# Global Solutions for 3D Quadratic Schrödingier Equations

## by P. Germain, N. Masmoudi and J. Shatah

Hyewon Yoon

2013 Participating School in Analysis of PDE

Oct, 2013

# Notation

- The Fourier transform $\widehat{f}$ of $f$ in $\mathbb{R}^d$ is defined by the formula

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\, dx.$$

- We use the short hand $L^p L^q = L_t^p([2,\infty), L_x^q(\mathbb{R}^3))$.
- We denote $L^{p,q}$ for usual Lorentz space.

# 3D Quadratic NLS

We consider the IVP

$$\begin{cases} \partial_t u + i\Delta u = \alpha u^2, \\ u\big|_{t=2} = u_2 = e^{-2i\Delta} u_*, \end{cases}$$

where $\alpha \in \mathbb{C}$ and $u$ is a $\mathbb{C}$-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^3$.
The first step is to take as the new unknown function

$$f(t) := e^{it\Delta} u(t), \quad \text{or equivalently,} \quad \widehat{f}(t, \xi) = e^{-it|\xi|^2} \widehat{u}(t, \xi)$$

in the Fourier side. Then, by Duhamel's formula,

$$\widehat{f}(t, \xi) = \widehat{u_*}(\xi) + \frac{\alpha}{(2\pi)^{3/2}} \int_2^t \int_{\mathbb{R}^3} e^{is\Phi(\xi,\eta)} \widehat{f}(s, \eta) \widehat{f}(s, \xi - \eta)\, d\eta ds,$$

where the phase function $\Phi(\xi, \eta) = -|\xi|^2 + |\eta|^2 + |\xi - \eta|^2$.

# Space-time resonance

- On the set of **time resonances** $\mathcal{T} := \{(\xi,\eta) : \Phi(\xi,\eta) = 0\}$, the phase is stationary in $s$.
- On the set of **space resonances** $\mathcal{S} := \{(\xi,\eta) : \partial_\eta \Phi(\xi,\eta) = 0\}$, the phase is stationary in $\eta$.
- On the set of **space-time resonances** $\mathcal{R} := \mathcal{T} \cap \mathcal{S}$, the phase is stationary in both $s$ and $\eta$.

Since the set of space-time resonances in our case is a point $\mathcal{R} = \{(0,0)\}$, we can take advantage of the oscillation of the phase in the Duhamel's formula

$$\widehat{f}(t,\xi) = \widehat{u_*}(\xi) + \frac{\alpha}{(2\pi)^{3/2}} \int_2^t \int_{\mathbb{R}^3} e^{is\Phi(\xi,\eta)} \widehat{f}(s,\eta) \widehat{f}(s,\xi-\eta)\, d\eta ds$$

by integrating parts in either $s$ or $\eta$.

In order to implement this strategy, we notice that $\partial_s e^{is\Phi} = i\Phi e^{is\Phi}$ and $\partial_\eta e^{is\Phi} = is(\partial_\eta \Phi)e^{is\Phi}$ thus for any $P$,

$$\frac{1}{iZ}\left(\partial_s + \frac{P}{s}\partial_\eta\right) e^{is\Phi} = e^{is\Phi},$$

where $Z := \Phi + P \cdot \partial_\eta \Phi$. We pick a $P$ such that $Z$ vanishes only at $(0,0)$. Among the functions $P$ that will do the trick, we choose

$$P = -\eta + \frac{1}{2}\xi.$$

For this specific $P$, we have
$Z = \Phi + P \cdot \partial_\eta \Phi = -2|\eta|^2 - |\xi|^2 + 2\xi \cdot \eta$, which vanishes only at the point where $\Phi$ and $\partial_\eta \Phi$ are zero, which is $(\xi, \eta) = (0,0)$. To deal with the singularity of $\frac{1}{Z}$, we also consider the smoothened version

$$\frac{1}{\frac{1}{s} + iZ}\left(\frac{1}{s} + \partial_s + \frac{P}{s}\partial_\eta\right) e^{is\Phi} = e^{is\Phi}.$$

# Main Result

Define the Banach space $X$ by its norm

$$\|u\|_X := \left\|\widehat{f}\right\|_{L_t^\infty L_\xi^\infty} + \|f\|_{L_t^\infty L_x^2} + \left\|\frac{x}{\log t} f\right\|_{L_t^\infty L_x^2} + \left\|\frac{x^2}{\sqrt{t}} f\right\|_{L_t^\infty L_x^2} + \left\|t^{\frac{3}{2}} u\right\|_{L_t^\infty L_x^\infty}$$

where $f$ is the profile of $u$, namely $\widehat{f}(t,\xi) = e^{-i|\xi|^2 t}\widehat{u}(t,\xi)$.

- The choice of initial data $t = 2$ is made to avoid having singularities at $t = 0$ and $t = 1$ in the norm of $X$. If we choose the data to be given at $t = 0$, then, in the definition of $\|\cdot\|_X$, $t$ should be replaced by $\langle t \rangle$.

The solution $u$ will be constructed using Picard's iteration. If we show that

$$\begin{aligned} \widehat{f}(t,\xi) &\mapsto \widehat{u_*}(\xi) + \frac{\alpha}{(2\pi)^{3/2}} \int_2^t \int_{\mathbb{R}}^3 e^{is\Phi(\xi,\eta)} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds \\ &=: \widehat{u_*}(\xi) + \alpha \widehat{B}(f,f)(t,\xi) \end{aligned}$$

is a contraction on a neighborhood of the origin in the Banach space $X$, then we can get the result:

### Theorem

*For data $u_*$ such that $\left\|e^{-it\Delta}u_*\right\|_X$ is small enough, there exists a solution of IVP*

$$\begin{cases} \partial_t u + i\Delta u = \alpha u^2, \\ u\big|_{t=2} = u_2 = e^{-2i\Delta}u_*, \end{cases} \tag{NLS}$$

*in $X$. Furthermore, $f(t)$ has a limit in $L^2$ as $t \to \infty$.*

So we need to prove the estimate

$$\|B(f,f)\|_X \lesssim \|f\|_X^2 .$$

The scattering follows immediately from the fact that $f \in X$.

# Stationary Phase Lemma

First we prove the $L^1 \to L^\infty$ decay of solutions to the Schr$\ddot{\text{d}}$ingier equation:

## Lemma (Stationary Phase Lemma)

*The Schrödingier semigroup satisfies*

$$(e^{-it\Delta}g)(x) = \frac{1}{(-2it)^{3/2}} e^{-\frac{x^2}{4t}} \widehat{g}\left(-\frac{x}{2t}\right) + \frac{1}{t^{7/4}} O(\left\|x^2 g\right\|_{L^2})$$

*with the convention that* $\frac{1}{(-1)^{3/2}} = e^{i\frac{3\pi}{4}}$*. In particular,*

$$\left\|e^{-it\Delta}g\right\|_{L^\infty} \lesssim \frac{1}{t^{3/2}} \|\widehat{g}\|_{L^\infty} + \frac{1}{t^{7/4}} \left\|x^2 g\right\|_{L^2}.$$

*Proof.* Note that

$$
\begin{aligned}
(e^{-it\Delta}g)(x) &= \frac{1}{(-4i\pi t)^{3/2}} \int_{\mathbb{R}^3} e^{-i\frac{|x-y|^2}{4t}} g(y)\,dy \\
&= \frac{1}{(-2it)^{3/2}} e^{-i\frac{|x|^2}{4t}} \widehat{g}\left(-\frac{x}{2t}\right) \\
&+ \frac{1}{(-4i\pi t)^{3/2}} e^{-i\frac{x^2}{4t}} \int_{\mathbb{R}^3} e^{i\frac{xy}{2t}} \left(e^{-i\frac{y^2}{4t}} - 1\right) g(y)\,dy.
\end{aligned}
$$

In order to prove the lemma, it suffices to bound the second term in the last line.

$$
\begin{aligned}
&\left| e^{-i\frac{x^2}{4t}} \int_{\mathbb{R}^3} e^{i\frac{xy}{2t}} \left(e^{-i\frac{y^2}{4t}} - 1\right) g(y)\,dy \right| \\
&\leq \int_{|y|\leq\sqrt{t}} \frac{y^2}{4t} |g(y)|\,dy + \int_{|y|\geq\sqrt{t}} |g(y)|\,dy \leq \frac{C}{t^{1/4}} \left\| y^2 g \right\|_{L^2}
\end{aligned}
$$

□

# Gagliardo-Nirenberg Type Inequality

## Lemma

*The following inequality holds*

$$\left\| e^{-it\Delta}(xf) \right\|_{L^4}^2 \leq \left\| e^{-it\Delta} f \right\|_{L^\infty} \left\| e^{-it\Delta}(x^2 f) \right\|_{L^2}.$$

*Proof.* Define $J := x - 2it\nabla$. Observe that $e^{-it\Delta}x = Je^{-it\Delta}$, and that $J = 2ite^{-i\frac{x^2}{4t}}\nabla e^{i\frac{x^2}{4t}}$. We have

$$\begin{aligned}
\left\| e^{-it\Delta}(xf) \right\|_{L^4}^2 = \left\| Je^{-is\Delta} f \right\|_{L^4}^2 &= 4t^2 \left\| e^{-i\frac{x^2}{4t}} \nabla e^{i\frac{x^2}{4t}} e^{-it\Delta} f \right\|_{L^4}^2 \\
&\lesssim t^2 \left\| e^{-it\Delta} f \right\|_{L^\infty} \left\| \Delta e^{i\frac{x^2}{4t}} e^{-it\Delta} f \right\|_{L^2} \\
&\lesssim \left\| e^{-it\Delta} f \right\|_{L^\infty} \left\| J^2 e^{it\Delta} f \right\|_{L^2} \\
&\lesssim \left\| e^{-it\Delta} f \right\|_{L^\infty} \left\| e^{-it\Delta}(x^2 f) \right\|_{L^2}.
\end{aligned}$$

□

# Coifman-Meyer Theorem

We consider the operators

$$T_m(f,g) = \mathcal{F}^{-1} \int m(\xi,\eta)\widehat{f}(\eta)\widehat{g}(\xi-\eta)\, d\eta.$$

## Theorem (Coifman-Meyer)

*Suppose that a multiplier m satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \le \frac{C}{(|\xi|+|\eta|)^{|\alpha|+|\beta|}}$$

*for sufficiently many multi-indices $(\alpha,\beta)$. Then $T_m : L^p \times L^q \to L^r$ is bounded for*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 1 < p, q \le \infty \quad \textit{and} \quad 0 < r < \infty.$$

- If $m$ is homogeneous of degree 0, and of class $C^\infty$ on a $(\xi, \eta)$-sphere, then the condition for the above holds.

- If $m(\xi, \eta)$ is a Coifman-Meyer multiplier, so is $m_t(\xi, \eta) = m(t\xi, t\eta)$ for a real number $t$. Furthermore, the bounds of $|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)|$ are independent of $t$, and consequently so are the norms of $T_{m_t}$ as an operator from $L^p \times L^q$ to $L^r$, for $(p, q, r)$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $1 < p, q \leq \infty$ and $0 < r < \infty$.

# Fractional Integration

Let $\Lambda^{\beta} := (-\Delta)^{\beta/2}$ and $\Lambda_t^{\beta} := (\frac{1}{t} - \Delta)^{\beta/2}$.

## Lemma

- *If* $\alpha \geq 0$, $1 < p, q < \infty$, *and* $\frac{1}{q} - \frac{1}{p} = \frac{\alpha}{3}$, *then*
$$\left\|\Lambda^{-\alpha} f\right\|_{L^p} \lesssim \|f\|_{L^q} .$$
- *If* $\alpha \geq 0$, *then*
$$\left\|\Lambda^{-\alpha} f\right\|_{L^\infty} \lesssim \|f\|_{L^{\frac{3}{\alpha},1}} .$$
- *If* $\alpha \geq 0$, $1 \leq p, q \leq \infty$, *and* $0 \leq \frac{1}{q} - \frac{1}{p} < \frac{\alpha}{3}$, *then*
$$\left\|\Lambda_t^{-\alpha} f\right\|_{L^p} \lesssim t^{\frac{\alpha}{2} + \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^q} .$$

# Multiplier Estimate

We can bound $Z$ and $\frac{1}{t} + Z$ in the denominator in the following manner:

## Lemma

*Let $Z = \Phi + P \cdot \partial_\eta \Phi$ and let $P_\ell$ denote a homogeneous polynomial in $(\xi, \eta)$ of degree $\ell$. Suppose that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ as in the C-M theorem. Then*

- *The multiplier $m(\xi, \eta) = \frac{P_{2k-1}(\xi,\eta)}{Z^k}$ satisfies*

$$\| T_m(f, g)\|_{L^r} \lesssim \left\|\Lambda^{-1} f\right\|_{L^p} \|g\|_{L^q} + \|f\|_{L^p} \left\|\Lambda^{-1} g\right\|_{L^q} .$$

- *The multiplier $m_t(\xi, \eta) = \frac{P_\ell(\xi,\eta)}{\left(\frac{1}{t}+Z\right)^k}$ satisfies*

$$\| T_m(f, g)\|_{L^r} \lesssim \left\|\Lambda_t^{\ell-2k} f\right\|_{L^p} \|g\|_{L^q} + \|f\|_{L^p} \left\|\Lambda_t^{\ell-2k} g\right\|_{L^q} .$$

*Proof.* Let $\psi_1$ and $\psi_2$ be two functions of $\xi$ and $\eta$, homogeneous of degree 0 and $C^\infty$ outside $(0,0)$, such that

$$\begin{aligned} \psi_1(\xi,\eta)+\psi_2(\xi,\eta) = 1 \quad & \text{for any } (\xi,\eta), \\ \psi_1(\xi,\eta) \equiv 0 \quad & \text{if } |\eta| \geq \frac{1}{4}|\xi-\eta|, \\ \psi_2(\xi,\eta) \equiv 0 \quad & \text{if } |\xi-\eta| \geq \frac{1}{4}|\eta|. \end{aligned}$$

- To prove the first statement, we decompose the Fourier multiplier $m$ into two pieces

  $$m(\xi,\eta) = \psi_1(\xi,\eta)m(\xi,\eta)+\psi_2(\xi,\eta)m(\xi,\eta) =: m_1(\xi,\eta)+m_2(\xi,\eta)$$

  We rewrite $m_1$ as

  $$m(\xi,\eta) = \psi_1(\xi,\eta)\frac{P_{2k-1}(\xi,\eta)|\eta|}{Z^k}\frac{1}{\eta}.$$

Since $\psi_1(\xi,\eta)\frac{P_{2k-1}(\xi,\eta)|\eta|}{Z^k}$ satisfies the hypothesis of the Coifman-Meyer theorem, we have

$$\|T_{m_1}(f,g)\|_{L^r} = \left\| T_{\frac{P_{2k-1}(\xi,\eta)|\eta|}{Z^k}}(\Lambda^{-1}f, g) \right\|_{L^r} \lesssim \left\|\Lambda^{-1}f\right\|_{L^p} \|g\|_{L^q} .$$

The estimate for $m_2$ can be obtained similarly by permuting the roles of $f$ and $g$.

- To prove the second statement, we similarly decompose $m_t$ into $m_{t1} + m_{t2}$ and rewrite $m_{t1}$ as

$$m_{t1}(\xi,\eta) = \underbrace{\psi_1(\xi,\eta)\frac{P_\ell(\xi,\eta)\left(\frac{1}{t}+\eta^2\right)^{k-\frac{\ell}{2}}}{\left(\frac{1}{t}+Z\right)^k}}_{=:\mu(t\xi,t\eta)} \frac{1}{\left(\frac{1}{t}+\eta^2\right)^{k-\frac{\ell}{2}}}.$$

Note that $\mu(\xi,\eta)$ satisfies the hypothesis of the Coifman-Meyer theorem. So we have

$$\|T_{m_{t1}}(f,g)\|_{L^r} = \left\| T_{\mu(t\xi,t\eta)}\left(\Lambda_t^{\ell-2k} f, g\right)\right\|_{L^r} \lesssim \left\|\lambda_t^{\ell-2k} f\right\|_{L^p} \|g\|_{L^q}.$$

The case for $m_{t2}$ is entirely similar. □

# Control of $\left\|\widehat{B}(f,f)\right\|_{L_t^\infty L_\xi^\infty}$

By change of variable $\eta \mapsto \frac{\xi}{2} + \zeta$, we write the bilinear term $\widehat{B}$ as

$$\widehat{B}(f,f)(t,\xi) = \frac{1}{(2\pi)^{3/2}} \int_2^t e^{-i\frac{|\xi|^2}{2}s} \int_{\mathbb{R}^3} e^{2i|\zeta|^2 s} \widehat{f}\left(s, \frac{\xi}{2} + \zeta\right) \widehat{f}\left(s, \frac{\xi}{2} - \zeta\right) \, d\eta ds.$$

We bound $e^{-\frac{|\xi|^2}{2}s}$ by 1, and the inner integral by stationary phase lemma.

$$\left|\widehat{B}(f,f)(t,\xi)\right| \lesssim \int_2^t \left( \frac{1}{s^{3/2}} \left| \widehat{f}\left(s, \frac{\xi}{2}\right) \right|^2 + \frac{1}{s^{7/4}} \left\| \partial_\zeta^2 \left( \widehat{f}\left(s, \frac{\xi}{2} + \zeta\right) \widehat{f}\left(s, \frac{\xi}{2} - \zeta\right) \right) \right\|_{L_\zeta^2} \right) ds$$

Using Gagliardo-Nirenberg type inequality mentioned before, we have

$$\begin{aligned}
\left|\widehat{B}(f,f)(t,\xi)\right| &\lesssim \int_2^t \frac{1}{s^{3/2}} \|f\|_X^2 \; ds \\
&+ \int_2^t \frac{1}{s^{7/4}} \left( \left\| \partial_\zeta^2 \widehat{f}(s) \right\|_{L^2_\zeta} \left\| \widehat{f}(s) \right\|_{L^\infty_\zeta} + \left\| \partial_\zeta \widehat{f}(s) \right\|_{L^4_\zeta}^2 \right) ds \\
&\lesssim \|f\|_X^2 + \int_2^t \left\| \partial_\zeta^2 \widehat{f}(s) \right\|_{L^2_\zeta} \left\| \widehat{f}(s) \right\|_{L^\infty_\zeta} \; ds \\
&\lesssim \|f\|_X^2 + \|f\|_X^2 \int_2^t \frac{s^{1/2}}{s^{7/4}} \, ds \lesssim \|f\|_X^2 .
\end{aligned}$$

# Control of $\|B(f,f)\|_{L^\infty L^2}$

For this norm, we can give a simple energy estimate

$$\|B(f,f)\|_{L^\infty L^2} \lesssim \int_2^\infty \frac{1}{s^{3/2}} \left\| t^{3/2} u \right\|_{L^\infty L^\infty} \|f\|_{L^\infty L^2} \, ds \lesssim \|f\|_X^2 .$$

# Control of $\left\| \frac{x}{\log t} B(f, f) \right\|_{L^\infty L^2}$

Applying $\partial_\xi$, we have

$$\partial_\xi \widehat{B}(f, f) = \int_2^t \int_{\mathbb{R}^3} is(\partial_\xi \Phi) e^{is\Phi} \widehat{f}(s, \eta) \widehat{f}(s, \xi - \eta)\, d\eta ds$$
$$+ \int_2^t \int_{\mathbb{R}^3} e^{is\Phi} \widehat{f}(s, \eta) \partial_\xi \widehat{f}(s, \xi - \eta)\, d\eta ds =: I + II.$$

By Hölder inequality,

$$\|II\|_{L^2} = \left\| \int_2^t e^{is\Delta} \left( e^{-is\Delta} f\, e^{-is\Delta}(xf) \right) ds \right\|_{L^2}$$
$$\lesssim \int_2^t \|u\|_{L^\infty} \|xf\|_{L^2}\, ds \lesssim \int_2^t \frac{\log s}{s^{3/2}} \|f\|_X^2\, ds \lesssim \|f\|_X^2 .$$

First, we observe that, interpolating between the different components of the $X$-norm,

$$\|u\|_{L^{3,1}} \lesssim \frac{1}{\sqrt{t}} \quad \text{and} \quad \|f\|_{L^{6/5}} \lesssim t^{\epsilon}$$

for any $\epsilon > 0$. In order to estimate $I$, we integrate by parts using

$$\frac{1}{iZ}\left(\partial_s + \frac{P}{s}\partial_\eta\right) e^{is\Phi} = e^{is\Phi}.$$

$$\begin{aligned}
I &= \int_2^t \int_{\mathbb{R}^3} is(\partial_\xi \Phi) \frac{1}{iZ}\left(\partial_s + \frac{P}{s}\partial_\eta\right) e^{is\Phi} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds \\
&= \int_2^t \int_{\mathbb{R}^3} \frac{s(\partial_\xi \Phi)}{Z} \partial_s \left(e^{is\Phi}\right) \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds \\
&+ \int_2^t \int_{\mathbb{R}^3} \frac{P(\partial_\xi \Phi)}{Z} \partial_\eta \left(e^{is\Phi}\right) \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds
\end{aligned}$$

$$
\begin{aligned}
I = &+ \int_{\mathbb{R}^3} \frac{t(\partial_\xi \Phi)}{Z} e^{it\Phi} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta && (I\text{-}1)\\
&- 2\int_{\mathbb{R}^3} \frac{\partial_\xi \Phi}{Z} e^{i2\Phi} \widehat{u_*}(\eta)\widehat{u_*}(\xi-\eta)\, d\eta && (I\text{-}2)\\
&- \int_2^t \int_{\mathbb{R}^3} \frac{\partial_\xi \Phi}{Z} e^{is\Phi} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds && (I\text{-}3)\\
&- \int_2^t \int_{\mathbb{R}^3} \frac{s(\partial_\xi \Phi)}{Z} e^{is\Phi} \left(\partial_s \widehat{f}(s,\eta)\right) \widehat{f}(s,\xi-\eta) d\eta ds + \text{s.t.} && (I\text{-}4)\\
&- \int_2^t \int_{\mathbb{R}^3} \partial_\eta \left(\frac{P(\partial_\xi \Phi)}{Z}\right) e^{is\Phi} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds && (I\text{-}5)\\
&- \int_2^t \int_{\mathbb{R}^3} \frac{P(\partial_\xi \Phi)}{Z} e^{is\Phi} \left(\partial_\eta \widehat{f}(s,\eta)\right) \widehat{f}(s,\xi-\eta) d\eta ds + s.t. && (I\text{-}6)
\end{aligned}
$$

By the multiplier estimates that we showed before,

$$\begin{aligned}\|(I\text{-}1)\|_{L^2} &= t\left\| e^{it\Delta} T_{\frac{\partial_\xi \Phi}{Z}}(e^{-is\Delta} f, e^{-is\Delta} f)\right\|_{L^2} \\ &\lesssim t\left\|\Lambda^{-1} e^{-it\Delta} f\right\|_{L^2}\left\| e^{-it\Delta} f\right\|_{L^\infty} \lesssim t\,\|f\|_{L^{6/5}}\,\|u\|_{L^\infty} \lesssim \|f\|_X^2 .\end{aligned}$$

We can estimate $(I\text{-}2)$, $(I\text{-}3)$, and $(I\text{-}5)$ similarly.

For $(I\text{-}6)$, we apply the Coifman-Meyer theorem as follows.

$$\begin{aligned}\|(I\text{-}6)\|_{L^2} &\lesssim \int_2^t s\left\| e^{is\delta} T_{\frac{P(\partial_\xi \Phi)}{Z}}\left(e^{-is\Delta} f, e^{-s\Delta}(xf)\right)\right\|_{L^2} ds \\ &\lesssim \int_2^t \|xf\|_{L^2}\,\|u\|_{L^\infty} \lesssim \|f\|_X^2 \int_2^t \frac{\log s}{s^{3/2}}\, ds \lesssim \|f\|_X^2 .\end{aligned}$$

For $(I\text{-}4)$, remind that $e^{i|\xi|^2 t}\partial_t \widehat{f} = \alpha \widehat{u^2}$.

$$
\begin{aligned}
\|(I\text{-}4)\|_{L^2} &\lesssim \int_2^t s \left\| e^{it\Delta} T_{\frac{\partial_\xi \Phi}{Z}}(u^2, e^{-is\Delta} f) \right\|_{L^2} \\
&\lesssim \int_2^t s(\left\|\Lambda^{-1} u\right\|_{L^2} \left\|u^2\right\|_{L^\infty} + \left\|\Lambda^{-1} u^2\right\|_{L^\infty} \|u\|_{L^2}) ds \\
&\lesssim \int_2^t s(\|u\|_{L^{3,1}} \|u\|_{L^\infty} \|u\|_{L^2})\, ds \\
&\|f\|_X^3 \int_2^t \frac{ds}{s} \lesssim \|f\|_X^3 \log t.
\end{aligned}
$$

# Control of $\left\| t^{3/2} e^{-it\Delta} B(f,f) \right\|_{L^\infty L^\infty}$

Using

$$\frac{1}{\frac{1}{s} + iZ} \left( \frac{1}{s} + \partial_s + \frac{P}{s} \partial_\eta \right) e^{is\Phi} = e^{is\Phi},$$

we have

$$\begin{aligned}\widehat{B}(f,f) &= \int_2^t \int_{\mathbb{R}^3} \frac{1}{\frac{1}{s} + iZ} \left( \frac{1}{s} + \partial_s + \frac{P}{s} \partial_\eta \right) e^{is\Phi} \widehat{f}(s,\eta) \widehat{f}(s,\xi-\eta)\, d\eta ds \\ &=: \widehat{g}(\xi) + \widehat{h}(\xi),\end{aligned}$$

where

$$\begin{aligned}\widehat{g}(\xi) &= \widehat{g_1}(\xi) + \widehat{g_2}(\xi)\\ &:= \int_{\mathbb{R}^3} \frac{1}{\frac{1}{t} + iZ} e^{it\Phi} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\\ &\quad - \frac{1}{\frac{1}{2} + iZ} e^{i2\Phi} \widehat{u_*}(\eta)\widehat{u_*}(\xi-\eta)\, d\eta,\end{aligned}$$

and

$$\begin{aligned}\widehat{h}(\xi) &:= \int_2^t \int_{\mathbb{R}^3} \frac{\frac{1}{s} + P\partial_\eta \Phi}{\frac{1}{s} + iZ} e^{is\Phi} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta) d\eta ds\\ &\quad - i \int_2^t \int_{\mathbb{R}^3} \frac{1}{s^2} \frac{1}{\left(\frac{1}{s} + iZ\right)^2} e^{is\Phi} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds\\ &\quad - 2 \int_2^t \int_{\mathbb{R}^3} \frac{1}{\frac{1}{s} + iZ} e^{is\Phi} \partial_s \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds.\end{aligned}$$

We first focus on $g_1$ ($g_2$ is more easy to deal with since it is constant in time):

$$\widehat{e^{-it\Delta}g_1}(\xi) = \int_{\mathbb{R}^3} \frac{1}{\frac{1}{t} + iZ} \widehat{u}(t,\eta)\widehat{u}(t,\xi-\eta)\, d\eta.$$

By the Coifman-Meyer theorem, we have

$$\begin{aligned}
\left\| e^{-it\Delta} g_1 \right\|_{L^\infty} &= \left\| \mathcal{F}^{-1} \frac{1}{\frac{1}{t} + \xi^2} \int_{\mathbb{R}^3} \frac{\frac{1}{t} + \xi^2}{\frac{1}{t} + Z} \widehat{u}(s,\eta)\widehat{u}(s,\xi-\eta)\, d\eta \right\|_{L^\infty} \\
&= \left\| \Lambda_t^{-2} T_{\frac{\frac{1}{t}+\xi^2}{\frac{1}{t}+Z}}(u,u) \right\|_{L^\infty} \lesssim t^{3/4} \left\| T_{\frac{\frac{1}{t}+\xi^2}{\frac{1}{t}+Z}}(u,u) \right\|_{L^6} \\
&\lesssim t^{3/4} \left\| u \right\|_{L^6} \left\| u \right\|_{L^\infty} \lesssim t^{-7/4} \left\| f \right\|_X^2
\end{aligned}$$

Also notice that the norm of $\widehat{g_1}$ in $L^\infty$ is bounded:

$$|\widehat{g_1}|_{L^\infty_\xi} \leq \int_{\mathbb{R}^3} \frac{1}{|\eta|^2} \left|\widehat{f}(s,\eta)\right| \left|\widehat{f}(s,\xi-\eta)\right| \, d\eta \lesssim \left\|\widehat{f}\right\|^2_{L^\infty_\eta \cap L^2_\eta} \lesssim \|f\|_X^2 .$$

By stationary phase lemma,

$$\left\|e^{-it\Delta} h\right\|_{L^\infty} \lesssim \frac{1}{t^{3/2}} \left\|\widehat{f}\right\|_{L^\infty} + \frac{1}{t^{7/4}} \left\|x^2 h\right\|_{L^2} .$$

If we show that $\left\|x^2 h(t)\right\|_{L^2} \lesssim t^\epsilon$ with $\epsilon$ a constant arbitrarily small, then

$$\begin{aligned} \left\|e^{-it\Delta} h\right\|_{L^\infty} &\lesssim \frac{1}{t^{3/2}} \left(\left\|\widehat{B}(f,f)\right\|_{L^\infty} + \|\widehat{g}\|_{L^\infty}\right) + \frac{1}{t^{7/4}} \left\|x^2 h\right\|_{L^2} \\ &\lesssim \frac{1}{t^{3/2}} \|f\|_X^2 . \end{aligned}$$

So, to complete the proof, the only thing which is left is the proof of the estimate

$$\left\|x^2 h(t)\right\|_{L^2} \lesssim t^\epsilon.$$

## Control of $\left\| \frac{x^2}{t^\epsilon} h \right\|_{L^\infty L^2}$

If we apply $\partial_\xi^2$ to $\widehat{h}(\xi)$, the following types of terms are produced.

$$\int_2^t \int_{\mathbb{R}^3} \frac{1}{s^j} \frac{P_{2k-4-2j}}{\left(\frac{1}{s} + iZ\right)^k} e^{is\Phi} \partial_s \widehat{f}(s,\eta) \widehat{f}(s,\xi-\eta) \, d\eta ds. \qquad (II\text{-}1)$$

with $k \geq 0$ and $k - 2 \geq j \geq -2$.

$$\int_2^t \int_{\mathbb{R}^3} \frac{1}{s^j} \frac{P_{2k-3-2j}}{\left(\frac{1}{s} + iZ\right)^k} e^{is\Phi} \partial_s \widehat{f}(s,\eta) \partial_\xi \widehat{f}(s,\xi-\eta) \, d\eta ds. \qquad (II\text{-}2)$$

with $k \geq 0$ and $k - \frac{3}{2} \geq j \geq -1$.

$$\int_2^t \int_{\mathbb{R}^3} \frac{1}{s^j} \frac{P_{2k-2-2j}}{\left(\frac{1}{s} + iZ\right)^k} e^{is\Phi} \partial_s \widehat{f}(s,\eta) \partial_\xi^2 \widehat{f}(s,\xi-\eta) \, d\eta ds. \qquad (II\text{-}3)$$

with $k \geq 0$ and $k - 1 \geq j \geq 0$.

These three terms can be handled in a similar fashion. Here I illustrate the estimate on ($II$-1).

$$
\begin{aligned}
\|(II\text{-}1)\|_{L^2} &\leq \int_2^t \frac{1}{s^j} \left\| e^{is\Delta} T_{\frac{P_{2k-4-2j}}{\left(\frac{1}{s}+iZ\right)^k}} \left( e^{-is\Delta} f, u^2 \right) \right\|_{L^2} ds \\
&\lesssim \int_2^t \frac{1}{s^j} \left( \left\| \Lambda_s^{-2j-4} e^{-is\Delta} f \right\|_{L^2} \left\| u^2 \right\|_{L^\infty} \right. \\
&\left. + \left\| e^{-is\Delta} f \right\|_{L^2} \left\| \Lambda^{-2j-4} u^2 \right\|_{L^\infty} \right) ds \\
&\lesssim \int_2^t \frac{1}{s^j} s^{j+2} \left\| u^2 \right\|_{L^\infty} \|f\|_{L^2} \, ds \\
&\lesssim \|f\|_X^2 \int_2^t \frac{s^2}{s^3} \, ds \lesssim \log t \, \|f\|_X^2 .
\end{aligned}
$$

$$\int_2^t \int_{\mathbb{R}^3} \frac{1}{s^j} \frac{P_{2k-2-2j}}{\left(\frac{1}{s} + iZ\right)^k} e^{is\Phi} \widehat{f}(s,\eta) \widehat{f}(s,\xi-\eta)\, d\eta ds. \qquad (II\text{-}4)$$

with $k \geq 0$ and $k-1 \geq j \geq 0$.

$$\begin{aligned}
\left\|(II\text{-}4)\right\|_{L^2} &\leq \int_2^t \frac{1}{s^j} \left\| e^{is\Delta} T_{\frac{P_{2k-2-2j}}{\left(\frac{1}{s}+iZ\right)^k}} \left(e^{-is\Delta} f, e^{-is\Delta} f\right) \right\|_{L^2} ds \\
&\lesssim \int_2^t \frac{1}{s^j} \left\| \Lambda_s^{-2j-2} e^{-is\Delta} f \right\|_{L^2} \left\| e^{-is\Delta} f \right\|_{L^\infty} ds \\
&\lesssim \int_2^t \frac{1}{s^j} s^{j+\frac{1}{2}} \left\| u \right\|_{L^\infty} \left\| f \right\|_{L^{6/5}} ds \\
&\lesssim \left\| f \right\|_X^2 \int_2^t \frac{\sqrt{s} s^\epsilon}{s^{3/2}}\, ds \lesssim t^\epsilon \left\| f \right\|_X^2 .
\end{aligned}$$

$$\int_2^t \int_{\mathbb{R}^3} \frac{1}{s^j} \frac{P_{2k-1-2j}}{\left(\frac{1}{s} + iZ\right)^k} e^{is\Phi} \widehat{f}(s,\eta) \partial_\xi \widehat{f}(s, \xi - \eta)\, d\eta ds. \qquad (II\text{-}5)$$

with $k \geq 0$ and $k - \frac{1}{2} \geq j \geq 0$.

$$\begin{aligned}
\|(II\text{-}5)\|_{L^2} &\leq \int_2^t \frac{1}{s^j} \left\| e^{is\Delta} T_{\frac{P_{2k-1-2j}}{\left(\frac{1}{s}+iZ\right)^k}} \left( e^{-is\Delta} f, e^{-is\Delta}(xf) \right) \right\|_{L^2} ds \\
&\lesssim \int_2^t \frac{1}{s^j} \left( \left\| \Lambda_s^{-2j-1} e^{-is\Delta}(xf) \right\|_{L^2} \left\| e^{-is\Delta} f \right\|_{L^\infty} \right. \\
&+ \left. \left\| \Lambda_s^{-2j-1} e^{-is\Delta} f \right\|_{L^\infty} \left\| e^{-is\Delta}(xf) \right\|_{L^2} \right) ds \\
&\lesssim \int_2^t \frac{1}{s^j} s^{j+\frac{1}{2}} \|u\|_{L^\infty} \|xf\|_{L^2}\, ds \\
&\lesssim \|f\|_X^2 \int_2^t \frac{\sqrt{s} \log s}{s^{3/2}}\, ds \lesssim t^\epsilon \|f\|_X^2 .
\end{aligned}$$

$$\int_2^t \int_{\mathbb{R}^3} \frac{1}{s^j} \frac{P_{2k-2j}}{\left(\frac{1}{s} + iZ\right)^k} e^{is\Phi} \widehat{f}(s,\eta) \partial_\xi^2 \widehat{f}(s,\xi-\eta)\, d\eta ds. \qquad (II\text{-}6)$$

with $k \geq 0$ and $k-1 \geq j \geq 0$.

$$\begin{aligned}
\|(II\text{-}6)\|_{L^2} &\leq \int_2^t \frac{1}{s^j} \left\| e^{is\Delta} T_{\frac{P_{2k-2j}}{\left(\frac{1}{s}+iZ\right)^k}} \left( e^{-is\Delta} f, e^{-is\Delta}(x^2 f) \right) \right\|_{L^2} ds \\
&\lesssim \int_2^t \frac{1}{s^j} \left( \left\| \Lambda_s^{-2j} e^{-is\Delta}(x^2 f) \right\|_{L^2} \left\| e^{-is\Delta} f \right\|_{L^\infty} \right. \\
&+ \left. \left\| \Lambda_s^{-2j} e^{-is\Delta} f \right\|_{L^\infty} \left\| e^{-is\Delta}(x^2 f) \right\|_{L^2} \right) ds \\
&\lesssim \int_2^t \frac{1}{s^j} s^j \|u\|_{L^\infty} \left\| x^2 f \right\|_{L^2} ds \\
&\lesssim \|f\|_X^2 \int_2^t \frac{\sqrt{s}}{s^{3/2}}\, ds \lesssim \log t \, \|f\|_X^2 .
\end{aligned}$$

$$\int_2^t \int_{\mathbb{R}^3} s \frac{P_{2k}}{\left(\frac{1}{s} + iZ\right)^k} e^{is\Phi} \widehat{f}(s,\eta) \widehat{f}(s,\xi-\eta) \, d\eta ds. \qquad (II\text{-}7)$$

In order to deal with this term, we need to integrate by parts using the identity

$$\frac{1}{\frac{1}{s} + iZ} \left( \frac{1}{s} + \partial_s + \frac{P}{s} \partial_\eta \right) e^{is\Phi} = e^{is\Phi}.$$

We rewrite ($II$-7) as

$$\int_2^t \int_{\mathbb{R}^3} s \frac{P_{2k}}{\left(\frac{1}{s} + iZ\right)^k} \frac{1}{\frac{1}{s} + iZ} \left( \frac{1}{s} + \partial_s + \frac{P}{s} \partial_\eta \right) e^{is\Phi} \widehat{f}(s,\eta) \widehat{f}(s,\xi-\eta) d\eta ds$$

and perform integrations by parts in $s$ and $\eta$. All terms that appear in this procedure are of the form ($II$-1)-($II$-6), except boundary integral term, which is easy to estimate. So we have

$$\|(II\text{-}7)\|_{L^2} \lesssim t^\epsilon \|f\|_X^2 .$$

$$\int_2^t \int_{\mathbb{R}^3} s \frac{P_{2k+1}}{\left(\frac{1}{s} + iZ\right)^k} e^{is\Phi} \widehat{f}(s,\eta) \partial_\xi \widehat{f}(s,\xi-\eta)\, d\eta ds. \qquad (II\text{-}8)$$

Similary, we can rewrite ($II$-8) as

$$\int_2^t \int_{\mathbb{R}^3} s \frac{P_{2k+1}}{\left(\frac{1}{s} + iZ\right)^k} \frac{1}{\frac{1}{s} + iZ} \left( \frac{1}{s} + \partial_s + \frac{P}{s} \partial_\eta \right) e^{is\Phi} \widehat{f}(s,\eta) \partial_\xi \widehat{f}(s,\xi-\eta) d\eta ds$$

and perform integrations by parts in $s$ and $\eta$. All terms that appear in this procedure are of the form ($II$-1)-($II$-6), except two terms. We estimate these two terms to show our desired estimate

$$\|(II\text{-}8)\|_{L^2} \lesssim t^{\epsilon} \|f\|_X^2 .$$

- The first one is

$$\int_2^t \int_{\mathbb{R}^3} \frac{P_{2k+1}P}{\left(\frac{1}{s}+iZ\right)^{k+1}} e^{is\Phi} \partial_\eta \widehat{f}(s,\eta) \partial_\xi \widehat{f}(s,\xi-\eta)\, d\eta ds.$$

By Coifman-Meyer theorem and Gagliardo-Nirenberg type inequality, its $L^2$-norm can be bounded by

$$\int_2^t \left\| e^{-it\Delta} f \right\|_{L^\infty} \left\| e^{-it\Delta}(x^2 f) \right\|_{L^2} \, ds.$$

- The second

$$\int_2^t \int_{\mathbb{R}^3} s \frac{P_{2k+1}}{\left(\frac{1}{s}+iZ\right)^{k+1}} e^{is\Phi} \widehat{f}(s,\eta) \partial_s \partial_\xi \widehat{f}(s,\xi-\eta)\, d\eta ds,$$

for which we remove the $\partial_\xi$ integral from the last term, using the fact that $\partial_\xi \widehat{f}(s,\xi-\eta) = -\partial_\eta \widehat{f}(s,\xi-\eta)$, and then integrating by parts in $\eta$. Resulting terms are of the form $(II\text{-}1)$-$(II\text{-}6)$.

$$\int_2^t \int_{\mathbb{R}^3} s^2(\partial_\xi \Phi)^2 \frac{P(\partial_\xi \Phi)}{\frac{1}{s} + iZ} e^{is\Phi} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds. \qquad (II\text{-}9)$$

We rewrite ($II$-9) as

$$(II\text{-}9) = \int_2^t \int_{\mathbb{R}^3} s(\partial_\xi \Phi)^2 \frac{P}{\frac{1}{s} + iZ} \left(\partial_\eta e^{is\Phi}\right) \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta ds.$$

By performing the integration by parts in $\eta$, one can get terms of the form ($II$-7) and ($II$-8).

The estimate of $\left\| \frac{x^2}{t^{1/2}} g \right\|_{L^\infty L^2}$ is straightforward. Thus if the initial data $\left\| e^{-it\Delta} u_* \right\|_X$ is small enough, then the map $f \mapsto u_* + B(f, f)$ is a contraction, and it completes the proof.

End of the slides. Thank you.