Endpoint Strichartz estimates

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Abstract of the paper

We prove an abstract Strichartz estimate, which implies previously unknown endpoint Strichartz estimates for the wave equation (in dimension $n \geq 4$) and the Schrödinger equation (in dimension $n \geq 3$).

Three other applications are discussed: local existence for a nonlinear wave equation; and Strichartz-type estimates for more general dispersive equations and for the kinetic transport equation.

Let: (X, dx) measure space, H Hilbert space,

Suppose: $\forall t \in \mathbb{R}$, we have an operator $U(t) : H \to L^2(X)$ which obeys

- Energy estimate:

$$\left\| U(t) f \right\|_{L^2} \leq C \left\| f \right\|_H, \quad \forall t, \forall f \in H, \tag{1}$$

- Decay estimate (one of the following): $\exists \sigma > 0$ s.t.

$$\left\| U(t) U(s)^* g \right\|_{L^\infty} \leq C |t - s|^{-\sigma} \left\| g \right\|_{L^1} \quad \forall t \neq s, \forall g \in L^1 \cap L^2(X) \tag{2}$$

($\mathit{untruncated\ decay}$), or

$$\left\| U(t) U(s)^* g \right\|_{L^\infty} \leq C (1 + |t - s|)^{-\sigma} \left\| g \right\|_{L^1} \quad \forall t, s, \forall g \in L^1 \cap L^2(X) \tag{3}$$

($\mathit{truncated\ decay}$), where $U(s)^* : L^2(X) \to H$ is the adjoint of $U(s)$.

In particular, we consider the following cases:

$$X = \mathbb{R}^n, \quad H = L^2(\mathbb{R}^n), \quad \text{and}$$

(i) Schrödinger case:

$$[U(t)f](x) = [e^{it\Delta} f](x) = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4it}} f(y)\, dy.$$

(ii) Wave case:

$$[U(t)f](x) = [e^{-it|\nabla|} P_N f](x) = \mathcal{F}^{-1}\left[e^{-it|\cdot|} \phi_N \mathcal{F} f\right](x),$$

where $N \in 2^{\mathbb{Z}}$ and P_N is a Littlewood-Paley projection to $\{|\xi| \sim N\}$.

- $U(t)$ satisfies (1), and

[Schrödinger case] for $n \geq 1$, $U(t)$ satisfies (2) with $\sigma = \frac{n}{2}$.

[wave case] for $n \geq 2$, $U(t)$ satisfies (3) with $\sigma = \frac{n-1}{2}$.

Definition 1.1. We say that the exponent pair (q, r) is σ-admissible if

$$2 \leq q, r \leq \infty, \quad (q, r, \sigma) \neq (2, \infty, 1), \quad \text{and}$$

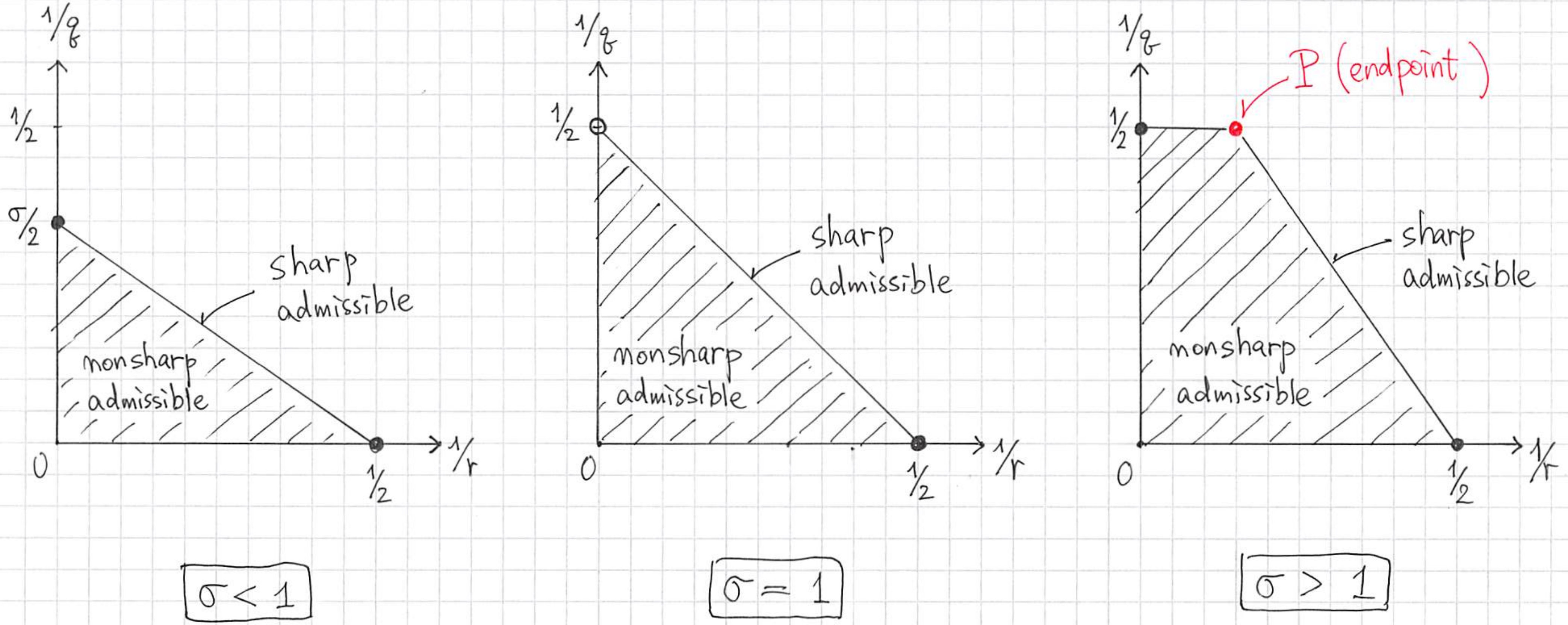
$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}. \tag{4}$$

If equality holds in (4) we say that (q, r) is sharp σ-admissible, otherwise nonsharp σ-admissible.

In particular, when $\sigma > 1$ the endpoint

$$P = (2, \frac{2\sigma}{\sigma - 1})$$

is sharp σ-admissible.


1/q
1/2
σ/2
sharp admissible
nonsharp admissible
0
1/2
1/r
σ < 1
1/q
1/2
sharp admissible
nonsharp admissible
0
1/2
1/r
σ = 1
1/q
P (endpoint)
1/2
sharp admissible
nonsharp admissible
0
1/2
1/r
σ > 1

Theorem 1.2. If $U(t)$ obeys (1) and (2), then the estimates

$$\|U(t)f\|_{L_t^q L_x^r} \leq C\|f\|_H, \tag{5}$$

$$\|\int_{\mathbb{R}} U(s)^* F(s)\, ds\|_H \leq C\|F\|_{L_t^{q'} L_x^{r'}}, \tag{6}$$

$$\|\int_{-\infty}^{t} U(t)U(s)^* F(s)\, ds\|_{L_t^q L_x^r} \leq C\|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{7}$$

hold for all *sharp* σ*-admissible* exponent pairs (q, r), $(\tilde{q}, \tilde{r})$, where q' is the Hölder conjugate of q (i.e. $\frac{1}{q} + \frac{1}{q'} = 1$).

Furthermore, if the decay hypothesis is strengthened to (3), then (5)–(7) hold for all *(sharp and nonsharp)* σ*-admissible* (q, r) and $(\tilde{q}, \tilde{r})$.

- Result on endpoint cases (i.e. $\sigma > 1$ and (q, r) or $(\tilde{q}, \tilde{r}) = P$) is new.

Corollary 1.3 (wave case). Suppose that $n \geq 2$ and (q, r) and $(\widetilde{q}, \widetilde{r})$ are $\frac{n-1}{2}$-admissible pairs with $r, \widetilde{r} < \infty$. If u is a (weak) solution to the problem

$$\begin{cases} (-\partial_t^2 + \Delta) u(t,x) = F(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g \end{cases}$$

for some data f, g, F and time $0 < T < \infty$, then

$$\begin{aligned} \|u\|_{L^q([0,T];L^r)} + \|u\|_{C([0,T];\dot{H}^\gamma)} + \|\partial_t u\|_{C([0,T];\dot{H}^{\gamma-1})} \\ \leq C\big(\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\widetilde{q}'}([0,T];L^{\widetilde{r}'})}\big), \end{aligned} \tag{9}$$

under the assumption

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma = \frac{1}{\widetilde{q}'} + \frac{n}{\widetilde{r}'} - 2. \tag{10}$$

The constant $C > 0$ in (9) is independent of f, g, F, T. Conversely, if (9) holds for all f, g, F, T, then (q, r) and $(\widetilde{q}, \widetilde{r})$ must be $\frac{n-1}{2}$-admissible and (10) must hold.

Furthermore, when $r = \infty$ the estimate (9) holds with the L^r norm replaced with the Besov norm $\dot{B}^0_{r,2}$, and similarly for $\widetilde{r} = \infty$.

Corollary 1.4 (Schrödinger case). Suppose that $n \geq 1$ and (q,r) and $(\widetilde{q},\widetilde{r})$ are sharp $\frac{n}{2}$-admissible pairs. If u is a (weak) solution to the problem

$$\begin{cases} (i\partial_t + \Delta)u(t,x) = F(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,\cdot) = f \end{cases}$$

for some data f, F and time $0 < T < \infty$, then

$$\|u\|_{L^q([0,T];L^r)} + \|u\|_{C([0,T];L^2)} \leq C\big(\|f\|_{L^2} + \|F\|_{L^{\widetilde{q}'}([0,T];L^{\widetilde{r}'})}\big), \qquad (11)$$

where the constant $C > 0$ is independent of f, F, T. Conversely, if (11) holds for all f, F, T, then (q,r) and $(\widetilde{q},\widetilde{r})$ must be sharp $\frac{n}{2}$-admissible.

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§3. Proof of (5) and (6) for $(q, r) \neq P$

First of all, we see that the estimate (5) follows from (6) by duality.

$(\because)$ For any test function $G : \mathbb{R} \times X \to \mathbb{C}$, we have

$$
\begin{aligned}
&\Big| \int_{\mathbb{R}} \int_X [U(t)f](x) \overline{G(t,x)} \, dx \, dt \Big| \\
&= \Big| \int_{\mathbb{R}} \langle U(t)f, \, G(t) \rangle_{L^2} \, dt \Big| = \Big| \int_{\mathbb{R}} \langle f, \, U(t)^* G(t) \rangle_H \, dt \Big| \\
&= \Big| \langle f, \int_{\mathbb{R}} U(t)^* G(t) \, dt \rangle_H \Big| \leq \|f\|_H \Big\| \int_{\mathbb{R}} U(t)^* G(t) \, dt \Big\|_H \\
&\leq C \|f\|_H \|G\|_{L_t^{q'} L_x^{r'}},
\end{aligned}
$$

which implies (5). $\square$

Since

$$\Big\| \int_{\mathbb{R}} U(s)^* F(s)\, dt \Big\|_H^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \langle U(s)^* F(s),\, U(t)^* F(t) \rangle_H \, ds\, dt,$$

(6) follows from the bilinear form estimate

$$\Big| \int_{\mathbb{R}} \int_{\mathbb{R}} \langle U(s)^* F(s),\, U(t)^* G(t) \rangle_H \, ds\, dt \Big| \le C \big\| F \big\|_{L_t^{q'} L_x^{r'}} \big\| G \big\|_{L_t^{q'} L_x^{r'}}. \qquad (13)$$

In fact, (6) is equivalent to (13). (TT^* method)

It then suffices to prove (13) for

$2 \le q, r \le \infty$, $\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}$, $(q, r) \neq (2, \frac{2\sigma}{\sigma - 1})$ under (1), (2) [Untruncated],

$2 \le q, r \le \infty$, $\frac{1}{q} + \frac{\sigma}{r} \le \frac{\sigma}{2}$, $(q, r) \neq (2, \frac{2\sigma}{\sigma - 1})$ under (1), (3) [Truncated].

Case (i): Untruncated decay (2)

By the energy estimate (1), we have $\|U(t)^*F(t)\|_H \le C\|F(t)\|_{L^2}$ uniformly in t, which implies

$$\big\|U(t)U(s)^*F(s)\big\|_{L^2} \le C\big\|F(s)\big\|_{L^2}.$$

Using *Riesz-Thorin theorem* to interpolate this inequality and (2), we have

$$\|U(t)U(s)^*F(s)\|_{L^r} \le C|t-s|^{-\sigma(1-\frac{2}{r})}\big\|F(s)\big\|_{L^{r'}}$$

for any $2 \le r \le \infty$. Therefore,

$$\begin{aligned}
\text{LHS of (13)} &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \big|\langle U(t)U(s)^*F(s),\, G(t)\rangle_{L^2}\big|\, ds\, dt \\
&\le \int_{\mathbb{R}}\int_{\mathbb{R}} \big\|U(t)U(s)^*F(s)\big\|_{L^r}\big\|G(t)\big\|_{L^{r'}}\, ds\, dt \\
&\le C\int_{\mathbb{R}}\int_{\mathbb{R}} |t-s|^{-\sigma(1-\frac{2}{r})}\big\|F(s)\big\|_{L^{r'}}\big\|G(t)\big\|_{L^{r'}}\, ds\, dt. \qquad \text{(A)}
\end{aligned}$$

- Hardy-Littlewood-Sobolev inequality (cf. Stein [2], Section V.1.2)

Let $1 < p_1, p_2 < \infty$, $0 < \lambda < n$ be such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{\lambda}{n} = 2$. Then,

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(\xi)g(\eta)}{|\xi - \eta|^\lambda} \, d\xi \, d\eta \right| \leq C \big\| f \big\|_{L^{p_1}} \big\| g \big\|_{L^{p_2}}.$$

Now, since (q, r) is sharp σ-admissible, we see that

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2} \qquad \Leftrightarrow \qquad \frac{1}{q'} + \frac{1}{q'} + \sigma\Big(1 - \frac{2}{r}\Big) = 2.$$

If $q < \infty$, the nonendpoint assumption $(q, r) \neq (2, \frac{2\sigma}{\sigma-1})$ implies

$$0 < \sigma\Big(1 - \frac{2}{r}\Big) < 1.$$

We apply the Hardy-Littlewood-Sobolev inequality to (A) and obtain (13).

The case $q = \infty$ follows directly from (A). $\square$

Case (ii): Truncated decay (3)

Since (3) $\Rightarrow$ (2), proof is reduced to Case (i) if (q, r) is *sharp σ-admissible*.

We consider *nonsharp σ-admissible* exponents, namely,

$$2 \le q, r \le \infty, \qquad \frac{1}{q} + \frac{\sigma}{r} < \frac{\sigma}{2}.$$

The same argument as Case (i) shows

$$\text{LHS of (13)} \le C \int_{\mathbb{R}} \int_{\mathbb{R}} \big(1 + |t-s|\big)^{-\sigma(1-\frac{2}{r})} \big\|F(s)\big\|_{L^{r'}} \big\|G(t)\big\|_{L^{r'}} \, ds \, dt$$

for any $2 \le r \le \infty$ instead of (A). Applying *Young's inequality*, we have

$$\text{LHS of (13)} \le C \big\|(1 + |\cdot|)^{-\sigma(1-\frac{2}{r})}\big\|_{L^{q/2}(\mathbb{R})} \big\|F\big\|_{L_t^{q'} L_x^{r'}} \big\|G\big\|_{L_t^{q'} L_x^{r'}}$$

whenever $\sigma(1 - \frac{2}{r})\frac{q}{2} > 1 \ \Leftrightarrow \ \frac{1}{q} + \frac{\sigma}{r} < \frac{\sigma}{2}$.

This concludes the proof of (5) and (6) when $(q, r) \neq P$. $\square$

§4. Proof of (5) and (6) for endpoint cases: Step I

Now, we consider the remaining *endpoint case*

$$(q, r) = P = (2, \frac{2\sigma}{\sigma - 1}), \qquad \sigma > 1. \tag{20}$$

Note that $2 < r < \infty$. Since P is sharp σ-admissible and (3) implies (2), we only consider the case of untruncated decay (2).

- The same argument as in §3 is not valid. In fact, the Hardy-Littlewood-Sobolev inequality is not applicable because $\sigma(1 - \frac{2}{r}) = 1$.

To show (13), we first decompose LHS dyadically as

$$\text{LHS of (13)} \leq \sum_{j \in \mathbb{Z}} \Big| \iint_{2^j \leq |s-t| < 2^{j+1}} \langle U(s)^* F(s), U(t)^* G(t) \rangle_H \, ds \, dt \Big|.$$

By symmetry it suffices to show

$$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \le C\|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{q'} L_x^{r'}}, \tag{22}$$

where

$$T_j(F,G) = \iint_{t-2^{j+1}<s\le t-2^j} \langle U(s)^* F(s),\, U(t)^* G(t)\rangle_H \, ds\, dt. \tag{21}$$

The goal of **Step I** is the following two-parameter family of estimates:

Lemma 4.1. Assume (20). The estimate

$$|T_j(F,G)| \le C 2^{-j\{\sigma-1-\sigma(\frac{1}{a}+\frac{1}{b})\}} \|F\|_{L_t^2 L_x^{a'}} \|G\|_{L_t^2 L_x^{b'}} \tag{23}$$

holds (uniformly) for all $j \in \mathbb{Z}$ and all $(\frac{1}{a}, \frac{1}{b})$ in a neighborhood of $(\frac{1}{r}, \frac{1}{r})$.

- Since $\sigma - 1 - \sigma(\frac{1}{r} + \frac{1}{r}) = 0$, we have (22) with $\sum_j$ replaced by $\sup_j$.

- First of all, we note that for the estimate of $T_j(F,G)$ we may assume that F, G are supported on a time interval of length $O(2^j)$.

$(\because)$ We decompose F as $F(s) = \sum_{l\in\mathbb{Z}} \chi_{[l2^j,(l+1)2^j)}(s)F(s)$, and assume that (23) holds (uniformly) for F, G supported in time on an $O(2^j)$ interval. Then, t is restricted to $[(l+1)2^j, (l+3)2^j)$ whenever $s \in [l2^j, (l+1)2^j)$, since the integral in s, t is restricted to $\{t-2^{j+1} < s \le t-2^j\}$. Therefore,

$$
\begin{aligned}
&|T_j(F,G)| \le \sum_{l\in\mathbb{Z}} \left|T_j(\chi_{[l2^j,(l+1)2^j)}F, \chi_{[(l+1)2^j,(l+3)2^j)}G)\right| \\
&\le C2^{-j\{\sigma-1-\sigma(\frac{1}{a}+\frac{1}{b})\}} \sum_{l\in\mathbb{Z}} \left\|\chi_{[l2^j,(l+1)2^j)}F\right\|_{L_t^2 L_x^{a'}} \left\|\chi_{[(l+1)2^j,(l+3)2^j)}G\right\|_{L_t^2 L_x^{b'}} \\
&\le C2^{-j\{\dots\}} \left(\sum_{l\in\mathbb{Z}} \left\|\chi_{[l2^j,(l+1)2^j)}F\right\|_{L_t^2 L_x^{a'}}^2\right)^{1/2} \left(\sum_{l\in\mathbb{Z}} \left\|\chi_{[(l+1)2^j,(l+3)2^j)}G\right\|_{L_t^2 L_x^{b'}}^2\right)^{1/2} \\
&\le C2^{-j\{\sigma-1-\sigma(\frac{1}{a}+\frac{1}{b})\}} \|F\|_{L_t^2 L_x^{a'}} \|G\|_{L_t^2 L_x^{b'}}. \quad \square
\end{aligned}
$$

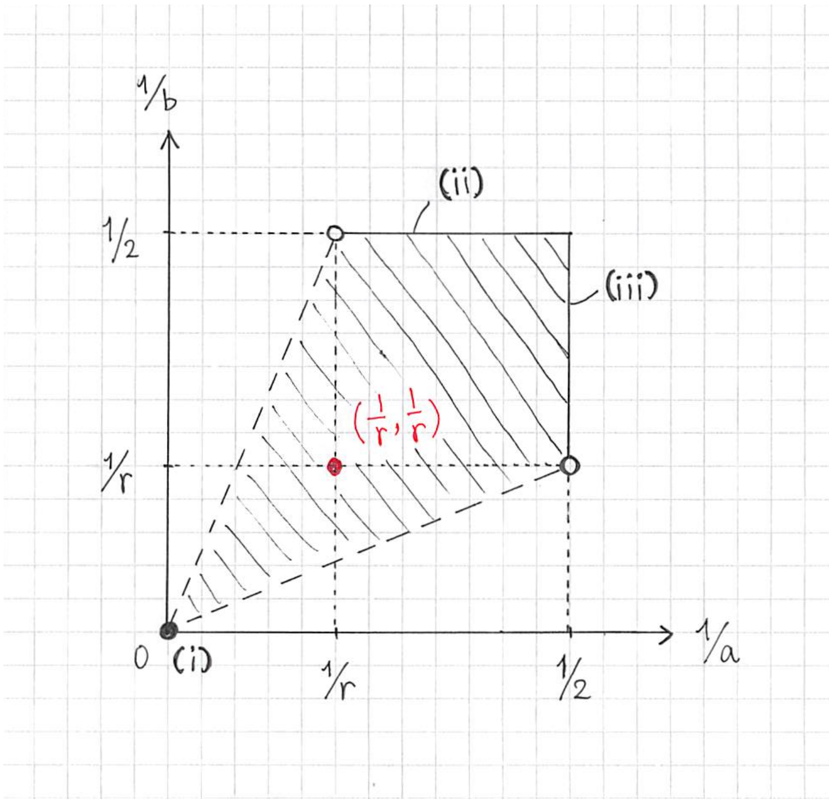
- We shall prove (23) for the exponents

(i) $a = b = \infty$,

(ii) $2 \leq a < r$, $b = 2$

(iii) $2 \leq b < r$, $a = 2$

The lemma will then follow by interpolation and the fact that $2 < r < \infty$.



Case (i) $a = b = \infty$

From the estimate (A) (with $r = \infty$) and the restriction to $\{t - 2^{j+1} < s \le t - 2^j\}$, we have

$$|T_j(F, G)| \le C 2^{-\sigma j} \|F\|_{L^1_t L^1_x} \|G\|_{L^1_t L^1_x}.$$

Recall that F, G are restricted in time to an interval of length $O(2^j)$. We apply *Hölder's inequality* in time to obtain

$$|T_j(F, G)| \le C 2^{-(\sigma - 1)j} \|F\|_{L^2_t L^1_x} \|G\|_{L^2_t L^1_x},$$

which is the desired estimate.

Case (ii) $2 \le a < r$**,** $b = 2$

Note that Case (iii) is parallel to (ii).

We bring the integration in s inside the inner product in (21) to obtain

$$
\begin{aligned}
|T_j(F,G)| &\le \int_{\mathbb{R}} \Big| \langle \int_{t-2^{j+1}}^{t-2^j} U(s)^* F(s)\, ds,\ U(t)^* G(t) \rangle_H \Big|\, dt \\
&\le \int_{\mathbb{R}} \Big\| \int_{t-2^{j+1}}^{t-2^j} U(s)^* F(s)\, ds \Big\|_H \big\| U(t)^* G(t) \big\|_H\, dt \\
&\le \sup_{t\in\mathbb{R}} \Big\| \int_{\mathbb{R}} U(s)^* \big[\chi_{(t-2^{j+1}, t-2^j]}(s) F(s) \big]\, ds \Big\|_H \cdot \int_{\mathbb{R}} \big\| U(t)^* G(t) \big\|_H\, dt.
\end{aligned}
$$

Since $2 \le a < r$, we can take $q(a)$ such that $(q(a), a)$ is *sharp* σ*-admissible* and $(q(a), a) \neq P$. By the nonendpoint Strichartz estimate (6) proved in §3 and Hölder's inequality in t, we obtain

$$\Big\| \int_{\mathbb{R}} U(s)^* \big[\chi_{(t-2^{j+1},t-2^j]}(s)F(s)\big]\,ds \Big\|_H$$
$$\leq C \big\| \chi_{(t-2^{j+1},t-2^j]} F \big\|_{L_t^{q(a)'} L_x^{a'}} \leq C 2^{j(\frac{1}{q(a)'} - \frac{1}{2})} \big\| F \big\|_{L_t^2 L_x^{a'}},$$

uniformly in t.

By the energy estimate (1) and Hölder's inequality in t, we have

$$\int_{\mathbb{R}} \big\| U(t)^* G(t) \big\|_H \, dt \leq C \big\| G \big\|_{L_t^1 L_x^2} \leq C 2^{j/2} \big\| G \big\|_{L_t^2 L_x^2}.$$

Combining these estimates, we have

$$|T_j(F,G)| \leq C 2^{j/q(a)'} \big\| F \big\|_{L_t^2 L_x^{a'}} \big\| G \big\|_{L_t^2 L_x^2}.$$

This is nothing but (23), since

$$\frac{1}{q(a)'} = 1 - \frac{1}{q(a)} = 1 - \sigma\Big(\frac{1}{2} - \frac{1}{a}\Big) = -\Big(\sigma - 1 - \sigma\Big(\frac{1}{a} + \frac{1}{2}\Big)\Big). \quad \square$$

§5. Proof of (5) and (6) for endpoint cases: Step II

- If we apply Lemma 4.1 directly for $a = b = r$, then we obtain

$$|T_j(F,G)| \le C\|F\|_{L_t^2 L_x^{r'}} \|G\|_{L_t^2 L_x^{r'}} \tag{25}$$

for each $j \in \mathbb{Z}$ (uniformly), which clearly *won't* sum to give (22).

Observation: To see how to sum up in j, we begin with the model case.

Assume that F and g have the special form

$$F(t,x) = f(t) \cdot 2^{-k/r'} \chi_{E(t)}(x), \quad G(t,x) = g(t) \cdot 2^{-\widetilde{k}/r'} \chi_{\widetilde{E}(t)}(x),$$

where $k, \widetilde{k} \in \mathbb{Z}$ and $E(t)$, $\widetilde{E}(t)$ are sets of measure 2^k and $2^{\widetilde{k}}$ respectively for each t. Note that $\|F\|_{L_t^2 L_x^{r'}} \sim \|f\|_{L^2}$, $\|G\|_{L_t^2 L_x^{r'}} \sim \|g\|_{L^2}$.

By Lemma 4.1, it holds that

$$\begin{aligned}|T_j(F,G)| &\le C2^{-j\{\sigma-1-\sigma(\frac{1}{a}+\frac{1}{b})\}}\|F\|_{L_t^2L_x^{a'}}\|G\|_{L_t^2L_x^{b'}}\\ &\le C2^{-j\sigma\{\frac{\sigma-1}{\sigma}-(\frac{1}{a}+\frac{1}{b})\}}\cdot 2^{\frac{k}{a'}-\frac{k}{r'}}\|f\|_{L^2}\cdot 2^{\frac{\widetilde{k}}{b'}-\frac{\widetilde{k}}{r'}}\|g\|_{L^2}\end{aligned}$$

(uniformly) for all $j\in\mathbb{Z}$ and $(\frac{1}{a},\frac{1}{b})$ in a neighborhood of $(\frac{1}{r},\frac{1}{r})$. Note that

$$\frac{\sigma-1}{\sigma}=\frac{2}{r},\qquad \frac{1}{a'}-\frac{1}{r'}=\frac{1}{r}-\frac{1}{a},\qquad \frac{1}{b'}-\frac{1}{r'}=\frac{1}{r}-\frac{1}{b}.$$

Then, the above estimate is simplified to

$$|T_j(F,G)|\le C2^{(k-j\sigma)(\frac{1}{r}-\frac{1}{a})}2^{(\widetilde{k}-j\sigma)(\frac{1}{r}-\frac{1}{b})}\|F\|_{L_t^2L_x^{r'}}\|G\|_{L_t^2L_x^{r'}}. \tag{26}$$

Take $\varepsilon>0$ sufficiently small so that the estimate is valid for $\frac{1}{a},\frac{1}{b}\in\{\frac{1}{r}\pm\varepsilon\}$. Now, for each $j\in\mathbb{Z}$ we choose $\frac{1}{a},\frac{1}{b}\in\{\frac{1}{r}\pm\varepsilon\}$ appropriately to obtain

$$|T_j(F,G)|\le C2^{-\varepsilon|k-j\sigma|}2^{-\varepsilon|\widetilde{k}-j\sigma|}\|F\|_{L_t^2L_x^{r'}}\|G\|_{L_t^2L_x^{r'}},$$

which *does* imply (22). $\square$

- This observation suggests that (25) is only sharp when F and G are both concentrated in a set of size $2^{j\sigma}$. However, such functions can only be critical for one scale of j. That's why we expect to obtain (22) for general F, G from Lemma 4.1.

- Also note that this argument requires a two-parameter family of estimates as Lemma 4.1, while the Strichartz estimates for *nonendpoint* case was obtained from a one-parameter family of estimates (namely, $a = b$).

To apply the above argument in the general case, we use the following lemma to decompose F, G so that each piece has a form similar to the above.

Lemma 5.1. Let $0 < p < \infty$ and $f \in L^p$. Then there exist $\{c_k\}_{k\in\mathbb{Z}} \subset [0,\infty)$, $\{\chi_k\}_{k\in\mathbb{Z}} \subset L^\infty$ such that

(i) $f(x) = \sum_{k\in\mathbb{Z}} c_k \chi_k(x)$,

(ii) $\|\chi_k\|_{L^\infty} \leq 2^{-k/p}$ and $\mathrm{meas}\{\, x \,|\, \chi_k(x) \neq 0 \,\} \leq 2 \cdot 2^k$,

(iii) $\|c_k\|_{\ell^p} \leq 2^{1+1/p} \|f\|_{L^p}$.

By applying Lemma 5.1 with $p = r'$ to $F(t)$ and $G(t)$, we have

$$F(t,x) = \sum_{k\in\mathbb{Z}} c_k(t)\chi_k(t,x), \qquad G(t,x) = \sum_{\widetilde{k}\in\mathbb{Z}} \widetilde{c}_{\widetilde{k}}(t)\widetilde{\chi}_{\widetilde{k}}(t,x), \tag{29}$$

where for each $t \in \mathbb{R}$ and $k \in \mathbb{Z}$ the function $\chi_k(t,\cdot)$ satisfies

$$\big\|\chi_k(t,\cdot)\big\|_{L^\infty} \leq 2^{-k/r'}, \qquad \mathrm{meas}\{\, x \,|\, \chi_k(t,x) \neq 0 \,\} \leq 2 \cdot 2^k,$$

and similarly for $\widetilde{\chi}_{\widetilde{k}}$. Moreover, $c_k(t)$ and $\widetilde{c}_{\widetilde{k}}(t)$ satisfy the inequalities

$$\big\| \|c_k(t)\|_{\ell^{r'}_k} \big\|_{L^2_t} \leq C \big\|F\big\|_{L^2_t L^{r'}_x}, \qquad \big\| \|\widetilde{c}_{\widetilde{k}}(t)\|_{\ell^{r'}_{\widetilde{k}}} \big\|_{L^2_t} \leq C \big\|G\big\|_{L^2_t L^{r'}_x}. \tag{30}$$

We are now ready to prove (22). By the decomposition (29) we have

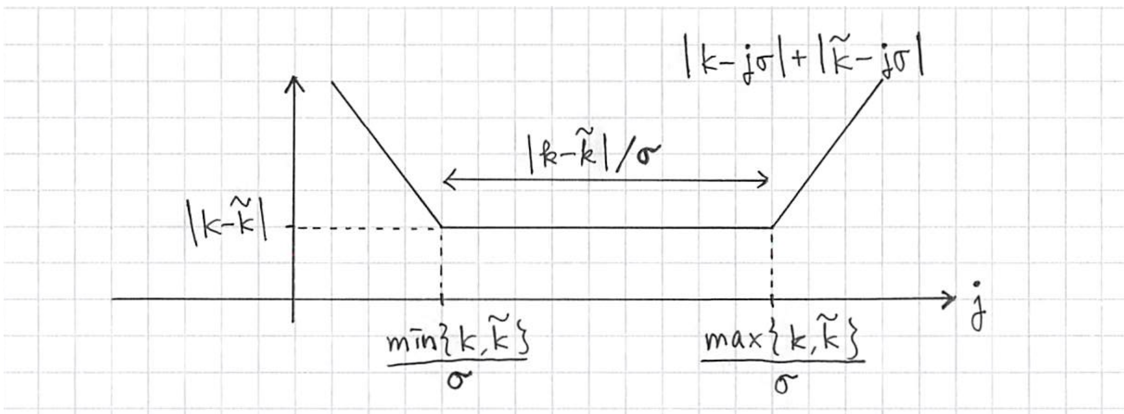
$$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \le \sum_{k,\widetilde{k}\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left|T_j(c_k\chi_k,\, \widetilde{c}_{\widetilde{k}}\widetilde{\chi}_{\widetilde{k}})\right|.$$

But by **Observation** at the start of §5, Lemma 4.1 gives

$$\left|T_j(c_k\chi_k,\, \widetilde{c}_{\widetilde{k}}\widetilde{\chi}_{\widetilde{k}})\right| \le C2^{-\varepsilon(|k-j\sigma|+|\widetilde{k}-j\sigma|)} \left\|c_k\right\|_{L^2} \left\|\widetilde{c}_{\widetilde{k}}\right\|_{L^2}$$

for some $\varepsilon > 0$. Summing in j, we have

$$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \le C \sum_{k,\widetilde{k}\in\mathbb{Z}} \big(1+|k-\widetilde{k}|\big)2^{-\varepsilon|k-\widetilde{k}|} \left\|c_k\right\|_{L^2} \left\|\widetilde{c}_{\widetilde{k}}\right\|_{L^2}.$$



Note that the quantity $w_k := (1+|k|)2^{-\varepsilon|k|}$ is summable, and RHS of the above estimate has the form $\sum_k \|c_k\|_{L^2}\big(w_{(\cdot)} * \|\widetilde{c}_{(\cdot)}\|_{L^2}\big)_k$.

We apply *Young's inequality*: $\big|\sum_k f_k (w*g)_k\big| \le \|w\|_{\ell^1}\|f\|_{\ell^2}\|g\|_{\ell^2}$ to obtain

$$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \le C \big\| \|c_k\|_{L^2} \big\|_{\ell^2_k} \big\| \|\widetilde{c}_{\widetilde{k}}\|_{L^2} \big\|_{\ell^2_{\widetilde{k}}}.$$

Interchanging the L^2 and ℓ^2 norms and using $\ell^{r'} \hookrightarrow \ell^2$, we obtain

$$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \le C \big\| \|c_k(t)\|_{\ell^{r'}_k} \big\|_{L^2_t} \big\| \|\widetilde{c}_{\widetilde{k}}(t)\|_{\ell^{r'}_{\widetilde{k}}} \big\|_{L^2_t}.$$

(22) then follows from (30), concluding the proof of (5), (6) for endpoint. □

We now proceed to the **proof of Lemma 5.1**. Let $f \in L^p$, $0 < p < \infty$.

Define the *distribution function* $\lambda(\alpha)$ of f for $\alpha \geq 0$ by

$$\lambda(\alpha) = \mathrm{meas}\{\, x \,\big|\, |f(x)| > \alpha \,\}.$$

Note that $\lambda(\alpha)$ is non-increasing and right-continuous.

For each $k \in \mathbb{Z}$, we set

$$\alpha_k = \inf \left\{\, \alpha > 0 \,\middle|\, \lambda(\alpha) < 2^k \,\right\}.$$

From definition we see that

$$0 \leq \alpha_k < \infty, \; \alpha_k \text{ is non-increasing in } k, \; \lim_{k \to -\infty} \alpha_k = \big\| f \big\|_{L^\infty} \in [0, \infty],$$

$$\lambda(\alpha_k) \leq 2^k, \text{ and } \lambda(\alpha_k - 0) \geq 2^k \text{ if } \alpha_k > 0. \tag{B}$$

Finally, we define

$$c_k = 2^{k/p}\alpha_k,$$

$$\chi_k(x) = \begin{cases} c_k^{-1}\chi_{(\alpha_{k+1},\alpha_k]}(|f(x)|)f(x) & \text{if } \alpha_k > 0, \\ 0 & \text{if } \alpha_k = 0 \ (= \alpha_{k+1}). \end{cases}$$

Property (i) is straightforward. For (ii), the L^∞ bound is easily verified. Since

$$\{\chi_k \neq 0\} \subset \{|f(x)| > \alpha_{k+1}\},$$

we have

$$\text{meas}\{\chi_k \neq 0\} \leq \lambda(\alpha_{k+1}) \leq 2^{k+1},$$

where we have used (B).

It remains to verify (iii). If we know *a priori* that

$$\sum_{k\in\mathbb{Z}} c_k^p = \sum_{k\in\mathbb{Z}} 2^k \alpha_k^p < \infty,$$

then we have

$$\sum_{k\in\mathbb{Z}} 2^k \alpha_k^p = \sum_{k\in\mathbb{Z}} (2^{k+1} - 2^k)\alpha_k^p = \sum_{k\in\mathbb{Z}} 2^{k+1}(\alpha_k^p - \alpha_{k+1}^p).$$

Let us take a non-increasing sequence $\{\alpha'_k\} \subset [0,\infty)$ such that

$$\begin{cases} \alpha_k > \alpha'_k > \alpha_{k+1}, \quad \alpha'_k \geq \alpha_k/2 & \text{if } \alpha_k > \alpha_{k+1}, \\ \alpha'_k = \alpha_k & \text{if } \alpha_k = \alpha_{k+1}. \end{cases}$$

Note that $\alpha_k \geq \alpha'_k \geq \alpha_{k+1}$ and $\alpha_k \leq 2\alpha'_k$ for all $k \in \mathbb{Z}$. Furthermore, from (B) we have $2^k \leq \lambda(\alpha'_k) \leq 2^{k+1}$ whenever $\alpha_k > \alpha_{k+1}$, which implies

$$\sum_{k\in\mathbb{Z}} 2^k(\alpha_k^p - \alpha_{k+1}^p) \le \sum_{k\in\mathbb{Z}} \lambda(\alpha_k')(\alpha_k^p - \alpha_{k+1}^p) \le \sum_{k\in\mathbb{Z}} 2^{k+1}(\alpha_k^p - \alpha_{k+1}^p).$$

Since RHS is absolutely summable, we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} 2^k \alpha_k^p &\le 2\sum_{k\in\mathbb{Z}} \lambda(\alpha_k')(\alpha_k^p - \alpha_{k+1}^p) \\
&= 2\sum_{k\in\mathbb{Z}} \big(\lambda(\alpha_k') - \lambda(\alpha_{k-1}')\big)\alpha_k^p \\
&\le 2\cdot 2^p \sum_{k\in\mathbb{Z}} \big(\lambda(\alpha_k') - \lambda(\alpha_{k-1}')\big)(\alpha_k')^p \qquad (\because \alpha_k \le 2\alpha_k') \\
&= 2^{1+p} \sum_{k\in\mathbb{Z}} (\alpha_k')^p \int_{\{\alpha_k' < |f(x)| \le \alpha_{k-1}'\}} dx \\
&\le 2^{1+p} \big\| f \big\|_{L^p}^p,
\end{aligned}$$

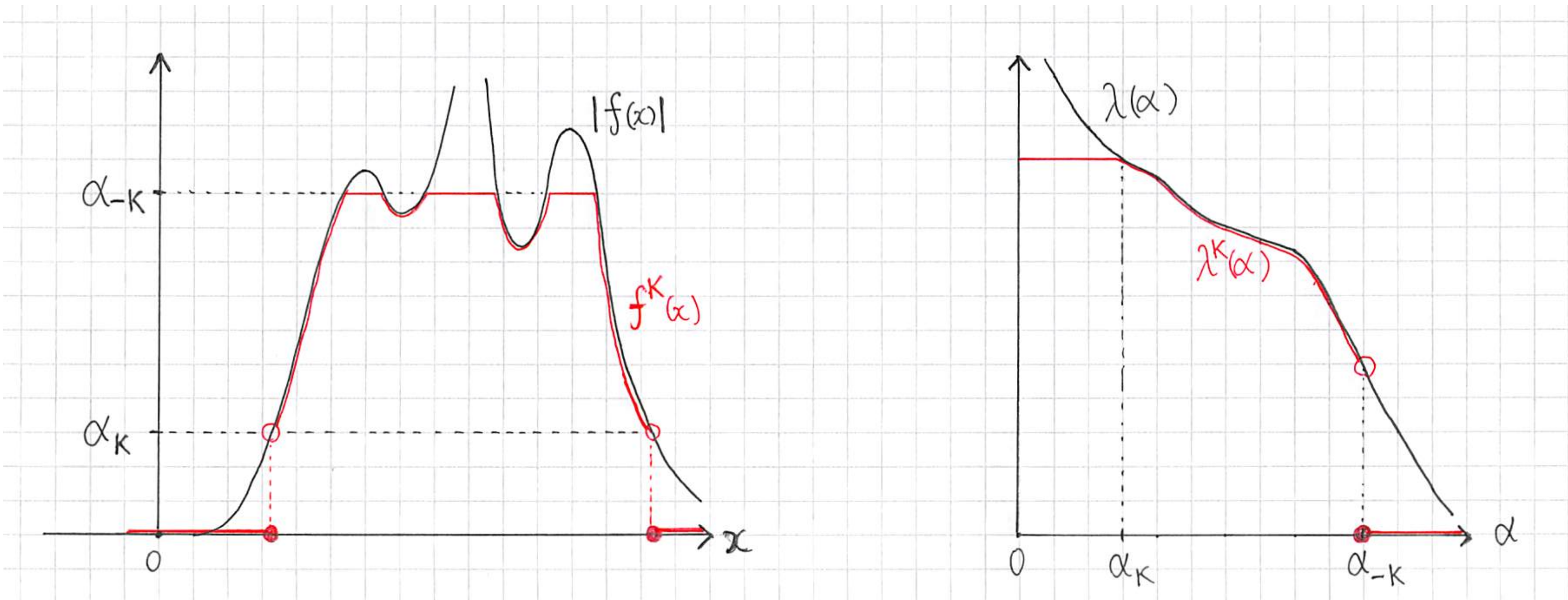
which shows (iii).

In the general case, we first define a non-negative function f^K for $K \in \mathbb{N}$ by

$$f^K(x) = \begin{cases} \alpha_{-K} & \text{if } |f(x)| > \alpha_{-K}, \\ |f(x)| & \text{if } \alpha_{-K} \geq |f(x)| > \alpha_K, \\ 0 & \text{if } |f(x)| \leq \alpha_K. \end{cases}$$

It is easy to see that

$$f^K(x) \leq |f(x)|, \quad K \in \mathbb{N}.$$



If we define the distribution function $\lambda^K(\alpha)$ and the sequence $\{\alpha_k^K\}_{k\in\mathbb{Z}}$ for each f^K as before, then

$$\lambda^K(\alpha) = \begin{cases} 0 & \text{if } \alpha \geq \alpha_{-K}, \\ \lambda(\alpha) & \text{if } \alpha_{-K} > \alpha \geq \alpha_K, \\ \lambda(\alpha_K) & \text{if } \alpha < \alpha_K, \end{cases} \qquad \alpha_k^K = \begin{cases} 0 & \text{if } k > K, \\ \alpha_k & \text{if } K \geq k > -K, \\ \alpha_{-K} & \text{if } k \leq -K, \end{cases}$$

so that

$$\sum_{k\in\mathbb{Z}} 2^k (\alpha_k^K)^p = \sum_{-K<k\leq K} 2^k \alpha_k^p + \alpha_{-K} \sum_{k\leq -K} 2^k < \infty$$

for any $K \in \mathbb{N}$. Therefore, we can apply the previous argument and obtain

$$\sum_{-K<k\leq K} 2^k \alpha_k^p \leq \sum_{k\in\mathbb{Z}} 2^k (\alpha_k^K)^p \leq 2^{1+p} \|f^K\|_{L^p}^p \leq 2^{1+p} \|f\|_{L^p}^p.$$

Letting $K \to \infty$ verifies (iii), which concludes the proof of Lemma 5.1. $\square$

§6. Alternate proof for Step II

In §5, we have deduced the *strong* (ℓ_j^1) *summability* in $L_t^2 L_x^{r'} \times L_t^2 L_x^{r'}$ from the *weak* (ℓ_j^∞) *summability* in $L_t^2 L_x^{a'} \times L_t^2 L_x^{b'}$ with (a, b) around (r, r).

Such a situation is similar to the *Marcinkiewicz interpolation theorem*, which asserts that the *strong boundedness*

$$T : L^p \longrightarrow L^p$$

of an operator T is deduced from the *weak boundedness*

$$T : L^{p_1} \longrightarrow L^{p_1,\infty}, \qquad T : L^{p_2} \longrightarrow L^{p_2,\infty}$$

if $1 \leq p_1 < p < p_2 \leq \infty$.

Marcinkiewicz interpolation theorem is one of the simplest consequences in real interpolation theory. In fact, we can rephrase the previous derivation of (22) from Lemma 4.1 by using existing results in real interpolation theory.

Real interpolation method

Let (A_0, A_1) be a pair of "compatible" Banach spaces. For parameters $0 < \theta < 1$ and $1 \leq q \leq \infty$, we define the real interpolation spaces $(A_0, A_1)_{\theta,q}$ as the spaces of all the elements a in

$$A_0 + A_1 = \left\{ a_0 + a_1 \,\middle|\, a_0 \in A_0,\, a_1 \in A_1 \right\}$$

such that the norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta} \inf_{a=a_0+a_1} \left(\|a_0\|_{A_0} + t\|a_1\|_{A_1} \right) \right]^q \frac{dt}{t} \right)^{1/q}$$

is finite, with the usual modification when $q = \infty$.

We will need the following two interpolation space identities.

- [Triebel [3], Sections 1.18.2 and 1.18.6]

$$\left(L_t^2 L_x^{p_0},\, L_t^2 L_x^{p_1}\right)_{\theta,2} = L_t^2 L_x^{p,2}$$

for $1 \le p_0 \ne p_1 \le \infty$, $\min\{p_0, p_1\} < 2$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

- [Bergh, Löfström [1], Section 5.6]

$$\left(\ell_{s_0}^{\infty},\, \ell_{s_1}^{\infty}\right)_{\theta,1} = \ell_s^1$$

for $s_0 \ne s_1 \in \mathbb{R}$ and $s = (1-\theta)s_0 + \theta s_1$.

Here, $L^{p,q}$ denotes *Lorentz spaces* which satisfy

$$L^{p,p} = L^p, \quad L^{p,q_1} \hookrightarrow L^{p,q_2} \qquad \text{for } 1 \le p \le \infty,\ 1 \le q_1 \le q_2 \le \infty,$$

and ℓ_s^q stands for *weighted sequence spaces* defined via the norm

$$\left\|\{a_j\}_{j\in\mathbb{Z}}\right\|_{\ell_s^q} = \left\|\{2^{sj} a_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q}.$$

With the notion of weighted sequence spaces ℓ^q_s, the estimate (23) can be regarded as boundedness

$$T : L^2_t L^{a'}_x \times L^2_t L^{b'}_x \to \ell^{\infty}_{\beta(a,b)}, \tag{31}$$

where $T = \{T_j\}_{j\in\mathbb{Z}}$ is the vector-valued bilinear operator and

$$\beta(a,b) = \sigma - 1 - \sigma\Big(\frac{1}{a} + \frac{1}{b}\Big),$$

and the claimed estimate (22) is rewritten as

$$T : L^2_t L^{r'}_x \times L^2_t L^{r'}_x \to \ell^1_0. \tag{C}$$

We will use the following bilinear interpolation theorem:

Lemma 6.1. ([1], Exercise 5(b) of Section 3.13)

Let (A_0, A_1), (B_0, B_1), (C_0, C_1) be "compatible" pairs of Banach spaces.

Suppose that the bilinear operator T is bounded as

$$T : A_0 \times B_0 \to C_0, \quad A_1 \times B_0 \to C_1, \quad A_0 \times B_1 \to C_1.$$

Then, we have boundedness

$$T : \big(A_0, A_1\big)_{\theta_A, p_A q} \times \big(B_0, B_1\big)_{\theta_B, p_B q} \to \big(C_0, C_1\big)_{\theta, q}$$

whenever

$$0 < \theta_A, \theta_B < \theta = \theta_A + \theta_B < 1, \quad 1 \leq p_A, p_B, q \leq \infty, \quad \frac{1}{p_A} + \frac{1}{p_B} \geq 1.$$

We set, with $\varepsilon > 0$ sufficiently small,

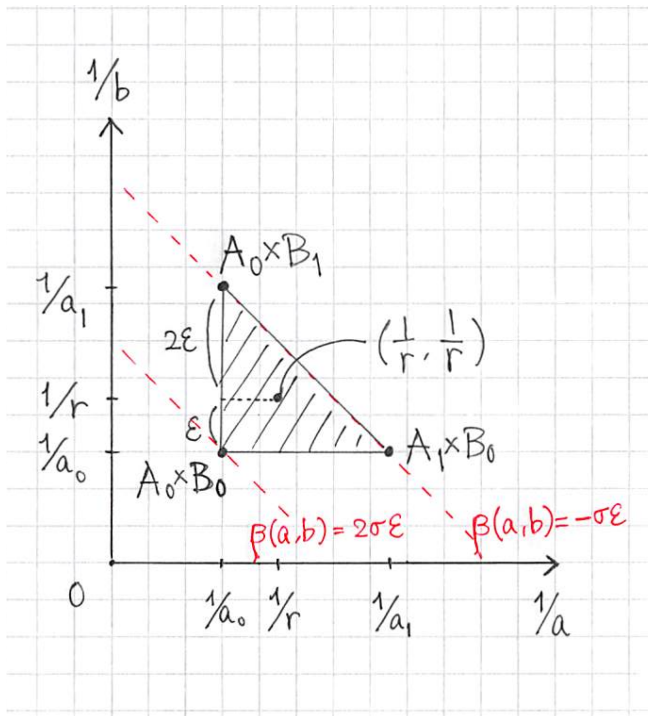
$$A_0 = B_0 = L_t^2 L_x^{a_0'}, \quad A_1 = B_1 = L_t^2 L_x^{a_1'}, \quad \frac{1}{a_0} = \frac{1}{r} - \varepsilon, \quad \frac{1}{a_1} = \frac{1}{r} + 2\varepsilon.$$

Then, by Lemma 4.1 (or (31)) the assumption in Lemma 6.1 is satisfied with

$$C_0 = \ell^\infty_{2\sigma\varepsilon}, \qquad C_1 = \ell^\infty_{-\sigma\varepsilon}.$$

We apply Lemma 6.1 with

$$\theta_A = \theta_B = \frac{1}{3}, \qquad p_A = p_B = 2, \qquad q = 1$$



to obtain the boundedness

$$T : L_t^2 L_x^{r',2} \times L_t^2 L_x^{r',2} \to \ell_0^1,$$

which implies the claim (C) because of the embedding $L^{r'} \hookrightarrow L^{r',2}$. $\square$

- There is also a *one-parameter* real interpolation theorem as follows:

Lemma. ([1], Exercise 4 of Section 3.13)

If the bilinear operator T is bounded as

$$T : A_0 \times B_0 \to C_0, \quad A_1 \times B_1 \to C_1,$$

then we have

$$T : \left(A_0, A_1\right)_{\theta,q} \times \left(B_0, B_1\right)_{\theta,q} \to \left(C_0, C_1\right)_{\theta,q}$$

whenever

$$0 < \theta < 1, \quad 1 \leq q \leq \infty.$$

However, we only deduce from this lemma

$$T : L_t^2 L_x^{r'} \times L_t^2 L_x^{r'} \to \ell_0^{r'},$$

and this is not sufficient because $r' = \frac{2\sigma}{\sigma+1} > 1$.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Springer-Verlag, New York, 1976.

[2] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[3] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, New York, 1978.

Thank you for your attention !