Talk about "Burq, N., Gérard, P., and Tzvetkov, N., Two singular dynamics of the nonlinear Schrödinger equation on a plane domain, Geometric And Functional Analysis, 13(1), 1-19".

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1 introduction

In this talk, we study "Burq, N., Gérard, P., and Tzvetkov, N., Two singular dynamics of the nonlinear Schrödinger equation on a plane domain, Geometric And Functional Analysis, 13(1), 1-19.".

We consider the following the cubic, focusing nonlinear Schrödinger equation (NLS), posed on Ω

$$(i\partial_t + \Delta)u = -|u|^2 u, \quad \text{in } \mathbb{R} \times \Omega \tag{NLS}$$

with initial data

$$u(0, x) = u_0(x), \quad x \in \Omega,$$

and Dirichlet boundary conditions

$$u(t, x) = 0 \quad (t, x) \in \mathbb{R} \times \partial\Omega,$$

where Ω is a domain of $\mathbb{R}^2$.

First, we define the local well-posedness in $H^s(\Omega)$ with uniformly continuous flow map for data in any ball of $H^s(\Omega)$.

Definition 1.1. (see [5, 9]) We say that the Cauchy problem (NLS) is locally well-posed in $H^s(\Omega)$ with uniformly continuous flow map for data in any ball of $H^s(\Omega)$ if for any $R > 0$ there exist $T > 0$ and a functional space X_T continuously embedded in $C([-T, T], H^s(\Omega))$ such that for every

$$u_0 \in B_R := \{u_0 \in H^s(\Omega) : \|u_0\|_{H^s} < R\}$$

the Cauchy problem (NLS) has a unique solution $u \in X_T$. Moreover

1. The map $u_0 \to u$ is uniformly continuous form B_R to $C([-T, T], H^s(\Omega))$.
2. If $u_0 \in H^1(\Omega)$, $u \in C([-T, T], H^1(\Omega))$ and satisfies the usual conservation laws

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2},$$
$$\|\nabla u(t)\|_{L^2}^2 - \frac{1}{2}\|u(t)\|_{L^4}^4 = const.$$

Next, we define the ground state Q on $\mathbb{R}^2$ as the unique positive radial solution of

$$(-\Delta_{\mathbb{R}^2} + 1)Q = |Q|^2 Q \quad \text{in } \mathbb{R}^2,$$

(see [1, 10].)

2 Known results

1. the case of $\Omega = \mathbb{R}^2$

   (a) If $s > 0$, then (NLS) is locally well-posed in $H^s(\mathbb{R}^2)$ with uniformly continuous flow map for data in any ball of $H^s(\mathbb{R}^2)$ (see Cazenave-Weissler [7]).

   (b) (NLS) is globally well-posed for initial data with L^2 norm smaller than the L^2 norm of the ground state Q (see Weinstein [12].)

   (c) If $\|\varphi\|_{L^2} = \|Q\|_{L^2}$ and the solution u with $u(0) = \varphi$ is blow-up in finite time $T > 0$, then there exist $\theta \in \mathbb{R}$, $\omega > 0$, $x_0 \in \mathbb{R}^2$ and $x_1 \in \mathbb{R}^2$ such that for $t < T$
   $$u(t,x) = \frac{\omega}{T-t} e^{i\theta + i|x-x_0|^2/4(t-T) - i\omega^2/(t-T)} Q\left(\frac{\omega}{T-t}(x-x_0) - x_1\right),$$
   and $\|\nabla u(t,\cdot)\|_{L^2}^2 = \frac{\omega^2}{(T-t)^2}\|\nabla Q\|_{L^2}^2 + \mathrm{Re}\int_{\mathbb{R}^2} \frac{ix}{T-t}\cdot(\nabla Q)Q + \int_{\mathbb{R}^2}\frac{|x|^2}{4\omega^2}Q^2 dx$. (see [11]).

2. If $\Omega = \mathbb{T}^2$ or a square and $s > 0$, then (NLS) is locally well-posed in $H^s(\Omega)$ with uniformly continuous flow map for data in any ball of $H^s(\Omega)$ (see Bourgain [3] and Burq-Gérard-Tzvetkov [6]).

3. If Ω is a compact 2-dimensional smooth Riemannian manifold with boundary and $s > 1/2$, then (NLS) is locally well-posed in $H^2(\mathbb{R}^2)$ with uniformly continuous flow map for data in any ball of $H^s(\Omega)$ (see Burq-Gérard-Tzvetkov [4] and Blair-Smith-Sogge [2].)

4. If $\Omega = S^2$ and $s \in [0, \frac{1}{4}[$, then (NLS) is not locally well-posed in $H^s(\Omega)$ with uniformly continuous flow map for data in any ball of $H^s(\Omega)$ (see Burq-Gérard-Tzvetkov [5].)

3 Main results

We show the following results.

Theorem 3.1. *([6]) Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let $x_0 \in \Omega$ with $\psi \in C_0^\infty(\Omega)$, $\psi = 1$ in a neighborhood of x_0. Then there exist $\kappa > 0$, $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ there exist $T_\lambda > 0$ and a family $\{r_\lambda\}$ of functions define on $[0, T_\lambda[\times\Omega$ satisfying*
$$\|r_\lambda(t,\cdot)\|_{H^2} \le ce^{-\frac{\kappa}{\lambda(T_\lambda - t)}}, \quad t \in [0, T_\lambda[\tag{3.1}$$
such that
$$u_\lambda(t,x) = \frac{1}{\lambda(T_\lambda - t)}\psi(x) e^{\frac{i(4-\lambda^2(x-x_0)^2)}{4\lambda^2(T_\lambda - t)}} Q\left(\frac{x - x_0}{\lambda(T_\lambda - t)}\right) + r_\lambda(t,x), \quad x \in \Omega, t \in [0, T_\lambda[\tag{3.2}$$
are solutions of (NLS), satisfying the Dirichlet boundary conditions, which blow-up at x_0 in time T_λ in the energy space H^1 with blow up speed $(T_\lambda - t)^{-1}$. Moreover, $\|u_\lambda(t,\cdot)\|_{L^2} = \|Q\|_{L^2}$.

Remark 1. If $u_0 \in H^1(\Omega)$ and $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the local solution of (NLS) with initial data u_0 can be extended to the whole real line in time t.

Theorem 3.2. *([6]) Let $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disc in $\mathbb{R}^2$ and Δ_D be the Laplace operator on D with Dirichlet boundary conditions. Fix $\kappa > 0$ and $s \in]1/5, 1/2[$. Then there exists a sequence $\phi_n(x)$ of eigenfunctions of $-\Delta_D$ with corresponding eigenvalues ζ_n ($\lim_{n\to\infty}\zeta_n = \infty$) such that $\|\phi_n\|_{H^s} \approx 1$ and equation (NLS) with Cauchy data $\kappa\phi_n(x)$, has, for $n \gg 1$, a unique global solution $u_n(t,x)$ which can be represented as*
$$u_n(t,x) = \kappa e^{-it(\zeta_n - \kappa^2\omega_n)}(\phi_n(x) + r_n(t,x)), \tag{3.3}$$

where $\omega \approx n^{\frac{2}{3}-2s}$ *and* $r_n(t,x)$ *satisfies for any* $T>0$, *n large enough and* $t \in [0,T]$
$$\|r_n(t,\cdot)\|_{H^s} \leq Cn^{-\delta} \tag{3.4}$$
where $\delta > 0$ *and* C *is independent of* n. *Moreover, if* $0<\kappa<1$
$$\|u_n\|_{L^\infty(\mathbb{R},H^s(D))} \leq C\kappa. \tag{3.5}$$

Remark 2. By Theorem 3.2, the Cauchy problem associated to (NLS) is not locally well-posed in $H^s(D)$ with uniformly continuous flow map for data in any ball of $H^s(D)$ for $s \in]1/5, 1/3[$. Indeed, we fix $s \in]1/5,1/3[$, $\kappa > 0$ and a sequence $\{\kappa_n\}$ tending to κ. Denote by $u_{\kappa,n}$ (resp. $u_{\kappa_n,n}$) the solution of (NLS) with initial data $\kappa\phi_n$ (resp. $\kappa_n\phi_n$.) Then, $\|\kappa\phi_n - \kappa_n\phi_n\|_{H^s} \to 0$ as $n\to\infty$. On the other hand,
$$\|u_{\kappa,n}(t,\cdot) - u_{\kappa_n,n}(t,\cdot)\|_{H^s} \geq C|e^{it\omega_n(\kappa^2-\kappa_n^2)} - 1| - Cn^{-\delta},$$
where $\delta > 0$. Then, if $\kappa_n^2 = \kappa^2 + |\omega_n|^{-1/2}$, then for $n \gg 1$ there exists $c>0$ such that $\|u_{\kappa,n}(t_n,\cdot) - u_{\kappa_n,n}(t_n,\cdot)\|_{H^s} > c$, where $t_n = \pi/(2\omega_n(\kappa^2-\kappa_n^2))$. Since $\omega_n \to \infty$ as $n\to\infty$, then $t_n \to 0$ as $n \to \infty$ and the flow map is not uniformly continuous.

4 Proof of Theorem 3.1

Let $T>0$ and $\lambda > 0$. We define
$$\tilde{R}_\lambda(t,x) = \frac{1}{\lambda(T-t)} e^{\frac{i(4-\lambda^2(x-x_0)^2)}{4\lambda^2(T-t)}} Q\left(\frac{x-x_0}{\lambda(T-t)}\right), \quad (t,x) \in [0,T[\times\mathbb{R}^2.$$
We set $R_\lambda(t,x) := \psi(x)\tilde{R}_\lambda(t,x)$. Constructing a smooth correction $r_\lambda(t,x)$ and choosing T , we make a solution $R_\lambda(t,x) + r_\lambda(t,x)$ of (NLS) by using the contraction mapping principle. Since $\tilde{R}_\lambda(t,x)$ is a solution of (NLS) on $\mathbb{R}^2$, we have
$$(i\partial_t + \Delta)R_\lambda = -\psi|\tilde{R}_\lambda|^2\tilde{R}_\lambda + 2\nabla\psi\nabla\tilde{R}_\lambda + (\Delta\psi)\tilde{R}_\lambda.$$
Then, we look for a solution $v \in C([0,T[, H^2(\Omega)\cap H_0^1(\Omega))$ such that
$$\begin{cases} (i\partial_t+\Delta)v = -|R_\lambda+v|^2(R_\lambda+v) + \psi|\tilde{R}_\lambda|^2\tilde{R}_\lambda - 2\nabla\psi\nabla\tilde{R}_\lambda - (\Delta\psi)\tilde{R}_\lambda, \\ v(t) \to 0 \text{ as } t\to T(t<T) \text{ in } H^2(\Omega)\cap H_0^1(\Omega). \end{cases} \tag{4.1}$$
Set
$$-|R_\lambda+v|^2(R_\lambda+v) + \psi|\tilde{R}_\lambda|^2\tilde{R}_\lambda - 2\nabla\psi\nabla\tilde{R}_\lambda - \Delta\psi\tilde{R}_\lambda = Q_0 + Q_1(v) + Q_2(v) + Q_3(v),$$
where
$$\begin{cases} Q_0 = \psi(1-|\psi|^2)|\tilde{R}_\lambda|^2\tilde{R}_\lambda - 2\nabla\psi\nabla\tilde{R}_\lambda - (\Delta\psi)\tilde{R}_\lambda, \\ Q_1(v) = -R_\lambda^2\bar{v} - 2|R_\lambda|^2 v, \\ Q_2(v) = -\bar{R}_\lambda v^2 - 2R_\lambda|v|^2, \\ Q_3(v) = -|v|^2 v. \end{cases}$$

Then, since there exists $\delta_0 > 0$ such that $\left\|e^{\delta_0|\cdot|}Q\right\|_{W^{\infty,3}(\mathbb{R}^2)} < \infty$, there exists $C,\delta > 0$ such that
$$\|Q_0(t,\cdot)\|_{H^2(\Omega)} \leq Ce^{-\frac{\delta}{\lambda(T-t)}}. \tag{4.2}$$
Indeed, for example there exists $\varepsilon > 0$ such that
$$\begin{aligned}\left\|\psi(1-|\psi|^2)|\tilde{R}_\lambda|^2\tilde{R}_\lambda\right\|^2_{L^2(\Omega)} &\leq \frac{C}{(\lambda(T-t))^6}\int_\varepsilon^\infty re^{\frac{-6\delta_0 r}{\lambda(T-t)}}dr, \\ &\leq \frac{C}{(\lambda(T-t))^5}e^{\frac{-3\delta_0\varepsilon}{\lambda(T-t)}}, \\ &\leq Ce^{\frac{-\delta_0\varepsilon}{\lambda(T-t)}}.\end{aligned}$$

Here we used that for $c, \varepsilon > 0$

$$|x|e^{-|cx|} \leq C_{c,\varepsilon} e^{-\frac{|cx|}{1+\varepsilon}}, \tag{4.3}$$

where $C_{c,\varepsilon}$ is independent of x. We look for solutions of (4.1) in the space

$$X_T = \{v \in C([0,T[, H^2(\Omega) \cap H_0^1(\Omega)) : \|v\|_{X_T} < \infty\}$$

where

$$\|v\|_{X_T} = \sup_{t \in [0,T[} \{e^{\frac{\delta}{2\lambda(T-t)}} \|v(t)\|_{L^2(\Omega)} + e^{\frac{\delta}{3\lambda(T-t)}} \|v(t)\|_{H^2(\Omega)}\}.$$

We define

$$\Phi(v)(t) = \int_t^T S(t-\tau) Q_0(\tau) d\tau + \sum_{j=1}^3 \int_t^T S(t-\tau) Q_j(v(\tau)) d\tau,$$

$$I_0(t) = \int_t^T S(t-\tau) Q_0(\tau) d\tau,$$

and for $j = 1, 2, 3$

$$I_j(v)(t) = \int_t^T S(t-\tau) Q_j(v(\tau)) d\tau,$$

where $S(t)$ is the unitary group which defines the free evolution of the Schrödinger equation on Ω with Dirichlet boundary conditions. Then, we estimate $\|I_j\|_{X_T}$ for $j = 0, 1, 2, 3$ and $\|I_j(v) - I_j(w)\|$ for $j = 1, 2, 3$.

Estimate for I_0. By the estimate of the source therm (4.2),

$$\|I_0\|_{X_T} \leq CT. \tag{4.4}$$

Estimate for $I_1(v)$. Recall $Q_1(v) = -R_\lambda^2 \bar{v} - 2|R_\lambda|^2 v$. For $t \in [0, T[$

$$\begin{aligned}
\|I_1(v)(t)\|_{L^2} \leq & C \int_t^T \|R_\lambda(\tau)\|_{L^\infty}^2 \|v(\tau)\|_{L^2} d\tau, \\
\leq & C \|v\|_{X_T} \int_t^T \frac{1}{(\lambda(T-\tau))^2} e^{-\frac{\delta}{2\lambda(T-\tau)}} d\tau, \\
\leq & C \frac{\|v\|_{X_T}}{\lambda^2} \frac{2\lambda}{\delta} e^{-\frac{\delta}{2\lambda(T-t)}} = \frac{C}{\lambda} e^{-\frac{\delta}{2\lambda(T-t)}} \|v\|_{X_T}.
\end{aligned}$$

For $t \in [0, T[$

$$\begin{aligned}
\|I_1(v)(t)\|_{H^2} \leq & C \int_t^T \left\|\nabla^2 Q_1(v(\tau))\right\|_{L^2} d\tau + C \|I_1(t)\|_{L^2}, \\
\leq & C \int_t^T \left\|\nabla^2 (R_\lambda^2)(\tau)\right\|_{L^\infty} \|v(\tau)\|_{L^2} d\tau \\
& + C \int_t^T \left\|\nabla (R_\lambda^2)(\tau)\right\|_{L^\infty} \|\nabla v(\tau)\|_{L^2} d\tau \\
& + C \int_t^T \left\|(R_\lambda^2)(\tau)\right\|_{L^\infty} \left\|\nabla^2 v(\tau)\right\|_{L^2} d\tau + \frac{C}{\lambda} e^{-\frac{\delta}{2\lambda(T-t)}} \|v\|_{X_T}.
\end{aligned}$$

Since

$$\nabla \tilde{R}_\lambda(\tau, x) = \frac{-i(x-x_0)}{2(T-\tau)} \tilde{R}_\lambda(\tau, x) + \frac{1}{(\lambda(T-\tau))^2} e^{\frac{i(4-\lambda^2(x-x_0)^2)}{4\lambda^2(T-\tau)}} (\nabla Q) \left(\frac{x-x_0}{\lambda(T-\tau)}\right),$$

we have

$$\left\|\nabla^k ((R_\lambda)^2(\tau))\right\|_{L^\infty} \leq \frac{C(1+\lambda^k)}{(\lambda(T-\tau))^{k+2}}, \quad k = 0, 1, 2, \quad \tau \in [0, T[. \tag{4.5}$$

Using the inequalities

$$\|\nabla v(\tau)\|_{L^2} \le \left\|\nabla^2 v(\tau)\right\|_{L^2}^{1/2} \|v(\tau)\|_{L^2}^{1/2},$$

, (4.3) and (4.5), we obtain for $\lambda \ge 1$ and $T \le 1$,

$$\begin{aligned}
\|I_1(v)(t)\|_{H^2} \le & C\|v\|_{X_T}\left\{\frac{1}{\lambda}e^{-\frac{\delta}{2\lambda(T-t)}} + \int_t^T \frac{1+\lambda^2}{(\lambda(T-\tau))^4} e^{-\frac{\delta}{2\lambda(T-\tau)}}d\tau \right.\\
& \left. + \int_t^T \frac{1+\lambda}{(\lambda(T-\tau))^3} e^{-\frac{1}{2}(\frac{\delta}{2\lambda(T-\tau)}+\frac{\delta}{3\lambda(T-\tau)})}d\tau + \int_t^T \frac{1}{(\lambda(T-\tau))^2} e^{-\frac{\delta}{3\lambda(T-\tau)}}d\tau \right\},\\
\le & C\|v\|_{X_T}\left\{\frac{1}{\lambda}e^{-\frac{\delta}{2\lambda(T-t)}} + \int_t^T \frac{1+\lambda^2}{(\lambda(T-\tau))^2} e^{-\frac{\delta}{3\lambda(T-\tau)}}d\tau \right.\\
& \left. + \int_t^T \frac{1+\lambda}{(\lambda(T-\tau))^2} e^{-\frac{\delta}{3\lambda(T-\tau)}}d\tau + \int_t^T \frac{1}{(\lambda(T-\tau))^2} e^{-\frac{\delta}{3\lambda(T-\tau)}}d\tau \right\}\\
\le & C\left(\frac{1}{\lambda} + \lambda^2 T\right) e^{-\frac{\delta}{3\lambda(T-t)}}\|v\|_{X_T}.
\end{aligned}$$

Therefore,

$$\|I_1(v)\|_{X_T} \le C\left(\frac{1}{\lambda} + T^{1/2}\right)\|v\|_{X_T}, \tag{4.6}$$

provided $\lambda^2 T^{1/2} \le 1$, $\lambda \ge 1$.

Estimate for $I_1(v) - I_1(w)$. Similarly, we obtain

$$\|I_1(v) - I_1(w)\|_{X_T} \le C\left(\frac{1}{\lambda} + T^{1/2}\right)\|v-w\|_{X_T}, \tag{4.7}$$

provided $\lambda^2 T^{1/2} \le 1$, $\lambda \ge 1$.

Estimate for $I_2(v)$. Recall that $Q_2(v) = -\bar{R}_\lambda^2 - 2R_\lambda|v|^2$. Using the inequality

$$\|R_\lambda(\tau)\|_{H^2} \le \frac{C(1+\lambda^2)}{(\lambda(T-\tau))^2},$$

and (4.3), we obtain

$$\begin{aligned}
\|I_2(v)(t)\|_{H^2} \le & C\int_t^T \|R_\lambda(\tau)\|_{H^2}\|v(\tau)\|_{H^2}^2 d\tau,\\
\le & C\left(\int_t^T \frac{1+\lambda^2}{(\lambda(T-\tau))^2} e^{-\frac{2\delta}{3\lambda(T-\tau)}}d\tau\right)\|v\|_{X_T}^2,\\
\le & CT(1+\lambda^2)e^{-\frac{\delta}{2\lambda(T-t)}}\|v\|_{X_T}^2.
\end{aligned}$$

Hence,

$$\|I_2(v)\|_{X_T} \le CT^{1/2}\|v\|_{X_T}^2, \tag{4.8}$$

provided $\lambda^2 T^{1/2} \le 1$, $\lambda \ge 1$.

Estimate for $I_2(v) - I_2(w)$. Similarly, we obtain

$$\|I_2(v) - I_2(w)\|_{X_T} \le CT^{1/2}(\|v\|_{X_T} + \|w\|_{X_T})\|v-w\|_{X_T}, \tag{4.9}$$

provided $\lambda^2 T^{1/2} \le 1$, $\lambda \ge 1$.

Estimate for $I_3(v)$. Recall that $Q_3(v) = -|v|^2 v$. We obtain

$$\begin{aligned}\|I_3(t)\|_{H^2} \leq& C\int_t^T \|v(\tau)\|_{H^2}^3 d\tau,\\ \leq& \|v\|_{X_T}^3 \int_t^T e^{-\frac{\delta}{\lambda(T-\tau)}} d\tau,\\ \leq& CTe^{-\frac{\delta}{3\lambda(T-t)}}\|v\|_{X_T}^3.\end{aligned}$$

Thus,

$$\|I_3(v)\|_{X_T} \leq CT\|v\|_{X_T}^3. \tag{4.10}$$

Estimate for $I_3(v) - I_3(w)$. Similarly, we obtain

$$\|I_3(v) - I_3(w)\|_{X_T} \leq CT(\|v\|_{X_T}^2 + \|w\|_{X_T}^2)\|v - w\|_{X_T}. \tag{4.11}$$

By (4.4)-(4.11), Φ is a contraction map if $\lambda \gg 1$, $T \ll 1$ and $\lambda^2 T^{1/2} \leq 1$. Therefore, there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ there exist $T_\lambda > 0$ and the unique solution r_λ of (4.1) such that

$$\|r_\lambda(t,\cdot)\|_{H^2} \leq ce^{-\frac{\kappa}{\lambda(T_\lambda - t)}}, \quad t \in [0, T_\lambda[,$$

and $u_\lambda(t,x) := R_\lambda(t,x) + r_\lambda(t,x)$ is a solution of (NLS). Moreover, for $t_0 \in [0, T_\lambda[$

$$\|u_\lambda(t_0,\cdot)\|_{L^2} = \lim_{t\to T_\lambda} \|u_\lambda(t,\cdot)\|_{L^2} = \lim_{t\to T_\lambda} \|R_\lambda(t,\cdot)\|_{L^2} = \|Q\|_{L^2}.$$

5 Proof of Theorem 2

The Bessel function of order n is defined as follows

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^\infty \frac{(-1)^k (\frac{z}{2})^{2k}}{k!(n+k)!}.$$

Let $z_{n1} < z_{n2} < z_{n3} < \cdots$ be the sequence of the positive zeros of $J_n(z)$. In the following lemma, we get the order of a positive zero of $J_n(z)$.

Lemma 5.1. *Let $n \geq 1$. Then there exists a constant α, independent of n, such that $z_{n1} = n + \alpha n^{1/3} + O(n^\lambda)$, for any $\lambda > 1/6$. Moreover $z_{n2} - z_{n1} \geq Cn^{1/3}$.*

In this talk, we do not prove this lemma (see [6, 8].)

We consider the equation (NLS) on $\Omega = D := \{x \in \mathbb{R}^2 : |x| < 1\}$. By introducing polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, we can get that $f_{nk}(r,\theta) := J_n(z_{nk}r)e^{in\theta}$ is an orthogonal basis of $L^2(D)$ of eigenfunctions for $-\Delta_D$ with corresponding eigenvalue z_{nk}^2. Then for $s \in [0, 1/2[$ and $u \in L^2(D)$, there exists $c_{nk} \in \mathbb{C}$ such that $u = \sum_{n,k} c_{nk} f_{nk}$ and $u \in H^s(D)$ if and only if

$$\|u\|_{H^s(D)} \approx \left\{\sum_{n,k} z_{nk}^{2s}|c_{nk}|^2\right\}^{1/2} < \infty. \tag{5.1}$$

If $s \geq 1/2$, the space defined by (5.1) will be denote by $H_0^s(D)$. Moreover, for a positive integer n and $u \in L^2$

$$(\forall (r,\theta) \in D,\ u(r,\theta) = e^{in\theta}u(r,0)) \Leftrightarrow u = \sum_{k\geq 1} c_{nk} f_{nk} \tag{5.2}$$

The following lemma show the asymptotics for the L^p norms of f_{n1}.

Lemma 5.2. *Let $p \in [2, \infty]$. Then $\|f_{n1}\|_{L^p(D)} \approx n^{-\frac{2}{3p}-\frac{1}{3}}$.*

To prove this lemma, we use the properties of the Bessel function. In this talk, we do not show this lemma (see [6, 8].)

Let $\kappa > 0$ and $s \in]1/5, 1/2[$. Set

$$\phi_n = n^{\frac{2}{3}-s} f_{n1},$$

and

$$\omega_n = \frac{\|\phi_n\|_{L^4}^4}{\|\phi_n\|_{L^2}^2}.$$

By Lemma 5.1 and Lemma 5.2, we have that

$$\|\phi_n\|_{H^s} \approx 1, \quad \|\phi_n\|_{L^2} \approx n^{-s}, \quad \omega_n \approx n^{\frac{2}{3}-2s}.$$

If $n \gg 1$, then there exists a unique global solution $u_n(t)$ of (NLS) with the initial data $\kappa\phi_n$. By the L^2 conservation law, we have $\|u_n(t,\cdot)\|_{L^2} = C\kappa n^{-s}$. By the energy conservation law and the Gagliardo-Nirenberg inequality,

$$\begin{aligned}\|\nabla u_n(t,\cdot)\|_{L^2}^2 &\le -\|\nabla u_n(t,\cdot)\|_{L^2}^2 + \|u_n(t,\cdot)\|_{L^4}^4 + 2\|\nabla\kappa\phi_n\|_{L^2}^2 - \|\kappa\phi_n\|_{L^4}^4, \\ &\le -\|\nabla u_n(t,\cdot)\|_{L^2}^2 + C\|\nabla u_n(t,\cdot)\|_{L^2}^2\|u_n(t,\cdot)\|_{L^2}^2 + 2\|\nabla\kappa\phi_n\|_{L^2}^2 - \|\kappa\phi_n\|_{L^4}^4, \\ &\le -\|\nabla u_n(t,\cdot)\|_{L^2}^2 + Cn^{-2s}\kappa^2\|\nabla u_n(t,\cdot)\|_{L^2}^2 + C\kappa^2 n^{2-2s} + C\kappa^4 n^{\frac{2}{3}-4s}.\end{aligned}$$

Thus, if $n \gg 1$, then $\|u_n(t,\cdot)\|_{H_0^1} \le C\kappa n^{1-s}$. By an interpolation,

$$\|u_n(t,\cdot)\|_{H^s} \le \|u_n(t,\cdot)\|_{H_0^1}^s \|u_n(t,\cdot)\|_{L^2}^{1-s} \le C\kappa.$$

Since $|\phi_n(r,\theta)|^2\phi_n(r,\theta) = e^{in\theta}|\phi_n(r,0)|^2\phi_n(r,0)$, by (5.2) $|\phi_n|^2\phi_n = \omega_n\phi_n + r_n$, where $r_n = \sum_{k\ge 2} c_k f_{nk}$. Set

$$u_n(t,x) = \kappa\exp(-it(z_{n1}^2 - \kappa^2\omega_n))(\phi_n(x) + w_n(t,x)). \tag{5.3}$$

Then w_n satisfied the following equation

$$\begin{cases}(i\partial_t + \Delta + z_{n1}^2 - \kappa^2\omega_n)w_n = -\kappa(|\phi_n + w_n|^2(\phi_n + w_n) - |\phi_n|^2\phi_n + r_n) \\ w_n(0,x) = 0, \quad x \in D\end{cases} \tag{5.4}$$

Here we estimate the H^s norms of w_n. We decompose

$$w_n(t,x) = \lambda_n(t)\phi_n(x) + q_n(t,x),$$

where

$$q_n(t,x) = \sum_{k\ge 2} c_k(t) f_{nk}(x). \tag{5.5}$$

Estimate of $\|q_n\|_{H^s}$. By conservation law of u_n, we have

$$|1 + \lambda_n(t)|^2\|\phi_n\|_{L^2}^2 + \|q_n(t)\|_{L^2}^2 = \|\phi_n\|_{L^2}^2, \tag{5.6}$$

while by the energy conservation law we have

$$|1 + \lambda_n(t)|^2\|\nabla\phi_n\|_{L^2}^2 + \|\nabla q_n(t)\|_{L^2}^2 - \frac{1}{2\kappa^2}\|u_n(t)\|_{L^4}^4 = \|\nabla\phi_n\|_{L^2}^2 - \frac{\kappa^2}{2}\|\phi_n\|_{L^4}^4. \tag{5.7}$$

Using $\|\nabla\phi_n\|_{L^2}^2 = z_{n1}^2\|\phi_n\|_{L^2}^2$ and calculating (5.7)$-z_{n1}^2$(5.6), we have

$$\|\nabla q_n(t)\|_{L^2}^2 - z_{n1}^2\|q_n(t)\|_{L^2}^2 = \frac{1}{2\kappa^2}\|u_n(t)\|_{L^4}^4 - \frac{\kappa^2}{2}\|\phi_n\|_{L^4}^4 \le \frac{1}{2\kappa^2}\|u_n(t)\|_{L^4}^4.$$

By (5.6), we have $|\lambda_n(t)| \le 2$. Thus, by the equation (5.3)

$$|u_n(t,x)|^4 \le C\kappa^4(|\phi_n(x)|^4 + |q_n(t,x)|^4), \quad t \in \mathbb{R}, \quad x \in D. \tag{5.8}$$

Due to Lemma 5.1, for $k \ge 2$

$$z_{nk}^2 - z_{n1}^2 = (z_{nk} - z_{n1})(z_{nk} + z_{n1}) \ge Cn^{1/3} z_{nk}. \tag{5.9}$$

Moreover,

$$\|\phi_n\|_{L^4}^4 = \omega_n\|\phi_n\|_{L^2}^2 \approx n^{\frac{2}{3}-4s} \tag{5.10}$$

Using (5.8)-(5.10), we have

$$
\begin{aligned}
n^{1/3}\|q_n(t)\|^2_{H_0^{1/2}} &\leq Cn^{1/3}\sum_{k\geq 2} z_{nk}\|c_k(t)f_{nk}\|^2_{L^2},\\
&\leq C\sum_{k\geq 2}(z_{nk}^2 - z_{n1}^2)\|c_k(t)f_{nk}\|^2_{L^2},\\
&\leq C(\|\nabla q_n(t)\|^2_{L^2} - z_{n1}^2\|q_n(t)\|^2_{L^2}),\\
&\leq \frac{C}{2\kappa^2}\|u_n(t)\|^4_{L^4},\\
&\leq C\kappa^2(\|\phi_n\|^4_{L^4} + \|q_n(t)\|^4_{L^4}) \leq C\kappa^2 n^{\frac{2}{3}-4s} + C\kappa^2\|q_n(t)\|^4_{H_0^{1/2}}.
\end{aligned}
$$

Since $q_n(0,\cdot) = 0$, by a bootstrap argument we obtain

$$\|q_n(t)\|_{H_0^{1/2}} \leq C\kappa n^{\frac{1}{6}-2s}. \tag{5.11}$$

By the Sobolev embedding

$$\|q_n(t)\|_{L^4} \leq C\kappa n^{\frac{1}{6}-2s}. \tag{5.12}$$

Moreover, by (5.5) and (5.11)

$$\|q_n(t)\|_{L^2} \leq Cz_{n1}^{-1/2}\|q_n(t)\|_{H_0^{1/2}} \leq C\kappa n^{-\frac{1}{3}-2s}. \tag{5.13}$$

By an interpolation between (5.11) and (5.13), we obtain the estimate

$$\|q_n(t)\|_{H^s} \leq C\|q_n(t)\|^{2s}_{H_0^{1/2}}\|q_n(t)\|^{1-2s}_{L^2} \leq C\kappa n^{-\frac{1}{3}-s}. \tag{5.14}$$

Estimate of $\lambda_n(t)$. We project the equation (5.4) on ϕ_n. Then we have

$$
\begin{cases}
(i\partial_t - \omega_n\kappa^2)\lambda_n = -\frac{\kappa}{\|\phi_n\|_{L^2}}\{(|\phi_n + w_n|^2(\phi_n + w_n), \phi_n)_{L^2} - (|\phi_n|^2\phi_n, \phi_n)_{L^2}\}\\
\lambda_n(0) = 0.
\end{cases} \tag{5.15}
$$

Since $w_n = \lambda_n\phi_n + q_n$, $|\phi_n|^2\phi_n = \omega_n\phi_n + r_n$ and

$$
\begin{aligned}
&(|\phi_n + w_n|^2(\phi_n + w_n), \phi_n)_{L^2} - (|\phi_n|^2\phi_n, \phi_n)_{L^2}\\
&= -\int(2|\phi_n|^2 w_n + \phi_n^2\bar{w}_n)\bar{\phi}_n + \int(2\mathrm{Re}(\bar{\phi}_n w_n)w_n\phi_n + |w_n|^2|\phi_n|^2 + |w_n|^2|\phi_n|^2 + |w_n|^2 w_n\bar{\phi}_n),
\end{aligned}
$$

we decompose the nonlinear term of (5.15) as follows

$$-\frac{\kappa}{\|\phi_n\|_{L^2}}\{(|\phi_n + w_n|^2(\phi_n + w_n), \phi_n)_{L^2} - (|\phi_n|^2\phi_n, \phi_n)_{L^2}\} = L_1 + L_2 + L_3,$$

where

$$
\begin{aligned}
L_1 &= -\kappa^2(2\omega_n\lambda_n + \omega_n\bar{\lambda}_n),\\
L_2 &= \frac{\kappa^2}{\|\phi_n\|^2_{L^2}}O\left(|\lambda_n|^2\int|\phi_n|^4 + |\lambda_n|^3\int|\phi_n|^4\right),\\
L_3 &= \frac{\kappa^2}{\|\phi_n\|^2_{L^2}}O\left(\int|q_n|^3|\phi_n| + \int|q_n|^2|\phi_n|^2 + |(q_n, r_n)_{L^2}|\right).
\end{aligned}
$$

First, we estimate the source term L_3. By (5.12)

$$\frac{\int|q_n|^3|\phi_n|}{\|\phi_n\|^2_{L^2}} \leq Cn^{2s}\|q_n\|^3_{L^4}\|\phi_n\|_{L^4} \leq Cn^{2s}n^{\frac{1}{2}-6s}n^{\frac{1}{6}-s} = Cn^{\frac{2}{3}-5s}.$$

Similarly, using (5.12) and (5.13), we obtain

$$\frac{\int|q_n|^2|\phi_n|^2}{\|\phi_n\|^2_{L^2}} \leq Cn^{2s}\|q_n\|^2_{L^2}\|\phi_n\|^2_{L^\infty} \leq Cn^{-4s},$$

and by $|\phi_n|^2\phi_n = \omega_n\phi_n + r_n$

$$\frac{|(q_n, r_n)_{L^2}|}{\|\phi_n\|_{L^2}^2} \leq \frac{|(q_n, |\phi_n|^2\phi_n)_{L^2}|}{\|\phi_n\|_{L^2}^2} \leq Cn^{\frac{1}{3}-3s}.$$

Since $s \in]1/5, 1/2[$, the equation for λ_n can be written as

$$\begin{cases} i\partial_t\lambda_n = -2\omega_n\kappa^2\mathrm{Re}(\lambda_n) + O(\omega_n|\lambda_n|^2 + \omega_n|\lambda_n|^3 + n^{\frac{1}{3}-3s}), \\ \lambda_n(0) = 0. \end{cases} \tag{5.16}$$

The L^2 conservation law (5.6) yields

$$-2\mathrm{Re}(\lambda_n) - |\lambda_n|^2 = 1 - |1 + \lambda_n|^2 = \frac{\|q_n(t)\|_{L^2}^2}{\|\phi_n\|_{L^2}^2} = O(n^{-\frac{1}{3}-3s}),$$

so we have

$$\begin{cases} i\partial_t\lambda_n = O(\omega_n|\lambda_n|^2 + \omega_n|\lambda_n|^3 + n^{\frac{1}{3}-3s}), \\ \lambda_n(0) = 0. \end{cases} \tag{5.17}$$

Set $\gamma_n = n^{\frac{1}{3}-3s}$ and $M_n(T) := \sup_{t\in[0,T[} \gamma_n^{-1}|\lambda_n|$. Then by integrating (5.17), we have that

$$\begin{aligned} M_n(T) &\leq CT(1 + \gamma_n\omega_n(M_n(T))^2 + \gamma_n^2\omega_n(M_n(T))^3)), \\ &\leq CT(1 + n^{1-5s}((M_n(T))^2 + n^{\frac{3}{4}-7s}(M_n(T))^3)). \end{aligned}$$

Since $s > 1/5$ and $M_n(0) = 0$, we obtain that $M_n(T) \leq CT$ for n large enough with respect to T. Thus,

$$|\lambda_n(t)| \leq CTn^{1-5s}. \tag{5.18}$$

Therefore, by (5.14) and (5.18) the proof Theorem 3.2 is complete.

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