$, 3, or 4 and $u_0 \in H^1 := H^1(\mathbb{R}, \mathbb{R})$. It is known that (gKdV) is global well-posed in the energy space H^1, that is, for $u_0 \in H^1(\mathbb{R})$, there exists a unique solution $u \in C(\mathbb{R}, H^1)$ of (gKdV). Moreover, the solution u satisfies the following two conservation laws:

$$\int u(t)^2 = \int u_0^2,$$

$$E(u(t)) := \frac{1}{2}\int u_x(t)^2 - \frac{1}{p+1}\int u(t)^{p+1} = E(u_0).$$

Eq. (gKdV) has explicit traveling wave solutions, called solitons, of the form

$$u_c(t,x) = Q_c(x - ct),$$

where $c > 0$ is the speed of the soliton, and

$$Q_c(x) = c^{1/(p-1)} Q(\sqrt{c}\, x), \quad Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2}\, x)} \right)^{1/(p-1)}.$$

Note that Q_c is the only positive solution in H^1 of

$$Q_{xx} + Q^p = cQ.$$

It is known that for $c > 0$, the soliton $Q_c(x - ct)$ is stable in the following sense: $\forall \delta_0 > 0$, $\exists \alpha_0 > 0$ s.t.

$$\|u_0 - Q_c\|_{H^1} < \alpha_0 \;\Rightarrow\; \exists x\colon [0, +\infty) \to \mathbb{R} \text{ s.t. } \|u(t) - Q_c(\cdot - x(t))\|_{H^1} < \delta_0$$

(see [1, 6]). Moreover, Martel and Merle proved the asymptotic stability of the family of solitons $\{\, Q_c(\cdot - x_0 - ct) \mid c > 0,\ x_0 \in \mathbb{R} \,\}$ in the following sense: $\forall c > 0$, $\exists \alpha_0 > 0$ s.t.

$$\|u_0 - Q_c\|_{H^1} < \alpha_0 \;\Rightarrow\; \exists c_{+\infty} > 0,\ \exists x\colon [0, +\infty) \to \mathbb{R} \text{ s.t. } u(t, \cdot + x(t)) \rightharpoonup Q_{+\infty} \text{ in } H^1.$$

In [5], Martel, Merle, and Tsai proved the stability and asymptotic stability of the sum of N solitons in the following sense.

Theorem 1 ([5]). *Let $p = 2$, 3, or 4. Let $0 < c_1^0 < \cdots < c_N^0$. Then $\exists \gamma_0$, A_0, L_0, $\alpha_0 > 0$ s.t. the following is true: Let $u_0 \in H^1$, $L > L_0$, $\alpha < \alpha_0$, and $x_1^0, \dots, x_N^0 \in \mathbb{R}$ satisfy*

$$\Big\| u_0 - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j^0) \Big\|_{H^1} < \alpha, \quad \text{and} \quad \forall j = 2, \dots, N, \quad x_j^0 > x_{j-1}^0 + L.$$

Let $u(t)$ be the solution of (gKdV). *Then $\exists x_1, \dots, x_N\colon [0, +\infty) \to \mathbb{R}$ s.t. the following is true.*

(i) (Stability of the sum of N decoupled solitons).

$$\forall t \geqslant 0, \quad \Big\| u(t) - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j(t)) \Big\|_{H^1} < A_0(\alpha + e^{-\gamma_0 L}).$$

(ii) (Asymptotic stability of the sum of N solitons). $\exists c_1^{+\infty}, \dots, c_N^{+\infty} > 0$ with $|c_j^{+\infty} - c_j^0| < A_0(\alpha + e^{-\gamma_0 L})$ s.t. $\forall j = 1, \dots, N$,

$$u(t, \cdot + x_j(t)) - \sum_{k=1}^{N} Q_{c_k^{+\infty}}(\cdot - (x_k(t) - x_j(t))) \rightharpoonup 0 \quad \text{in } H^1, \quad \dot{x}_j(t) \to c_j^{+\infty}, \tag{1}$$

$$\Big\| u(t) - \sum_{j=1}^{N} Q_{c_j^{+\infty}}(\cdot - x_j(t)) \Big\|_{L^2(x > c_1^0 t/10)} \to 0 \tag{2}$$

as $t \to +\infty$.

After this paper, Martel and Merle [4] improved Theorem 1 (ii), that is, they showed that

$$\Big\| u(t) - \sum_{j=1}^{N} Q_{c_j^{+\infty}}(\cdot - x_j(t)) \Big\|_{H^1(x > c_1^0 t/10)} \to 0$$

as $t \to +\infty$. In this talk, we do not prove (2), only treat Theorem 1 (i) and (ii) (1).

2 Outline of Proof of Theorem 1

Let $0 < c_1^0 < \cdots < c_N^0$, and let

$$\sigma_0 := \frac{1}{2} \min\{c_1^0, c_2^0 - c_1^0, c_3^0 - c_2^0, \ldots, c_N^0 - c_{N-1}^0\}.$$

For α, $L > 0$, define

$$\mathcal{U}(\alpha, L) := \Big\{ v \in H^1 \ \Big| \inf_{y_j > y_{j-1} + L} \Big\| v - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - y_j) \Big\|_{H^1} < \alpha \Big\}.$$

The following lemma is a very useful tool to examine the behavior of solutions close to the sum of N solitons.

Lemma 1 (Decomposition of the solution). *$\exists L_1$, α_1, $K_1 > 0$ s.t. the following is true: If $L > L_1$, $0 < \alpha < \alpha_1$, and $t_0 > 0$ satisfy*

$$\forall t \in [0, t_0], \quad u(t) \in \mathcal{U}(\alpha, L),$$

then $\exists! C^1$-functions $c_1, \ldots, c_N : [0, t_0] \to (0, +\infty)$, $x_1, \ldots, x_N : [0, t_0] \to \mathbb{R}$ s.t.

$$\varepsilon(t, x) := u(t, x) - \sum_{j=1}^{N} R_j(t, x), \quad \textit{where} \quad R_j(t, x) := Q_{c_j(t)}(x - x_j(t)),$$

satisfies $\forall t \in [0, t_0]$, $\forall j = 1, \ldots, N$,

$$\int R_j(t)\varepsilon(t) = \int (R_j)_x(t)\varepsilon(t) = 0,$$

$$\|\varepsilon(t)\|_{H^1} + \sum_{j=1}^{N} |c_j(t) - c_j^0| \leqslant K_1\alpha, \quad x_j(t) > x_{j-1}(t) + L/2,$$

$$|\dot{c}_j(t)| + |\dot{x}_j(t) - c_j(t)| \leqslant K_1 \left(\int e^{-\sqrt{\sigma_0}|x - x_j(t)|/2} \varepsilon(t)^2 \right)^{1/2} + K_1 e^{-\sqrt{\sigma_0}(L + \sigma_0 t)/4}. \quad (3)$$

2.1 Proof of Stability in the Energy Space

Let $\gamma_0 := \sqrt{\sigma_0}/16$. For $A_0, L, \alpha > 0$, we define

$$\begin{aligned}\mathcal{V}_{A_0}(L,\alpha) &:= \mathcal{U}\left(A_0(\alpha + e^{-\gamma_0 L/2}), \frac{L}{2}\right)\\ &= \left\{ v \in H^1 \;\middle|\; \inf_{y_j > y_{j-1}+L/2} \Big\| v - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - y_j) \Big\|_{H^1} < A_0(\alpha + e^{-\gamma_0 L/2}) \right\}.\end{aligned}$$

Theorem 1 follows from the following proposition.

Proposition 1 (A priori estimate)**.** *$\exists A_0, L_0, \alpha_0 > 0$ s.t. if $u_0 \in H^1$, $L > L_0$, $\alpha < \alpha_0$, and $x_1^0, \ldots, x_N^0$ satisfy*

$$\Big\| u_0 - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j^0) \Big\|_{H^1} < \alpha, \quad \textit{and} \quad \forall j = 2, \ldots, N, \quad x_j^0 > x_{j-1}^0 + L, \tag{4}$$

and if $t^* > 0$ satisfies

$$\forall t \in [0, t^*], \quad u(t) \in \mathcal{V}_{A_0}(L, \alpha), \tag{5}$$

then

$$\forall t \in [0, t^*], \quad u(t) \in \mathcal{V}_{A_0/2}(L, \alpha).$$

Assuming this proposition, we prove Theorem 1 (i).

Proof of Theorem 1. Let $A_0, L_0, \alpha_0 > 0$ be chosen as in Proposition 1, and let u_0, L, α, and $x_1^0, \ldots, x_N^0$ satisfy the assumptions of Theorem 1. Then by continuity of $u(t)$ in H^1, $\exists \tau_0 > 0$ s.t. $\forall t \in (0, \tau_0)$, $u(t) \in \mathcal{V}_{A_0}(L, \alpha)$. Let

$$t^* := \sup\{\, t \geqslant 0 \mid \forall t' \in [0, t],\ u(t') \in \mathcal{V}_{A_0}(L, \alpha) \,\}.$$

Assume $t^* < +\infty$. Then by Proposition 1, we have $u(t) \in \mathcal{V}_{A_0/2}(L, \alpha)$. Therefore, by continuity of $u(t)$ in H^1, $\exists \tau > 0$ s.t. $\forall t \in [0, t^* + \tau]$, $u(t) \in \mathcal{V}_{2A_0/3}(L, \alpha)$, which contradicts the definition of t^*. Hence, the stability result follows. $\square$

Next, we prove Proposition 1.

Proof of Proposition 1. Let $A_0 > 0$ to be fixed later. First, note that $\exists \alpha_2(A_0) > \alpha_1$, $\exists L_2(A_0) > L_1$ s.t. $\forall \alpha \in (0, \alpha_2(A_0))$, $\forall L > L_2(A_0)$,

$$A_0(\alpha + e^{-\gamma_0 L/2}) \leqslant \alpha_1,$$

where α_1 and L_1 are defined in Lemma 1. Therefore, by (5) and Lemma 1, there exist $c_j : [0, t^*] \to (0, +\infty)$, $x_j : [0, t^*] \to \mathbb{R}$ s.t.

$$\varepsilon(t,x) = u(t,x) - \sum_{j=1}^{N} R_j(t,x), \quad \text{where} \quad R_j(t,x) = Q_{c_j(t)}(x - x_j(t)),$$

satisfies $\exists C(A_0) > 0$ s.t. $\forall j$, $\forall t \in [0, t^*]$,

$$\int R_j(t)\varepsilon(t) = \int (R_j)_x(t)\varepsilon(t) = 0, \tag{6}$$

$$|c_j(t) - c_j^0| + |\dot{c}_j(t)| + |\dot{x}_j(t) - c_j^0| + \|\varepsilon(t)\|_{H^1} \leqslant C(A_0)K_1(\alpha_0 + e^{-\gamma_0 L_0/2}). \tag{7}$$

Note that by (4) and Lemma 1 at $t = 0$,

$$\|\varepsilon(0)\|_{H^1} + \sum_{j=1}^{N} |c_j(0) - c_j^0| \leqslant K_1\alpha, \quad x_j(0) - x_{j-1}(0) \geqslant \frac{L}{2}. \tag{8}$$

From (7) and (8), $\exists \alpha_0(A_0) \in (0, \alpha_2(A_0))$, $\exists L_3(A_0) > L_2(A_0)$ s.t. $\forall \alpha \in (0, \alpha_0)$, $\forall L > L_3$, $\forall t \in [0, t^*]$,

$$c_1(t) \geqslant \sigma_0, \quad \dot{x}_1(t) \geqslant \sigma_0, \quad c_j(t) - c_{j-1}(t) \geqslant \sigma_0, \quad \dot{x}_j(t) - \dot{x}_{j-1}(t) \geqslant \sigma_0, \tag{9}$$

$$x_j(t) - x_{j-1}(t) \geqslant L/2 + \sigma_0 t, \quad \|\varepsilon(t)\|_{H^1} \leqslant \frac{1}{2}\left(\frac{\sigma_0}{8}\right)^{\frac{1}{p-1}}. \tag{10}$$

Throughout this talk, we assume (9) and (10). Now, we give a uniform upper bound on $|c_j(t) - c_j(t)|$ and $\|\varepsilon(t)\|_{H^1}$ on $[0, t^*]$ improving (7) for A_0 large enough.

Lemma 2 (Quadratic control of the variation of $c_j(t)$)**.** $\exists K_2 > 0$ *independent of* A_0*,* $\exists L_4 > L_3$ *s.t.* $\forall L > L_4$*,* $\forall t \in [0, t^*]$*,*

$$\sum_{j=1}^{N} |c_j(t) - c_j(0)| \leqslant K_2(\|\varepsilon(t)\|_{H^1}^2 + \|\varepsilon(0)\|_{H^1}^2 + e^{-\gamma_0 L}).$$

Lemma 3 (Control of $\|\varepsilon(t)\|_{H^1}$)**.** $\exists K_3 > 0$ *independent of* A_0*,* $\exists L_0 > L_4$ *s.t.* $\forall L > L_0$*,* $\forall t \in [0, t^*]$*,*

$$\|\varepsilon(t)\|_{H^1}^2 \leqslant K_3(\|\varepsilon(0)\|_{H^1}^2 + e^{-\gamma_0 L}).$$

We give proofs of Lemmas 2 and 3 in Section 3.1.

Now, we finish the proof of Proposition 1. By the regularity of $c \mapsto Q_c$, (8), and Lemmas 2 and 3, we have

$$\begin{aligned}
&\Big\| u(t) - \sum_{j=1}^{N} Q_{c_j^0}(x - x_j(t)) \Big\|_{H^1} \\
&\leqslant \Big\| u(t) - \sum_{j=1}^{N} R_j(t) \Big\|_{H^1} + \Big\| \sum_{j=1}^{N} R_j(t) - \sum_{j=1}^{N} Q_{c_j^0}(x - x_j(t)) \Big\|_{H^1} \\
&\leqslant \|\varepsilon(t)\|_{H^1} + C\sum_{j=1}^{N} |c_j(t) - c_j^0|
\end{aligned}$$

$$\begin{aligned}&\leqslant \|\varepsilon(t)\|_{H^1} + C\sum_{j=1}^{N}|c_j(t) - c_j(0)| + C\sum_{j=1}^{N}|c_j(0) - c_j^0|\\&\leqslant \|\varepsilon(t)\|_{H^1} + CK_2(\|\varepsilon(0)\|_{H^1}^2 + e^{-\gamma_0 L}) + CK_1\alpha\\&\leqslant K_4(\alpha + e^{-\gamma_0 L/2}),\end{aligned}$$

where $K_4 > 0$ is a constant independent of A_0.

Choosing $A_0 = 2K_4$, we complete the proof of Proposition 1 and thus the proof of Theorem 1 (i). □

2.2 Proof of Asymptotic Stability Result

In this subsection, we prove the following asymptotic result on $\varepsilon(t)$ as $t \to +\infty$.

Proposition 2 (Convergence around solitons, $p = 2$, 3, or 4)**.** *Under the assumptions of Theorem 1, the following is true:*

(i) $\forall j \in \{1, \dots, N\}$,

$$\varepsilon(t, \cdot + x_j(t)) \rightharpoonup 0 \quad \textit{weakly in } H^1(\mathbb{R}) \textit{ as } t \to +\infty. \tag{11}$$

(ii) There exist $0 < c_1^{+\infty} < \cdots < c_N^{+\infty}$ *s.t.*

$$c_j(t) \to c_j^{+\infty}, \quad \dot{x}_j(t) \to c_j^{+\infty} \quad \textit{as } t \to +\infty.$$

The asymptotic stability of sum of N solitons follows from the Liouville property close to solitons.

Theorem (Liouville property close to a soliton for $p = 2$, 3, or 4 [2])**.** *Let* $p = 2$, 3, *or* 4, *and let* $c_0 > 0$ *and* $x_0 \in \mathbb{R}$. *Then* $\exists \alpha_0 > 0$ *s.t. if* $\|u_0 - Q_{c_0}(\cdot - x_0)\|_{H^1} < \alpha_0$, *and if there exists* $y\colon \mathbb{R} \to \mathbb{R}$ *s.t.*

$$\forall \delta_0 > 0,\ \exists A_0 > 0 \textit{ s.t. } \forall t \in \mathbb{R}, \quad \int_{|x|>A_0} u(t, x + y(t))^2 \leqslant \delta_0, \qquad (L^2\text{-compactness})$$

then $\exists c^* > 0$, $\exists x^* \in \mathbb{R}$ *s.t.* $\forall (t, x) \in \mathbb{R}^2$, $u(t, x) = Q_{c^*}(x - x^* - c^* t)$.

Proof of Proposition 2 (i). We prove by contradiction. Assume that (11) does not hold for some j. Since $\|\varepsilon(t)\|_{H^1}$ and $c_j(t)$ is bounded by the stability results, $\exists \tilde{\varepsilon}_0 \in H^1 \setminus \{0\}$, $\exists \tilde{c}_0 > 0$, $\exists (t_n)$ with $t_n \to +\infty$ s.t.

$$\varepsilon(t, . + x_j(t)) \rightharpoonup \tilde{\varepsilon}_0 \quad \text{weakly in } H^1(\mathbb{R}), \quad c_j(t_n) \to \tilde{c}_0 \quad \text{as } t \to +\infty.$$

Moreover, by weak convergence and the stability result, $\|\tilde{\varepsilon}_0\|_{H^1} \leqslant \sup_{t\geqslant 0} \|\varepsilon(t)\|_{H^1} \leqslant A_0(\alpha_0 + e^{-\gamma_0 L_0})$, and therefore $\|\tilde{\varepsilon}_0\|_{H^1}$ is as small as we want by taking α_0 small and L_0 large.

Let $\tilde{u}(t)$ be the global solution of (gKdV) with $\tilde{u}(0) = Q_{\tilde{c}_0} + \tilde{\varepsilon}_0$. Let $\tilde{x}(t)$ and $\tilde{c}(t)$ be the geometrical parameters associated to the solution $\tilde{u}(t)$ (apply the modulation theory for a solution close to a single soliton $Q_{\tilde{c}_0}$). Note that $\tilde{x}(t)$ and $\tilde{c}(t)$ is defined in $\mathbb{R}$ by the stability of the single soliton $Q_{\tilde{c}_0}$.

We claim that the solution $\tilde{u}(t)$ is L^2-compact.

Lemma 4 (L^2-compactness of the asymptotic solution)**.**

$$\forall \delta_0 > 0, \exists A_0 > 0 \ s.t. \ \forall t \in \mathbb{R}, \quad \int_{|x|>A_0} \tilde{u}(t, x + \tilde{x}(t))^2 \leqslant \delta_0.$$

Assuming this lemma, we finish the proof of Proposition 2 (i). Indeed, by choosing α_0 small enough and L_0 large enough, we can apply the Liouville theorem to $\tilde{u}(t)$. Therefore, $\exists c^* > 0$, $\exists x^* \in \mathbb{R}$ s.t. $\tilde{u}(t) = Q_{c^*}(\cdot - x^* - c^* t)$. In particular, $Q_{\tilde{c}_0} + \tilde{\varepsilon}_0 = \tilde{u}(0) = Q_{c^*}(\cdot - x^*)$. Since by (6) and weak convergence, $0 = \int \tilde{\varepsilon}_0 (Q_{\tilde{c}_0})_x = \int Q_{c^*}(x - x^*)(Q_{\tilde{c}_0})_x$, we have $x^* = 0$. Next, since $0 = \int \tilde{\varepsilon}_0 Q_{\tilde{c}_0} = \int (Q_{c^*} - Q_{\tilde{c}_0}) Q_{\tilde{c}_0}$, we have $c^* = \tilde{c}_0$, and so $\tilde{\varepsilon}_0 = 0$. This is a contradiction. □

We prove Lemma 4 in Section 3.2.

We define

$$\phi(x) := cQ(\sqrt{\sigma_0} x/2), \quad \psi(x) := \int_{-\infty}^{x} \phi(y)\, dy, \quad \text{where } c := \left(\frac{2}{\sqrt{\sigma_0}} \int_{-\infty}^{\infty} Q \right)^{-1}.$$

Note that $\forall x \in \mathbb{R}$, $\psi' > 0$, $0 < \psi(x) < 1$, and $\lim_{x \to -\infty} \psi(x) = 0$, $\lim_{x \to +\infty} \psi(x) = 1$. To prove Lemma 4 and Proposition 2 (ii), we introduce for $y_0 > 0$,

$$J_{\mathrm{L}}(t) := \int (1 - \psi(\cdot - (x_j(t) - y_0))) u(t)^2, \quad J_{\mathrm{R}}(t) := \int \psi(\cdot - (x_j(t) + y_0)) u(t)^2.$$

Lemma 5 (Monotonicity on the right and on the left of a soliton)**.** $\exists C_1, y_1 > 0$ *s.t. the following is true:* $\forall y_0 > y_1$, $\exists T = T(y_0) > 0$ *s.t. if* $T < t' < t$, *then*

$$J_{\mathrm{L}}(t) \geqslant J_{\mathrm{L}}(t') - C_1 e^{-\gamma_0 y_0}, \quad J_{\mathrm{R}}(t) \leqslant J_{\mathrm{R}}(t') + C_1 e^{-\gamma_0 y_0}.$$

Proof of Proposition 2 (ii). Let $\delta > 0$ be arbitrary. Since $\int R_j^2(t) = c_j^{\frac{5-p}{2(p-1)}}(t) \int Q^2$ and $\varepsilon(t, . + x_j(t)) \to 0$ in L^2_{loc} as $t \to +\infty$, $\exists T_1(\delta) > 0$, $\exists y_1(\delta)$ s.t. $\forall t > T_1(\delta)$, $\forall y_0 > y_1(\delta)$,

$$\left| \int (\psi(x - (x_j(t) - y_0)) - \psi(x - (x_j(t) + y_0))) u(t,x)^2 - c_j(t)^{\frac{5-p}{2(p-1)}} \int Q^2 \right| \leqslant \delta.$$

By Lemma 5, $\exists y_2(\delta) > y_1(\delta)$, $\exists T_2(\delta) > T_1(\delta)$ s.t. if $T_2(\delta) < t' < t$, then

$$J_{\mathrm{L}}(t) \geqslant J_{\mathrm{L}}(t') - \delta, \quad J_{\mathrm{R}}(t) \leqslant J_{\mathrm{R}}(t') + \delta.$$

It follows that $\exists T_3(\delta) > T_2(\delta)$, $\exists J_{\mathrm{L}}^{+\infty} \geqslant 0$, $\exists J_{\mathrm{R}}^{+\infty} \geqslant 0$ s.t. $\forall t \geqslant T_3(\delta)$,

$$|J_{\mathrm{L}}(t) - J_{\mathrm{L}}^{+\infty}| \leqslant 2\delta, \quad |J_{\mathrm{R}}(t) - J_{\mathrm{R}}^{+\infty}| \leqslant 2\delta.$$

Therefore, by conservation of L^2-norm, if $T_3 < t' < t$, then

$$|c_j(t)^{\frac{5-p}{2(p-1)}} - c_j(t')^{\frac{5-p}{2(p-1)}}| \leqslant C\,\delta.$$

Since δ is arbitrary, it follows that $c_j(t)^{\frac{5-p}{2(p-1)}}$ has a limit as $t \to +\infty$, that is, $\exists c_j^{+\infty} > 0$ s.t. $c_j(t) \to c_j^{+\infty}$ as $t \to +\infty$. The fact that $\dot{x}_j(t) \to c_j^{+\infty}$ is a direct consequence of (3). □

3 Proof of Lemmas

3.1 Proof of Lemmas 2 and 3.

First, we prove Lemma 2. Put

$$R(t,x) := \sum_{j=1}^{N} R_j(t,x) = \sum_{j=1}^{N} Q_{c_j(t)}(x - x_j(t)).$$

The following lemma follows from the decay properties of Q and the conservation of the energy E.

Lemma 6 (Energy bounds)**.** *$\exists K_4 > 0$ s.t. $\forall L > L_2$, $\forall t \in [0, t^*]$,*

$$\left|\sum_{j=1}^{N}[E(R_j(t)) - E(R_j(0))] + \frac{1}{2}\int(\varepsilon_x^2 - pR^{p-1}\varepsilon^2)(t)\right| \leqslant K_4(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\sqrt{\sigma_0}L/4}).$$

Recall that

$$\phi(x) = cQ(\sqrt{\sigma_0}x/2), \quad \psi(x) = \int_{-\infty}^{x}\phi(y)\,dy, \quad \text{where } c = \left(\frac{2}{\sqrt{\sigma_0}}\int_{-\infty}^{\infty} Q\right)^{-1}.$$

For $j \geqslant 2$, let

$$\mathcal{I}_j(t) := \int u(t,x)^2\psi(x - m_j(t)), \quad m_j(t) := \frac{x_{j-1}(t) + x_j(t)}{2}.$$

The following lemma is essential for the proof of the stability of sum of N solitons.

Lemma 7 (Almost monotonicity of the mass on the right of each soliton)**.** *$\exists K_5 > 0$, $\exists L_4 > L_3$ s.t. $\forall j \geqslant 2$, $\forall L > L_4$, $\forall t \in [0, t^*]$,*

$$\mathcal{I}_j(t) - \mathcal{I}_j(0) \leqslant K_5 e^{-\sqrt{\sigma_0}L/16}.$$

Proof of Lemma 2. Let $\beta = \frac{2}{p-1}$. First, we show that $\exists C>0$ s.t.

$$\left|\sum_{j=1}^{N} c_j(0)[c_j(t)^{\beta-1/2} - c_j(0)^{\beta-1/2}]\right| \leqslant C(\|\varepsilon(t)\|_{H^1}^2 + \|\varepsilon(0)\|_{H^1}^2 + e^{-\gamma_0 L}) + C\sum_{j=1}^{N}[c_j(t)-c_j(0)]^2. \tag{12}$$

Let us prove (12). By linearization, we have $c_j^{\beta+1/2}(t) - c_j^{\beta+1/2}(0) = \frac{2\beta+1}{2\beta-1}c_j(0)[c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0)] + O([c_j(t)-c_j(0)]^2)$. Since $E(Q_c) = -\frac{\kappa}{2}c^{\beta+1/2}\int Q^2$, where $\kappa = \frac{5-p}{p+3}$, we have

$$\begin{aligned}
&-\sum_{j=1}^{N}[E(R_j(t)) - E(R_j(0))]\\
&= \frac{\kappa}{2}\left(\int Q^2\right)\sum_{j=1}^{N}[c_j(t)^{\beta+1/2} - c_j(0)^{\beta+1/2}]\\
&= \frac{1}{2}\left(\int Q^2\right)\sum_{j=1}^{N} c_j(0)[c_j(t)^{\beta-1/2} - c_j(0)^{\beta-1/2}] + O\Big(\sum_{j=1}^{N}[c_j(t)-c_j(0)]^2\Big),
\end{aligned} \tag{13}$$

where we used $\frac{2\beta+1}{2\beta-1} = \frac{1}{\kappa}$. Therefore, by Lemma 6, we obtain (12).

Let

$$d_j(t) := \sum_{k=j}^{N} c_k(t)^{\beta-1/2}.$$

We claim $\forall j$,

$$\left(\int Q^2\right)|d_j(t) - d_j(0)| \leqslant -\left(\int Q^2\right)(d_j(t)-d_j(0)) + C\left[\int \varepsilon(0)^2 + e^{-\gamma_0 L}\right]. \tag{14}$$

Let us prove (14). Recall that by Lemma 7, we have $\forall j \geqslant 2$,

$$\mathcal{I}_j(t) \leqslant \mathcal{I}_j(0) + K_5 e^{-\gamma_0 L}, \quad \text{where} \quad \mathcal{I}_j(t) = \int \psi(x - m_j(t))u(t,x)^2.$$

Since $\int R_j^2(t) = c_j(t)^{\beta-1/2}\int Q^2$, $\int R_j(t)\varepsilon(t) = 0$, we have

$$\left|\mathcal{I}_j(t) - \left(\int Q^2\right)d_j(t) - \int \psi(.-m_j(t))\varepsilon(t)^2\right| \leqslant Ce^{-\gamma_0 L}.$$

Therefore,

$$\left(\int Q^2\right)(d_j(t)-d_j(0)) \leqslant \int \psi(.-m_j(0))\varepsilon(0)^2 - \int \psi(.-m_j(t))\varepsilon(t)^2 + Ce^{-\gamma_0 L} \tag{15}$$

$$\leqslant \int \psi(. - m_j(0))\varepsilon(0)^2 + Ce^{-\gamma_0 L}. \tag{16}$$

Note that by $\int u(t)^2 = \int u(0)^2$ and

$$\begin{aligned}\int u(t)^2 &= \int R(t)^2 + \int \varepsilon(t)^2 + 2\int R(t)\varepsilon(t)\\ &= \int R(t)^2 + \int \varepsilon(t)^2 = \left(\int Q^2\right) d_1(t) + \int \varepsilon(t)^2 + O(e^{-\gamma_0 L}),\end{aligned}$$

we obtain

$$\left(\int Q^2\right)(d_1(t) - d_1(0)) \leqslant \int \varepsilon(0)^2 - \int \varepsilon(t)^2 + Ce^{-\gamma_0 L} \tag{17}$$

$$\leqslant \int \varepsilon(0)^2 + Ce^{-\gamma_0 L}. \tag{18}$$

Therefore, (16) and (18) implies (14).

By Abel transform, we have

$$\begin{aligned}&\sum_{j=1}^{N} c_j(0)[c_j(t)^{\beta-1/2} - c_j(0)^{\beta-1/2}]\\ &\quad = \sum_{j=1}^{N-1} c_j(0)[d_j(t) - d_{j+1}(t) - (d_j(0) - d_{j+1}(0))] + c_N(0)[d_N(t) - d_N(0)]\\ &\quad = c_1(0)[d_1(t) - d_1(0)] + \sum_{j=2}^{N}(c_j(0) - c_{j-1}(0))(d_j(t) - d_j(0)).\end{aligned} \tag{19}$$

Therefore, by (12),

$$\begin{aligned}&\left| c_1(0)[d_1(t) - d_1(0)] + \sum_{j=2}^{N}(c_j(0) - c_{j-1}(0))(d_j(t) - d_j(0)) \right|\\ &\qquad \leqslant C(\|\varepsilon(t)\|_{H^1}^2 + \|\varepsilon(0)\|_{H^1}^2 + e^{-\gamma_0 L}) + C\sum_{j=1}^{N}[c_j(t) - c_j(0)]^2.\end{aligned} \tag{20}$$

Since $c_1(0) \geqslant \sigma_0$, $c_j(0) - c_{j-1}(0) \geqslant \sigma_0$, by (14) and (20), we have

$$\begin{aligned}\sigma_0 \sum_{j=1}^{N} |d_j(t) - d_j(0)| &\leqslant c_1(0)|d_1(t) - d_1(0)| + \sum_{j=2}^{N}(c_j(0) - c_{j-1}(0))|d_j(t) - d_j(0)|\\ &\leqslant -\left[c_1(0)[d_1(t) - d_1(0)] + \sum_{j=2}^{N}(c_j(0) - c_{j-1}(0))(d_j(t) - d_j(0)) \right]\end{aligned}$$

$$
\begin{aligned}
&+C\int \varepsilon(0)^2 + Ce^{-\gamma_0 L}\\
&\leqslant C(\|\varepsilon(t)\|_{H^1}^2 + \|\varepsilon(0)\|_{H^1}^2 + e^{-\gamma_0 L}) + C\sum_{j=1}^{N}[c_j(t)-c_j(0)]^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
|c_j(t)-c_j(0)| &\leqslant C|c_j(t)^{\beta-1/2} - c_j(0)^{\beta-1/2}|\\
&\leqslant C(|d_j(t)-d_j(0)| + |d_{j+1}(t)-d_{j+1}(0)|),
\end{aligned}
$$

we obtain,

$$
\sum_{j=1}^{N}|c_j(t)-c_j(0)| \leqslant C(\|\varepsilon(t)\|_{H^1}^2 + \|\varepsilon(0)\|_{H^1}^2 + e^{-\gamma_0 L}) + C\sum_{j=1}^{N}[c_j(t)-c_j(0)]^2.
$$

Choosing a smaller $\alpha_0(A_0)$ and a larger $L_0(A_0)$, by (7), we assume $C|c_j(t)-c_j(0)| \leqslant 1/2$, and thus, Lemma 2 is proved. □

Next, we prove Lemma 3. It is well known that $\exists\lambda_1 > 0$ s.t. if $v \in H^1(\mathbb{R})$ satisfies $\int Qv = \int Q_x v = 0$, then

$$
\int (v_x^2 - pQ^{p-1}v^2 + v^2) \geqslant \lambda_1 \|v\|_{H^1}^2 \tag{21}
$$

(See proof of Proposition 2.9 in Weinstein [7]). By using the local version of (21) (see [3]), we can obtain the following lemma.

Lemma 8 (Positivity of the quadratic form)**.** *$\exists L_0 > L_4$, $\exists\lambda_0 > 0$ s.t. if $\forall j$, $c_j(t) \geqslant \sigma_0$, $x_j(t) \geqslant x_{j-1}(t) + L_0$, then $\forall t \in [0, t_0]$,*

$$
\int (\varepsilon_x(t)^2 - pR(t)^{p-1}\varepsilon(t)^2 + c(t,x)\varepsilon(t)^2) \geqslant \lambda_0 \|\varepsilon(t)\|_{H^1}^2,
$$

where $c(t,x) = c_1(t) + \sum_{j=2}^{N}(c_j(t)-c_{j-1}(t))\psi(x-m_j(t))$.

Proof of Lemma 3. By Lemma 6, (13), (19), and Lemma 2, we have

$$
\begin{aligned}
&\frac{1}{2}\int (\varepsilon_x(t)^2 - pR(t)^{p-1}\varepsilon(t)^2)\\
&\leqslant -\sum_{j=1}^{N}[E(R_j(t)) - E(R_j(0))] + K_5(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\gamma_0 L})\\
&\leqslant \frac{1}{2}\left(\int Q^2\right)\sum_{j=1}^{N} c_j(0)[c_j(t)^{\beta-1/2} - c_j(0)^{\beta-1/2}] + C\sum_{j=1}^{N}[c_j(t)-c_j(0)]^2
\end{aligned}
$$

$$
\begin{aligned}
&+ K_5(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\gamma_0 L}) \\
&\leqslant \frac{1}{2}\left(\int Q^2\right)\left[c_1(0)(d_1(t)-d_1(0)) + \sum_{j=2}^{N}(c_j(0)-c_{j-1}(0))(d_j(t)-d_j(0))\right] \\
&+ C(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\gamma_0 L}).
\end{aligned}
$$

Therefore, by (15),(17), and Lemma 2, we have

$$
\begin{aligned}
&\int (\varepsilon_x(t)^2 - pR(t)^{p-1}\varepsilon(t)^2) \\
&\leqslant -\left(c_1(0)\int \varepsilon(t)^2 + \sum_{j=2}^{N}(c_j(0)-c_{j-1}(0))\int \psi(x-m_j(t))\varepsilon(t)^2\right) \\
&\quad + C(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\gamma_0 L}) \\
&\leqslant -\left(c_1(t)\int \varepsilon(t)^2 + \sum_{j=2}^{N}(c_j(t)-c_{j-1}(t))\int \psi(x-m_j(t))\varepsilon(t)^2\right) \\
&\quad + C\left(\int \varepsilon(t)^2\right)\sum_{j=1}^{N}|c_j(t)-c_j(0)| + C(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\gamma_0 L}) \\
&\leqslant -\int c(t,x)\varepsilon(t)^2 + C(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\gamma_0 L}) \qquad (22)
\end{aligned}
$$

where $c(t,x) = c_1(t) + \sum_{j=2}^{N}(c_j(t)-c_{j-1}(t))\psi(x-m_j(t))$. By Lemma 8 and (22), we obtain

$$
\|\varepsilon(t)\|_{H^1}^2 \leqslant C(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\gamma_0 L}).
$$

Choosing a smaller $\alpha_0(A_0)$ and a larger $L_0(A_0)$, by (7), we assume $C\|\varepsilon(t)\|_{H^1} \leqslant 1/2$, and thus, Lemma 3 is proved. □

3.2 Proof of Lemmas 4 and 5

We prove Lemmas 4 and 5.

Proof of Lemma 5. Since ψ is monotonically increasing, we have

$$
\int u(t)^2 - J_{\mathrm{L}}(t) = \int \psi(\cdot-(x_j(t)-y_0))u(t)^2 \leqslant \int \psi(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t')))u(t)^2. \quad (23)
$$

By using (gKdV), since $\psi''' \leqslant \frac{\sigma_0}{4}\psi'$ and $\dot{x}_j(t) \geqslant \sigma_0$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int \psi(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t')))u(t)^2 \\
&= \int \left(-3u_x(t)^2 - (\dot{x}_j(t)-\tfrac{\sigma_0}{2})u(t)^2 - \frac{2p}{p+1}u(t)^{p+1}\right)\psi'(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t')))
\end{aligned}
$$

$$
\begin{aligned}
&+\int u(t)^2\psi'''(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t'))) \\
&\leqslant -\frac{\sigma_0}{4}\int u(t)^2\psi'(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t'))) \\
&+\int u(t)^{p+1}\psi'(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t'))).
\end{aligned}
\tag{24}
$$

Since $\dot{x}_j(t)-\dot{x}_{j-1}(t)\geqslant\sigma_0$, $\exists T=T(y_0)>0$ s.t. if $T<t'<t$, then

$$
x_{j-1}(t)+y_0\leqslant x_j(t)-y_0-\frac{\sigma_0}{2}(t-t')\leqslant x_j(t)-y_0. \tag{25}
$$

Let $I:=[x_{j-1}(t)+y_0/2,x_j(t)-y_0/2]$. Then $\forall x\in I$, since $\forall k\in\{1,\dots,N\}$, $|x-x_k(t)|\geqslant y_0/2$, we have $|u(t,x)|^{p-1}\leqslant\frac{\sigma_0}{4}$ for large y_0, and so

$$
\int_I u^{p+1}\psi'(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t')))\leqslant\frac{\sigma_0}{4}\int_I u^2\psi'(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t'))). \tag{26}
$$

Moreover, since by $\dot{x}_j(t)-\dot{x}_{j-1}(t)\geqslant\sigma_0$ and (25), $x_j(t)-x_{j-1}(t)\geqslant 2y_0+\sigma(t-t')$, we have

$$
\sup_{I^c}\psi'(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t')))\leqslant\psi'\left(\frac{y_0}{2}+\frac{\sigma_0}{2}(t-t')\right)\leqslant Ce^{-\frac{\sqrt{\sigma}}{4}(y_0+\sqrt{\sigma_0}(t-t'))}. \tag{27}
$$

Therefore, by (24), (26), and (27), we have

$$
\begin{aligned}
\int\psi(\cdot-(x_j(t)-y_0-\tfrac{\sigma_0}{2}(t-t')))u(t)^2&\leqslant\int\psi(\cdot-(x_j(t')-y_0))u(t')^2+C_1e^{-\gamma_0y_0}\\
&=\int u(t')^2-J_{\mathrm{L}}(t')+C_1e^{-\gamma_0y_0},
\end{aligned}
$$

and so by $\int u(t)^2=\int u(t')^2$ and (23), we have the estimate for J_{L}. Similarly, we can obtain the estimate for J_{R}. □

Proof of Lemma 4. Recall from [2] that we have stability of (gKdV) by weak convergence in $H^1(\mathbb{R})$ in the following sense

$$
\forall t\in\mathbb{R},\quad u(t+t_n,\cdot+x_j(t+t_n))\to\tilde{u}(t,\cdot+\tilde{x}(t))\quad\text{in } L^2_{\mathrm{loc}}(\mathbb{R})\text{ as } n\to+\infty. \tag{28}
$$

We prove Lemma 4 by contradiction. Let

$$
m_0:=\int\tilde{u}(0)^2=\int\tilde{u}(t)^2,\quad m_1:=\int u(0)^2=\int u(t)^2.
$$

Assume that Lemma 4 does not hold. Then $\exists\delta_0>0$ s.t. $\forall y_0>0$, $\exists t_0(y_0)\in\mathbb{R}$, s.t.

$$
\int_{|x|<2y_0}\tilde{u}(t_0(y_0),x+\tilde{x}(t_0(y_0)))^2\leqslant m_0-\delta_0. \tag{29}
$$

Fix $y_0 > 0$ large enough so that

$$\int (\psi(x+y_0) - \psi(x-y_0))\tilde{u}(0,x)^2 \geqslant m_0 - \frac{1}{10}\delta_0, \tag{30}$$

$$(m_0+m_1) \sup_{|x|>2y_0} \{\psi(x+y_0) - \psi(x-y_0)\} \leqslant \frac{1}{10}\delta_0. \tag{31}$$

Assume that $t_0 = t_0(y_0) > 0$ and, by possibly considering a subsequence of (t_n), that $\forall n$, $t_{n+1} \geqslant t_n + t_0$.

Since $0 < \psi < 1$ and $\psi' > 0$, by (31) and (29), we have

$$\begin{aligned} &\int (\psi(x-(\tilde{x}(t_0)-y_0)) - \psi(x-(\tilde{x}(t_0)+y_0)))\tilde{u}(t_0,x)^2 \\ &\leqslant \int_{|x|<2y_0} \tilde{u}(t_0, x+\tilde{x}(t_0))^2 + m_0 \sup_{|x|>2y_0} \{\psi(x+y_0) - \psi(x-y_0)\} \\ &\leqslant \int_{|x|<2y_0} \tilde{u}(t_0, x+\tilde{x}(t_0))^2 + \frac{1}{10}\delta_0 \leqslant m_0 - \frac{9}{10}\delta_0. \end{aligned} \tag{32}$$

Then, by (30), (32) and (28), $\exists N_0 > 0$ s.t. $\forall n \geqslant N_0$,

$$\int (\psi(x-(x_j(t_n)-y_0)) - \psi(x-(x_j(t_n)+y_0)))u(t_n,x)^2 \geqslant m_0 - \frac{1}{5}\delta_0, \tag{33}$$

$$\int (\psi(x-(x_j(t_n+t_0)-y_0)) - \psi(x-(x_j(t_n+t_0)+y_0)))u(t_n+t_0,x)^2 \leqslant m_0 - \frac{4}{5}\delta_0. \tag{34}$$

By Lemma 5 and the choice of y_0, we have $J_{\mathrm{R}}(t_n+t_0) \leqslant J_{\mathrm{R}}(t_n) + \frac{1}{10}\delta_0$. Therefore, by conservation of the L^2-norm and (34), (33), we have

$$J_{\mathrm{L}}(t_n+t_0) \geqslant J_{\mathrm{L}}(t_n) + \frac{1}{2}\delta_0.$$

Since $J_{\mathrm{L}}(t_{n+1}) \geqslant J_{\mathrm{L}}(t_n+t_0) - \frac{1}{10}\delta_0$ by Lemma 5, we finally obtain

$$\forall n \geqslant N_0, \quad J_{\mathrm{L}}(t_{n+1}) \geqslant J_{\mathrm{L}}(t_n) + \frac{2}{5}\delta_0.$$

This contradicts the fact that $\forall t > 0$, $J_{\mathrm{L}}(t) \leqslant m_1$. This completes the proof. □

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