TENSOR PRODUCT AND CORRELATION ESTIMATES WITH APPLICATIONS TO NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this paper, we review Colliander, Grillakis and Tzirakis' paper [3]. We establish interaction Morawetz-type (correlation) estimates in one and two dimensions. We provide a proof in two different ways. Firstly, We follow the original approach of Lin and Strauss [12] applied to the tensor products of solutions. Secondly, we give the proof using commutator vector operators acting on the conservation laws of the equation. By using correlation estimates, we show the H^1 scattering for the L^2-supercritical nonlinear Schödinger equation. This result has already been proven by Nakanishi [13]. But the simplified proof is given in this paper.

1. Introduction

In this paper, we obtain new a priori estimates for solutions of the nonlinear Schrödinger equation (NLS) in one and two dimensions. We apply these estimates to study the global-in-time behavior of solutions to the following Cauchy problem:

$$\begin{cases} i\partial_t u + \triangle u = \mu |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $\mu = \pm 1$ and $p > 1$. (1.1) with $\mu = 1$ is the defocusing case and (1.1) with $\mu = -1$ is the focusing case. In the defocusing case, the Hamiltonian is positive definite. In this paper, we only deal with the defocusing NLS as follows:

$$i\partial_t u + \triangle u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{1.2}$$

Smooth solutions to (1.2) satisfy mass conservation

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}$$

and energy conservation

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t)|^2 \, dx + \frac{1}{p+1}\int |u|^{p+1} \, dx = E(u_0).$$

Now the scaling is given by

$$u^\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u(\lambda^{-2}t, \lambda^{-1}x)$$

for $\lambda \geq 1$. If u is a solution of (1.2), then u^λ is also a solution. The problem is called H^s-critical when $s = \frac{n}{2} - \frac{2}{p-1}$. Therefore L^2-critical and H^1-critical correspond to $p = 1 + \frac{4}{n}$ and $p = 1 + \frac{4}{n-2}$ respectively.

Conjecture. *Let $p = 1 + \frac{4}{n}$, $n \geq 1$ and $u_0 \in L^2(\mathbb{R}^n)$. L^2-critical equation (1.2) is globally well-posed in $L^2(\mathbb{R}^n)$. Moreover there exist $u_\pm \in L^2(\mathbb{R}^n)$ such that*

$$\left\| u(t) - e^{it\triangle} u_\pm \right\|_{L^2(\mathbb{R}^n)} \to 0, \ \text{ as } t \to \pm\infty.$$

In the subcritical case, the local time T only depends on the norm of initial data. But in the critical case the local time depends not only on the norm of initial data but also the profile. This conjecture has recently been solved by Dondon in [8], [9] and [10]. He used the interaction Morawetz estimates and minimal blow-up solutions. For the details, see Dodson's papers.

In the present paper, we deal with L^2-supercritical (H^1-subcritical) problem in one and two dimensions. Our aim is to establish the global well-posedness and scattering for the L^2-supercritical NLS (1.2) in $H^1(\mathbb{R}^n)$ for $n = 1, 2$. Nakanishi [13] already have solved this problem. But his proof is very complicated. Then we give a simple proof by using the correlation estimates.

The interaction Morawetz inequalities help us to prove global well-posedness and scattering. Visan [14] and Visan and Zhang [15] established the following Morawetz inequality for $n \geq 4$

$$\left\| D^{-\frac{n-3}{2}}(|u|^2) \right\|_{L^2_t L^2_x} \lesssim \|u\|_{L^\infty_t L^2_x} \|u\|_{L^\infty_t \dot{H}^{1/2}_x}. \tag{1.3}$$

In the case $n = 3$, Collinader, Keel, Staffilani, Takaoka and Tao [6] obtained

$$\|u\|^2_{L^4_t L^4_x} \lesssim \|u\|_{L^\infty_t L^2_x} \|u\|_{L^\infty_t \dot{H}^{1/2}_x}.$$

Collecting these estimates, we have (1.3) for $n \geq 3$. Note that the distribution $-\triangle\triangle|x|$ is positive for $n \geq 3$ but not positive anymore for $n = 1, 2$. So we need to construct another approach. Then we use the commutator vector operators acting on the conservation laws and obtain the following interaction Morawetz inequalities in one and two dimensions.

Theorem 1.1. (Correlation Estimates in Two Dimensions) *Let u be an $H^{1/2}$ solution to (1.2) on the space-time slab $I \times \mathbb{R}^2$. Then the following estimate holds*

$$\left\| D^{1/2}(|u|^2) \right\|_{L^2_t L^2_x} \lesssim \|u\|_{L^\infty_t L^2_x} \|u\|_{L^\infty_t \dot{H}^{1/2}_x}. \tag{1.4}$$

Theorem 1.2. (Correlation Estimates in One Dimension) *Let u be an H^1 solution to (1.2) on the space -time slab $I \times \mathbb{R}$. Then the following estimate holds*

$$\left\|\partial_x(|u|^2)\right\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty L_x^2}^{3/2} \|u\|_{L_t^\infty \dot{H}_x^1}^{1/2}, \tag{1.5}$$

$$\|u\|_{L_t^{p+3} L_x^{p+3}}^{p+3} \lesssim \|u\|_{L_t^\infty L_x^2}^{3} \|u\|_{L_t^\infty \dot{H}_x^1}. \tag{1.6}$$

The correlation estimate in two dimensions (1.4) can be considered as the diagonal, nonlinear analog of the bilinear refinement of Strichartz established by Bourgain [1]. A weaker local-in-time estimate was obtained in [11]

$$\|u\|_{L_{t\in[0,T]}^4 L_x^4}^2 \lesssim T^{1/4} \|u_0\|_{L^2} \|u\|_{L_{t\in[0,T]}^\infty \dot{H}_x^{1/2}}.$$

This estimate is very useful since the $L_t^4 L_x^4$ norm is the Strichartz norm and enables us to extend local-in-time solutions to global ones. Moreover this estimate was improved to the following one by Colliander, Grillakis and Tzirakis [2].

$$\|u\|_{L_{t\in[0,T]}^4 L_x^4}^2 \lesssim T^{1/6} \|u_0\|_{L_x^2}^{4/3} \|u\|_{L_{t\in[0,T]}^\infty \dot{H}_x^{1/2}}. \tag{1.7}$$

A brief proof of (1.7) is given in the subsection 3.1.

Using the Sobolev embedding theorem and (1.4), we can control the $L_t^4 L_x^8$ norm in two dimensions as follows:

$$\|u\|_{L_t^4 L_x^8}^2 \lesssim \left\|D^{1/2}(|u|^2)\right\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty L_x^2} \|u\|_{L^\infty \dot{H}_x^{1/2}}.$$

We can use this estimate to obtain a simplified proof of the H^1 scattering result.

Theorem 1.3. (Asymptotic Completeness in $H^1(\mathbb{R}^2)$) *Let $u_0 \in H^1(\mathbb{R}^2)$. Then there exists a unique solution to (1.2) when $p > 1$. Moreover if $p > 3$, there exist $u_\pm \in H^1(\mathbb{R}^2)$ such that*

$$\left\|u(t) - e^{it\triangle} u_\pm\right\|_{H^1(\mathbb{R}^2)} \to 0, \ \ \text{as } t \to \pm\infty.$$

Combining the correlation estimates and the I-method introduced in [5], we obtain the asymptotic completeness blow $H^1(\mathbb{R}^2)$.

Theorem 1.4. *Let $u_0 \in H^s(\mathbb{R}^2)$. Then for each positive integer $k \geq 2$, there exists $s_k = 1 - \frac{1}{4k-3}$ such that the Cauchy problem*

$$\begin{cases} i\partial_t u + \triangle u = |u|^{2k} u, \\ u|_{t=0} = u_0 \end{cases}$$

is globally well-posed and scatters provided $s > s_k$. In particular, there exist $u_\pm \in H^s(\mathbb{R}^2)$ such that

$$\left\|u(t) - e^{it\triangle} u_\pm\right\|_{H^s(\mathbb{R}^2)} \to 0, \ \ \text{as } t \to \pm\infty.$$

In this paper, we omit the proof of Theorem 1.4. For the proof, see [3].

2. Preliminaries

In this section, we introduce some notations and the Strichartz estimates.

Definition. A pair of exponents (q,r) is called admissible pair if $(q,r,n) \neq (2,\infty,2)$

$$\frac{2}{q}+\frac{n}{r}=\frac{n}{2}, \quad 2 \leq r \leq \infty.$$

For the space-time slab $I \times \mathbb{R}^n$, we define the Strichartz norm $\|\cdot\|_{S^0(I)}$ as

$$\|f\|_{S^0(I)} := \sup_{(q,r):\text{admissible}} \|f\|_{L_t^q L_x^r(I\times\mathbb{R}^n)}.$$

Then we have the following inhomogeneous Strichartz estimate.

Lemma 2.1. *Let I be a compact time interval, $t_0 \in I$, $s \geq 0$, and let u be a solution to*

$$i\partial_t u + \triangle u = \sum_{i=1}^{m} F_i$$

for some functions $F_1, F_2, \cdots, F_m$. Then,

$$\||\nabla|^s u\|_{S^0(I)} \lesssim \|u_0\|_{\dot{H}_x^s} + \sum_{i=1}^{m} \left\||\nabla|^s F_i\right\|_{L_t^{q_i'} L_x^{r_i'}(I\times\mathbb{R}^n)}$$

for any admissible pair (q_i, r_i) and $1 \leq i \leq m$. Here p' denotes $1/p + 1/p' = 1$.

3. Correlation Estimates for all Dimensions

We consider solutions of the equation

$$i\partial_t u + \triangle u = |u|^{p-1}u, \quad (t,x) \in [0,T]\times\mathbb{R}^n. \tag{3.1}$$

This equation has the following momentum conservation laws

$$\vec{p}(t) = \int_{\mathbb{R}^n} \mathrm{Im}(u(t,x)\nabla u(t,x))\; dx = \vec{p}(0).$$

We define the Morawetz action as

$$M_a(t) := \int_{\mathbb{R}^n} \nabla a(x)\cdot \mathrm{Im}(u(t,x)\nabla u(t,x))\; dx$$

where a: $\mathbb{R}^n \to \mathbb{R}$ and is convex. Put $\rho = \frac{1}{2}|u|^2$ and $\vec{p} = \mathrm{Im}(u\nabla u)$ corresponding to the mass density and the momentum density respectively. Now we assume that a solution u to (3.1) is smooth since the embedding $\mathcal{S} \subset H^s$ is dense. For the details, see [7]. The equation (3.1) satisfies the following local conservation laws:

$$\partial_t \rho + \partial_j p^j = 0, \quad \text{(local mass conservation)},$$
$$\partial_t p_k + \partial_j \big\{ \sigma_k^j + \delta_k^j \big(-\triangle \rho + \frac{p-1}{p+1} |u|^{p+1} \big) \big\} = 0, \quad \text{(local momentum conservation)}$$

where

$$\sigma_{j,k} := \frac{1}{\rho} (p_j p_k + \partial_j \rho \partial_k \rho)$$

which is a stress tensor. $\sigma_{j,k}$ can be rewritten by

$$\sigma_{j,k} = \mathrm{Re}(\partial_j \bar{u} \partial_k u).$$

We are ready to the generalized Virial identity introduced by Lin and Strauss [12].

Proposition 3.1. *If a is convex and u is a smooth solution to (1.2). Then the following estimate holds:*

$$\int_0^T \int_{\mathbb{R}^n} (-\triangle \triangle a(x)) |u(t,x)|^2 \, dt dx \lesssim \sup_{[0,T]} |M_a(t)|. \tag{3.2}$$

Proof. The Morawetz action $M_a(t)$ can be rewritten into

$$M_a(t) = 2 \int_{\mathbb{R}^n} (\partial_k a) p_k \, dx.$$

By using the local momentum conservation laws, we have

$$\begin{aligned}
\partial_t M_a(t) &= 2 \int_{\mathbb{R}^n} (\partial_k a) \partial_t p_k \, dx \\
&= -2 \int_{\mathbb{R}^n} (\partial_k a) \partial_j \Big\{ \sigma_k^j + \delta_k^j \big(-\triangle \rho + \frac{p-1}{p+1} |u|^{p+1} \big) \Big\} \, dx \\
&= 2 \int_{\mathbb{R}^n} (\partial_j \partial_k a) \sigma_{j,k} \, dx - 2 \int_{\mathbb{R}^n} (\partial_k^2 a) \triangle \rho \, dx + \frac{2(p-1)}{p+1} \int_{\mathbb{R}^n} (\partial_k^2 a) |u|^{p+1} \, dx \\
&= 4 \int_{\mathbb{R}^n} (\partial_j \partial_k a) \ \mathrm{Re}(\partial_j \bar{u} \partial_k u) \, dx + \int_{\mathbb{R}^n} (-\triangle \triangle a) |u|^2 \, dx \\
&+ \frac{2(p+1)}{p-1} \int_{\mathbb{R}^n} \triangle a |u|^{p+1} \, dx.
\end{aligned}$$

Since a is convex,

$$\partial_j \partial_k a \ \mathrm{Re}(\partial_j \bar{u} \partial_k u) \geq 0.$$

In addition, the trace of the Hessian $\partial_j \partial_k a$, which is $\triangle a$, is positive. Therefore we obtain

$$\int_{\mathbb{R}^n} (-\triangle \triangle a) |u(t,x)|^2 \, dx \leq \partial_t M_a(t),$$

which shows the desired estimate (3.2). $\square$

3.1. **Interaction Morawetz Inequalities.** In this subsection, we want to establish the interaction Morawetz inequalities by using the tensor product of two solutions to (3.1). Now the derivative D denotes $D^2 = -\triangle$.

Proposition 3.2. *Let u be an $H^{1/2}$ solution to (3.1). Then the following inequality holds for $n \geq 3$*

$$\big\|D^{-\frac{n-3}{2}}(|u|^2)\big\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty L_x^2}\|u\|_{L_t^\infty \dot{H}_x^{1/2}}. \tag{3.3}$$

Define the tensor product $u := (u_1 \otimes u_2)(t,x)$ for x in

$$\mathbb{R}^{n_1+n_2} := \big\{(x_1, x_2) \ : \ x_1 \in \mathbb{R}^{n_1},\ x_2 \in \mathbb{R}^{n_2}\big\}$$

by the formula

$$(u_1 \otimes u_2)(t,x) = u_1(t,x_1)u_2(t,x_2).$$

Let u_i be solutions to $i\partial_t u_i + \triangle u_i = F_i$ where $F_i = |u_i|^{p-1}u_i$ for $i = 1,2$. Put $F = F_1 \otimes u_2 + F_2 \otimes u_1$. Then $u = u_1 \otimes u_2$ solves

$$i\partial_t u + \triangle u = F. \tag{3.4}$$

We have that $\rho = \frac{1}{2}|u|^2 = \frac{1}{2}|u_1|^2|u_2|^2$ and $p_k = \mathrm{Im}\big(\overline{u_1 u_2}\partial_k(u_1u_2)\big)$. The equation (3.4) has the following local conservation laws:

$$\begin{aligned}
&\partial_t \rho + \partial_j p^j = 0,\\
&\partial_t p_k + \partial_j\big\{\sigma_k^j + \delta_k^j(-\triangle\rho + G)\big\} = 0
\end{aligned}$$

where

$$G = \frac{p-1}{p+1}(F_1 \otimes |u_2|^2 + F_2 \otimes |u_1|^2) \geq 0.$$

We define the Morawetz action corresponding to $u_1 \otimes u_2$ as

$$\begin{aligned}
M_a^{\otimes_2}(t) &= 2\int_{\mathbb{R}^{n_1}\otimes\mathbb{R}^{n_2}} \nabla a \cdot \mathrm{Im}\big(\overline{u_1 \otimes u_2}\nabla(u_1 \otimes u_2)\big)\ dx\\
&= M_a(u_1(t))\|u_2\|_{L^2}^2 + M_a(u_2(t))\|u_1\|_{L^2}^2.
\end{aligned}$$

In this setting, $\nabla = (\nabla_{x_1}, \nabla_{x_2})$ and $\triangle = \triangle_{x_1} + \triangle_{x_2}$. If u is a solution to (3.1), by using Proposition 3.1, we obtain for a convex function a that

$$\int_0^T \int_{\mathbb{R}^n\otimes\mathbb{R}^n} (-\triangle\triangle a)|(u_1 \otimes u_2)(t,x)|^2\ dtdx \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|. \tag{3.5}$$

We choose $a(x) = a(x_1, x_2) = |x_1 - x_2|$ for $x = (x_1, x_2) \in \mathbb{R}^n \otimes \mathbb{R}^n$. A simple calculation yields that

$$-\triangle\triangle a(x) = \begin{cases} c_0 \delta_{\{x_1 = x_2\}} & \text{for } n = 3, \\ \frac{(n-1)(n-3)}{|x_1 - x_2|^3} & \text{for } n \geq 4 \end{cases}$$

where c_0 is some positive constant. Substituting $-\triangle\triangle a$ into (3.5) and choosing $u_1 = u_2$, we have that in the case $n = 3$

$$\int_0^T \int_{\mathbb{R}^3} |u(t,x)|^4 \, dtdx \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

In the case $n \geq 4$, we have

$$\int_0^T \int_{\mathbb{R}^n} \Big(|u|^2 * \frac{1}{|\cdot|^3} \Big)(x_1) |u(t,x_1)| \, dtdx_1 \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|. \tag{3.6}$$

We can write

$$D^{-(n-3)} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^3} \, dy$$

since for $n \geq 4$ the distribution Fourier transform of $|x|^{-3}$ is given by $\widehat{|\cdot|^{-3}}(\xi) = |\xi|^{-(n-3)}$. By using the Plancherel theorem, we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(t,x_2)|^2}{|x_1 - x_2|^3} |u(t,x_1)|^2 \, dtdx &= \int_0^T \int_{\mathbb{R}^n} D^{-(n-3)}(|u(t,x)|^2) |u(t,x)|^2 \, dx \\ &= \int_0^T \int_{\mathbb{R}^n} \Big| D^{-\frac{n-3}{2}} (|u|^2) \Big|^2 dtdx. \end{aligned}$$

Then we obtain for $n \geq 4$

$$\int_0^T \int_{\mathbb{R}^n} \Big| D^{-\frac{n-3}{2}} (|u|^2) \Big|^2 dtdx \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

We combine the two estimate and it follows that for $n \geq 3$

$$\big\| D^{-\frac{n-3}{2}} (|u|^2) \big\|_{L_t^2 L_x^2}^2 \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

On the other hand, we estimate the action term $M_a^{\otimes_2}(t)$. It is enough to estimate $M_a(u(t))$ since

$$M_a^{\otimes_2}(t) = M_a(u_1(t)) \|u_2\|_{L_x^2}^2 + M_a(u_2(t)) \|u_1\|_{L_x^2}^2.$$

Note that $\nabla a(x) = \frac{x}{|x|}$. The Cauchy Schwarz inequality shows that

$$|M_a(u(t))| \lesssim \|u\|_{\dot{H}_x^{1/2}} \Big\| \frac{x}{|x|} u \Big\|_{\dot{H}_x^{1/2}}.$$

By using the Hardy inequality, we have

$$\left\|\nabla\left(\frac{x}{|x|}\right)u\right\|_{L^2_x} \lesssim \|\nabla u\|_{L^2_x}.$$

Interpolating this estimate and $\left\|\frac{x}{|x|}u\right\|_{L^2} \lesssim \|u\|_{L^2}$, we obtain $\|\frac{x}{|x|}u\|_{\dot H^{1/2}} \lesssim \|u\|_{\dot H^{1/2}}$. Therefore it follows that

$$|M_a(u(t))| \lesssim \|u\|^2_{\dot H^{1/2}_x},$$

which implies that the following interaction Morawetz inequality for $n \geq 3$ holds

$$\left\|D^{-\frac{n-3}{2}}(|u|^2)\right\|_{L^2_t L^2_x} \lesssim \|u\|_{L^\infty_t L^2_x}\|u\|_{L^\infty_t \dot H^{1/2}_x}.$$

This estimate is appeared in [7] and [15].

We remark that the above method breaks down for $n \geq 3$ since the distribution $-\triangle\triangle|x|$ is not positive anymore. Then we introduce the function $f : [0,\infty) \to [0,\infty)$ such that

$$f(x) = \begin{cases} \frac{x^2}{2M}(1 - \log\frac{x}{M}) & \text{if } 0 \leq x < \frac{M}{\sqrt{e}}, \\ 100x & \text{if } x > M, \\ \text{smooth and convex for all } x \end{cases}$$

where M is a large parameter determined later. When we choose $a(x_1, x_2) = f(|x_1 - x_2|)$ for $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$, a is convex and

$$-\triangle\triangle a = \frac{4\pi}{M}\delta_{\{x_1=x_2\}} + O\left(\frac{1}{M^3}\right).$$

In this setting, we can use the analogous method in the case $n \geq 3$. Taking

$$M \sim T^{1/3}\left(\frac{\|u\|_{L^\infty_t L^2_x}}{\|u\|_{L^\infty_t \dot H^{1/2}_x}}\right)^{2/3},$$

we get the following interaction Morawetz inequality in two dimensions

$$\|u\|^2_{L^4_{t\in[0,T]}L^4_x} \lesssim T^{1/6}\|u\|^{4/3}_{L^\infty_t L^2_x}\|u\|^{2/3}_{L^\infty_t \dot H^{1/2}_x},$$

which is a better estimate than one established by Fang and Grillakis [11].

3.2. **Commutator Vector Operators and Correlation Estimates in Dimension $n \geq 2$.** We drive the correlation estimates by using commutator vector operators acting on the conservation laws of the equation (3.1). Recall the Morawetz action

$$M_a^{\otimes_2}(t) = 2\int_0^T \int_{\mathbb{R}^{n_1}\otimes\mathbb{R}^{n_2}} \nabla a \cdot \mathrm{Im}\big(u_1 \otimes u_2 \nabla(u_1 \otimes u_2)\big)\, dt dx_1 dx_2$$

for the tensor product of solutions $u = (u_1 \otimes u_2)(t, x)$ where $x = (x_1, x_2) \in \mathbb{R}^n \otimes \mathbb{R}^n$. We consider the special case $u_1 = u_2$, $a(x_1, x_2) = |x_1 - x_2|$ for $n \geq 2$ and observe that

$$\nabla_{x_1} a(x_1, x_2) = \frac{x_1 - x_2}{|x_1 - x_2|} = -\frac{x_2 - x_1}{|x_2 - x_1|} = -\nabla_{x_2} a(x_1, x_2).$$

We can view $M(t) := M_a^{\otimes_2}(t)$ as

$$M_a^{\otimes_2}(t) = 2\int_0^T \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{x_1 - x_2}{|x_1 - x_2|} \big\{\vec{p}(t, x_1)\rho(t, x_2) - \vec{p}(t, x_2)\rho(t, x_1)\big\} dx_1 dx_2 dt.$$

Note that

$$D^{-(n-1)} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|}\ dy.$$

Now we define the vector operator $\vec{X}$ as

$$\vec{X} = [x, D^{-(n-1)}].$$

We change notations and write $x_1 = x$ and $x_2 = y$. Then the action term $M(t)$ can be rewritten

$$M(t) = \langle [x, D^{-(n-1)}]\rho \mid \vec{p}\rangle = \langle \vec{X}\rho \mid \vec{p}\rangle.$$

We investigate the property of the commutator vector operator $\vec{X}$. Notice that

$$\vec{X} f(x) = \int_{\mathbb{R}^n} \frac{x - y}{|x - y|} f(y)\ dy.$$

We calculate the differentiation of $\vec{X}$ and have

$$\partial_j X^k f(x) = \int_{\mathbb{R}^n} \frac{|x - y|^2 \delta_{j,k} - (x_j - y_j)(x_k - y_k)}{|x - y|^3} f(y)\ dy =: \int_{\mathbb{R}^n} r_{j,k}(x, y) dy.$$

A direct calculation shows that

$$\langle r_{j,k}\ | q_j q_k\rangle = \frac{|x - y|^2 |\vec{q}|^2 - \big((x - y) \cdot \vec{q}\big)^2}{|x - y|^3} \geq 0,$$

which implies that $\partial_j X^k$ is a positive definite operator. Moreover it follows that $\langle \vec{X} f\ | g\rangle = -\langle f\ | \vec{X} g\rangle$ from the Fubini theorem. Thus $\vec{X}$ is an antisymmetric operator.

We recall that the local mass and momentum conservation laws:

$$\partial_t \rho + \partial_j p^j = 0,$$
$$\partial_t p_k + \partial_j \big\{\sigma_k^j + \delta_k^j(-\triangle\rho + \frac{p-1}{p+1}|u|^{p+1})\big\} = 0,$$

where $\sigma_{j,k} = \rho^{-1}(p_j p_k + \partial_k \rho \partial_j \rho)$. We differentiate the action term and have

$$\partial_t M(t) = -\langle \partial_t \rho \mid \vec{X} \cdot \vec{p}\rangle - \langle \rho \mid \vec{X} \cdot \partial_t \vec{p}\rangle.$$

We divide $\partial_t M(t)$ into four parts as follows:

$$\begin{aligned}\partial_t M(t) &= -\langle \partial_j X^k p_k \mid p_j\rangle + \langle \sigma_{j,k} \mid \partial_j X^k \rho\rangle \\ &\quad + \langle -\triangle \rho \mid \partial_k X^k \rho\rangle + \frac{p-1}{p+1}\langle |u|^{p+1} \mid \partial_k X^k \rho\rangle \\ &= P_1 + P_2 + P_3 + P_4\end{aligned}$$

where

$$\begin{aligned}P_1 &= \langle \rho^{-1}\partial_j \rho \partial_k \rho \mid \partial_j X^k \rho\rangle, \\ P_2 &= \langle \rho^{-1} p_j p_k \mid \partial_j X^k \rho\rangle - \langle p_j \mid \partial_j X^k p_k\rangle, \\ P_3 &= -\langle \triangle \partial_k X^k \rho\rangle = -\langle \triangle \rho \mid \nabla \cdot \vec{X} \rho\rangle, \\ P_4 &= \frac{p-1}{p+1}\langle |u|^{p+1} \mid \nabla \cdot \vec{X}\rho\rangle.\end{aligned}$$

Since $\partial_j X^k$ is positive definite, P_1 is positive. Note that

$$\nabla \cdot \vec{X} = \partial_j X^j = (n-1) D^{-(n-1)}.$$

By using the Plancherel theorem, we have

$$\begin{aligned}P_3 &= \langle D^2 \rho \mid (n-1) D^{-(n-1)} \rho\rangle = (n-1)\langle D^{-\frac{n-3}{2}} \rho \mid D^{-\frac{n-3}{2}} \rho\rangle \\ &= \frac{n-1}{4}\big\| D^{-\frac{n-3}{2}} (|u|^2)\big\|_{L^2_x}^2.\end{aligned}$$

Moreover

$$\begin{aligned}P_4 &= \frac{p-1}{p+1}\langle |u|^{p+1} \mid (n-1) D^{-(n-1)} \rho\rangle \\ &= \frac{(p-1)(n-1)}{2(p+1)} \int_{\mathbb{R}^n} \frac{|u(t,x)|^{p+1} |u(t,x)|^2}{|x-y|}\, dxdy \geq 0.\end{aligned}$$

By changing variables, we compute P_2 to obtain

$$\begin{aligned}P_2 &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \Big\{ \frac{\rho(y)}{\rho(x)} p_j(x) p_k(x) - p_j(y) p_k(x)\Big\} r_{j,k}(x,y)\, dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^n \times \mathbb{R}^n} \bigg\{ \frac{\rho(x)}{\rho(y)} p_j(x) p_k(x) + \frac{\rho(y)}{\rho(x)} p_j(y) p_k(y) \\ &\qquad - p_j(y) p_k(x) - p_j(x) p_k(y)\bigg\} r_{j,k}(x,y)\, dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^n \times \mathbb{R}^n} \bigg\{ \sqrt{\frac{\rho(x)}{\rho(y)}} p_k(x) - \sqrt{\frac{\rho(y)}{\rho(x)}} p_k(y)\bigg\} \\ &\qquad \times \bigg\{ \sqrt{\frac{\rho(x)}{\rho(y)}} p_j(x) - \sqrt{\frac{\rho(y)}{\rho(x)}} p_j(y)\bigg\} r_{j,k}(x,y)\, dxdy\end{aligned}$$

If we define the two-point momentum vector as

$$\vec{J} = \sqrt{\frac{\rho(y)}{\rho(x)}}\vec{p}(x) - \frac{\rho(x)}{\rho(y)}\vec{p}(y),$$

then we have

$$P_4 = \langle J^j J_k \mid (\partial_j X^k)\rangle \geq 0$$

since $\partial_j X^k$ is positive definition. Thus

$$\left\| D^{-\frac{(n-3)}{2}}(|u|^2)\right\|_{L_x^2}^2 \leq \partial_t M(t).$$

We integrate in time to have

$$\left\| D^{-\frac{(n-3)}{2}}(|u|^2)\right\|_{L_t^2 L_x^2}^2 \lesssim \sup_{[0,T]} |M(t)|.$$

On the other hand,

$$\begin{aligned} |M(t)| = |\langle [x, D^{-(n-1)}]_j \rho \mid p_j\rangle| &\leq \|p_j\|_{L^1} \|[x, D^{-(n-1)}]_j \rho\|_{L^\infty} \\ &\leq \|p_j\|_{L^1} \|[x, D^{-(n-1)}]_j\|_{L^1 \to L^\infty} \|\rho\|_{L^1}. \end{aligned}$$

Following the Hardy inequality and interpolation, $\|p_j\|_{L^1} \lesssim \|u\|_{\dot{H}_x^{1/2}}^2$. Moreover $\|[x, D^{-(n-1)}]\|_{L^1 \to L^\infty}$ is bounded by 1 since

$$[x, D^{-(n-1)}]f(x) = \int_{\mathbb{R}^n} \frac{x-y}{|x-y|} f(y)\, dy \quad \text{for } f \in L^1.$$

Therefore $|M(t)| \lesssim \|u(t)\|_{L_x^2}^2 \|u(t)\|_{\dot{H}_x^{1/2}}^2$. For all $n \geq 2$, we have the correlation estimate

$$\left\| D^{-\frac{n-3}{2}}(|u|^2)\right\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty L_x^2} \|u\|_{L_t^\infty \dot{H}_x^{1/2}}.$$

3.3. **Correlation Estimate in One Dimension.** We would like to prove the following correlation estimate in one dimension by using the Gauss-Weierstrauss summability method:

$$\|\partial_x(|u|^2)\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty L_x^2}^{3/2} \|u\|_{L_t^\infty \dot{H}_x^1}^{1/2}$$

for solutions of one dimension NLS $i\partial_t u + \triangle u = |u|^{p-1}u$. Recall the local mass and momentum conservation laws as follows:

$$\begin{aligned} &\partial_t \rho + \partial_x p = 0, \\ &\partial_t p + \partial_x \big\{\rho^{-1}(\rho^2 + p_x^2) - \rho_{xx} + \frac{p-1}{p+1}|u|^{p+1}\big\} = 0. \end{aligned}$$

We define the action term as

$$M(t) = \int_{\mathbb{R}\times\mathbb{R}} a(x-y)\rho(y)p(x)\ dxdy$$

where

$$a(x-y) = \mathrm{erf}\Big(\frac{x-y}{\varepsilon}\Big) = \int_0^{\frac{x-y}{\varepsilon}} e^{-t^2}\ dt$$

is the scaled error function. Its derivative is

$$\partial_x \mathrm{erf}\Big(\frac{x-y}{\varepsilon}\Big) = \frac{1}{\varepsilon}\exp\Big(-\frac{(x-y)^2}{\varepsilon^2}\Big) \geq 0,$$

which is the heat kernel in one dimension. Clearly,

$$\sup_t |M(t)| \lesssim \|u\|^3_{L_t^\infty L_x^2}\|u\|_{L_t^\infty \dot{H}_x^1} \tag{3.7}$$

We can write $M(t) = \langle X\rho \mid p\rangle$ where X is defined as

$$Xf(x) = \int_{\mathbb{R}} \mathrm{erf}\Big(\frac{x-y}{\varepsilon}\Big) f(y)\ dy.$$

We easily check $\langle X\rho \mid p\rangle = -\langle \rho \mid Xp\rangle$, which implies X is an antisymmetric operator. We calculate the time derivative of the action term to obtain

$$\partial_t M(t) = -\langle \partial_t\rho \mid Xp\rangle - \langle \rho \mid X\partial_t p\rangle \tag{3.8}$$

Applying the local mass and momentum conservation laws to (3.8), we have

$$\partial_t M(t) = P_1 + P_2 + P_3 + P_4$$

where

$$P_1 := \langle \rho^{-1}\rho_x^2 \mid X'\rho\rangle,\ \ P_2 := \langle \rho^{-1}p^2 \mid X'\rho\rangle - \langle p \mid X'p\rangle,$$
$$P_3 := \langle (-\rho_{xx}) \mid X'\rho\rangle,\ \ P_4 := \frac{p-1}{p+1}\langle |u|^{p+1} \mid X'\rho\rangle.$$

These terms can be rewritten

$$P_1 = \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\varepsilon} e^{-(\frac{x-y}{\varepsilon})^2}\frac{\rho(y)}{\rho(x)}\rho_x^2(x)\ dxdy \geq 0,$$
$$P_4 = \frac{p-1}{p+1}\int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\varepsilon} e^{-(\frac{x-y}{\varepsilon})^2}\rho(y)|u(x)|^{p+1}\ dxdy \geq 0,$$
$$P_2 = \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\varepsilon} e^{-(\frac{x-y}{\varepsilon})^2}\Big\{\frac{\rho(y)}{\rho(x)}p^2(x) - p(x)p(y)\Big\}\ dxdy.$$

By using change variables and symmetry, P_2 is rewritten into

$$\begin{aligned}P_2 =&\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}}\frac{1}{\varepsilon}e^{-(\frac{x-y}{\varepsilon})^2}\Big\{\frac{\rho(y)}{\rho(x)}p^2(x)+\frac{\rho(x)}{\rho(y)}p^2(y)-2p(x)p(y)\Big\}\ dxdy\\ =&\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}}\frac{1}{\varepsilon}e^{-(\frac{x-y}{\varepsilon})^2}\Big\{\sqrt{\frac{\rho(y)}{\rho(x)}}p(x)-\sqrt{\frac{\rho(x)}{\rho(y)}}p(y)\Big\}^2\ dxdy\geq 0\end{aligned}$$

Following the Plancherel's theorem, we have

$$\begin{aligned}P_3 &= \int_{\mathbb{R}\times\mathbb{R}}\frac{1}{\varepsilon}e^{-(\frac{x-y}{\varepsilon})^2}\rho(y)(-\rho_{xx}(x))\ dxdy\\ &= \int_{\mathbb{R}}\Big(\frac{1}{\varepsilon}e^{-(\frac{\cdot}{\varepsilon})^2}*\rho\Big)(x)(-\rho_{xx}(x))\ dx\\ &= \int_{\mathbb{R}}\exp\Big(-\frac{\varepsilon^2\xi^2}{4}\Big)\xi^2\hat{\rho}^2(\xi)\ d\xi\geq 0.\end{aligned}$$

Notice that

$$\lim_{\varepsilon\to 0}\Big(e^{-(\frac{\cdot}{\varepsilon})^2}*\rho\Big)(x)=\rho(x)$$

in the distribution sense. Thus we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}P_1 &= \int_{\mathbb{R}}\rho_x^2(x)\ dx=\frac{1}{4}\|\partial_x(|u|^2)\|_{L_x^2}^2,\\ \lim_{\varepsilon\to 0}P_3 &= \frac{p-1}{p+1}\int_{\mathbb{R}}\rho(x)|u(x)|^{p+1}dx=\frac{p-1}{2(p+1)}\|u\|_{L_x^{p+3}}^{p+3}.\end{aligned}$$

Collecting the above estimates, we have

$$\begin{aligned}\frac{1}{4}\|\partial_x(|u|^2)\|_{L_x^2}^2 &\leq \partial_t M(t),\\ \frac{p-1}{2(p+1)}\|u\|_{L_x^{p+3}}^{p+3} &\leq \partial_t M(t).\end{aligned}$$

Integrating in time and using (3.7), we obtain

$$\begin{aligned}\|\partial_x(|u|^2)\|_{L_t^2L_x^2}^2 &\lesssim \|u\|_{L_t^\infty L_x^2}^3\|u\|_{L_t^\infty\dot{H}_x^1},\\ \|u\|_{L_t^{p+3}L_x^{p+3}}^{p+3} &\lesssim \|u\|_{L_t^\infty L_x^2}^3\|u\|_{L_t^\infty\dot{H}_x^1}.\end{aligned}$$

Recalling that the scaling is

$$u^\lambda(t,x)=\lambda^{-\frac{2}{p-1}}u(\lambda^{-2}t,\lambda^{-1}x),$$

we can verify that a priori one dimensional estimate

$$\|u\|_{L_t^{p+3}L_x^{p+3}}^{p+3}\lesssim\|u\|_{L_t^\infty L_x^2}^3\|u\|_{L_t^\infty\dot{H}_x^1}.$$

is scale invariant.

4. The Proof of the H^1 Scattering Results

In this section, by using the correlation estimate, we prove the global well-posedness and the scattering of the Cauchy problem

$$\begin{cases} i\partial_t u + \triangle u = |u|^{p-1}u, \quad p>3, \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}^2). \end{cases} \tag{4.1}$$

This proof is very simplified compared to Nakanishi's one [13]. Combing (1.4) and the Sobolev embedding theorem, we obtain

$$\|u\|^2_{L_t^4 L_x^8} \lesssim \|u\|_{L_t^\infty L_x^2} \|u\|_{L_t^2 \dot{H}_x^{1/2}}.$$

By using the mass and energy conservation laws, we can control the $L_t^4 L_x^8$ norm as follows:

$$\|u\|_{L_t^4 L_x^8} \lesssim C(E(u_0)). \tag{4.2}$$

To prove the scattering, we have to control the following Strichartz norm:

$$\|u\|_{S^1} := \sup_{\frac{1}{q}+\frac{1}{r}=\frac{1}{2}} \|\langle \nabla \rangle u\|_{L_t^q L_x^r}.$$

We divide the real line into finitely many subintervals $\{I_j\}_{j=1}^L$ such that for each I_j we have

$$\|u\|_{L_t^4 L_x^8(I_j)} \sim \delta.$$

It is enough to show that

$$\|u\|_{S^1(I_j)} \lesssim \|u_0\|_{H^1}, \tag{4.3}$$

which implies that $\|u\|_{S^1} \lesssim \|u_0\|_{H^1}$.

We will suppress the I_j notation for what follows. By Lemma 2.1 and interpolation, we have

$$\begin{aligned} \|u\|_{S^1} \lesssim & \|u_0\|_{H^1} + \left\| \langle \nabla \rangle |u|^{p-1}u \right\|_{L_t^{4/3} L_x^{4/3}} \\ \lesssim & \|u_0\|_{H^1} + \|\langle \nabla \rangle u\|_{L_t^\infty L_x^2} \| |u|^{p-1} \|_{L_t^{4/3} L_x^4} \\ \lesssim & \|u_0\|_{H^1} + \|u\|_{S^1} \|u\|^{\varepsilon}_{L_t^4 L_x^8} \|u\|^{p-1-\varepsilon}_{L_t^{q_0} L_x^{r_0}} \end{aligned}$$

where

$$q_0 = \frac{4(p-1-\varepsilon)}{3-\varepsilon}, \ r_0 = \frac{8(p-1-\varepsilon)}{2-\varepsilon}.$$

Thus we have

$$\|u\|_{S^1} \lesssim \|u_0\|_{H^1} + \delta^\varepsilon \|u\|_{S^1} \|u\|^{p-1-\varepsilon}_{L_t^{q_0} L_x^{r_0}}.$$

By using the Sobolev embedding theorem, we observe that for $p > 3 + \frac{\varepsilon}{4}$

$$\|u\|_{L_t^{q_0} L_x^{r_0}} \lesssim \||\nabla|^{\alpha} u\|_{L_t^{q_0} L_x^{r_1}}$$

where

$$r_1 = \frac{4(p-1-\varepsilon)}{2p-5-\varepsilon},\ \alpha = \frac{p-3-\varepsilon/4}{p-1-\varepsilon}.$$

Since the pair of exponents (q_0, r_1) are admissible, it follows that

$$\|u\|_{L_t^{q_0} L_x^{r_0}} \lesssim \|\langle\nabla\rangle u\|_{L_t^{q_0} L_x^{r_1}} \lesssim \|u\|_{S^1}$$

Thus we have

$$\|u\|_{S^1(I_j)} \lesssim \|u_0\|_{H^1} + \delta^{\varepsilon} \|u\|_{S^1(I_j)}^{p-\varepsilon}.$$

and by a continuity argument for ε small we obtain (4.3).

We now prove that there exists $u_+ \in H^1(\mathbb{R}^2)$ such that

$$\|u(t) - e^{it\triangle} u_+\|_{H^1(\mathbb{R}^2)} \to 0,\ \text{ as } t \to \infty.$$

We define $v(t) = e^{-it\triangle} u(t)$ where u is an H^1 solution to (4.1). Then v satisfies

$$v(t) = u_0 - i \int_0^t e^{-is\triangle} (|u|^{p-1} u)(s)\ ds.$$

For any $0 < \tau < t$,

$$v(t) - v(\tau) = -i \int_\tau^t e^{-is\triangle} (|u|^{p-1} u)(s)\ ds.$$

By Lemma 2.1 and the Sobolev embedding, it follows that

$$\begin{aligned}\|v(t) - v(\tau)\|_{H^1(\mathbb{R}^2)} &\lesssim \left\|\langle\nabla\rangle |u|^{p-1} u\right\|_{L_t^{4/3} L_x^{4/3}([\tau,t]\times\mathbb{R}^2)} \\ &\lesssim \|u\|_{L^4_{t\in[\tau,t]} L_x^8}^{\varepsilon} \|u\|_{S^1([t,\tau])}^{p-\varepsilon},\end{aligned}$$

which implies that $\|v(t) - v(\tau)\|_{H^1}$ is bounded from (4.2) and (4.3). Therefore

$$\|v(t) - v(\tau)\|_{H^1(\mathbb{R}^2)} \to 0,\ \text{ as } t, \tau \to \infty.$$

This shows that $u_+ \in H^1(\mathbb{R}^2)$ is well-defined as follows:

$$u_+ = u_0 - i \int_0^\infty e^{-is\triangle} (|u|^{p-1} u)(s) ds.$$

Remark. By using the correlation estimates in one dimension (1.5) and (1.6), an analogue simplified proof of the scattering for the L^2-supercritical NLS in one dimension appeared in [4].

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