Transfer of energy frequencies in the cubic defocusing nonlinear Schrödinger equation II

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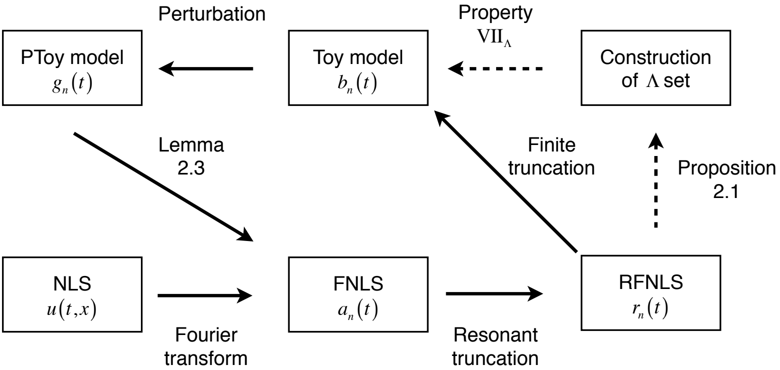
Contents

Review

Periodic defocusing cubic nonlinear Schrödinger equation,

$$\begin{cases} -i\partial_t u + \Delta u = |u|^2 u, \\ u(0,x) := u_0(x). \end{cases} \qquad \text{(NLS)}$$



Contents

Properties of Λ

- Initial data : $r_n(0) = 0$ whenever $n \notin \Lambda$
- Closure : $(n_1, n_2, n_3) \in \Gamma_{res}(n), n_1, n_2, n_3 \in \Lambda \Rightarrow n \in \Lambda$
- $\exists!$ of spouse and children
- $\exists!$ of sibling and parents
- Nondegeneracy : Sibling of $n \neq$ spouse of n
- Faithfullness
- Intragenerational equality : $n, n' \in \Lambda_j \Rightarrow r_n(0) = r_{n'}(0)$

Remark

One can choose the initial data which satisfies the first and last properties.

Proposition 1

Given parameters $\delta \ll 1$, $K \gg 1$, we can find an $N \gg 1$ and a set of frequencies $\Lambda \subset \mathbb{Z}^2$ with

$$\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_N \qquad \text{disjoint union}$$

which satisfies Property II_Λ – Property VI_Λ and also,

$$\frac{\sum_{n \in \Lambda_N} |n|^{2s}}{\sum_{n \in \Lambda_1} |n|^{2s}} \gtrsim \frac{K^2}{\delta^2}.$$

In addition, given any $\mathcal{R} \gg C(K, \delta)$, we can ensure that Λ consists of $N \cdot 2^{N-1}$ disjoint frequencies n satisfying $|n| \geq \mathcal{R}$.

- Identify the frequency $n \in \mathbb{Z}^2$ with the Gaussian integers $\mathbb{Z}[i] \subset \mathbb{C}$

$$n = (n_1, n_2) \leftrightarrow n_1 + in_2.$$

- $S = \{0, 1, 1+i, i\}$, $S_1 = \{1, i\}$ and $S_2 = \{0, 1+i\}$.
- For $1 \leq j \leq N$, define $\Sigma_j \subset \mathbb{C}^{N-1}$

$$\Sigma_j = S_2^{j-1} \times S_1^{N-j}$$

Definition

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_N$$

We call Σ_j the *j-th generation of* Σ.

Combinatorial Nuclear Family

$F \subset \Sigma_j \cup \Sigma_{j+1}$: Combinatorial nuclear family connecting generations Σ_j, Σ_{j+1}

$$F := \{(z_1, \cdots, z_{j-1}, w, z_{j+1}, \cdots, z_{N-1}) : w \in S\}$$

- For each F, we call F_1, F_i : Parents, F_0, F_{1+i} : Children
- For any $1 \leq j < N$ and any $x \in \Sigma_j$,

$$\exists! \text{ spouse in } \Sigma_j \text{ and } \exists! \text{ two children in } \Sigma_{j+1}.$$

- For any $1 \leq j < N$ and any $y \in \Sigma_{j+1}$,

$$\exists! \text{ sibling in } \Sigma_{j+1} \text{ and } \exists! \text{ two parents in } \Sigma_j.$$

- The sibling of an element $x \in \Sigma_j$ is never equal to its spouse.

Consider $\Sigma \hookrightarrow \mathbb{Z}^2$

- (Placement of initial generation) $f_1 : \Sigma_1 \to \mathbb{C}$
- (Angle of each nuclear family) For each $1 \leq j \leq N$ and F, an angle $\theta(F) \in \mathbb{R}/2\pi\mathbb{Z}$.

Placement Functions f_j

If $1 \leq j < N$ and $f_j : \Sigma \to \mathbb{C}$ has already been constructed. Then we define $f_{j+1} : \Sigma_{j+1} \to \mathbb{C}$ by

$$f_{j+1}(F_{1+i}) = \frac{1 + e^{i\theta(F)}}{2} f_j(F_1) + \frac{1 - e^{i\theta(F)}}{2} f_j(F_i)$$

$$f_{j+1}(F_0) = \frac{1 + e^{i\theta(F)}}{2} f_j(F_1) - \frac{1 - e^{i\theta(F)}}{2} f_j(F_i)$$

Complete Placement Function f

$$f(x) := f_j(x), \qquad \text{if } x \in \Sigma_j.$$

Example

Let $\mathcal{R}$ be a large integer and all the angles $\theta(F) = \pi/2$. Then,

- First generation

$$f_1(z_1, \cdots, z_{N-1}) := \mathcal{R} z_1 \cdots z_{N-1} \in \{\mathcal{R}, i\mathcal{R}, -\mathcal{R}, -i\mathcal{R}\}$$

- Second generation

$$\begin{aligned} f_2(F_{1+i}) &= \frac{1+i}{2} f_1(F_1) - \frac{1-i}{2} f_j(F_i) \\ &= \mathcal{R}\frac{1+i}{2} \cdot 1 \cdot z_2 \cdots z_{N-1} + \mathcal{R}\frac{1-i}{2} \cdot i \cdot z_2 \cdots z_{N-1} \\ &= \mathcal{R}(1+i) z_2 \cdots z_{N-1} = \bar{z}_1 (1+i) \mathcal{R} z_1 \cdots z_{N-1} \end{aligned}$$

Similarly,

$$f_2(F_0) = \mathcal{R} \cdot 0 \cdot z_2 \cdots z_{N-1} = 0.$$

Example - Continued

- Inductively, j-th generation

$$f_j(z_1, \cdots, z_{N-1}) := \mathcal{R} z_1 \cdots z_{N-1}$$
$$\in \left\{0, (1+i)^{j-1}\mathcal{R}, i(1+i)^{j-1}\mathcal{R}, -(1+i)^{j-1}\mathcal{R}, -i(1+i)^{j-1}\mathcal{R}\right\}$$

Analysis of Example

- For each nuclear family $\{F_0, F_1, F_{1+i}, F_i\}$,
  - $|f(F_1)| = |f(F_i)|$
  - $F_0 \mapsto 0$, but $|f(F_{1+i})| = \sqrt{2}|f(F_1)|$
- The only single element $(1+i, \cdots, 1+i) \in \Sigma_N \mapsto (1+i)^{N-1}\mathcal{R}$.

Consequently, for solution to R$\mathcal{F}$NLS,

$$\text{mass } : F_1, F_i \longrightarrow F_0, F_{1+i} \text{ but energy } : F_1, F_i \longrightarrow F_{1+i}$$

From simple calculation, we have

$$\sum_{n\in f(\Sigma_{N-2})} |n|^{2s} = \sharp \text{ of nonzero } n \cdot 2^{s(N-3)}\mathcal{R}^{2s} = 2^{s(N-3)+2}\mathcal{R}^{2s},$$

and

$$\sum_{n\in f(\Sigma_3)} |n|^{2s} = \sharp \text{ of nonzero } n \cdot 2^{2s}\mathcal{R}^{2s} = 2^{2s+N-3}\mathcal{R}^{2s}.$$

In this example, there is a norm by

$$\frac{\sum_{n\in f(\Sigma_{N-2})} |n|^{2s}}{\sum_{n\in f(\Sigma_3)} |n|^{2s}} = 2^{s(N-3)+2-(2s+N-3)} = 2^{(s-1)(N-5)}.$$

Theorem - Good Placement Function

Let $N \geq 3$, $s > 1$, and $\mathcal{R} := \mathcal{R}(N)$ be a large integer. Then, $\exists f_1 : \Sigma_1 \to \mathbb{C}$ and $\exists \theta(F)$ for each F, with the properties :

- (Nondegenerarcy) f : injective
- (Integrality) $f(\Sigma) \subset \mathbb{Z}[i]$
- (Magnitude) $|f(x)| \sim_N \mathcal{R}$ for all $x \in \Sigma$
- (Closure and Faithfullness) x, y, z are distinct

$$f(x), f(y), f(z) : \text{ form a right-angled triangle } \Rightarrow \{x, y, z\} \subset F$$

- (Norm explosion)

$$\sum_{n \in f(\Sigma_{N-2})} |n|^{2s} \geq \frac{1}{2} 2^{(s-1)(N-5)} \sum_{n \in f(\Sigma_3)} |n|^{2s}$$

Proof

Remark

We see that $\Lambda := f(\Sigma)$ with generations $\Lambda_j := f_j(\Sigma_j)$, obey all the required properties.

Reduction of conditions

- Integrality condition : $f(\Sigma) \subset \mathbb{Z}[i] \Longrightarrow f(\Sigma) \subset \mathbb{Q}[i]$
- Magnitude condition :

$$|f(x)| \sim_N \mathcal{R} \Longrightarrow f(x) \neq 0, \quad \forall x \in \Sigma$$

Now the following remains:

- Injectivity
- $f(x) \neq 0$ for all $x \in \Sigma$
- Closure and Faithfulness
- Norm explosion

- Norm explosion

Since

$$\left\{ (f_1, \theta(F)) : \frac{\sum_{n \in f(\Sigma_{N-2})} |n|^{2s}}{\sum_{n \in f(\Sigma_{N_3})} |n|^{2s}} > \frac{1}{2} \cdot 2^{(s-1)(N-5)} \right\}$$

is open set, it is either empty or has positive measure.

Norm explosion

By perturbation argument in the previous example, we do not need to consider this condition, and it suffices to show that the set where the other conditions fail is a measure zero set.

Injectivity

Need to show that for $x \in \Sigma_j, y \in \Sigma_{j'}$, $j \geq j'$

$$x, y \in \Sigma \text{ distinct } \implies f(x) \neq f(y)$$

Use the mathematical induction on j

- $j = 1$: Clear
- $j > 1$
  - $x \in \Sigma_j \Rightarrow \exists! F := \{x, x', p, p'\}$, $p, p' \in \Sigma_{j-1}$: parents of x
  - Induction hypothesis $\Rightarrow f(p) \neq f(p')$
  - Definition of $f \Rightarrow f(x)$ lies on the circle depending on $f(p), f(p')$, and $\theta(F)$
  - $y \neq x' \Rightarrow \theta(F)$ does not influence the value of $f(y)$ $\Rightarrow$ One can choose a point $f(y)$ on some circle unequal to $f(x)$
  - $y = x' \Rightarrow f(y) = f(x')$ is diametrically opposite to $f(x)$
  - Hence, $f(x) \neq f(y)$

Non-zero / Closure and Faithfulness

Non-zero

- $x \in \Sigma_1 \Rightarrow f_1(x) \neq 0$ for almost every f_1
- $x \in \Sigma_j$ for $j > 1$
  $\Rightarrow f(x)$ lies on the circle depending on two parents and $\theta(F)$
  $\Rightarrow f(x) \neq 0$ for almost every angle $\theta(F)$

Closure and Faithfulness

For $j_x \geq j_y \geq j_z$, $x \in \Sigma_{j_x}, y \in \Sigma_{j_y}, z \in \Sigma_{j_z}$: distinct,
Need to show $\{x, y, z\} \not\subset F$
$\Rightarrow f(x), f(y), f(z)$ do not form a right-angled triangle

Closure and Faithfulness

Use the mathematical induction

- $j_x = 1$: Clear
- $j_x > 1$
  - $x \in \Sigma_{j_x} \Rightarrow \exists! F := \{x, x', p, p'\}$, $p, p' \in \Sigma_{j_x - 1}$: parents of x
  - Injectivity and Definition of $f \Rightarrow f(x)$ freely lies on the circle C with diameter $f(p), f(p')$, with the location on this circle determined by the angle $\theta(F)$
  - Either y or z in F (WLOG $y \in F$) $\Rightarrow y = x'$ or not
    - $y \neq x' \Rightarrow f(z)$ lies outside of $C \Rightarrow \theta(F)$ does not influence the value of $f(z)$
    - $y = x' \Rightarrow f(z)$ should lie on C to form a right-angled triangle

Closure and Faithfulness

Lemma

The circle C contains no elements of $\Lambda_1 \cup \cdots \cup \Lambda_{j_x}$ other than $f(F)$

- $u \in \Lambda_1 \cup \cdots \cup \Lambda_{j_x - 1} \Rightarrow f(u)$ does not lie on C
  $\because f(p), f(p'), f(u)$ form a right-angled triangle
  (Contradiction to the induction hypothesis)
- $u \in \Lambda_{j_x} \Rightarrow \exists! F' := \{q, q', u, u'\}$ such that $f(F')$ lies on C'
  $\Rightarrow f(q), f(q')$ does not lie on C by induction hypothesis
  $\Rightarrow C'$ is not coincident to C
  $\Rightarrow$ for almost every $\theta(F')$, $f(u)$ does not lie on C

Contents

Consider the Toy Model System

$$\partial_t b_j = -i|b_j|^2 b_j + 2i\overline{b_j}(b_{j-1}^2 + b_{j+1}^2) \qquad \text{(Toy Model)}$$

General Remarks about (Toy Model)

- (Toy Model) is globally well-posed in $l^2(\mathbb{Z})$
- (Toy Model) has a number of symmetries
  - Phase invariance : $b_j(t) \leftarrow e^{i\theta} b_j(t)$
  - Scaling symmetry : $b_j(t) \leftarrow \lambda b(\lambda^2 t)$
  - Translation symmetry : $b_j(t) \leftarrow b_j(t-\tau)$
  - Space translation symmetry : $b_j(t) \leftarrow b_{j-j_0}(t)$
  - Time reversal symmetry : $b_j(t) \leftarrow \overline{b_j}(-t)$
  - Space reflection symmetry : $b_j(t) \leftarrow b_{-j}(t)$
  - Sign symmetry : $b_j(t) \leftarrow \epsilon_j b_j(t)$, $\epsilon_j = \pm 1$

Unstable behavior in (Toy Model)

Let $N \geq 1$ be fixed integer and let $\Sigma \subset \mathbb{C}^N$

$$\Sigma := \left\{ x \in \mathbb{C}^N : |x|^2 = 1 \right\}$$

Assume that $b_j(t_0) = 0$ for $j \leq 0$ and $j > N$

- $S(t) : \Sigma \to \Sigma$: smooth flow defined by

$$S(t)b_j(t_0) := b_j(t + t_0)$$

- $\mathbb{T}_j := \{(b_1, \cdots, b_N) \in \Sigma : |b_j| = 1; b_k = 0 \text{ for all } k \neq j\}$

Theorem - Instability for (Toy Model)

Let $N \geq 6$. Given any $\varepsilon > 0$, there exists a point x_3 within ε of $\mathbb{T}_3$ (using the usual metric on Σ), a point x_{N-2} within ε of $\mathbb{T}_{N-2}$, and a time $t \geq 0$ such that $S(t)x_3 = x_{N-2}$.

2D Cubic NLS

Consider the 2D cubic NLS

$$\begin{cases} i\partial_t b_1 = |b_1|^2 b_1 - 2\bar{b}_1 b_2^2, \\ i\partial_t b_2 = |b_2|^2 b_2 - 2\bar{b}_2 b_1^2. \end{cases}$$

Hamilonian system

$$H(b_1, b_2, \bar{b}_1, \bar{b}_2) = \frac{1}{2}|b_1|^4 + \frac{1}{2}|b_2|^4 - (b_1^2 \bar{b}_2^2 + \bar{b}_1^2 b_2^2)$$

Since $b_j = |b_j| e^{i\theta_j}$,

$$H(|b_1|^2, \theta_1, |b_2|^2, \theta_2) = \frac{1}{2}|b_1|^4 + \frac{1}{2}|b_2|^4 - 2|b_1|^2|b_2|^2 \cos(2\theta_1 - 2\theta_2)$$

2D Cubic NLS

Mass conservation law : $M = |b_1|^2 + |b_2|^2$
New coordinates change

$$\begin{pmatrix} |b_1|^2 \\ |b_2|^2 \end{pmatrix} \longrightarrow \begin{pmatrix} K_0 \\ K_1 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |b_1|^2 \\ |b_2|^2 \end{pmatrix} = \begin{pmatrix} |b_1|^2 \\ M \end{pmatrix}$$

and

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \theta_1 - \theta_2 \\ \theta_2 \end{pmatrix}$$

Then the Hamiltonian is rewritten as

$$\begin{aligned} H(K_0, K_1, \varphi_0, \varphi_1) &= H(K_0, K_1, \varphi_0) \\ &= \frac{1}{2}[K_0^2 + (K_1 - K_0)^2] - 2K_0(K_1 - K_0)\cos 2\varphi_0 \end{aligned}$$

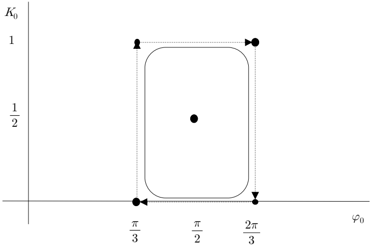
2D Cubic NLS

In new variables,

- $\dot{\varphi}_0 = -\frac{\partial H}{\partial K_0} = -2(2K_0 - 1)(\frac{1}{2} + \cos 2\varphi_0)$
- $\dot{\varphi}_1 = -\frac{\partial H}{\partial K_1} = -1 + 2K_0(\frac{1}{2} + \cos 2\varphi_0)$
- $\dot{K}_0 = \frac{\partial H}{\partial \varphi_0} = 4K_0(1 - K_0)\sin 2\varphi_0$
- $\dot{K}_1 = \frac{\partial H}{\partial \varphi_1} = 0(\Rightarrow K_1 \overset{\text{set}}{=} 1)$

and we also have

- $\dot{\varphi}_0 = 0 \Leftrightarrow K_0 = \frac{1}{2}$ or $\varphi_0 = \frac{\pi}{3}$ or $\frac{2\pi}{3}$
- $\dot{K}_0 = \frac{\partial H}{\partial \varphi_0} = 0 \Leftrightarrow K_0 = 0$ or 1 or $\varphi_0 = \frac{\pi}{2}$



2D Cubic NLS

When $\varphi_0 = \frac{2\pi}{3}$ ($\Rightarrow \sin 2\varphi_0 = -\frac{\sqrt{3}}{2}$)

$$\dot{K}_0 = -2\sqrt{3}K_0(1-K_0) \Rightarrow \frac{dK_0}{K_0(1-K_0)} = -2\sqrt{3}dt$$

$$\Rightarrow \frac{K_0}{1-K_0} = C_1 e^{-2\sqrt{3}t}$$

$$\Rightarrow K_0 = \frac{1}{1+C_2 e^{2\sqrt{3}t}} \quad (C_2 = C_1^{-1})$$

Going back to the original solutions,

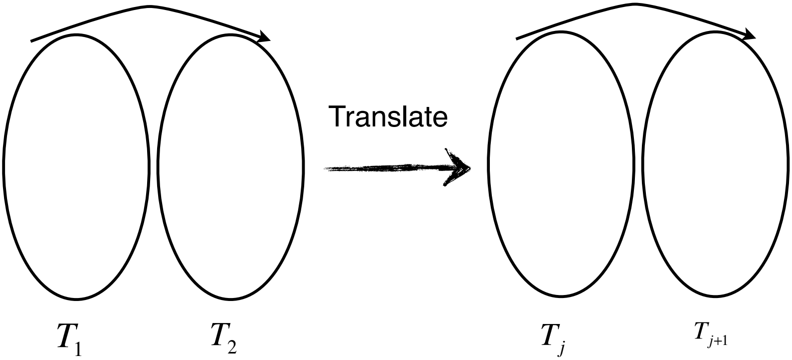
$$\dot{\varphi}_1 = -1 \Rightarrow \varphi_1 = \theta_2 = -t, \quad \theta_1 = \frac{2\pi}{3} - t$$

Hence, from $K_0 = |b_1|^2$ ($C_2 = 1 \Leftrightarrow |b_1(0)|^2 = \frac{1}{2}$),

$$b_1(t) = \frac{e^{-it}\omega}{\sqrt{1+e^{2\sqrt{3}t}}}, \quad b_2(t) = \frac{e^{-it}\omega^2}{\sqrt{1+e^{-2\sqrt{3}t}}},$$

where $\omega = e^{2\pi i/3}$ is a cube root of unity.

Slider solution



Target

- A target (M, d, R) : M : subset of Σ, d : semi-metric on Σ, $R > 0$: radius
- $(M_1, d_1, R_1) \twoheadrightarrow (M_2, d_2, R_2) \Leftrightarrow \forall x_2 \in M_2 \; \exists x_1 \in M_1$ such that $d_1(x_1, y_1) < R_1$ for all $y_1 \in \Sigma$ and $\exists y_2 \in \Sigma$ with $d_2(x_2, y_2) < R_2$ such that y_1 hits y_2
- *Incoming target* (M_j^-, d_j^-, R_j^-) : located near the stable manifold of $\mathbb{T}_j$
- *Ricochet target* (M_j^0, d_j^0, R_j^0) : located very near $\mathbb{T}_j$ itself
- *Outgoing target* (M_j^+, d_j^+, R_j^+) : located near the unstable manifold of $\mathbb{T}_j$

Ingredients : See the reference

- (Transitivity) $(M_1, d_1, R_1) \twoheadrightarrow (M_2, d_2, R_2)$, $(M_2, d_2, R_2) \twoheadrightarrow (M_3, d_3, R_3) \Rightarrow (M_1, d_1, R_1) \twoheadrightarrow (M_3, d_3, R_3)$
- (Sec. 3.6) $(M_j^-, d_j^-, R_j^-) \twoheadrightarrow (M_j^0, d_j^0, R_j^0)$, $3 < j \leq N-2$
- (Sec. 3.8) $(M_j^0, d_j^0, R_j^0) \twoheadrightarrow (M_j^+, d_j^+, R_j^+)$, $3 \leq N < N-2$
- (Sec. 3.10) $(M_j^+, d_j^+, R_j^+) \twoheadrightarrow (M_{j+1}^-, d_{j+1}^-, R_{j+1}^-)$, $3 \leq N < N-2$

Hence, we have

$$(M_3^0, d_3^0, R_3^0) \twoheadrightarrow (M_{N-2}^0, d_{N-2}^0, R_{N-2}^0), \qquad (*)$$

which implies directly the proof of Theorem

Reference

- J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation*, Inventiones mathematicae, 181(1), 39-113.

Thank You
for Your Kind Attention!!