On the normal form reduction (The 2nd talk on the paper "On the regularization mechanism for the periodic Kortewegde Vries equation")

Kotaro TSUGAWA

Nagoya University

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Plan of the talk

As Hirayama mentioned yesterday, the main idea used in [BIT] is the normal form reduction. In this talk, I would like to talk on related topics as follows:

- time resonances set ([GMS], [S])
- I-method ([Iteam])
- modified energy method ([K], [P])

We will focus on how it works in these method. The first and second topics are reviewed by other speakers. So, I will mention them briefly and talk mainly on the third topic.

Mechanism of the normal form reduction(1)

First, let us recall the mechanism of the normal form reduction. As an example, we consider the following j th order dispersive equation with k th power nonlinearity.

$$\partial_t u + i(\partial_x/i)^j u = u^k, \qquad t \in \mathbf{R}, x \in \mathbf{R}\,(\text{or}\,\mathbf{T}),$$

where $j, k \geq 2$ and in $\mathbf{N}$.
(Step 1)
Put $v(t) := U(-t)u(t) := \mathcal{F}^{-1} e^{it\xi^j} \widehat{u}(t)$ (profile). Then,

$$e^{-it\xi^j} \partial_t \widehat{v} = \{e^{-it\xi^j}\widehat{v}\} * \{e^{-it\xi^j}\widehat{v}\} * \cdots * \{e^{-it\xi^j}\widehat{v}\}.$$

Therefore,

$$\partial_t \widehat{v}(\xi) = \int_{\xi_1+\cdots+\xi_k=\xi} e^{it\Phi} \widehat{v}(\xi_1)\cdots\widehat{v}(\xi_k)$$

where $\Phi := (\xi_1 + \cdots + \xi_k)^j - (\xi_1^j + \cdots + \xi_k^j)$ (oscillation function).

Mechanism of the normal form reduction(2)

(Step 2)
Since $e^{it\Phi} = \partial_t e^{it\Phi}/i\Phi$, we have

$$\partial_t \widehat{v}(\xi) = \int_{\xi_1+\cdots+\xi_k=\xi} \partial_t\left(e^{it\Phi}/i\Phi\right)\widehat{v}(\xi_1)\cdots\widehat{v}(\xi_k)$$

By Leibniz's rule,

$$\begin{aligned}\partial_t \widehat{v}(\xi) =&\partial_t \int_{\xi_1+\cdots+\xi_k=\xi} \frac{e^{it\Phi}}{i\Phi}\widehat{v}(\xi_1)\cdots\widehat{v}(\xi_k) \\ &- \int_{\xi_1+\cdots+\xi_k=\xi} \frac{e^{it\Phi}}{i\Phi}\partial_t\left(\widehat{v}(\xi_1)\cdots\widehat{v}(\xi_k)\right)\end{aligned}$$

Mechanism of the normal form reduction(3)

(Step 3)
Substituting

$$\partial_t \widehat{v}(\xi) = \int_{\xi_1+\cdots+\xi_k=\xi} e^{it\Phi}\widehat{v}(\xi_1)\cdots\widehat{v}(\xi_k)$$

for the second term, we obtain

$$\begin{aligned}\partial_t \widehat{v}(\xi) =& \partial_t \int_{\xi_1+\cdots+\xi_k=\xi} \frac{e^{it\Phi}}{i\Phi}\widehat{v}(\xi_1)\cdots\widehat{v}(\xi_k) \\ &- k\int_{\xi_1+\cdots+\xi_{2k-1}=\xi} \frac{e^{it\Phi^*}}{i\Phi^*}\widehat{v}(\xi_1)\cdots\widehat{v}(\xi_{2k-1})\end{aligned}$$

where $\Phi^*(\xi_1, \cdots, \xi_{2k-1}) := \Phi(\xi_1, \cdots, \xi_{k-1}, \xi_k + \cdots + \xi_{2k-1})$.

Mechanism of the normal form reduction(4)

(Step 4)
In some cases, we also use the following integral form:

$$
\begin{aligned}
&\widehat{v}(t,\xi) - \widehat{v}(0,\xi) \\
=&\Big[\int_{\xi_1+\cdots+\xi_k=\xi} \frac{e^{is\Phi}}{i\Phi} \widehat{v}(s,\xi_1)\cdots\widehat{v}(s,\xi_k)\, d\xi_1\cdots d\xi_{k-1} \Big]_0^t \\
&-k\int_0^t \int_{\xi_1+\cdots+\xi_{2k-1}=\xi} \frac{e^{is\Phi^*}}{i\Phi^*} \widehat{v}(s,\xi_1)\cdots\widehat{v}(s,\xi_{2k-1})\, d\xi_1\cdots d\xi_{2k-2} ds. \\
=&(\text{no time integral, } \Phi \text{ gain}) + (\text{time integral, higher order, } \Phi^* \text{ gain})
\end{aligned}
$$

Time resonances(1)

Germain, Masmoudi and Shatah introduced "space-time resonances" in [GMS] to study global well-posedness of quadratic Schrödinger equations with small initial data.

- time resonances set

$$\mathcal{T} := \{(\xi, \xi_1) \,|\, \Phi(\xi_1, \xi_2) = 0, \xi = \xi_1 + \xi_2\}$$

- space resonances set

$$\mathcal{S} := \{(\xi, \xi_1) \,|\, \partial_{\xi_1}\Phi(\xi_1, \xi_2) = 0, \xi = \xi_1 + \xi_2\}$$

For (ξ, ξ_1) away from $\mathcal{T} \cap \mathcal{S}$,

$$\frac{1}{i(\Phi + P\partial_{\xi_1}\Phi)}\left(\partial_t + \frac{P}{t}\partial_{\xi_1}\right)e^{it\Phi} = e^{it\Phi}.$$

If we focus only on "time resonances", then it is the normal form reduction.

Time resonances(2)

The normal form reduction (time resonances) is a method to study ordinary differential equations originally. As far as I know, the first application to partial differential equations is by Shatah in '85. In [S], Shatah used it to study the small data global well-posed problem of quadratic nonlinear Klein-Gordon equations. He used the normal form reduction not to gain Φ, but to transform the quadratic nonlinear terms to higher order terms. In the study of small data global well-posedness problems, decay estimates of solutions with respect to t play an important role. We can expect faster decay with t for higher order terms.

I-method (1)

The key idea in [Iteam] is to obtain a priori estimate of H^s norm of the solution of the KdV equation:

$$(KdV)\begin{cases}\partial_t u - \partial_x^3 u = \partial_x(u^2),\ t \in \mathbf{R}, x \in \mathbf{R}\\ u(0,x) = \varphi(x) \in H^s,\end{cases}$$

where $0 > s > -3/4$. We put

$$\widehat{Iu}(\xi) := m(\xi)\widehat{u}(\xi), \quad m(\xi) := \begin{cases}1,\ |\xi| < N\\ N^{-s}|\xi|^s,\ |\xi| > 2N\end{cases}.$$

Then, $\|u\|_{H^s} \lesssim \|Iu\|_{L^2} \lesssim N^{-s}\|u\|_{H^s}$. Therefore, we will show a priori estimate of $\|Iu\|_{L^2}$ for sufficiently large N.

I-method (2)

$\int I(\text{equation}) \times Iu\,dx$ yields

$$\frac{1}{2}\partial_t \|Iu(t)\|_{L^2}^2 = \int I\partial_x(u^2)Iu\,dx.$$

We can treat the right-hand side as an error term when $s > -3/10$. Thus, we obtain

$$\|Iu(t)\|_{L^2}^2 = \|Iu(0)\|_{L^2}^2 + (\text{error}).$$

When $-3/10 \geq s > -3/4$, we can not treat the right-hand side as an error term. So, we need to add correction terms. For that purpose, we use the normal form reduction.

I-method (3)

Put $v(t) := \mathcal{F}^{-1} e^{it\xi^3} \widehat{u}(t)$. Then,

$$\partial_t \widehat{v}(\xi) = \int_{\xi_1+\xi_2=\xi} e^{it\Phi} (i\xi) \widehat{v}(\xi_1) \widehat{v}(\xi_2),$$

where $\Phi = (\xi_1 + \xi_2)^3 - \xi_1^3 - \xi_2^3 = 3\xi_1\xi_2(\xi_1 + \xi_2)$. Therefore,

$$\partial_t \|Iu\|_{L^2}^2 = \partial_t \|Iv\|_{L^2}^2 = 2 \int m(\xi) \partial_t \widehat{v}(\xi) m(-\xi) \widehat{v}(-\xi)\, d\xi$$

$$=2 \int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi} i(\xi_1 + \xi_2) m(\xi_1 + \xi_2) \widehat{v}(\xi_1) \widehat{v}(\xi_2) m(\xi_3) \widehat{v}(\xi_3),$$

where we put $\xi_3 = -\xi$.

I-method (4)

By Leibniz's rule,

$$
\begin{aligned}
=&2\int_{\xi_1+\xi_2+\xi_3=0}\left(\partial_t\frac{e^{it\Phi}}{i\Phi}\right)i(\xi_1+\xi_2)m(\xi_1+\xi_2)\widehat{v}(\xi_1)\widehat{v}(\xi_2)m(\xi_3)\widehat{v}(\xi_3)\\
=&2\partial_t\int_{\xi_1+\xi_2+\xi_3=0}\frac{e^{it\Phi}}{i\Phi}i(\xi_1+\xi_2)m(\xi_1+\xi_2)m(\xi_3)\widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3)\\
&-2\int_{\xi_1+\xi_2+\xi_3=0}\frac{e^{it\Phi}}{i\Phi}i(\xi_1+\xi_2)m(\xi_1+\xi_2)m(\xi_3)\partial_t\left(\widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3)\right)\\
=&(\text{correction term})+(\text{error}).
\end{aligned}
$$

In [Iteam], they used this argument twice. The style of the proof in [Iteam] looks different from above and they do not use the word "normal form". But, it is essentially same as the normal form reduction.

Modified energy method(1)

We consider the fifth order KdV type equations:

$$(5\text{KdV})\begin{cases}\partial_t u + \partial_x^5 u = u\partial_x^3 u,\ t\in\mathbf{R}, x\in\mathbf{R},\\ u(0,x)=\varphi(x)\in H^m.\end{cases}$$

The contraction mapping argument and the standard energy method do not work for (5KdV) because of the loss of 3 derivatives. [P]'93 Ponce ($m\ge 4$), [K]'07 Kwon ($m>5/2$) [GKK]Guo, Kwak and Kwon ($m>5/4$) proved the local well-posedness of (5KdV). In [P], [K], they use the modified energy:

$$E(t):=\|D^m u(t)\|_{L^2}^2+\|u(t)\|_{L^2}^2 + c\int u(t)D_x^{m-2}\partial_x u(t)D_x^{m-2}\partial_x u(t)\,(=:\text{correction term}).$$

Modified energy method(2)

(Outline of the proof) For simpleness, we consider only the case $m \geq 4, \in \mathbf{N}$. By the scaling argument, we can assume that $\|u_0\|_{H^m} << 1$. First, we consider the parabolic regularization of (5KdV):

$$\text{(p5KdV)} \begin{cases} \partial_t u_\varepsilon + \partial_x^5 u_\varepsilon - \varepsilon \partial_x^6 u_\varepsilon = u_\varepsilon \partial_x^3 u_\varepsilon, \ t \in \mathbf{R}, x \in \mathbf{R}, \\ u_\varepsilon(0, x) = \rho_\varepsilon(x) * \varphi(x), \end{cases}$$

where $\varepsilon > 0, \rho_\varepsilon$ is a mollifier. We can easily prove the existence of the local solution u_ε of (p5KdV) by the contraction mapping argument. By Bona-Smith's argument and taking $\varepsilon \to 0$, we have a solution u of (5KdV) as the limit of u_ε. In this process, the following a priori estimate plays an important role:

$$sup_{0 \leq t \leq 1} \|u_\varepsilon(t)\|_{H^m} \leq C(\|u_0\|_{H^m}) \tag{1}$$

where C does not depend on ε.

Modified energy method(3)

(1) follows from the following proposition and Gronwall's lemma.

Proposition 1

Let $m \geq 4$ and u_ε be sol. of (p5KdV). Then,

$$\partial_t E(t) \lesssim \|u_\varepsilon(t)\|_{H^m}^3 + \|u_\varepsilon(t)\|_{H^m}^4 \tag{2}$$

where

$$\begin{aligned} E(t) &:= \|\partial_x^m u_\varepsilon(t)\|_{L^2}^2 + \|u_\varepsilon(t)\|_{L^2}^2 \\ &\quad + c \int u_\varepsilon(t) \partial_x^{m-1} u_\varepsilon(t) \partial_x^{m-1} u_\varepsilon(t)\, dx. \end{aligned}$$

In the rest of the talk, I will explain the proof of Proposition 1 with $\varepsilon = 0$.

Modified energy method(4)

Put $v(t) := \mathcal{F}^{-1} e^{it\xi^5}\widehat{u}(t)$. Then, we have

$$\partial_t \widehat{v}(\xi) = \int_{\xi_1+\xi_2=\xi} e^{it\Phi}\widehat{v}(\xi_1)\widehat{v}(\xi_2)\, d\xi_1 \qquad (3)$$

where $\Phi := (\xi_1+\xi_2)^5 - (\xi_1^5+\xi_2^5)$. Note that $\Phi = c\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_2^2+(\xi_1+\xi_2)^2)$. The estimate of $\partial_t\|u(t)\|_{L^2}^2$ is easy. So, we will estimate only

$$\partial_t\|\partial_x^m u(t)\|_{L^2}^2 = c\int \xi^m \partial_t\widehat{v}(\xi)\xi^m\widehat{v}(-\xi)\, d\xi$$

Substitute (3) and put $\xi = -\xi_3$. Then,

$$= c\int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}(\xi_1+\xi_2)^m\widehat{v}(\xi_1)\widehat{v}(\xi_2)\xi_3^m\widehat{v}(\xi_3)$$

Modified energy method(5)

Since
$$(\xi_1+\xi_2)^m = -\xi_3(\xi_1+\xi_2)^{m-1}$$
$$= -\xi_3(\xi_2^{m-1} + c\xi_2^{m-2}\xi_1 + c\xi_2^{m-3}\xi_1^2 + \cdots),$$

we have

$$\int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}(\xi_1+\xi_2)^m \widehat{v}(\xi_1)\xi_2^3\widehat{v}(\xi_2)\xi_3^m\widehat{v}(\xi_3)$$
$$=c\int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\widehat{v}(\xi_1)\xi_2^{m+2}\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3)(:= R_1)$$
$$+c\int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\xi_1\widehat{v}(\xi_1)\xi_2^{m+1}\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3)(:= R_2)$$
$$+c\int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\xi_1^2\widehat{v}(\xi_1)\xi_2^{m}\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3)d\xi_1 d\xi_3(:= R_3)$$
$$+\sum_{\substack{0\le p,q,r\le m \\ p+q+r=2m+3}} c\int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\xi_1^p\widehat{v}(\xi_1)\xi_2^q\widehat{v}(\xi_2)\xi_3^r\widehat{v}(\xi_3)(:= R_4).$$

Modified energy method(6)

- R_4 have no derivative loss. So, by the Sobolev, $|R_4| \lesssim \|v\|_{H^m}^3 = \|u\|_{H^m}^3$.
- By symmetry, we can change the role of ξ_2 and ξ_3 to have

$$
\begin{aligned}
R_1 &= \int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\widehat{v}(\xi_1)\xi_2^{m+2}\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3) \\
&= \int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\widehat{v}(\xi_1)\xi_2^{m+1}\widehat{v}(\xi_2)\xi_3^{m+2}\widehat{v}(\xi_3) = R_1^*,
\end{aligned}
$$

$$
\begin{aligned}
2R_1 =& R_1 + R_1^* \\
=& \int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\widehat{v}(\xi_1)(\xi_2+\xi_3)\xi_2^{m+1}\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3) \\
=& -\int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\xi_1\widehat{v}(\xi_1)\xi_2^{m+1}\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3) = -R_2.
\end{aligned}
$$

Modified energy method(7)

- In the same manner, we have

$$R_3 = \int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\xi_1^2\widehat{v}(\xi_1)\xi_2^m\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3)d\xi_1 d\xi_3$$
$$= -\int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\xi_1^3\widehat{v}(\xi_1)\xi_2^m\widehat{v}(\xi_2)\xi_3^m\widehat{v}(\xi_3)d\xi_1 d\xi_3.$$

  Thus, we have $|R_3| \lesssim \|u\|_{H^m}$.
- The argument above is same as the standard energy method.
- Only R_2 is difficult to estimate. So, we apply the normal form reduction to R_2.

Modified energy method(8)

$$R_2 = \int_{\xi_1+\xi_2+\xi_3=0} e^{it\Phi}\xi_1\widehat{v}(\xi_1)\xi_2^{m+1}\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3)$$

$$= \partial_t \int_{\xi_1+\xi_2+\xi_3=0} \frac{e^{it\Phi}}{i\Phi}\xi_1\widehat{v}(\xi_1)\xi_2^{m+1}\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3)(:= \partial_t R_5)$$

$$- \int_{\xi_1+\xi_2+\xi_3=0} \frac{e^{it\Phi}}{i\Phi}\partial_t\Big(\xi_1\widehat{v}(\xi_1)\xi_2^{m+1}\widehat{v}(\xi_2)\xi_3^{m+1}\widehat{v}(\xi_3)\Big)(:= R_6).$$

Recall $\Phi = c\xi_1\xi_2\xi_3(\xi_1^2+\xi_2^2+\xi_3^2)$.

- Since $(\xi_1^2+\xi_2^2+\xi_3^2)+2\xi_2\xi_3 = 2\xi_1^2$, we have $\Phi \approx c\xi_1\xi_2^2\xi_3^2$ when $|\xi_1| << |\xi_2| \sim |\xi_3|$. Thus, $R_5 \approx c\int u\partial_x^{m-1}u\partial_x^{m-1}u\,dx$($=$ correction term of $E(t)$).
- Since $\partial_t v$ has 3 derivatives loss, R_6 has $3-2=1$ derivative loss, which is removed in the same manner as R_3 (the standard energy method).

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End

Thank you for paying attention!