Periodic nonlinear Schrödinger equations and invariant measures

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The nonlinear Schrödinger equation

Jean Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys. **166** (1994), no.1, 1-26.

The Cauchy problem for the nonlinear Schrödinger equation :

$$\text{(S)} \qquad \begin{cases} iu_t + u_{xx} + u|u|^{p-2} = 0, \\ u(x,0) = \phi(x) \in H^s(\mathbb{T}). \end{cases}$$

- $u = u(x,t) : \mathbb{T} \times I \to \mathbb{C}$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, I is an interval.
- ϕ is a given function.
- $H^s(\mathbb{T}) := \{\phi \in L^2(\mathbb{T}) : \sum_{n\in\mathbb{Z}}(1+|n|)^{2s}|\widehat{\phi}(n)|^2 < 1\}$.
- $p > 2$.
- The scale transformation: $u_\lambda(x,t) = \lambda^{2/(p-2)}u(\lambda x, \lambda^2 t)$.
- The scale critical index $s_* := 1/2 - 2/(p-2)$.
- The focusing case.

The nonlinear Schrödinger equation (sequel)

Conservation quantities: L^2-norm $N(\phi)$ and Hamiltonian (energy) $H(\phi)$:

$$N(u(t)) = N(\phi), \quad H(u(t)) = H(\phi),$$

$$N(\phi) := \|\phi\|_{L^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} |\phi(x)|^2 dx,$$

$$H(\phi) := \|\partial_x \phi\|_{L^2}^2 - \frac{2}{p}\|\phi\|_{L^p}^p = \frac{1}{2\pi}\int_0^{2\pi} |\partial_x \phi(x)|^2 dx - \frac{1}{\pi p}\int_0^{2\pi} |\phi(x)|^p dx.$$

Theorem 1 (Bourgain '93)

Let $p = 4$. Then, (S) is global well-posed (GWP) in $H^s(\mathbb{T})$ with $s \geq 0$.

Theorem 2 (Bourgain '93)

Let $p > 4$. Then, (S) is local well-posed (LWP) in $H^s(\mathbb{T})$ with $s > \max(s_*, 0)$, where $s_* := 1/2 - 2/(p-2)$.

Main result

Theorem 3

Let $4 \le p \le 6$. The (L^2-truncated) Gibbs measure of (S) is invariant under the flow.

Corollary 4

Let $4 < p \le 6$ and $0 < s < 1/2$. Then, (S) with almost every $\phi \in H^s(\mathbb{T})$ is GWP.

Strategies

1. Prove LWP of (S) and (TS).
2. Construct the Gibbs measures μ and μ_N of (S) and (TS) respectively.
3. Show invariance of μ_N.
4. Prove GWP of (S) for almost all $\phi \in H^s(\mathbb{T})$ when $4 < p \le 6$.
5. Prove invariance of μ.

Notations I

- $\widehat{\phi}$ denotes the Fourier coefficients. $\widehat{\phi}(n) := \frac{1}{2\pi}\int_0^{2\pi} e^{-inx}\phi(x)dx$.
- $L^q(\mathbb{T}) := \{\phi : \int_{\mathbb{T}} |u(x)|^q dx < \infty\}$, $\|\phi\|_{L^q} := (\frac{1}{2\pi}\int_0^{2\pi} |\phi(x)|^q dx)^{1/q}$.
- We (sometimes) abbreviate $\|\cdot\|_{L^q}$ as $\|\cdot\|_q$.
- q' denotes the Hölder conjugate of q, i.e., $1/q + 1/q' = 1$.
- $H^s(\mathbb{T}) := \{\phi \in L^2(\mathbb{T}) : \sum_{n\in\mathbb{Z}}(1+|n|)^{2s}|\widehat{\phi}(n)|^2 < \infty\}$, $\|\phi\|_{H^s} := (\sum_{n\in\mathbb{Z}}(1+|n|)^{2s}|\widehat{\phi}(n)|^2)^{1/2}$.
- $H_0^s(\mathbb{T}) := \{\phi \in H^s(\mathbb{T}) : \widehat{\phi}(0) = 0\}$.
- $B_R^{(s)} := \{\phi \in H^s(\mathbb{T}) : \|\phi\|_{H^s} \leq R\}$.
- $E_N := \text{span}\{e^{inx} : |n| \leq N\} \cong \mathbb{C}^{2N+1} \cong \mathbb{R}^{2(2N+1)}$.
- $E_{N,0} := \text{span}\{e^{inx} : 0 \neq |n| \leq N\} \cong \mathbb{C}^{2N} \cong \mathbb{R}^{4N}$.
- For $\phi \in E_N$, we identify ϕ and $a^N := \{a_n\}_{|n|\leq N}$ through $\phi(x) = \sum_{|n|\leq N} e^{inx}\widehat{\phi}(n)$.
- $e^{it\partial_x^2}$ denotes the free propagation of the Schrödinger equation, i.e., $u(x,t) = e^{it\partial_x^2}\phi$ solves $(i\partial_t + \partial_x^2)u = 0$, $u(x,0) = \phi(x)$. $e^{it\partial_x^2}\phi(x) := \sum_{n\in\mathbb{Z}} e^{i(nx-n^2t)}\widehat{\phi}(n)$.

Notations II

- $X \lesssim Y$ means $X \le CY$ for some $C > 1$.
- $X \ll Y$ means $\frac{1}{C} X \le Y$.
- $X \sim Y$ means $X \lesssim Y$ and $Y \lesssim X$.
- The capital letters L, M, M_1, N denote dyadic numbers, e.g., $L,\ M,\ M_1,\ N \in 2^{\mathbb{N}}$.
- $S_N \phi(x) := \sum_{|n| \le N} e^{inx} \widehat{\phi}(n)$. Define $S_{1/2}\phi := 0$.
- $P_N := S_N - S_{N/2}$.
- $\Delta_I := \sum_{n \in I} e^{inx} \widehat{\phi}(n)$ for some interval $I \subset \mathbb{R}$.
- $\Lambda_{L,N} := \{(n, \lambda) \in \mathbb{Z} \times \mathbb{R} : N/2 < |n| \le N,\ \langle \lambda + n^2 \rangle \sim L\}$.
- $\Lambda_{L,I} := \{(n, \lambda) \in I \times \mathbb{R} : \langle \lambda + n^2 \rangle \sim L\}$ for some interval $I \subset \mathbb{R}$.
- $\Phi(t)$ and $\Phi_N(t)$ denote the flow map of (S) and (TS) respectively.

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NLS and truncated NLS

The Cauchy problem for the NLS:

(S)
$$\begin{cases} iu_t + u_{xx} + u|u|^{p-2} = 0, \\ u(x,0) = \phi(x) \in H^s(\mathbb{T}). \end{cases}$$

The finite dimensional model equation:

(TS)
$$\begin{cases} iu_t + u_{xx} + S_N(u|u|^{p-2}) = 0, \\ u(x,0) = \phi(x) \in E_N. \end{cases}$$

First of all, we show that these Cauchy problems are locally well-posed. There is essentially no change in the argument (cf. Bourgain '93).

1. The Strichartz estimates on $\mathbb{T}$.
2. The Bourgain spaces.
3. Related estimates.
4. Proof of LWP.

The Strichartz estimates I

Proposition 5 (L^4 Strichartz estimate)

$$\|e^{it\partial_x^2}\phi\|_{L^4_{t,x}(\mathbb{T}^2)} \lesssim \|\phi\|_{L^2} \quad \textit{for } \phi \in L^2(\mathbb{T}).$$

Proof.

$$(2\pi)^2\|e^{it\partial_x^2}\phi\|^4_{L^4_{t,x}(\mathbb{T}^2)} = \int_{\mathbb{T}^2}(e^{it\partial_x^2}\phi)^2 \cdot \overline{(e^{it\partial_x^2}\phi)^2}dtdx$$

$$= \sum_{n_1,\dots,n_4} \iint e^{i(n_1+n_2-n_3-n_4)x}e^{-i(n_1^2+n_2^2-n_3^2-n_4^2)t}\widehat{\phi}(n_1)\widehat{\phi}(n_2)\overline{\widehat{\phi}(n_3)\widehat{\phi}(n_4)}dtdx.$$

Here, $n_1+n_2-n_3-n_4=0$ and $n_1^2+n_2^2-n_3^2-n_4^2=0$ are equivalent to $(n_1=n_3$ and $n_2=n_4)$ or $(n_1=n_4$ and $n_2=n_3)$. Thus, it is equal to

$$\sum_{n_1,n_2}|\widehat{\phi}(n_1)|^2|\widehat{\phi}(n_2)|^2 = \|\phi\|^4_{L^2}.$$

□

The Strichartz estimates II

Proposition 6 (Almost L^6 Strichartz estimate)

$$\|S_N e^{it\partial_x^2}\phi\|_{L^6_{t,x}(\mathbb{T}^2)} \lesssim N^{\varepsilon}\|\phi\|_{L^2} \quad \textit{for } \phi \in L^2(\mathbb{T}).$$

We can not remove the loss N^{ε}, which cause difficulties on $\mathbb{T}$.

Proof.

$$\|e^{it\partial_x^2}S_N\phi\|^6_{L^6_{t,x}(\mathbb{T}^2)} = \|e^{it\partial_x^2}S_N\phi|e^{it\partial_x^2}S_N\phi|^2\|^2_{L^2_{t,x}(\mathbb{T}^2)}$$

$$= \Big\| \sum_{\substack{n_1,n_2,n_3,\\ |n_j|\le N}} e^{i(n_1-n_2+n_3)x}e^{-i(n_1^2-n_2^2+n_3^2)t}\widehat{\phi}(n_1)\overline{\widehat{\phi}(n_2)}\widehat{\phi}(n_3)\Big\|^2_{L^2_{t,x}}$$

$$= \sum_{n,\lambda\in\mathbb{Z}} \Big| \sum_{(n_1,n_2)\in\Gamma(n,\lambda)} \widehat{\phi}(n_1)\overline{\widehat{\phi}(n_2)}\widehat{\phi}(n-n_1+n_2)\Big|^2,$$

where $\Gamma(n,\lambda) := \left\{(n_1,n_2)\in\mathbb{Z}^2 : \begin{array}{l} |n_j|\le N,\ |n-n_1+n_2|\le N, \\ \lambda = -n_1^2+n_2^2-(n-n_1+n_2)^2 \end{array}\right\}.$

The Strichartz estimates III

Proof (sequel).

From Hölder's inequality,

$$\|e^{it\partial_x^2} S_N \phi\|_{L^6_{t,x}(\mathbb{T}^2)}^6 \leq \sup_{(n,\lambda)\in\mathbb{Z}^2} \#\Gamma(n,\lambda) \|\phi\|_{L^2}^6.$$

Note that for every $n, \lambda \in \mathbb{Z}$ and $(n_1, n_2) \in \Gamma(n,\lambda)$

$$\lambda + n^2 = -2(n_1 - n)(n_1 - n_2).$$

By Lemma 7 (below), $\sup_{(n,\lambda)\in\mathbb{Z}^2} \#\Gamma(n,\lambda) \lesssim e^{c\frac{\log N}{\log\log N}} \lesssim N^{\varepsilon}$, which concludes the proof. □

Lemma 7 (divisor counting (see Theorem 317 in Hardy and Wright "An introduction to the Theory of numbers"))

$$\text{“the number of divisors of } A\text{”} \leq C \exp\left(\frac{c\log A}{\log\log A}\right) \text{ for } A \in \mathbb{N}.$$

The Bourgain spaces I

Definition 8 (The Bourgain spaces)

For $s, b \in \mathbb{R}$, *we define*

$$\|u\|_{X^{s,b}} := \Big(\sum_{n \in \mathbb{Z}} \int \langle n \rangle^{2s} \langle \lambda + n^2 \rangle^{2b} |\widehat{u}(n,\lambda)|^2 d\lambda \Big)^{1/2}.$$

For an interval $I \subset \mathbb{R}$, *we define* $\|u\|_{X_I^{s,b}} := \inf\{\|v\|_{X^{s,b}} : u = v \ \textit{on}\ I\}$.

Proposition 9 (Linear estimates)

Let $s \in \mathbb{R}$, $1/2 < b \leq 1$, $0 < T \leq 1$. *Also, let* $0 \leq \delta \leq 1 - b$. *Then,*

$$\|e^{it\partial_x^2}\phi\|_{X^{s,b}} \lesssim \|\phi\|_{H^s}, \quad \Big\| \int_0^t e^{i(t-t')\partial_x^2} G(x,t') dt' \Big\|_{X^{s,b}_{[-T,T]}} \lesssim T^\delta \|G\|_{X^{s,b-1+\delta}_{[-T,T]}}.$$

See, for the proof, Lemma 2.11 and Proposition 2.12 in T. Tao "Nonlinear Dispersive Equations, local and global analysis, CBMS 106."

The Bourgain spaces II

Thanks to Proposition 9, it suffices to prove

$$\|u|u|^{p-2}\|_{X^{s,-b_1}} \lesssim \|u\|_{X^{s,b}}^{p-1}, \tag{1}$$

$$\|u|u|^{p-2} - v|v|^{p-2}\|_{X^{s,-b_1}} \lesssim (\|u\|_{X^{s,b}}^{p-2} + \|v\|_{X^{s,b}}^{p-2})\|u-v\|_{X^{s,b}} \tag{2}$$

for $0 < b_1 < 1/2 < b$.
(S) is equivalent to

$$u(x,t) = e^{it\partial_x^2}\phi(x) + i\int_0^t e^{i(t-t')\partial_x^2}(u|u|^{p-2})(x,t')dt'. \tag{3}$$

Let $\Theta(u)$ denote the right hand side of (3). (1) and (2) imply

$$\|\Theta(u)\|_{X^{s,b}_{[-\tau,\tau]}} \leq C\|\phi\|_{H^s} + \tau^{\delta}\|u\|^{p-1}_{X^{s,b}_{[-\tau,\tau]}},$$

$$\|\Theta(u)-\Theta(v)\|_{X^{s,b}_{[-\tau,\tau]}} \leq \tau^{\delta}(\|u\|_{X^{s,b}_{[-\tau,\tau]}} + \|v\|_{X^{s,b}_{[-\tau,\tau]}})^{p-2}\|u-v\|_{X^{s,b}_{[-\tau,\tau]}}.$$

Θ is a contraction mapping on $\mathcal{X}_\tau$, where

$$\mathcal{X}_\tau := \{u \in X^{s,b}_{[-\tau,\tau]} : \|u\|_{X^{s,b}_{[-\tau,\tau]}} \leq 2C\|\phi\|_{H^s}\},\ (2C\|\phi\|_{H^s})^{p-2}\tau^{\delta} < \frac{1}{4},$$

and b is chosen closely to $1/2$.

The case $p = 4$ I

Proposition 10

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,3/8}}.$$

Proof.

Write

$$u = \sum_L Q_L u, \quad \mathcal{F}[Q_L u](n,\lambda) = u\chi_{\langle \lambda + n^2 \rangle \sim L}.$$

$$\|u\|^2_{L^4_{t,x}} = \|uu\|_{L^2_{t,x}} \lesssim \sum_{L_1} \sum_{L_2 \le L_1} \|Q_{L_1}u \cdot Q_{L_2}u\|_{L^2_{x,t}}.$$

Let $L_1 = 2^l L_2$. It is reduced to show that

$$(4) \qquad \|Q_L u \cdot Q_{2^l L}u\|_{L^2_{t,x}} \lesssim 2^{-\varepsilon l} L^{3/8} \|Q_L u\|_{L^2_{t,x}} (2^l L)^{3/8} \|Q_{2^l L} u\|_{L^2_{t,x}}$$

for some $\varepsilon > 0$. Put $U_L := Q_L u / \|Q_L u\|_{L^2_{t,x}}$. (4) is equivalent to

$$\Big\| \sum_{n_1 \in \mathbb{Z}} \int \widehat{U_L}(n_1, \lambda_1) \widehat{U_{2^l L}}(n - n_1, \lambda - \lambda_1) d\lambda_1 \Big\|_{L^2_{n,\lambda}} \lesssim 2^{(3/8-\varepsilon)l} L^{3/4}.$$

The case $p = 4$ II

Proof (sequel).

By the Cauchy-Schwartz inequality, it is sufficient to show

$$\sup_{(n,\lambda)\in\mathbb{Z}\times\mathbb{R}} \#\Gamma(n,\lambda) \lesssim 2^{(3/4-2\varepsilon)l} L^{3/2}, \tag{5}$$

$\Gamma(n,\lambda) := \{(n_1,\lambda_1) \in \mathbb{Z}\times\mathbb{R} : |\lambda_1 + n_1^2| \lesssim L,\ |\lambda - \lambda_1 + (n-n_1)^2| \lesssim 2^l L\}.$

Since

$$\lambda = \lambda_1 + (\lambda - \lambda_1) = -2n_1^2 + 2nn_1 - n^2 + O(2^l L),$$

$$n_1 = \frac{n \pm \sqrt{-n^2 - 2\lambda}}{2} + O(2^{l/2} L^{1/2}),$$

we have

$$\#\Gamma(n,\lambda) \lesssim 2^{l/2} L^{3/2},$$

which shows (5) with $\varepsilon = 1/8$. □

The case $p = 4$ III

The L^4-Strichartz estimate and the transference principle[1] show

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,b}} \tag{6}$$

for any $b > 1/2$.

Thanks to Proposition 10, (6) and the fractional chain rule, we obtain

$$\begin{aligned}\|u|u|^2\|_{X^{s,-b_1}} &\le \|\partial_x^s(u|u|^2)\|_{X^{0,-3/8}} \lesssim \|\partial_x^s(u|u|^2)\|_{L^{4/3}_{t,x}} \\ &\lesssim \|u\|^2_{L^4_{t,x}} \|\partial_x^s u\|_{L^4_{t,x}} \lesssim \|u\|^2_{X^{0,b}} \|u\|_{X^{s,b}}\end{aligned}$$

if $3/8 \le b_1 < 1/2 < b$. Similarly,

$$\|u|u|^2 - v|v|^2\|_{X^{s,-b_1}} \lesssim (\|u\|_{X^{0,b}} + \|v\|_{X^{0,b}})^2 \|u - v\|_{X^{s,b}},$$

which shows LWP for $p = 4$.

By the L^2-conservation law, we can extend the local solution obtained above to global one.

[1] See Lemma 2.9 in T. Tao "Nonlinear Dispersive Equations, local and global analysis, CBMS 106."

Related estimates I

Lemma 11

For $b > 1/2$ *and* $q \geq 6$,

$$\|P_N u\|_{L^q_{t,x}} \lesssim N^{1/2-3/q+\varepsilon}\|P_N u\|_{X^{0,b}}.$$

Proof.

It suffices to show the following:

$$\|e^{it\partial_x^2}P_N\phi\|_{L^q_{t,x}} \lesssim N^{1/2-3/q+\varepsilon}\|P_N\phi\|_{L^2} \tag{7}$$

because of the transference principle. Interpolating Proposition 6 with

$$\|e^{it\partial_x^2}P_N\phi\|_{L^\infty_{t,x}} \lesssim N^{1/2}\|e^{it\partial_x^2}P_N\phi\|_{L^\infty_t L^2_x} \lesssim N^{1/2}\|P_N\phi\|_{L^2},$$

we obtain (7). □

Related estimates II

Lemma 12

For $2 < q < 6$ *and* $b > 1/2$,

$$\|P_N u\|_{L^q_{t,x}} \lesssim N^{\varepsilon(1-\theta)} \|P_N u\|_{X^{0,b}}, \tag{8}$$

$$\|P_{\Lambda_{L,N}} u\|_{L^2_{t,x}} \lesssim L^{(1-\theta)/2} N^{\varepsilon(1-\theta)} \|u\|_{L^{q'}}, \tag{9}$$

where $\theta = 3/q - 1/2$, $1/q' = 1 - 1/q$,

$$\mathcal{F}[P_{\Lambda_{L,N}} u](\lambda, n) = \widehat{u}(\lambda, n) \chi_{N<|n|\leq 2N} \chi_{\langle \lambda - n^2 \rangle \sim L}.$$

The estimate (9) follows from duality argument.

Proof of Theorem 2 I

We only consider the case where $p > 4$ is even for simplicity.
Put $w := u|u|^{p-2}$. Write

$$w = \sum_M \left(S_M u |S_M u|^{p-2} - S_{M/2} u |S_{M/2} u|^{p-2} \right) \tag{10}$$

Without loss of generality, we may assume $M \gtrsim N$. One may write for complex values z, w

$$z|z|^{p-2} - w|w|^{p-2} = (z-w)\varphi_1(z,w) + (\overline{z} - \overline{w})\varphi_2(z,w),$$

where φ_j satisfy $|\nabla \varphi_j| \lesssim (|z| + |w|)^{p-3}$. Substituting in (10) with $z = S_M u$ and $w = S_{M/2} u$, we get

$$w = \sum_M \left(P_M u \cdot \varphi_1(S_M u, S_{M/2} u) + \overline{P_M u} \cdot \varphi_2(S_M u, S_{M/2} u) \right).$$

Proof of Theorem 2 II

Putting $v_M := \varphi_1(S_M u, S_{M/2} u)$ and $v_{1/2} := 0$, we write again

$$v_M = (v_M - v_{M/2}) + \cdots + (v_1 - v_{1/2}) = \sum_{M_1 \leq M} (v_{M_1} - v_{M_1/2}).$$

Since φ_1 is Lipschitz continuous, we have

$$v_{M_1} - v_{M_1/2} = P_{M_1} u \cdot \psi_1 + \overline{P_{M_1} u} \cdot \psi_2 + P_{M_1/2} u \cdot \psi_3 + \overline{P_{M_1/2} u} \cdot \psi_4,$$

where $\psi_j = \psi_j(S_{M_1} u, S_{M_1/2} u, S_{M_1/4} u)$ satisfy
$|\psi_j| \lesssim (|S_{M_1} u| + |S_{M_1/2} u| + |S_{M_1/4} u|)^{p-3}$.
Hence, we have to estimate

$$\sum_{L,N} \sum_{M \gtrsim N} \sum_{M_1 \leq M} N^s L^{-b_1} \left\| P_{\Lambda_{L,N}} (P_M u \cdot P_{M_1} u \cdot \psi) \right\|_{L^2_{t,x}},$$

where ψ denotes one of ψ_1 and ψ_2.
Decompose the interval $[M/2, M]$ as follows:

$$[\tfrac{M}{2}, M] = \bigcup_{k=1}^{M/2M_1} I_k, \quad I_k := [\tfrac{M}{2} + (k-1)M_1, \tfrac{M}{2} + kM_1].$$

Proof of Theorem 2 III

Then, one has $P_M u = \sum_{k=1}^{M/2M_1} \Delta_{I_k} u$, where $\Delta_I \phi := \sum_{n\in I} e^{inx}\widehat{\phi}(n)$.
The functions

$$w_{I_k} := \Delta_{I_k} u \cdot P_{M_1} u \cdot \psi(S_{M_1} u, S_{M_1/2} u, S_{M_1/4} u)$$

have essentially disjoint supported Fourier transform of varying k.
Thus, one has to estimate the following:

$$\sum_{L,N} \sum_{M \gtrsim N} \sum_{M_1 \leq M} N^s L^{-b_1} \Big(\sum_{k=1}^{M/2M_1} \|\widehat{w_{I_k}}\|_{L^2(\Lambda_{L,I_k})}^2 \Big)^{1/2}.$$

Choose $2 < p_1 < 6$. From (9) in Lemma 12 with $q = p_1$,

$$\|\widehat{w_{I_k}}\|_{L^2(\Lambda_{L,I_k})} \lesssim L^{(1-\theta)/2} M_1^{\varepsilon(1-\theta)} \|w_{I_k}\|_{L^{p_1'}}, \quad \theta = \frac{3}{p_1} - \frac{1}{2}.$$

Thanks to Hölder's inequality, we get

$$\|w_{I_k}\|_{L^{p_1'}} \leq \|\Delta_{I_k} u\|_{L^{p_1}} \|P_{M_1} u \cdot \psi\|_{p_1 p_1'/(p_1 - p_1')}.$$

Proof of Theorem 2 IV

By the orthogonality and (8) in Lemma 12, we have

$$\Big(\sum_{k=1}^{M/2M_1} \|\Delta_{I_k} u\|_{L^2}^2 \Big)^{1/2} \lesssim M_1^{\varepsilon(1-\theta)} \Big(\sum_{k=1}^{M/2M_1} \|\Delta_{I_k} u\|_{X^{0,b}}^2 \Big)^{1/2}$$
$$\lesssim M_1^{\varepsilon(1-\theta)} \|P_M u\|_{X^{0,b}} \sim M_1^{\varepsilon(1-\theta)} M^{-s} \|P_M u\|_{X^{s,b}}.$$

Let $p_2 \geq 6$ and $1 > 2/p_1 + 1/p_2$. Hölder's inequality and Lemma 11 imply

$$\|P_{M_1} u \cdot \psi\|_{p_1 p_1'/(p_1 - p_1')} \lesssim \|P_{M_1} u\|_{L^{p_2}} \|\psi\|_{(1-2/p_1-1/p_2)^{-1}}$$
$$\lesssim M_1^{1/2-3/p_2+\varepsilon-s} \|P_{M_1} u\|_{X^{s,b}} \|S_{M_1} u\|_{(p-3)(1-2/p_1-1/p_2)^{-1}}^{p-3}.$$

Taking p_3 such that $p_3 \geq 6$ and $(p-3)/p_3 \leq 1 - 2/p_1 - 1/p_2$, we have

$$\|S_{M_1} u\|_{(p-3)(1-2/p_1-1/p_2)^{-1}}$$
$$\lesssim \sum_{M_2 \leq M_1} \|P_{M_2} u\|_{p_3} \lesssim \sum_{M_2 \leq M_1} M_2^{1/2-3/p_3+\varepsilon-s} \|P_{M_2} u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}}$$

provided that $s > 1/2 - 3/p_3$.

Proof of Theorem 2 V

We therefore have

$$\|P_{M_1} u \cdot \psi\|_{p_1 p_1'/(p_1 - p_1')} \lesssim M_1^{1/2-3/p_2+\varepsilon-s} \|P_{M_1} u\|_{X^{s,b}} \|u\|_{X^{s,b}}^{p-3}.$$

Combing it with above estimates, we obtain

$$\|w\|_{X^{s,-b_1}} \lesssim \|u\|_{X^{s,b}} \|u\|_{X^{s,b}} \|u\|_{X^{s,b}}^{p-3},$$

provided that

$$2 < p_1 < 6, \quad p_2 \geq 6, \quad p_3 \geq 6,$$
$$1 > \frac{2}{p_1} + \frac{1}{p_2}, \quad \frac{p-3}{p_3} \leq 1 - \frac{2}{p_1} - \frac{1}{p_2},$$
$$s > \frac{1}{2} - \frac{3}{p_3}, \quad s > \frac{1}{2} - \frac{3}{p_2}.$$

We can choose p_1, p_2, p_3 which satisfy the all conditions if $s > \max(1/2 - 2/(p-2), 0)$.

Remarks on well-posednes

- The existence time τ depends on $\|\phi\|_{H^s}$. More precisely, one has
$$\tau > \frac{C(p,s)}{(1+\|\phi\|_{H^s})^{C_1(p,s)}}.$$
- This existence time τ does not depend on N even if we consider the truncated equation (TS).
- The constant $C_1(p,s)$ does not depend on s for $p<6$. This fact is however not of importance for the sequel.
- For $p \geq 4$, $s, s_0 > \max(s_*, 0)$, the same calculation shows
$$\|u|u|^{p-2}\|_{X^{s,-b_1}} \lesssim \|u\|_{X^{s,b}} \|u\|^{p-2}_{X^{s_0,b}}.$$

Remark

In general case, some more technicalities are needed because the nonlinear term $u|u|^{p-2}$ is not smooth.

The finite dimensional approximation I

Lemma 13

Let $4 \leq p \leq 6$, $s > 0$, $\phi \in H^s(\mathbb{T})$, $\|\phi\|_{H^s} \leq A$. *Assume the solution* u_N *of (TS) with data* $S_N\phi$ *satisfies*

$$\|u_N(t)\|_{H^s} \leq A \text{ for } |t| \leq T.$$

Then, (S) is WP on $[-T, T]$ *and there is the approximation for* $|t| \leq T$ *and* $0 < s_0 < s$

$$\|u(t) - u_N(t)\|_{H^{s_0}} < \exp(C(p,s)(1+A)^{C_1(p,s)}T)AN^{s_0 - s} \tag{11}$$

provided that the expression on the right hand side of (11) remains < 1.

We will only consider $t > 0$. Let τ be the existence time given by LWP. Note that

$$\tau > \frac{C(p,s)}{(1+A)^{C_1(p,s)}}.$$

Assume for $t \leq t_0$ we obtain

$$\|u(t) - u_N(t)\|_{H^{s_0}} < \delta < 1.$$

The finite dimensional approximation II

Thanks to LWP, the IVPs

$$i\partial_t u + \partial_x^2 u + u|u|^{p-2} = 0, \quad u|_{t=t_0} = u(t_0),$$

$$i\partial_t v + \partial_x^2 v + v|v|^{p-2} = 0, \quad v(t_0) = u_N(t_0)$$

are WP for $t \in [t_0, t_0 + \tau]$. Moreover, we have

(12) $$\|u(t) - v(t)\|_{H^{s_0}} \le 2\|u(t_0) - v(t_0)\|_{H^{s_0}} < 2\delta.$$

Compare u_N and v on $[t_0, t_0 + \tau]$. From LWP, one has

$$\|v\|_X \lesssim \|v(t_0)\|_{H^{s_0}} \lesssim CA, \quad \|u_N\|_X \lesssim \|u_N(t_0)\|_{H^{s_0}} \lesssim CA,$$

where $X := X^{s_0,b}_{[t_0,t_0+\tau]}$ for some $b > 1/2$. Write

$$v(t) - u_N(t) = i\int_0^t e^{i(t-t')\partial_x^2}\Gamma(t')dt', \quad \Gamma := v|v|^{p-2} - S_N(u_N|u_N|^{p-2}).$$

By $\Gamma = (v|v|^{p-2} - S_N(v|v|^{p-2})) + S_N(v|v|^{p-2} - u_N|u_N|^{p-2})$, the same argument in the proof of LWP implies

$$\|v - u_N\|_X \lesssim \tau^\delta(\|v - S_N v\|_X\|v\|_X^{p-2} + \|v - u_N\|_X(\|v\|_X + \|u_N\|_X)^{p-2}).$$

The finite dimensional approximation III

Thanks to the choice of τ, we get

$$(13) \qquad \|v - u_N\|_X < \|v - S_N v\|_X \lesssim N^{s_0 - s}\|v(t_0)\|_{H^s} \leq CAN^{s_0 - s}.$$

From (12) and (13), we obtain

$$\|v(t) - u_N(t)\|_{H^{s_0}} \leq 2\delta + CAN^{s_0 - s}, \quad t_0 \leq t \leq t_0 + \tau.$$

Break the interval $[0, t]$ up in subintervals of length τ. For $t_j := j\tau$ $(j = 0, \dots, [T/\tau])$, denoting $\|u(t_j) - u_N(t_j)\|_{H^{s_0}}$ by δ_j, we have

$$\delta_0 < N^{s_0 - s}A, \quad \delta_j < 2\delta_{j-1} + CAN^{s_0 - s},$$

which implies

$$\delta_j < 2^j \delta_0 + (2^j - 1)CAN^{s_0 - s} < C2^j AN^{s_0 - s}.$$

By the lower bound of τ, we obtain

$$\|u(t) - u_N(t)\|_{H^{s_0}} \leq \exp(C(1 + A)^{C_1(p,s)}T)AN^{s_0 - s}, \quad 0 \leq t \leq T.$$

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Basic terminology of probability theory

- We call a measure space $(\Omega, \mathcal{F}, P)$ a *probability space* if $P(\Omega) = 1$.
- Let $X : \Omega \to \mathbb{R}$ be a random variable (i.e., X is measurable.). We define its *expected value* to be $E[X] := \int_\Omega X(\omega) dP(\omega)$ if $X \in L^1(\Omega)$. We call a measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ the *distribution* of X if $\lambda(A) = P(X^{-1}(A))$ for all $A \in \mathcal{B}(\mathbb{R})$.
- We call a r.v. *real Gaussian* if its distribution is given by $e^{-x^2/2}/\sqrt{2\pi}dx$.
- The r.v.s X, Y are *independent* if for all $A, B \in \mathcal{B}(\mathbb{R})$ the following equality holds:

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

- We call a r.v. g *complex Gaussian* if there exist independent real Gaussian g_1, g_2 such that $g = g_1 + ig_2$.
- If X, Y are independent r.v.s, one has $E[XY] = E[X]E[Y]$.
- If X, Y are independent r.v. and g, h are measurable, then $g(X, Y)$ and $h(X, Y)$ are independent. Especially, e^X and e^Y are independent.

The Gibbs measure of (S) I

Recall our Cauchy problem:

$$\text{(S)} \qquad \begin{cases} iu_t + u_{xx} + u|u|^{p-2} = 0, \\ u(x,0) = \phi(x) \in H^s(\mathbb{T}). \end{cases}$$

The Hamiltonian is *formally* conserved:

$$H(\phi) = \|\partial_x \phi\|_{L^2}^2 - \frac{2}{p}\|\phi\|_{L^p}^p = \frac{1}{2\pi}\int_0^{2\pi} |\partial_x \phi|^2 dx - \frac{1}{\pi p}\int_0^{2\pi} |\phi|^p dx.$$

For $s < 1/2$, we denote by $H_0^s(\mathbb{T})$ the set $\{\phi \in H^s(\mathbb{T}) : \widehat{\phi}(0) = 0\}$.
Let ρ be the image measure under the map $X : \Omega \to H_0^s(\mathbb{T})$

$$\omega \mapsto X(\omega) := \sum_{n \neq 0} \frac{g_n(\omega)}{n} e^{inx},$$

where the $\{g_n\}$ are independent complex Gaussian random variables.

Since the random Fourier series $\sum_{n\neq 0} \frac{g_n(\omega)}{n} e^{inx}$ is in $\mathcal{H}_0 := \bigcap_{s<1/2} H_0^s(\mathbb{T})$ a.s., we may consider ρ as a measure on $\mathcal{H}_0$.

The Gibbs measure of (S) II

Proposition 14

Let $s < 1/2$*. There exist constants* $C, c > 0$ *such that for all* $N_0 \in 2^{\mathbb{N} \cup \{0\}}$, $\lambda \geq 1$ *one has*

$$\rho(\{\phi \in H_0^s(\mathbb{T}) : \|(1 - S_{N_0})\phi\|_{H^s} > \lambda\}) \leq C \exp(-c\lambda^2 N_0^{2(1-s)}).$$

In particular, $\rho(H^s(\mathbb{T})) = 1$*.*

Proof.

For $N_0 \geq 1$, we set

$$A_{N_0} := \{\omega \in \Omega : \|(1 - S_{N_0})X(\omega)\|_{H^s} > \lambda\}.$$

Note that $\rho(\{\phi \in H_0^s(\mathbb{T}) : \|(1 - S_{N_0})\phi\|_{H^s} > \lambda\}) = P(A_{N_0})$.
Let θ be a real number such that $0 < \theta < 1/2 - s$. Next, we set

$$A'_N := \{\omega \in \Omega : \|P_N X(\omega)\|_{H^s} > \tfrac{\lambda}{2}(N^{-\theta} + (N^{-1}N_0)^{1-s})\}.$$

Then, $A_{N_0} \subset \bigcup_{N \geq N_0} A'_N$ holds.

Proof (sequel).

From Lemma 15 (below), we have

$$P(A_{N_0}) \le \sum_{N \ge N_0} P(A'_N) \le \sum_{N \ge N_0} C \exp(c_1 N - c_2 \lambda^2 (N^{2(1-s)-2\theta} + N_0^{2(1-s)})).$$

The choice of θ implies $1 < 2(1-s) - 2\theta$ and thus

$$P(A_{N_0}) \le C \exp(-c\lambda^2 N_0^{2(1-s)}).$$

□

Lemma 15

Let Λ be a finite subset of $\mathbb{N}$. For $\lambda > 0$, we have

$$P(\sum_{n \in \Lambda} |g_n(\omega)|^2 > \lambda) < e^{c_1 \# \Lambda - c_2 \lambda}.$$

Noting that $E[e^{|g_n|^2/4}] = 2$, we have
$P(\sum_{n\in\Lambda} |g_n(\omega)|^2 > \lambda) = P(\prod_{n\in\Lambda} e^{|g_n(\omega)|^2/4} > e^{\lambda/4})$
$\le e^{-\lambda/4} E[\prod_{n\in\Lambda} e^{|g_n|^2/4}] = e^{-\lambda/4} \prod_{n\in\Lambda} E[e^{|g_n|^2/4}] < e^{-\lambda/4} 2^{\#\Lambda}.$

The Gibbs measure of (S) IV

Let us define

$$f(\phi) := \exp(\tfrac{1}{p}\|\phi\|_{L^p}^p)\chi_{\{\|\phi\|_{L^2}\le B\}},$$

where B is the L^2-cutoff.

Lemma 16

Let $1 \le q \le 2$. Then, we have $f \in L^q(d\rho)$ for $p < 6$ and arbitrary B and for $p = 6$ and sufficiently small B.

We set

$$d\mu(\phi) := f(\phi)d^2a_0 d\rho(\phi).$$

where a_0 denotes $\widehat{\phi}(0)$ and $d^2a_0 := da_0 d\overline{a_0}$.

- Lemma 16 shows that this measure ρ is well-defined and a measure on $\mathcal{H} := \bigcap_{0<s<1/2} H^s(\mathbb{T})$.
- If $p < 6$, $f \in L^q(d\rho)$ for all $1 \le q < \infty$.

The Gibbs measure of (S) V

Proposition 17

Let $0 \le s < 1/2$. *There exist constants* $C, c > 0$ *such that for all* $\lambda \ge 1$ *one has*

$$\mu(\{\phi \in H^s(\mathbb{T}) : \|\phi\|_{H^s} > \lambda\}) \le Ce^{-c\lambda^2}.$$

Proof.

Set $A_\lambda := \{\phi \in H^s(\mathbb{T}) : \|\phi\|_{H^s} > \lambda\}$. Using Proposition 14 and Lemma 16, we can write

$$\begin{aligned}
\mu(A_\lambda) &= \int_{A_\lambda} d\rho = \int_{A_\lambda} f(\phi) da_0 d\rho(\phi) \\
&\le Be^B \Big(\int_{A_\lambda \cap H_0^s(\mathbb{T})} f^2(\phi) d\rho(\phi) \Big)^{1/2} \Big(\int_{A_\lambda \cap H_0^s(\mathbb{T})} d\rho \Big)^{1/2} \\
&\le C\rho(A_\lambda \cap H_0^s(\mathbb{T})) \le Ce^{-c\lambda^2}.
\end{aligned}$$

□

Proof of Lemma 16 I

Note that

$$|f|^q \in L^1(d\rho) \Leftrightarrow \int_0^\infty \rho(\{\phi \in \mathcal{H} : |f|^q > \lambda\})d\lambda < \infty \Leftrightarrow \int_1^\infty g(\lambda)d\lambda < \infty,$$

where $g(\lambda) := P(\{\omega \in \Omega : \|X(\omega)\|_{L^p} > \gamma, \ \|X(\omega)\|_{L^2} \leq B\})$,
$X(\omega) = \sum_{n\neq 0} g_n(\omega)e^{inx}/n$, and $\gamma := (p(\log \lambda)/q)^{1/p}$.
Let $s := 1/2 - 1/p$. By $H^s(\mathbb{T}) \hookrightarrow L^p(\mathbb{T})$, we have

$$g(\lambda) \leq P(\{\omega \in \Omega : \|X(\omega)\|_{H^s} > \gamma/C_s, \ \|X(\omega)\|_{L^2} \leq B\})$$

Set $N_0 := \kappa\gamma^{1/s}$, where $\kappa > 0$ is small number to be fixed. Then,

$$\{\omega \in \Omega : \|X(\omega)\|_{H^s} > \gamma/C_s, \ \|X(\omega)\|_{L^2} \leq B\} \subset A_1 \cup A_2$$

with

$$A_1 := \{\omega \in \Omega : \|S_{N_0}X(\omega)\|_{H^s} > \gamma/4C_s, \ \|X(\omega)\|_{L^2} \leq B\},$$
$$A_2 := \{\omega \in \Omega : \|(1 - S_{N_0})X(\omega)\|_{H^s} > \gamma/4C_s\}.$$

Proof of Lemma 16 II

Since

$$\|S_{N_0}X(\omega)\|_{H^s} \le CN_0^s\|X(\omega)\|_{L^2} \le C\kappa^s\gamma B,$$

$A_1 = \emptyset$ if $\kappa = (5C_sC)^{-1/s}B^{-1/s}$. This fixes the parameter κ.
On the other hand, thanks to Proposition 14,

$$\begin{aligned} P(A_2) &= \rho(\{\phi \in H_0^s(\mathbb{T}) : \|(1-S_{N_0})\phi\|_{H^s} > \gamma/4C_s\}) \\ &\le C\exp(-c\gamma^2 N_0^{2(1-s)}) = C\exp(-c\gamma^{2/s}B^{-2(1-s)/s}). \end{aligned}$$

Therefore, we obtain

$$g(\lambda) \le C\exp(-c(p/q)^{4/(p-2)}(\log\lambda)^{4/(p-2)}B^{-2(p+2)/(p-2)}).$$

If $2 < p < 6$, by $4/(p-2) > 1$, $g(\lambda)$ is integrable on $[1,\infty)$ for all $B > 0$.
If $p = 6$, $g(\lambda)$ is bounded by $C\lambda^{-c/qB^4}$. Thus, for sufficiently small B, $g(\lambda)$ is integrable on $[1,\infty)$.

The Gibbs measure of (TS) I

(TS)
$$\begin{cases} iu_t + u_{xx} + S_N(u|u|^{p-2}) = 0, \\ u(x,0) = \phi(x) \in E_N. \end{cases}$$

We identify $\phi \in E_N$ and $a^N := \{a_n\}_{|n|\le N} \in \mathbb{C}^{2N+1}$ through $\phi(x) = \sum_{|n|\le N} e^{inx} a_n$, where $a_n := \widehat{\phi}(n)$.
The Hamiltonian of (TS) is given by

$$H_N(\phi) := \frac{1}{2\pi}\int_0^{2\pi} |\partial_x \phi|^2 dx - \frac{1}{\pi p}\int_0^{2\pi} |\phi|^p dx,$$

$$H_N(a^N, \overline{a^N}) = \sum_{|n|\le N} n^2 |a_n|^2 - \frac{1}{\pi p}\int_0^{2\pi} \left| \sum_{|n|\le N} e^{inx} a_n \right|^p dx.$$

Since (TS) is ODE, this Hamiltonian is *rigorously* conserved.

The Gibbs measure of (TS) II

As in NLS, we define the following measures. Let ρ_N be image measure on $E_{N,0} := \mathrm{span}\{e^{inx} : 0 \neq |n| \leq N\} \cong \mathbb{C}^{2N} \cong \mathbb{R}^{4N}$ under the map

$$\omega \mapsto X_N(\omega) := \sum_{0 \neq |n| \leq N} \frac{g_n(\omega)}{n} e^{inx}.$$

This measure also has the following explicit formula:

$$d\rho_N = \frac{e^{-\frac{1}{2}\sum_{0\neq|n|\leq N} n^2|a_n|^2} d^2a_1 \dots d^2a_N}{\int_{\mathbb{C}^{2N}} e^{-\frac{1}{2}\sum_{0\neq|n|\leq N} n^2|a_n|^2} d^2a_1 \dots d^2a_N}.$$

Remark

If we replace the distribution of real and imaginary parts of g_n with $\frac{1}{\sqrt{\pi}}e^{-x^2}$, (namely $\Re g_n, \Im g_n = N(0, 1/\sqrt{2})$) then

$$d\rho_N = \frac{e^{-\sum_{0\neq|n|\leq N} n^2|a_n|^2} d^2a_1 \dots d^2a_N}{\int_{\mathbb{C}^{2N}} e^{-\sum_{0\neq|n|\leq N} n^2|a_n|^2} d^2a_1 \dots d^2a_N}.$$

We may replace the coefficient $1/2$ with 1.

The Gibbs measure of (TS) III

Let $V_N := \prod_{0\neq|n|\leq N}(-\infty,\alpha_n]\times(-\infty,\beta_n]$ and
$U_N = \{\phi\in E_{N,0} : \prod_{0\neq|n|\leq N}(\Re\widehat{\phi}(n),\Im\widehat{\phi}(n))\in V_N\}$. The independence implies

$$\begin{aligned}
&\rho_N(U_N)\\
&= P\Big(\bigcap_{0\neq|n|\leq N}\{\omega\in\Omega : \Re g_n(\omega)/n\in(-\infty,\alpha_n], \Im g_n(\omega)/n\in(-\infty,\beta_n]\}\Big)\\
&= \prod_{0\neq|n|\leq N} P(\Re g_n/n<\alpha_n)P(\Im g_n/n<\beta_n)\\
&= \prod_{0\neq|n|\leq N}\frac{n^2}{2\pi}\int_{(-\infty,\alpha_n]\times(-\infty,\beta_n]} e^{-\frac{n^2}{2}(x_n^2+y_n^2)}dx_n dy_n\\
&= \kappa_N\int_{V_N} e^{-\frac{1}{2}\sum_{0\neq|n|\leq N}n^2|a_n|^2}d^2a_1\ldots d^2a_N,\quad \kappa_N := (2\pi)^{-2N}\prod_{j=1}^N j^4.
\end{aligned}$$

We have used the equality: $P(\Re g_n/n<\alpha_n) = \frac{|n|}{\sqrt{2\pi}}\int_{-\infty}^{\alpha_n} e^{-\frac{n^2}{2}x^2}dx$.

The Gibbs measure of (TS) IV

We set

$$d\mu_N(\phi) := f(\phi)d^2a_0 d\rho_N(\phi) = \kappa_N e^{-H_N(a^N,\overline{a^N})}\chi_{\{\|a^N\|_{l^2}\le B\}}\prod_{|n|\le N} d^2a_n.$$

Recall that $f(\phi) = \exp(\frac{1}{p}\|\phi\|_{L^p}^p)\chi_{\{\|\phi\|_{L^2}\le B\}}$.

- Since (TS) is ODE, μ_N is invariant under the flow (Proposition 19 below).
- The measures ρ_N and μ_N are natural restrictions to E_N of ρ and μ, respectively. Thus, for $U \in \mathcal{H}_0$ and $V \in \mathcal{H}$, we have

$$\rho(S_N^{-1}U) = \rho_N(U \cap E_{N,0}), \quad \mu(S_N^{-1}V) = \mu_N(V \cap E_N),$$
$$S_N^{-1}U := \{\phi \in \mathcal{H}_0 : S_N\phi \in U\}.$$

Lemma 18

Let $0 \le s < 1/2$. If U is an open set in $H^s(\mathbb{T})$, one has $\mu(U) \le \liminf_{N\to\infty} \mu_N(U \cap E_N)$. Moreover, if V is a closed set in $H^s(\mathbb{T})$, one has $\mu(V) \ge \limsup_{N\to\infty} \mu_N(V \cap E_N)$.

The Gibbs measure of (TS) V

Proof of Lemma 18.

Define $U_N = S_N^{-1}U := \{u \in H^s(\mathbb{T}) : S_N u \in U\}$. The inclusion $U \subset \liminf_{N\to\infty} U_N := \bigcup_{N\geq 1}\bigcap_{M\geq N} U_M$ holds because U is open set. Let f_N be $f_N := \chi_{U_N} \cdot f$. Then, $\liminf_{N\to\infty} f_N \geq \chi_U \cdot f$. By Fatou's lemma, one gets

$$\begin{aligned}\liminf_{N\to\infty} \mu_N(U \cap E_N) &= \liminf_{N\to\infty} \mu(U_N) = \liminf_{N\to\infty} \int_{H^s} f_N d^2a_0 d\rho \\ &\geq \int_{H^s} \liminf_{N\to\infty} f_N d^2a_0 d\rho \geq \int_U f d^2 a_0 d\rho = \mu(U).\end{aligned}$$

Defining $V_N := \{u \in H^s(\mathbb{T}) : S_N u \in V\}$, one has $V \supset \limsup_{N\to\infty} V_N := \cap_{N\geq 1} \cup_{M\geq N} V_M$ because V is closed. The desired estimate follows from a similar argument. □

Invariance of the measure μ_N I

Proposition 19

The measure μ_N is invariant under the flow $\Phi_N(t)$ of (TS).

Proof.

Set $a^N(t) := \{a_n(t)\}_{|n|\leq N}$, where $u(x,t) = \sum_{|n|\leq N} e^{inx} a_n(t)$. (TS) can be written as

$$(14) \qquad i\partial_t a_n(t) - n^2 a_n(t) + \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} S_N(u|u|^{p-2})(x,t)dx = 0.$$

(14) can be written in a Hamiltonian format as follows:

$$\partial_t a_n = -i\frac{\partial H_N}{\partial \overline{a_n}}, \ \partial_t \overline{a_n} = i\frac{\partial H_N}{\partial a_n}$$

with

$$H_N(a^N, \overline{a^N}) = \sum_{|n|\leq N} n^2 |a_n|^2 - \frac{1}{\pi p}\int_0^{2\pi} \Big| \sum_{|n|\leq N} e^{inx} a_n \Big|^p dx.$$

Invariance of the measure μ_N II

Proof (sequel).

Since

$$\sum_{|n|\leq N}\Big(\frac{\partial}{\partial a_n}(-i\frac{\partial H_N}{\partial \overline{a_n}})+\frac{\partial}{\partial \overline{a_n}}(i\frac{\partial H_N}{\partial a_n})\Big)=0,$$

we can apply the Liouville theorem for Hamiltonian to conclude that the measure $da^N d\overline{a^N}$ is invariant under the flow of (TS).

Let A be a Borel set of E_N. Then,

$$\mu_N(A)=\kappa_N\int_A e^{-\frac{1}{2}H_N(a^N,\overline{a^N})}\chi_{\{\|a^N\|_{l^2}\leq B\}}da^N d\overline{a^N},\ \kappa_N:=(2\pi)^{-2N}\prod_{j=1}^N j^4.$$

We can write

$$\Phi(t)(A)=\{(a^N,\overline{a^N}):(a^N,\overline{a^N})=\Phi_N(t)(b^N,\overline{b^N}),\exists(b^N,\overline{b^N})\in A\}.$$

By change of variables $(a^N,\overline{a^N})=\Phi(t)(b^N,\overline{b^N})$ and the invariance of $da^N d\overline{a^N}$ under $\Phi_N(t)$, we get the Jacobian of this variable change is one.

Invariance of the measure μ_N III

Proof (sequel).

Thanks to the conservation laws

$$H_N(\Phi_N(t)(b^N,\overline{b^N})) = H_N(b^N,\overline{b^N}), \quad \|\Phi(t)b^N\|_{l^2} = \|b^N\|_{l^2}.$$

We therefore obtain

$$\begin{aligned}\mu_N(\Phi(t)(A)) &= \kappa_N \int_{\Phi(t)(A)} e^{-\frac{1}{2}H_N(a^N,\overline{a^N})}\chi_{\{\|a^N\|_{l^2}\le B\}}da^N d\overline{a^N} \\ &= \kappa_N \int_A e^{-\frac{1}{2}H_N(b^N,\overline{b^N})}\chi_{\{\|b^N\|_{l^2}\le B\}}db^N d\overline{b^N} \\ &= \mu_N(\Phi(t)(A)),\end{aligned}$$

which completes the proof. □

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Invariance of μ $(p = 4)$ I

Let $\{s_j\}_{j\in\mathbb{N}}$ be a increasing sequence of real numbers such that $s_1 > 0$ and $\lim_{j\to\infty} s_j = 1/2$. Note that $\mathcal{H} = \bigcap_{j=1}^{\infty} H^{s_j}(\mathbb{T})$.

Theorem 20

Let $p = 4$. The measure μ is invariant nuder the flow of (S). More precisely, for every μ-measurable A, $\mu(\Phi(t)A) = \mu(A)$ holds.

By the reversibility of the flow, it suffices to prove for every $t \in \mathbb{R}$ and every μ-measurable set $A \subset \mathcal{H}$, one has the inequality

$$\mu(\Phi(t)(A)) \geq \mu(A). \tag{15}$$

It suffices to prove (15) for closed sets of $H^s(\mathbb{T})$.
Indeed, by the regularity of the bounded Borel measure, $\exists\{V_n\}$ such that

$$V_n \text{ is a closed set of } H^s(\mathbb{T}),\ V_n \subset A,\ \mu(A) = \lim_{n\to\infty} \mu(V_n).$$

Hence, if we can prove (15) for the sets V_n, we have

$$\mu(A) = \lim_{n\to\infty} \mu(V_n) \leq \limsup_{n\to\infty} \mu(\Phi(t)V_n) \leq \mu(\Phi(t)A).$$

Invariance of μ $(p = 4)$ II

Fix s_0, s with $s_0 < s$ and $s, s_0 \in \{s_j\}_{j\in\mathbb{N}}$. Let us next show that it suffices to prove (15) for subsets of $\mathcal{H}$ which are bounded in $H^s(\mathbb{T})$ and are compacts of $H^{s_0}(\mathbb{T})$.

Indeed, from Proposition 17, for every closed set A of $\mathcal{H}$, one has

$$\begin{aligned} 0 \le \mu(A) - \mu(A \cap B_R^{(s)}) &= \mu(A \cup B_R^{(s)}) - \mu(B_R^{(s)}) \\ &\le \mu(H^s(\mathbb{T})) - \mu(B_R^{(s)}) = \mu(H^s(\mathbb{T}) \backslash B_R^{(s)}) \le Ce^{-cR^2}, \end{aligned}$$

which implies

$$\mu(A) = \lim_{R\to\infty} \mu(A \cap B_R^{(s)}).$$

$A \cap B_R^{(s)}$ is compact of $H^{s_0}(\mathbb{T})$. If we can prove (15) for compacts which are bounded in $H^s(\mathbb{T})$ then

$$\mu(A) \le \limsup_{R\to\infty} \mu(\Phi(t)(A \cap B_R^{(s)}) \le \mu(\Phi(t)(A)).$$

Thus, it suffices to prove (15) for subsets of $\mathcal{H}$ which are compacts in $H^{s_0}(\mathbb{T})$ and bounded in $H^s(\mathbb{T})$.

Invariance of μ ($p = 4$) III

Let us now fix $t \in \mathbb{R}$ and $K \subset \mathcal{H}$, a bounded set of $H^s(\mathbb{T})$ which is a compact in $H^{s_0}(\mathbb{T})$. Fix $\varepsilon > 0$. Thanks to GWP and Lemma 13, we have

(16) $\Phi_N(t)((K+B_\varepsilon^{(s_0)})\cap E_N) \subset \Phi_N(t)(S_N K)+B_{C\varepsilon}^{(s_0)} \subset \Phi(t)(K)+B_{2C\varepsilon}^{(s_0)}$,

provided that $N \gg 1$.

Since $\Phi(t)(K)$ is compact of $H^{s_0}(\mathbb{T})$ and $B_{2C\varepsilon}^{(s_0)}$ is closed, $\Phi(t)(K) + B_{2C\varepsilon}^{(s_0)}$ is a closed set of $H^{s_0}(\mathbb{T})$.

By Lemma 18, (16), and Proposition 19, we obtain

$$\begin{aligned}\mu(\Phi(t)(K) + B_{2C\varepsilon}^{(s_0)}) &\geq \limsup_{N\to\infty} \mu_N((\Phi(t)(K) + B_{2C\varepsilon}^{(s_0)}) \cap E_N)\\ &\geq \liminf_{N\to\infty} \mu_N(\Phi_N(t)((K + B_\varepsilon^{(s_0)}) \cap E_N)\\ &= \liminf_{N\to\infty} \mu_N((K + B_\varepsilon^{(s_0)}) \cap E_N)\\ &\geq \mu(K + B_\varepsilon^{(s_0)}) \geq \mu(K).\end{aligned}$$

By letting $\varepsilon \to 0$, we obtain the desired inequality $\mu(\Phi(t)(K)) \geq \mu(K)$.

Improved bounds for (TS) I

Let us denote by $\Phi_N(t)$ the smooth flow map of (TS) which is defined globally.

Proposition 21

For $\forall\ i \geq 1,\ 0 < s < 1/2,\ \exists$ *a set* $\Xi^i_{N,s} \subset E_N$ *such that*

$$\mu_N(E_N \backslash \Xi^i_{N,s}) \leq 2^{-i},$$

and for $\phi \in \Xi^i_{N,s}$ *one has the bound*

$$\|\Phi_N(t)\phi\|_{H^s} \leq C(i + \log(1 + |t|))^{1/2}.$$

Moreover, for $N_1 \leq N_2$*, we have the inclusion* $\Xi^i_{N_1,s} \subset \Xi^i_{N_2,s}$*.*

Proof.

We will consider only the positive value of t. The analysis for $t < 0$ is the same. For $0 < s < 1/2$, and $i, j \in \mathbb{Z}$, we set

$$B^{i,j}_{N,s}(D_s) := \{\phi \in E_N : \|\phi\|_{H^s} \leq D_s(i+j)^{1/2},\ \|\phi\|_{L^2} \leq B\},$$

where the number $D_s \gg 1$ will be fixed later.

Improved bounds for (TS) II

Proof (sequel).

Thanks to LWP, there exists $\tau \in (0,1]$ such that

$$\tau > \frac{C(p,s)}{(D_s(i+j)^{1/2})^{C_1(p,s)}}, \tag{17}$$

$$\Phi_N(t)(B^{i,j}_{N,s}(D_s)) \subset B^{i,j}_{N,s}(CD_s) \text{ for } 0 \le t \le \tau. \tag{18}$$

Next, we set

$$\Xi^{i,j}_{N,s}(D_s) := \bigcap_{k=0}^{[2^j/\tau]} \Phi_N(-k\tau)(B^{i,j}_{N,s}(D_s)).$$

Using Proposition 19 and (17), we can write

$$\begin{aligned} \mu_N(E_N \backslash \Xi^{i,j}_{N,s}(D_s)) &\le \sum_{k=0}^{[2^i/\tau]} \mu_N(E_N \backslash \Phi_N(-k\tau)(B^{i,j}_{N,s}(D_s))) \\ &= ([2^i/\tau]+1)\mu_N(E_N \backslash B^{i,j}_{N,s}(D_s)). \end{aligned}$$

Improved bounds for (TS) III

Proof (sequel).

Let us observe that

$$\mu_N(E_N \backslash B^{i,j}_{N,s}(D_s)) = \mu(\{\phi \in H^s(\mathbb{T}) : \|S_N\phi\|_{H^s} > D_s(i+j)^{1/2}\})$$
$$\leq \mu(\{\phi \in H^s(\mathbb{T}) : \|\phi\|_{H^s} > D_s(i+j)^{1/2}\}).$$

Using Proposition 17 and (17), we can write

(19)

$$\mu_N(E_N \backslash B^{i,j}_{N,s}(D_s)) \leq C2^i D_s^{C_1(p,s)}(i+i)^{C_1(p,s)/2} e^{-cD_s^2(i+j)} \leq 2^{-(i+j)},$$

provided that $D_s \gg 1$ depending on s, p but dependent of i, j, N.

Thanks to (18), for $\phi \in \Xi^{i,j}_{N,s}$, the solution $u(t)$ of (TS) with data ϕ satisfies

$$\|u(t)\|_{H^s} \leq CD_s(i+j)^{1/2}, \quad 0 \leq t \leq 2^j.$$

Next, we set $\Xi^i_{N,s} := \bigcap_{j=1}^\infty \Xi^{i,j}_{N,s}(D_s)$. From (19),

$$\mu_N(E_N \backslash \Xi^i_{N,s}) \leq \sum_{j=1}^\infty \mu_N(E_N \backslash \Xi^{i,j}_{N,s}(D_s)) \leq 2^{-i}.$$

□

Improved bounds for (TS) IV

Proposition 22

For every $0 < s < 1/2$, $0 < s_0 < s$, $t \in \mathbb{R}$, $i \in \mathbb{N}$, *there exists* $i_1 \in \mathbb{N}$ *such that for every* $N \geq 1$, *if* $\phi \in \Xi^i_{N,s}$ *then one has* $\Phi_N(t)\phi \in \Xi^{i+i_1}_{N,s_0}$.

Proof.

Again, we can suppose $t > 0$. Set $u(t) := \Phi_N(t)\phi$. If $\phi \in \Xi^i_{N,s}$, for $j \in \mathbb{N}$, we have

$$\|\Phi_N(t)\phi\|_{H^s} \leq C_s(i+j)^{1/2}, \quad 0 \leq t_1 \leq 2^j.$$

Let $j_0 \in \mathbb{N}$, depending on t, be such that for every $j \geq 1$, $2^j + t \leq 2^{j+j_0}$. Then, we get

(20)

$$\|\Phi_N(t_1)u(t)\|_{H^s} = \|\Phi_N(t+t_1)\phi\|_{H^s} \leq C_s(i+j+j_0)^{1/2}, \quad 0 \leq t_1 \leq 2^j.$$

Interpolating between (20) with and L^2-conservation implies

$$\|\Phi_N(t_1)u(t)\|_{H^{s_0}} \leq C(C_s(i+j+j_1))^{(1-\theta)/2}, \quad 1 \leq t_1 \leq 2^j,$$

where $\theta = 1 - s_0/s$.

Improved bounds for (TS) V

Proof (sequel).

Since $0 < \theta < 1$, for $j_0 \gg 1$,

$$C(C_s(i+j+j_1))^{(1-\theta)/2} \le D_{s_0}(i+j+j_0)^{1/2}.$$

Thus,

$$\|\Phi_N(t_1)u(t)\|_{H^{s_0}} \le D_{s_0}(i+j+j_0)^{1/2}, \quad 0 \le t_1 \le 2^j,$$

which implies $u(t) \in \Xi_{N,s_0}^{i+j_0,j}(D_{s_0})$ for every $j \ge 1$. Therefore, we obtain

$$u(t) \in \Xi_{N,s_0}^{i+j_0}.$$

□

Remark

The number i_1 in Proposition 22 is the same for every i, i.e., it depends only on t, s, s_1. This fact is however not of importance for the sequel.

A set is of full μ-measure

For every $i \in \mathbb{N}$ and $0 < s < 1/2$, we set

$$\Xi_s^i := \bigcup_{N \geq 1} \Xi_{N,s}^i.$$

By Lemma 18 and Proposition 21, we have

$$\mu(\overline{\Xi_s^i}) \geq \limsup_{N\to\infty} \mu_N(\Xi_{N,s}^i) = \limsup_{N\to\infty}(\mu_N(E_N) - 2^{-i}) = \mu(H^s(\mathbb{T})) - 2^{-i},$$

where $\overline{\Xi_s^i}$ denotes the closure of Ξ_s^i in $H^s(\mathbb{T})$. Next, we set

$$\Xi_s := \bigcup_{i=1}^{\infty} \overline{\Xi_s^i}.$$

Let $\{s_j\}_{j\in\mathbb{N}}$ be a increasing sequence of real numbers such that $s_1 > 0$ and $\lim_{j\to\infty} s_j = 1/2$. Then, we set

$$\Xi := \bigcap_{j=1}^{\infty} \Xi_{s_j}. \tag{21}$$

The set Ξ is of full μ-measure, since every Ξ_s is of full μ-measure and the intersection in (21) is countable.

Global well-posedness of NLS I

Proposition 23

For every $\phi \in \Xi$, the local solution of (S) given by Theorem 2 is globally defined. Moreover, for every $t \in \mathbb{R}$, $\Phi(t)(\Xi) = \Xi$.

Proof.

Let us fix $\phi \in \overline{\Xi^i_s}$, $0 < s < 1/2$, $0 < s_0 < s$, $T > 0$. Thus, there exists a sequence $\{\phi_k\}$ such that $\phi_k \in \Xi^i_{N_k,s}$ where N_k is tending to infinity, $\phi_k \to \phi$ in $H^s(\mathbb{T})$. Thanks to Proposition 21,

$$\|\Phi_{N_k}(t)\phi_k\|_{H^s} \leq C_s(i + \log(1 + |t|))^{1/2}.$$

Applying Lemma 13 with $A = C_s(i + \log(1 + T))^{1/2}$, we have

$$\|\Phi(t)\phi - \Phi_{N_k}(t)\phi_k\|_{H^{s_0}} < 1$$

provided that k is sufficiently large. It implies

$$\|\Phi(t)\phi\|_{H^{s_0}} < 2A = C(i + \log(1 + T))^{1/2},$$

which shows the global well-posedness of (S).

Global well-posedness of NLS II

Proof (sequel).

Let us show the inclusion

$$\Phi(t)(\Xi) \subset \Xi. \tag{22}$$

Fix $\phi \in \Xi$. It suffices to show that for every $s_0 \in \{s_j\}_{j\in\mathbb{N}}$, we have

$$\Phi(t)(\Xi) \subset \Xi_{s_0}.$$

Let us take $s \in \{s_j\}_{j\in\mathbb{N}}$ with $s_0 < s < 1/2$. By $\phi \in \Xi_s$, there exists $i \in \mathbb{N}$ such that $\phi \in \overline{\Xi_s^i}$. Let again $\phi_k \in \Xi^i_{N_k,s}$ be a sequence which tends to ϕ in $H^s(\mathbb{T})$. Thanks to Proposition 22, there is $i_1 \in \mathbb{N}$ such that $\Phi_{N_k}(t)\phi_k \in \Xi^{i+i_1}_{N_k,s_0}$. From Lemma 13, we obtain

$$\Phi(t)\phi \in \overline{\Xi^{i+i_1}_{s_0}}.$$

Hence, $\Phi(t)\phi \in \Xi_{s_0}$, which proves (22). Since the flow $\Phi(t)$ is reversible, (22) implies $\Phi(t)(\Xi) = \Xi$. □

Global well-posedness of NLS III

Proposition 24 (a continuity of $\Phi(t)$)

Let $\phi \in \Xi$ and $\{\phi_k\} \subset \Xi$ be a sequence such that $\phi_k \to \phi$ in $H^s(\mathbb{T})$. Then, for every $t \in \mathbb{R}$, $\Phi(t)\phi_k \to \Phi(t)\phi$ in $H^s(\mathbb{T})$. In particular, for every closed set A in $H^s(\mathbb{T})$, one has

$$\Phi(t)(A \cap \Xi) = \overline{\Phi(t)(A \cap \Xi)} \cap \Xi,$$

where $\overline{\Phi(t)(A \cap \Xi)}$ denotes the closure in $H^s(\mathbb{T})$ of $\Phi(t)(A \cap \Xi)$.

Proof.

Since $\phi \in \Xi$ and the construction of Ξ, for every $T > 0$ there exists $\Lambda \geq 1$ such that

$$\sup_{|t| \leq T} \|\Phi(t)\phi\|_{H^s} \leq \Lambda.$$

Let us denote by τ the local existence time in LWP associated Λ. Then, by the continuity of the flow on $[-\tau, \tau]$,

$$\Phi(t)\phi_k \to \Phi(t)\phi \quad \text{in } H^s(\mathbb{T}),\ |t| \leq \tau.$$

Global well-posedness of NLS IV

Proof (sequel).

Next, we cover the interval $[-T, T]$ by intervals of size τ and we apply the continuity of the flow established in LWP at each step. Therefore, we obtain that

$$\Phi(t)\phi_k \to \Phi(t)\phi \quad \text{in } H^s(\mathbb{T}),\ |t| \leq T.$$

Since $\Phi(t)(\Xi) \subset \Xi$, it is clear that

$$\Phi(t)(A \cap \Xi) \subset \overline{\Phi(t)(A \cap \Xi)} \cap \Xi.$$

Next, let us fix $u \in \overline{\Phi(t)(A \cap \Xi)} \cap \Xi$. Then, there exists a sequence $\{\phi_k\} \subset A \cap \Xi$ such that $u_k := \Phi(t)\phi_k \to u$ in $H^s(\mathbb{T})$. From $u_k, \Phi(-t)u \in \Xi$ and the continuity of $\Phi(t)$, $\phi_k = \Phi(-t)u_k \to \Phi(-t)u$ in $H^s(\mathbb{T})$. Since A is closed, $\Phi(-t)u \in A$. Thus, we get $u \in \Phi(t)(A \cap \Xi)$. □

Invariance of μ $(4 < p \leq 6)$ I

Theorem 25

Let $4 \leq p \leq 6$. The measure μ is invariant nuder the flow of the (S). More precisely, for every μ-measurable A, $\mu(\Phi(t)A) = \mu(A)$ holds.

As in the proof of Theorem 20, it suffices to prove the inequality

$$\mu(\Phi(t)(K)) \geq \mu(K). \tag{23}$$

for subsets K of Ξ which are compacts in $H^{s_0}(\mathbb{T})$ and bounded in $H^s(\mathbb{T})$. Let us now fix $t \in \mathbb{R}$ and $K \subset \Xi$, a bounded set of $H^s(\mathbb{T})$ which is a compact in $H^{s_0}(\mathbb{T})$.

Lemma 26

There exists R_0 such that $\{\Phi(t_1)(K) : |t_1| \leq |t|\} \subset B_{R_0}^{(s_0)}$.

Invariance of μ $(4 < p \leq 6)$ II

Proof of Lemma 26.

If not, then for all $k > 0$ there exists $t_k \in \mathbb{R}$ and $\phi_k \in K$ such that $|t_k| \leq |t|$ and $\|\Phi(t_k)\phi_k\|_{H^{s_0}} > k$. Since K is a compact set in $H^{s_0}(\mathbb{T})$, there exists a subsequence $\{\phi_{k_l}\} \subset \{\phi_k\}$ and $\phi \in K$ such that $\phi_{k_l} \to \phi$ in $H^{s_0}(\mathbb{T})$. Proposition 24 implies $\Phi(t_{k_l})\phi_{k_l} \to \Phi(t_{k_l})\phi$ in $H^{s_0}(\mathbb{T})$, which contradicts to the unboundedness of $\{\Phi(t_k)\phi_k\}$. □

Set

$$\tau_0 := \frac{C(p, s_0)}{(1 + R_0)^{C_1(p,s_0)}}.$$

It suffices to show that

$$\mu(K) \leq \mu(\Phi(t_1)K), \quad |t_1| \leq \tau_0. \tag{24}$$

Indeed, once (24) is established, it suffices to cover $[0, t]$ by intervals of size τ_0 and to apply (24) at each step.

The proof of (24) is the same as Theorem 20.