SUMMARY NOTES:

I-METHOD TO STUDY GLOBAL WELL-POSEDNESS FOR DISPERSIVE EQUATION IN LOW REGULARITY

1. Introduction

In the paper [3], the authors Colliander, Keel, Staffilani, Takaoka, and Tao(I-team) considered the Cauchy problem for the Korteweg-de Vries equations:

$$\begin{cases} \partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x(u^2) = 0, \quad (x,t) \in \mathbb{R} \times [0,T], \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}), \end{cases} \tag{1.1}$$

where u is an unknown real function defined on $\mathbb{R} \times [0,T]$. In this paper, they introduced the I-method to study the global well-posedness for dispersive equations in low regularity. They proved ***the global well-posedness of*** (1.1) ***in*** $H^s(\mathbb{R})$ ***for any*** $s > -\frac{3}{4}$ in this paper. This result is sharp expect the endpoint case. See [5, 7] for endpoint case, see [2] for ill-posedness for $s < -\frac{3}{4}$ in sense of uniformly continuous.

The problem (1.1) obeys the mass conservation law,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}.$$

However, for the global well-posedness in $H^s, s < 0$, we need *a priori* estimate on

$$\|u(t)\|_{H^s}.$$

To this end, we apply the I-operator.

Definition 1.1 (I-operator)**.** *Let $N \gg 1$ be fixed, and the Fourier multiplier operator $I = I_{N,s}$ be defined as*

$$\widehat{If}(\xi) = m(\xi)\hat{f}(\xi). \tag{1.2}$$

Here the multiplier $m = m_{N,s}(\xi)$ is a smooth, monotone function satisfying $0 < m(\xi) \leq 1$ and

$$m(\xi) = \begin{cases} 1, & |\xi| \leq N, \\ N^{1-s}|\xi|^{s-1}, & |\xi| > 2N. \end{cases} \tag{1.3}$$

One may find that

- $I : H^s(\mathbb{R}) \mapsto L^2(\mathbb{R})$. Indeed,

$$\|f\|_{H^s} \lesssim \|If\|_{L^2} \lesssim N^{-s}\|f\|_{H^s}. \tag{1.4}$$

- I is an identity approximate operator: $I \to Id$, as $N \to \infty$.

Using I-operator, we have

$$\begin{cases} \partial_t Iu + \partial_x^3 Iu + \frac{1}{2}\partial_x I(u^2) = 0, \\ Iu(0,x) = Iu_0(x) \in L^2(\mathbb{R}), \end{cases}$$

Also, it is reasonable to establish

$$\|Iu(t)\|_{L^2}^2 = \|Iu_0\|_{L^2}^2 + O(N^{-\beta}) \quad \text{for local time.} \quad \text{(Almost conservation law)} \tag{1.5}$$

If (1.5) is done, then the global result can be obtained by iteration. Moreover, larger β leads better global result.

2. Preliminary

2.1. Bourgain Space.

Definition 2.1 (Bourgain Space). *We define the space $X_{s,b}$ by the norm,*

$$\begin{aligned}\|u\|_{X_{s,b}} &\triangleq \left\| \langle\xi\rangle^s \langle\tau - \xi^3\rangle^b \hat{u}(\xi,\tau) \right\|_{L^2_{\xi\tau}} \\ &= \left\| e^{t\partial_{xxx}} u \right\|_{H^s_x H^b_t}.\end{aligned}$$

The definition is not important to study the global theory (in general), we only focus our attention on its properties.

Lemma 2.1.
- *(Strichartz estimate)For $\frac{1}{p} + \frac{1}{2q} = \frac{1}{4}, \alpha = \frac{2}{q} - \frac{1}{p}$, $1 \le p,q \le \infty$, $-\frac{1}{4} \le \alpha \le 1$,*

$$\left\| D^\alpha (e^{t\partial_{xxx}} u_0) \right\|_{L^p_x L^q_t} \lesssim \|u_0\|_{L^2}.$$

- *(Bilinear Strichartz estimate) Let*

$$\widehat{I(f,g)}(\xi) = \int_{\xi_1+\xi_2=\xi} \left| \xi_1^2 - \xi_2^2 \right|^{\frac{1}{2}} \hat{f}(\xi_1) g(\xi_2)\, d\xi_1,$$

then

$$\left\| I(e^{t\partial_{xxx}} u_0, e^{t\partial_{xxx}} v_0) \right\|_{L^2_{xt}} \lesssim \|u_0\|_{L^2} \|v_0\|_{L^2}.$$

Then we have

Corollary 2.1.

$$\left\| D^\alpha u \right\|_{L^p_x L^q_t} \lesssim \|u\|_{X_{0,\frac{1}{2}+}}; \tag{2.1}$$

$$\left\| I(u,v) \right\|_{L^2_{xt}} \lesssim \|u\|_{X_{0,\frac{1}{2}+}} \|v\|_{X_{0,\frac{1}{2}+}}. \tag{2.2}$$

2.2. Normalization.

Iu_0 is too large, indeed, by (1.4),

$$\|Iu_0\|_{L^2} \lesssim N^{-s} \|u_0\|_{H^s}.$$

We normalize it by rescaling. Note that if u is a solution of (1.1), then so is

$$u_\lambda(x,t) = \lambda^{-2} u\Big(\frac{x}{\lambda}, \frac{t}{\lambda^3}\Big), \quad \text{with } u_{0,\lambda}(x) = \lambda^{-2} u_0\Big(\frac{x}{\lambda}\Big). \tag{2.3}$$

Moreover, u exists on $[0,T]$ if and only if u_λ exists $[0,\lambda^3 T]$. Now by rescaling, and I-operator, we consider the following problem for instead,

$$\begin{cases} \partial_t I u_\lambda + \partial_x^3 I u_\lambda + \dfrac{1}{2} I \partial_x (u_\lambda^2) = 0, \\ I u_\lambda(0,x) = I u_{0,\lambda}(x) \in L^2(\mathbb{R}). \end{cases} \tag{2.4}$$

In particular,

$$\|Iu_{0,\lambda}\|_{L^2} \lesssim N^{-s} \|u_{0,\lambda}\|_{H^s} \lesssim N^{-s} \lambda^{-\frac{3}{2}-s}.$$

Therefore, fixing small $\varepsilon_0 > 0$, selecting suitably

$$\lambda \sim N^{-\frac{2s}{3+2s}}, \tag{2.5}$$

we have

$$\|Iu_{0,\lambda}\|_{L^2} \le \varepsilon_0.$$

From now on, we drop the subscript λ.

2.3. **Local theorey for rescaled problem** (2.4).

Lemma 2.2. *Let $s > -\frac{3}{4}$, $\|Iu(t_0)\|_{L^2} \le 2\varepsilon_0$, then the problem* (2.4) *is local well-posedness on the time interval $[t_0, t_0+1]$. Moreover,*

$$\|Iu\|_{X_{0,\frac{1}{2}+}([t_0,t_0+1])} \lesssim 1.$$

Proof. It is standard. One may prove it by multilinear estimates as the manner of the original problem (1.1), see cf. [6]. It may be necessary in the study of the problem lack of scaling invariance (fine estimate on the lifespan should be considered). One also may prove it by using the multilinear estimate from [6] and a general estimate on I obtained in [4]. $\square$

3. Almost conservation law

Consider (1.5). Indeed, by the equation (2.4), we have

$$\begin{aligned}
\frac{d}{dt}\|Iu(t)\|_{L^2}^2 &= \int I(u^2)\cdot \partial_x Iu\, dx \\
&= \int_{\xi_1+\xi_2+\xi_3=0} m(\xi_1)^2\xi_1 \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\, d\xi_1 d\xi_2 \\
&= \int_{\xi_1+\xi_2+\xi_3=0} \frac{1}{3}\big[m(\xi_1)^2\xi_1 + m(\xi_2)^2\xi_2 + m(\xi_3)^2\xi_3\big]\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\, d\xi_1 d\xi_2. \qquad (3.1)
\end{aligned}$$

Here we have symmetrized the multipliers, which is important. For simplicity, we denote it as $[\cdot]_{sym}$, for example,

$$[m(\xi_1)^2\xi_1]_{sym} = \frac{1}{3}\big[m(\xi_1)^2\xi_1 + m(\xi_2)^2\xi_2 + m(\xi_3)^2\xi_3\big].$$

Except symmetry, it has two other advantages:

- $[m(\xi_1)^2\xi_1]_{sym} = 0$, when $|\xi_1|, |\xi_2|, |\xi_3| \le N$.
- It gives a lower upper bound than non-symmetry form. Indeed, assume that $|\xi_1| \ge |\xi_2| \ge |\xi_3|$ by symmetries, then $\big|[m(\xi_1)^2\xi_1]_{sym}\big| \lesssim m^2(\xi_3)|\xi_3|$.

By (3.1), we have

$$\|Iu(t)\|_{L^2}^2 = \|Iu(t_0)\|_{L^2}^2 + \int_{t_0}^{t}\int_{\xi_1+\xi_2+\xi_3=0} [m(\xi_1)^2\xi_1]_{sym}\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\, d\xi_1 d\xi_2 ds.$$

Therefore, to establish the type estimate as (1.5), the key is to estimate

$$\int_{t_0}^{t_0+1}\int_{\xi_1+\xi_2+\xi_3=0} [m(\xi_1)^2\xi_1]_{sym}\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\, d\xi_1 d\xi_2 ds \triangleq II.$$

Lemma 3.1. *Suppose that* $\|Iu\|_{X_{0,\frac{1}{2}+}[t_0,t_0+1]} \lesssim 1$, *then*

$$|II| \lesssim N^{-\frac{3}{4}+}.$$

Proof. By several reductions (see [3] or [1]), it is sufficient to show

$$\Big| \int_{\substack{\xi_1+\xi_2+\xi_3=0\\ \tau_1+\tau_2+\tau_3=0}} [m(\xi_1)^2\xi_1]_{sym}\hat{u}(\xi_1,\tau_1)\hat{u}(\xi_2,\tau_2)\hat{u}(\xi_3,\tau_3)\, d\xi_1 d\xi_2\, d\tau_1 d\tau_2 \Big| \lesssim N^{-\frac{3}{4}+}\|Iu\|^3_{X_{0,\frac{1}{2}+}}. \quad (3.2)$$

We may assume $\hat{u}(\xi,\tau)$ is positive,since it is invariance under $X_{s,b}$-norm. Assume $|\xi_1| \geq |\xi_2| \geq |\xi_3|$ by symmetries and $|\xi_1| \sim |\xi_2| \gtrsim N$, then

$$\begin{aligned}
\text{LHS of (3.2)} &\lesssim \int_{\substack{\xi_1+\xi_2+\xi_3=0\\ \tau_1+\tau_2+\tau_3=0}} m^2(\xi_3)|\xi_3|\hat{u}(\xi_1,\tau_1)\hat{u}(\xi_2,\tau_2)\hat{u}(\xi_3,\tau_3)\, d\xi_1 d\xi_2\, d\tau_1 d\tau_2 \\
&\lesssim \sum_{N_1\geq N_2\geq N_3} N_3 \int_{\mathbb{R}^2} u_1(x,t)u_2(x,t)Iu_3(x,t)\, dxdt,
\end{aligned}$$

where we denote $u_j = P_{N_j}u$. Now we write

$$u_j = u_j^l + u_j^h, \qquad \text{where } u_j^h = u_j\chi_{|\tau-\xi^3|\gtrsim N_1N_2N_3}.$$

Note that we have

$$\|u_j^h\|_{L^2_{xt}} \lesssim (N_1N_2N_3)^{-\frac{1}{2}}\|u_j\|_{X_{0,\frac{1}{2}}}.$$

Since

$$|\tau_1 - \xi_1^3 + \tau_2 - \xi_2^3 + \tau_3 - \xi_3^3| = 3|\xi_1||\xi_2||\xi_3| \sim N_1N_2N_3.$$

there is at least one of $j = 1,2,3$ such that $u_j = u_j^h$. We only consider the case, $u_1 = u_1^h$, then by Strichartz estimates,

$$\begin{aligned}
\text{LHS of (3.2)} &\lesssim \sum_{N_1\geq N_2\geq N_3} N_3\|u_1^h\|_{L^2_{xt}}\|u_2\|_{L^4_{xt}}\|Iu_3\|_{L^4_{xt}} \\
&\lesssim \sum_{N_1\geq N_2\geq N_3} N_3(N_1N_2N_3)^{-\frac{1}{2}}\|u_1^h\|_{X_{0,\frac{1}{2}}}N_2^{-\frac{1}{8}}\|u_2\|_{X_{0,\frac{3}{8}+}}N_3^{-\frac{1}{8}}\|Iu_3\|_{X_{0,\frac{3}{8}+}} \\
&\lesssim N^{2s}\sum_{N_1\geq N_2\geq N_3} N_1^{-\frac{1}{2}-s}N_2^{-\frac{5}{8}-s}N_3^{\frac{3}{8}}\|u_1\|_{X_{0,\frac{1}{2}+}}\|u_2\|_{X_{0,\frac{1}{2}+}}\|Iu_3\|_{X_{0,\frac{1}{2}+}} \\
&\lesssim N^{-\frac{3}{4}+}.
\end{aligned}$$

□

Now suppose that

$$\sup_{t\in[0,J]} \|Iu(t)\|_{L^2} \leq 2\varepsilon_0, \quad (3.3)$$

then by Lemma 2.2, $\|Iu\|_{X_{0,\frac{1}{2}+}[t_0,t_0+1]} \lesssim 1$ for any $t_0 \in [0,J]$, and thus by Lemma 3.1,

$$\|Iu(t)\|^2_{L^2} \leq \|Iu(0)\|^2_{L^2} + CJN^{-\frac{3}{4}+}, \qquad \text{for any } J \leq t \leq J+1.$$

Thus for any $J \leq N^{\frac{3}{4}-}$,

$$\sup_{t\in[0,J+1]} \|Iu(t)\|_{L^2} \leq 2\varepsilon_0.$$

This extends (3.3). By iteration, we obtain that the solution of (2.4) exists on $[0, N^{\frac{3}{4}-}]$. This implies that the solution of (1.1) exists on $[0,T]$ for $T = \lambda^3N^{\frac{3}{4}-}$. Suppose that $T \geq N^{0+}$, then u exists for arbitrary time by choosing large N, and proves the global well-posedness. Now

$$T \geq N^{0+} \Leftrightarrow c\lambda^3 N^{3+\frac{3}{4}-} = cN^{-\frac{6s}{3+2s}} N^{\frac{3}{4}-} \geq N^{0+},$$

which holds for any $s > -\frac{3}{10}$. But this is far away from expectation. To improve the result, we need some new ideas.

4. Modified energies

Consider again,

$$\frac{d}{dt}\|Iu(t)\|_{L^2}^2 = \int_{\xi_1+\xi_2+\xi_3=0} \big[m(\xi_1)^2\xi_1\big]_{sym} \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\, d\xi_1 d\xi_2. \tag{4.1}$$

Let $f(x,t) = e^{-t\partial_{xxx}}u(x,t)$, then

$$\hat{u}(\xi,t) = e^{-i\xi^3 t}\hat{f}(\xi,t); \quad \partial_t f(x,t) = -e^{-t\partial_{xxx}}\frac{1}{2}\partial_x(u^2). \tag{4.2}$$

By this, we rewrite

$$\text{LHS of (4.1)} = \int_{\xi_1+\xi_2+\xi_3=0} [m(\xi_1)^2\xi_1]_{sym} e^{-i\alpha_3 t}\hat{f}(\xi_1)\hat{f}(\xi_2)\hat{f}(\xi_3)\, d\xi_1 d\xi_2, \tag{4.3}$$

where $\alpha_k = \xi_1^3 + \cdots + \xi_k^3$. In particular,

$$\alpha_3 = -3\xi_1\xi_2\xi_3;$$
$$\alpha_4 = 3(\xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_4 + \xi_1\xi_3\xi_4 + \xi_2\xi_3\xi_4) = 3(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_1+\xi_4).$$

Now we use the identity

$$e^{-i\alpha_3 t} = \frac{1}{-i\alpha_3}\partial_t\big(e^{-i\alpha_3 t}\big).$$

Then by (4.2), we have

$$\begin{aligned}
(4.3) = &\int_{\xi_1+\xi_2+\xi_3=0} \frac{[m(\xi_1)^2\xi_1]_{sym}}{-i\alpha_3}\partial_t\big(e^{-i\alpha_3 t}\big)\hat{f}(\xi_1)\hat{f}(\xi_2)\hat{f}(\xi_3)\, d\xi_1 d\xi_2 \\
=&\partial_t \int_{\xi_1+\xi_2+\xi_3=0} \frac{[m(\xi_1)^2\xi_1]_{sym}}{-i\alpha_3} e^{-i\alpha_3 t}\hat{f}(\xi_1)\hat{f}(\xi_2)\hat{f}(\xi_3)\, d\xi_1 d\xi_2 \\
&- \int_{\xi_1+\xi_2+\xi_3=0} \frac{[m(\xi_1)^2\xi_1]_{sym}}{-i\alpha_3} e^{-i\alpha_3 t}\partial_t\big[\hat{f}(\xi_1)\hat{f}(\xi_2)\hat{f}(\xi_3)\big]\, d\xi_1 d\xi_2 \\
=&\partial_t \int_{\xi_1+\xi_2+\xi_3=0} \frac{[m(\xi_1)^2\xi_1]_{sym}}{-i\alpha_3}\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\, d\xi_1 d\xi_2 \\
&+ \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} M_4(\xi_1,\xi_2,\xi_3,\xi_4)\hat{u}(\xi_1)\cdots\hat{u}(\xi_4)\, d\xi_1 d\xi_2 d\xi_3,
\end{aligned}$$

where

$$M_4(\xi_1,\cdots,\xi_4) = \Big[\frac{m(\xi_1)^2\xi_1 + m(\xi_2)^2\xi_2 + m(\xi_3+\xi_4)^2(\xi_3+\xi_4)}{i\alpha_3(\xi_1,\xi_2,\xi_3+\xi_4)}(\xi_3+\xi_4)\Big]_{sym}.$$

In this precess, one should check that

$$\frac{[m(\xi_1)^2\xi_1]_{sym}}{\alpha_3} \quad \text{makes sense!}$$

We define a new modified energy $E_I^2(t)$ by

$$E_I^2(t) = \|Iu(t)\|_{L^2}^2 - \int_{\xi_1+\xi_2+\xi_3=0} \frac{[m(\xi_1)^2\xi_1]_{sym}}{-i\alpha_3} \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\, d\xi_1 d\xi_2.$$

Then

$$\frac{d}{dt}E_I^2(t) = \int_{\xi_1+\cdots+\xi_4=0} M_4(\xi_1,\xi_2,\xi_3,\xi_4)\hat{u}(\xi_1)\cdots\hat{u}(\xi_4)\, d\xi_1 d\xi_2 d\xi_3.$$

Lemma 4.1.

$$\Big|\int_{\xi_1+\xi_2+\xi_3=0} \frac{[m(\xi_1)^2\xi_1]_{sym}}{-i\alpha_3} \hat{u}(\xi_1,t)\hat{u}(\xi_2,t)\hat{u}(\xi_3,t)\, d\xi_1 d\xi_2\Big| \lesssim N^{0-}\|Iu(t)\|_{L^2}^3.$$

Proof. Assume $|\xi_1| \geq |\xi_2| \geq |\xi_3|$ by symmetries and $|\xi_1| \sim |\xi_2| \gtrsim N$, then

$$\Big|\frac{[m(\xi_1)^2\xi_1]_{sym}}{-i\alpha_3}\Big| \lesssim \frac{m(\xi_3)^2|\xi_3|}{|\xi_1||\xi_2||\xi_3|} \lesssim m(\xi_3)|\xi_1|^{-1}|\xi_2|^{-1}.$$

We assume that $\hat{u}(\xi,t)$ is positive, then

$$\begin{aligned}
&\Big|\int_{\xi_1+\xi_2+\xi_3=0} \frac{[m(\xi_1)^2\xi_1]_{sym}}{-i\alpha_3} \hat{u}(\xi_1,t)\hat{u}(\xi_2,t)\hat{u}(\xi_3,t)\, d\xi_1 d\xi_2\Big| \\
\lesssim&\Big|\int \Big[P_{\gtrsim N}D^{-1}u(x,t)\Big]^2 Iu(x,t)\, dx\Big| \\
\lesssim&\|P_{\gtrsim N}D^{-1}u\|_{L_x^4}^2 \|Iu\|_{L_x^2} \\
\lesssim&N^{-\frac{1}{2}}\|Iu(t)\|_{L^2}^3.
\end{aligned}$$

∎

This lemma implies that

$$\Big|E_I^2(t) - \|Iu(t)\|_{L^2}^2\Big| \lesssim N^{0-}\|Iu(t)\|_{L^2}^3. \tag{4.4}$$

Lemma 4.2 (Pointwise estimate on Multiplier)**.**

$$\big|M_4(\xi_1,\cdots,\xi_4)\big| \lesssim \frac{|\alpha_4|}{(|\xi_1|+N)(|\xi_2|+N)(|\xi_3|+N)(|\xi_4|+N)}.$$

Proof. Since $\alpha_3 = -3\xi_1\xi_2\xi_3$, then

$$\begin{aligned}
M_4(\xi_1,\cdots,\xi_4) =& c\Big[\frac{m(\xi_1)^2\xi_1 + m(\xi_2)^2\xi_2 + m(\xi_3+\xi_4)^2(\xi_3+\xi_4)}{\xi_1\xi_2}\Big]_{sym} \\
=& 2c\Big[\frac{m(\xi_1)^2}{\xi_2}\Big]_{sym} + c\Big[\frac{m(\xi_3+\xi_4)^2(\xi_3+\xi_4)}{\xi_1\xi_2}\Big]_{sym}.
\end{aligned}$$

First,

$$\begin{aligned}\Big[\frac{m(\xi_1)^2}{\xi_2}\Big]_{sym} =&\frac{1}{3}\Big[m(\xi_1)^2\big(\frac{1}{\xi_2}+\frac{1}{\xi_3}+\frac{1}{\xi_4}\big)\Big]_{sym}\\ =&\frac{1}{3}\Big[m(\xi_1)^2\big(\frac{1}{\xi_1}+\frac{1}{\xi_2}+\frac{1}{\xi_3}+\frac{1}{\xi_4}\big)\Big]_{sym}-\frac{1}{3}\Big[\frac{m(\xi_1)^2}{\xi_1}\Big]_{sym}\\ =&\frac{1}{36}\frac{\alpha_4}{\xi_1\xi_2\xi_3\xi_4}\Big[m(\xi_1)^2+m(\xi_2)^2+m(\xi_3)^2+m(\xi_4)^2\Big]\\ &-\frac{1}{12}\Big[\frac{m(\xi_1)^2}{\xi_1}+\frac{m(\xi_2)^2}{\xi_2}+\frac{m(\xi_3)^2}{\xi_3}+\frac{m(\xi_4)^2}{\xi_4}\Big].\end{aligned}$$

Second,

$$\begin{aligned}\Big[\frac{m(\xi_3+\xi_4)^2(\xi_3+\xi_4)}{\xi_1\xi_2}\Big]_{sym} =&-\Big[\frac{m(\xi_1+\xi_2)^2(\xi_1+\xi_2)}{\xi_1\xi_2}\Big]_{sym}\\ =&-\frac{1}{2}\Big[\frac{m(\xi_1+\xi_2)^2(\xi_1+\xi_2)}{\xi_1\xi_2}+\frac{m(\xi_3+\xi_4)^2(\xi_3+\xi_4)}{\xi_3\xi_4}\Big]_{sym}\\ =&-\frac{1}{2}\Big[m(\xi_1+\xi_2)^2\big(\frac{1}{\xi_1}+\frac{1}{\xi_2}+\frac{1}{\xi_3}+\frac{1}{\xi_4}\big)\Big]_{sym}\\ =&-\frac{1}{18}\frac{\alpha_4}{\xi_1\xi_2\xi_3\xi_4}\Big[m(\xi_1+\xi_2)^2+m(\xi_1+\xi_3)^2+m(\xi_1+\xi_4)^2\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}M_4(\xi_1,\cdots,\xi_4)=&c\frac{\alpha_4}{\xi_1\xi_2\xi_3\xi_4}\Big[m(\xi_1)^2+m(\xi_2)^2+m(\xi_3)^2+m(\xi_4)^2-m(\xi_1+\xi_2)^2-m(\xi_1+\xi_3)^2\\ &-m(\xi_1+\xi_4)^2\Big]-3c\Big[\frac{m(\xi_1)^2}{\xi_1}+\frac{m(\xi_2)^2}{\xi_2}+\frac{m(\xi_3)^2}{\xi_3}+\frac{m(\xi_4)^2}{\xi_4}\Big].\end{aligned}$$

Using this formula, the desirable estimate becomes easy. Assume $|\xi_1|\geq|\xi_2|\geq\cdots\geq|\xi_4|$, and divide into several cases: 1, $|\xi_1|-|\xi_4|\ll|\xi_1|$; 2, $|\xi_1|-|\xi_4|\gtrsim|\xi_1|,|\xi_3|,|\xi_4|\gtrsim N$; 3. $|\xi_3|\gtrsim N,|\xi_4|\ll N$; 4. $|\xi_3|\ll N$. □

By this lemma, one may find that

$$\frac{M_4}{\alpha_4}\quad\text{makes sense!}$$

This means that we can argue as above and define a new modified energy. Indeed, one may define $E_I^3(t)$ as

$$E_I^3(t)=E_I^2(t)-\int_{\xi_1+\cdots+\xi_4=0}\frac{M_4(\xi_1,\cdots,\xi_4)}{-i\alpha_4}\hat{u}(\xi_1)\cdots\hat{u}(\xi_4)\,d\xi_1d\xi_2d\xi_3,$$

then

$$\frac{d}{dt}E_I^3(t)=\int_{\xi_1+\cdots+\xi_5=0}M_5(\xi_1,\cdots,\xi_5)\hat{u}(\xi_1)\cdots\hat{u}(\xi_5)\,d\xi_1\cdots d\xi_4.$$

where

$$M_5(\xi_1,\cdots,\xi_5)=\Big[\frac{M_4(\xi_1,\cdots,\xi_4+\xi_5)}{i\alpha_4(\xi_1,\xi_2,\xi_3,\xi_4+\xi_5)}(\xi_4+\xi_5)\Big]_{sym}.$$

Similar as (4.4), we also have

$$\big|E_I^3(t)-E_I^2(t)\big|\lesssim N^{0-}\|Iu(t)\|_{L^2}^4. \tag{4.5}$$

Moreover,

Lemma 4.3. *Assume that* $|\xi_1| \geq |\xi_2| \geq \cdots \geq |\xi_5|$. *Then*

$$|M_5| \lesssim \frac{1}{(N+|\xi_3|)(N+|\xi_4|)(N+|\xi_5|)}.$$

Using this lemma and by a similar manner as the proof in Lemma 3.1, we have

Lemma 4.4. *Suppose that* $\|Iu\|_{X_{0,\frac{1}{2}+}[t_0,t_0+1]} \lesssim 1$, *then*

$$\Big| \int_{t_0}^{t_0+1} \int_{\xi_1+\cdots+\xi_5=0} M_5(\xi_1,\cdots,\xi_5)\hat{u}(\xi_1)\cdots\hat{u}(\xi_5)\, d\xi_1\cdots d\xi_4 ds \Big| \lesssim N^{-3-\frac{3}{4}+}. \tag{4.6}$$

Moreover,

$$\left|E_I^3(t_0+1) - E_I^3(t_0)\right| \lesssim N^{-3-\frac{3}{4}+}.$$

Proof. By the reduction as the proof in Lemma 3.1, we have for any $s > -\frac{3}{4}$,

$$\begin{aligned}
\text{LHS of (4.6)} \lesssim & \sum_{N_1\geq\cdots\geq N_5} \frac{1}{(N+N_3)(N+N_4)(N+N_5)} \int_{\mathbb{R}^2} u_1(x,t)\cdots u_5(x,t)\, dxdt \\
\lesssim & \sum_{N_1\geq\cdots\geq N_5} \frac{1}{(N+N_3)(N+N_4)(N+N_5)} \|I(u_1,u_5)\|_{L^2_{tx}} \|u_2\|_{L^\infty_x L^2_t} \|u_3\|_{L^4_x L^\infty_t} \|u_4\|_{L^4_x L^\infty_t} \\
\lesssim & \sum_{N_1\geq\cdots\geq N_5} \frac{N_1^{-1}N_2^{-1}N_3^{\frac{1}{4}}N_4^{\frac{1}{4}}}{(N+N_3)(N+N_4)(N+N_5)} \|u_1\|_{X_{0,\frac{1}{2}+}} \cdots \|u_5\|_{X_{0,\frac{1}{2}+}} \\
\lesssim & N^{-3-\frac{3}{4}+} \|u\|^5_{X_{0,\frac{1}{2}+}}.
\end{aligned}$$

□

5. Conclusion

Suppose that

$$\sup_{t\in[0,J]} \|Iu(t)\|^2_{L^2} \leq 2\varepsilon_0. \tag{5.1}$$

Then by local theory Lemma 2.2, we have $\|Iu\|_{X_{0,\frac{1}{2}+}(t_0,t_0+1)} \lesssim 1$ for any $t_0 \in [0,J]$. Now by (4.4), (4.5) and Lemma 4.4, for any $t \in [J, J+1]$,

$$\begin{aligned}
\|Iu(t)\|^2_{L^2} \leq & E_I^3(t) + CN^{0-}(\|Iu(t)\|^3_{L^2} + \|Iu(t)\|^4_{L^2}) \\
\leq & E_I^3(0) + CN^{0-}(\|Iu(t)\|^3_{L^2} + \|Iu(t)\|^4_{L^2}) + J\cdot N^{-3-\frac{3}{4}+} \\
\leq & \frac{9}{8}\varepsilon_0 + CN^{0-}(\|Iu(t)\|^3_{L^2} + \|Iu(t)\|^4_{L^2}) + J\cdot N^{-3-\frac{3}{4}+}.
\end{aligned}$$

If

$$J\cdot N^{-3-\frac{3}{4}+} \leq \frac{1}{16}\varepsilon_0,$$

then

$$\|Iu(t)\|^2_{L^2} \leq \frac{5}{4}\varepsilon_0 + CN^{0-}(\|Iu(t)\|^3_{L^2} + \|Iu(t)\|^4_{L^2}).$$

Thus by continuity and choosing N large enough, we have

$$\|Iu(t)\|_{L^2}^2 \le 2\varepsilon_0.$$

This extends the assumption (5.1) to the interval $[0, J+1]$ until $J \le cN^{3+\frac{3}{4}-}$ for some small $c > 0$. Thus by iteration, we prove that

$$\sup_{t\in[0,cN^{3+\frac{3}{4}-}]} \|Iu(t)\|_{L^2}^2 \le 2\varepsilon_0,$$

which implies that the solution of (2.4) exists on $[0, cN^{3+\frac{3}{4}-}]$. Recall that $u = u_\lambda$ is the rescaled solution, which implies that the solution u of the original problem (1.1) exists on $[0, T]$ with $T = c\lambda^3 N^{3+\frac{3}{4}-}$. Suppose that $T \ge N^{0+}$, then u exists for arbitrary time by choosing large N, and proves the global well-posedness. Now

$$T \ge N^{0+} \Leftrightarrow c\lambda^3 N^{3+\frac{3}{4}-} = cN^{-\frac{6s}{3+2s}} N^{3+\frac{3}{4}-} \ge N^{0+},$$

which holds for any $s > -\frac{3}{4}$. This proves the theorem.

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