Approximation of solutions of the cubic nonlinear Schrödinger equations by finite-dimensional equations and nonsqueezing properties - Jean Bourgain

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Aug 26, 2013

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Equation

$$\begin{cases} iu_t + u_{xx} + u|u|^2 = 0, \\ u = u(x,t) \text{ periodic in } x \\ \text{with initial data } u(x,0) = \phi(x) \in L^2_x. \end{cases}$$

$$iu_t + u_{xx} + A(x,t)u + B(x,t)u|u|^2 = 0$$

with A, B real smooth functions in x, t, both periodic in x

Integral equation

$$u(x,t) = U(t)\phi + i\int_0^t U(t-\tau)w(\tau)d\tau,$$

where $U(t) = e^{i\partial_x^2}$ and $w := u|u|^2$.

$$e^{i\partial_x^2}\phi = \mathcal{F}^{-1}e^{-ik^2}\mathcal{F}\phi,$$

where $\mathcal{F}\phi = \int e^{-2\pi ixk}\phi(x)dx$

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Fixed point theorem.

Norm

$$\|u\| = \inf\{\sum_k \int (1+|\lambda - k^2|)^{1+\epsilon} |\hat{u}(k,\lambda)|^2 d\lambda\}$$

$$u(x,t) = \sum_k \int e^{i(kx+\lambda t)} \hat{u}(k,\lambda) d\lambda \text{ on } \mathbf{T} \times [0,T]$$

$$\tilde{u} = u \text{ on } \mathbf{T} \times [0,T]$$

Inequalities

$$\|\!|u|\!\| \leq \|\phi\|_2 + \gamma(T)\|\!|u|\!\|^3 \text{ for } t \in [0, T],$$
$$\|u(t)\|_2 \leq \|\phi\|_2 + \gamma(T)\|\!|u|\!\|^3 \text{ for } t \in [0, T],$$
$$\text{where } \gamma(T) \to 0 \text{ for } T \to 0.$$

For sufficiently small T,

$$\|\!|u|\!\|, \|u(t)\|_2 \lesssim \|\phi\|_2.$$

$$\begin{aligned}
\|u(t)\|_2 &= \|\mathcal{F}u(t)\|_{l^2} = \|\int e^{i\lambda t}\hat{u}(k,\lambda)d\lambda\|_2 \\
&= \|\int e^{i(\lambda+k^2)t}\hat{u}(k,\lambda+k^2)d\lambda\|_2 \\
&= \|\int (1+|\lambda|)^{-\frac{1}{2}-\frac{\epsilon}{2}}(1+|\lambda|)^{\frac{1}{2}+\frac{\epsilon}{2}} e^{i(\lambda+k^2)t}\hat{u}(k,\lambda+k^2)d\lambda\|_2 \\
&\lesssim \|\!|u|\!\|.
\end{aligned}$$

Contraction

$$\|u - v\| \leq \|\phi - \psi\|_2 + \gamma(T)(\|u\|^2 + \|v\|^2)\|u - v\| \text{ for } t \in [0, T],$$
$$\|u(t) - v(t)\|_2 \leq \|\phi - \psi\|_2 + \gamma(T)(\|u\|^2 + \|v\|^2)\|u - v\| \text{ for } t \in [0, T],$$
$$\text{where } \gamma(T) \to 0 \text{ for } T \to 0.$$

For sufficiently small T,

$$\|u - v\|, \|u(t) - v(t)\|_2 \lesssim \|\phi - \psi\|_2.$$

NLSE is locally wellposed on $[0, T(\|\phi\|_2)]$.

Conservation

$$\int_{\mathcal{T}} |u(t)|^2 dx = \int_{\mathcal{T}} |\phi|^2 dx.$$

- Local Wellposedness $\Rightarrow$ Global Wellposedness
- $\|u(t) - v(t)\|_2 \leq c(\|\phi\|_2, \|\psi\|_2)^{|t|} \|\phi - \psi\|_2$

$L^2 - L^4$ bound

$$\|\sum_k \int e^{i(kx+\lambda t)} \hat{u}(k,\lambda)\|_{L^4(\mathbf{T})\times[0,1]} \leq c(\sum_k \int (1+|\lambda - k^2|^{\frac{3}{4}})|\hat{u}(k,\lambda)|^2)^{\frac{1}{2}}$$

Restriction Estimates

$$\|\hat{u}(k,k^2)\|_{l^2} \lesssim \|u\|_{L^{4/3}_{x,t}}$$

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Modified Equation

$$\begin{cases} iv_t + v_{xx} + P_N(v|v|^2) = 0, \\ \text{with data } v(x,0) = \phi(x), \phi = P_N\phi, \\ \text{where } P_N\phi = \sum_{|n|\leq N} \hat{\phi}(n)e^{inx}. \end{cases}$$

- $v = \sum_{|n|\leq N} v_n(t)e^{inx}$
- Hamiltonian formulation

$$\begin{cases} \frac{dv}{dt} = i\frac{\partial H}{\partial \bar{v}} \\ H(\phi) = \frac{1}{2}\int_{\mathbf{T}} |\phi'|^2 - \frac{1}{4}\int_{\mathbf{T}} |\phi|^4. \end{cases}$$

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Proposition 1

Consider the solution u, v to the Cauchy problems

$$\begin{cases} iu_t + u_{xx} + u|u|^2 = 0, \\ u(x,0) = \phi(x) \end{cases}$$

$$\begin{cases} iv_t + v_{xx} + P_N(v|v|^2) = 0, \\ v(x,0) = \phi(x) \end{cases}$$

where $\phi = P_N\phi$. Fix a positive integer N' and a time t. Then one has an approximation

$$\|P_{N'}(u(t) - v(t))\|_2 \leq \epsilon$$

provided $N > N(N', |t|, \epsilon, \|\phi\|_2)$.

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$$B(r) = \{(p,q) | |p|^2 + |q|^2 < r^2\} \text{ (ball)}$$
$$\mathbf{T} = \{(p,q) | p_1^2 + q_1^2 < r^2\} \text{ (cylinder)}$$

Gromovs's squeezing Theorem

B_r : some ball in $L^2(\mathbf{T})$,

$T_r^{(k)}$: some cylinder in $L^2(\mathbf{T})$ w.r.t the kth coordinate ($|k| \leq N$),

$S_N(t)$: the flow map associated to modified equation,

$$\Downarrow$$

$$S_N(t)(B_r) \subset T_R^{(k)} \text{ implies } R \geq r.$$

$S(t)$: the flow map corresponding to the cubic NLSE

Proposition 1 $\Rightarrow \|P_k S(t) P_N - P_k S_N(t) P_N\| < \epsilon$ provided $N > N(k, |t|, \epsilon, B)$.

$$\Downarrow$$

$$P_k S_N(t)(P_N B_r) \longrightarrow P_k S(t)(P_N B_r) \text{ as } N \to \infty$$

Proposition 2

$$S(t)(B_r) \subset T_R^{(k)} \text{ implies } R \geq r$$

ball, cylinder : centered at the origin

L^2 conservation $\Rightarrow S(-t)(B_r - T_R^{(k)}) \subset B_r$

$$\Downarrow$$

$$S(t)(B_r) \subset T_R^{(k)} \text{ implies } R \geq r$$

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Lemma 3

Consider the solution u, v to the Cauchy problems

$$\begin{cases} iu_t + u_{xx} + u|u|^2 = 0, \\ u(x,0) = \phi(x) \end{cases}$$

$$\begin{cases} iv_t + v_{xx} + v|v|^2 = 0, \\ v(x,0) = \psi(x) \end{cases}$$

and assume $\|\phi\|_2 = \|\psi|_2$. Then for $|t| < T(\|\phi\|_2)$ one has

$$\|P_{N_0}(u(t) - v(t))\|_2 \leq \|P_{N_1}(\phi - \psi)\|_2 + \epsilon$$

provided $N_1 - N_0 > C_1 \epsilon^{-C_1}$.

Lemma 4

Consider the solution u, v to the Cauchy problems

$$\begin{cases} iu_t + u_{xx} + u|u|^2 = 0, \\ u(x,0) = \phi(x) \end{cases}$$

$$\begin{cases} iv_t + v_{xx} + P_N(v|v|^2) = 0, \\ v(x,0) = \phi(x) \end{cases}$$

where $\phi = P_N\phi$. Then for $|t| < T(\|\phi\|_2)$ one has

$$\|P_{N_0}(u(t) - v(t))\|_2 \leq \epsilon$$

provided $N - N_0 > C_1\epsilon^{-C_1}$.

$[0, T] = \cup_1^J [t_i, t_{i+1}]$, where $t_{i+1} - t_i < T(\|\phi\|_2)$
For fixed i, compare on $[t_i, t_{i+1}]$ the solutions to the initial value problems

$$\begin{cases} iu_t + u_{xx} + u|u|^2 = 0, \\ u(x, t_i) = u(t_i)(x) \end{cases}$$

$$\begin{cases} i\tilde{u}_t + \tilde{u}_{xx} + \tilde{u}|\tilde{u}|^2 = 0, \\ \tilde{u}(x, t_i) = v(t_i)(x) \end{cases}$$

$$\begin{cases} iv_t + v_{xx} + P_N(v|v|^2) = 0, \\ v(x, t_i) = v(t_i)(x) \end{cases}$$

Then

$$\|u(t_i)\|_2 = \|v(t_i)\|_2 = \|\phi\|_2.$$

$$N' = N_1 > N_2 > \cdots > N_J = N_0$$

Lemma 3

$$\Downarrow$$

$$\|P_{N_{i+1}}(u(t_{i+1}) - \tilde{u}(t_{i+1}))\|_2 \leq \|P_{N_i}(u(t_i) - v(t_i))\|_2 + (N_i - N_{i+1})^{-c_2} \text{ for some } c_2 > 0$$

Lemma 4

$$\Downarrow$$

$$\|P_{N_{i+1}}(\tilde{u}(t_{i+1}) - v(t_{i+1}))\|_2 \leq (N_i - N_{i+1})^{-c_2} \text{ for some } c_2 > 0$$

$$\|P_{N_{i+1}}(u(t_{i+1}) - v(t_{i+1}))\|_2 \leq \|P_{N_i}(u(t_i) - v(t_i))\|_2 + (N_i - N_{i+1})^{-c_2}$$

$$\|P_{N'}(u(t_{i+1}) - v(t_{i+1}))\|_2 \leq \sum (N_i - N_{i+1})^{-c_2}$$

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$$\|u_{N_0}\| \lesssim \|\phi_{N_1}\|_2 + (N_1 - N_0)^{-c_5}.$$

And

$$\|u_{N_0}(t)\|_2 \lesssim \|\phi_{N_1}\|_2 + (N_1 - N_0)^{-c_5}.$$

$$u_k(t) = U(t)\phi_K + i\int_0^t U(t-\tau)(P_K w)(\tau)d\tau$$

$$\|u_K\| \leq \|\phi_k\|_2 + T^{c_3}(\|u\|^2 + \|\phi\|_2^2)\|u_{K+\Delta}\| + T^{c_3}\Delta^{-c_3}\|u\|^3.$$

$$\int_0^t U(t-\tau)(P_K w)(u,u,u)(\tau)d\tau - \int_0^t U(t-\tau)P_K w(u_{K+\Delta}, u_{K+\Delta}, u_{K+\Delta})(\tau)d\tau$$

$$w(u,u,u) = u|u|^2 = \sum_{k=k_1-k_2+k_3} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i(kx+\lambda t)} \hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)$$

$$\sum_{k=k_1-k_2+k_3} =$$

$$\sum_{k=k_1-k_2+k_3, k_2\neq k_1,k_3} - \sum_{k=k_1-k_2+k_3, k_1=k_2=k_3} + \sum_{k=k_1-k_2+k_3, k_1=k_2} + \sum_{k=k_1-k_2+k_3, k_3=k_2}$$

$$\sum_{k=k_1-k_2+k_3, k_1=k_2=k_3} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i(kx+\lambda t)} \hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)$$

$$= \sum_k e^{ikx} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i\lambda t} \hat{u}(k,\lambda_1)\overline{\hat{u}(k,\lambda_2)}\hat{u}(k,\lambda_3)$$

$$\sum_{k=k_1-k_2+k_3, k_1=k_2} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i(kx+\lambda t)} \hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)$$

$$= \sum_{k,k_1} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i(kx+(\lambda_1-\lambda_2+\lambda_3)t)} \hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_1,\lambda_2)}\hat{u}(k,\lambda_3)$$

$$= \sum_{k,k_1} \mathcal{F}u(k_1,t)\overline{\mathcal{F}u(k_1,t)} \int_{\lambda_3} e^{i(kx+\lambda_3 t)} \hat{u}(k,\lambda_3)$$

$$= (\int |\phi|^2) \cdot \sum_k \int_\lambda e^{i(kx+\lambda t)} \hat{u}(k,\lambda)$$

$$
\begin{aligned}
w(u,u,u) = & \sum_{k=k_1-k_2+k_3, k_2\neq k_1, k_3} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i(kx+\lambda t)} \hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3) \\
& - \sum_k e^{ikx} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i\lambda t} \hat{u}(k,\lambda_1)\overline{\hat{u}(k,\lambda_2)}\hat{u}(k,\lambda_3) \\
& + 2\Big(\int |\phi|^2\Big) \cdot \sum_k \int_\lambda e^{i(kx+\lambda t)} \hat{u}(k,\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t U(t-\tau) w(u,u,u)(\tau)\,d\tau \\
&= \int_0^t e^{ik^2(t-\tau)} \sum_{k=k_1-k_2+k_3,\, k_2\neq k_1,k_3} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i(kx+\lambda\tau)} \hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)\,d\tau \\
&- \int_0^t e^{ik^2(t-\tau)} \sum_k e^{ikx} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{i\lambda\tau} \hat{u}(k,\lambda_1)\overline{\hat{u}(k,\lambda_2)}\hat{u}(k,\lambda_3)\,d\tau \\
&+ \int_0^t e^{ik^2(t-\tau)} 2\Big(\int |\phi|^2\Big) \cdot \sum_k \int_\lambda e^{i(kx+\lambda\tau)} \hat{u}(k,\lambda)\,d\tau \\
&= \sum_{k=k_1-k_2+k_3,\, k_2\neq k_1,k_3} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{ikx} \frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2} \hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3) \\
&- \sum_k e^{ikx} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} \frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2} \hat{u}(k,\lambda_1)\overline{\hat{u}(k,\lambda_2)}\hat{u}(k,\lambda_3) \\
&+ 2\Big(\int |\phi|^2\Big) \cdot \sum_k \int_\lambda e^{ikx} \frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2} \hat{u}(k,\lambda)
\end{aligned}
$$

$$\int_0^t U(t-\tau)(P_K w)(u,u,u)(\tau)d\tau$$

$$= \sum_{\substack{k=k_1-k_2+k_3, k_2\neq k_1,k_3,\\ |k|\leq K}} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{ikx}\frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)$$

$$-\sum_{k,|k|\leq K} e^{ikx}\int_{\lambda=\lambda_1-\lambda_2+\lambda_3}\frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\hat{u}(k,\lambda_1)\overline{\hat{u}(k,\lambda_2)}\hat{u}(k,\lambda_3)$$

$$+2(\int|\phi|^2)\cdot\sum_{k,|k|\leq K}\int_{\lambda_3} e^{ikx}\frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\hat{u}(k,\lambda)$$

For $\Delta \in \mathbb{N}$

$$\int_0^t U(t-\tau) P_K w(u_{K+\Delta}, u_{K+\Delta}, u_{K+\Delta})(\tau) d\tau$$

$$= \sum_{\substack{k=k_1-k_2+k_3, k_2 \neq k_1, k_3, \\ |k| \leq K, \min(|k_i|) \leq K+\Delta}} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{ikx} \frac{e^{i\lambda t} - e^{ik^2 t}}{\lambda - k^2} \hat{u}(k_1, \lambda_1) \overline{\hat{u}(k_2, \lambda_2)} \hat{u}(k_3, \lambda_3)$$

$$- \sum_{k, |k| \leq K} e^{ikx} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} \frac{e^{i\lambda t} - e^{ik^2 t}}{\lambda - k^2} \hat{u}(k, \lambda_1) \overline{\hat{u}(k, \lambda_2)} \hat{u}(k, \lambda_3)$$

$$+ 2\left(\int |u_{K+\Delta}|^2\right) \cdot \sum_{k, |k| \leq K} \int_{\lambda_3} e^{ikx} \frac{e^{i\lambda t} - e^{ik^2 t}}{\lambda - k^2} \hat{u}(k, \lambda)$$

$$\int_0^t U(t-\tau)(P_K w)(u,u,u)(\tau)d\tau - \int_0^t U(t-\tau)P_K w(u_{K+\Delta},u_{K+\Delta},u_{K+\Delta})(\tau)d\tau$$

$$= \sum_{\substack{k=k_1-k_2+k_3,k_2\neq k_1,k_3,\\ |k|\leq K,\max(|k_i|)>K+\Delta}} \int_{\lambda=\lambda_1-\lambda_2+\lambda_3} e^{ikx}\frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)$$

$$+ \int_0^t U(t-\tau)2(\int(|\phi|^2-|u_{K+\Delta}|^2)dx)u_K(\tau)d\tau$$

$$
\begin{aligned}
&\sum \int e^{ikx} \frac{e^{i\lambda t} - e^{ik^2 t}}{\lambda - k^2} \hat{u}(k_1, \lambda_1) \overline{\hat{u}(k_2, \lambda_2)} \hat{u}(k_3, \lambda_3) \\
&= \sum \int (1 - \psi_2(\lambda - k^2)) e^{ikx} \frac{e^{i\lambda t}}{\lambda - k^2} \hat{u}(k_1, \lambda_1) \overline{\hat{u}(k_2, \lambda_2)} \hat{u}(k_3, \lambda_3) \\
&+ \sum \int (1 - \psi_2(\lambda - k^2)) e^{ikx} \frac{-e^{ik^2 t}}{\lambda - k^2} \hat{u}(k_1, \lambda_1) \overline{\hat{u}(k_2, \lambda_2)} \hat{u}(k_3, \lambda_3) \\
&+ \sum \int \psi_2(\lambda - k^2) e^{ikx} \frac{e^{i\lambda t} - e^{ik^2 t}}{\lambda - k^2} \hat{u}(k_1, \lambda_1) \overline{\hat{u}(k_2, \lambda_2)} \hat{u}(k_3, \lambda_3)
\end{aligned}
$$

where $\psi_2 = 1$ on $[-1, 1]$ and $\psi_2 = 0$ on $\mathbb{R} - [-2, 2]$.
Define

$$
c(k, \lambda) = (1 + |\lambda - k^2|)^{\frac{1}{2} + \frac{\epsilon}{2}} |\hat{u}(k, \lambda)|.
$$

Then $\|c(k, \lambda)\|_{l^2 L^2} = \|c\|$.

$$\|\sum\int(1-\psi_2(\lambda-k^2))e^{ikx}\frac{e^{i\lambda t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)\|$$

$$=\sup_{\|a\|_{l^2L^2}\le 1}\int\int\sum\int(1-\psi_2(\lambda-k^2))e^{ikx}\frac{e^{i\lambda t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)$$

$$\widehat{(1+|\lambda-k^2|)^{\frac{1}{2}+\frac{\epsilon}{2}}a(k,\lambda)}dxdt$$

Let $G(x,t) := \widehat{(1+|\lambda-k^2|)^{\frac{1}{2}+\frac{\epsilon}{2}}a(k,\lambda)}$
Then $\int\int e^{ikx}e^{i\lambda t}G(x,t) = (1+|\lambda-k^2|)^{\frac{1}{2}+\frac{\epsilon}{2}}a(k,\lambda)$
Hence

$$\|\sum\int(1-\psi_2(\lambda-k^2))e^{ikx}\frac{e^{i\lambda t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)\|$$

$$\le\sup_{\|a\|_{l^2L^2}\le 1}\sum\int\frac{c(k_1,\lambda_1)}{\langle\lambda_1-k_1^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_2,\lambda_2)}{\langle\lambda_2-k_2^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_3,\lambda_3)}{\langle\lambda_3-k_3^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{a(k,\lambda)}{\langle\lambda-k^2\rangle^{\frac{1}{2}-\frac{\epsilon}{2}}}$$

$$\|\sum \int (1-\psi_2(\lambda-k^2))e^{ikx}\frac{e^{ik^2t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)\|$$

$$= \sup_{\|a\|_{l^2L^2}\leq 1}\int\int\sum\int(1-\psi_2(\lambda-k^2))e^{ikx}\frac{e^{ik^2t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)$$

$$\widehat{(1+|\lambda-k^2|)^{\frac{1}{2}+\frac{\epsilon}{2}}a(k,\lambda)}dxdt$$

Let $G(x,t) := \widehat{(1+|\lambda-k^2|)^{\frac{1}{2}+\frac{\epsilon}{2}}a(k,\lambda)}$. Then $\int\int e^{ikx}e^{ik^2t}G(x,t) = a(k,k^2)$.
And $\|a(k,k^2)\|_{l^2} \leq \|\hat{a}(x,t)\|_{L^{4/3}_{x,t}} \lesssim \|\hat{a}(x,t)\|_{L^2_{x,t}} = \|a(k,\lambda)\|_{l^2L^2}$

Hence

$$\|\sum \int (1-\psi_2(\lambda-k^2))e^{ikx}\frac{e^{ik^2t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)\|$$

$$\leq \sup_{\|b\|_{l^2}\leq 1}\sum\int\frac{c(k_1,\lambda_1)}{\langle\lambda_1-k_1^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_2,\lambda_2)}{\langle\lambda_2-k_2^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_3,\lambda_3)}{\langle\lambda_3-k_3^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{b(k)}{\langle\lambda-k^2\rangle}$$

$$\|\sum \int \psi_2(\lambda - k^2) e^{ikx} \frac{e^{i\lambda t} - e^{ik^2 t}}{\lambda - k^2} \hat{u}(k_1, \lambda_1)\overline{\hat{u}(k_2, \lambda_2)}\hat{u}(k_3, \lambda_3)\|$$

$$= \sup_{\|a\|_{l^2 L^2}} \int\int \sum \int \psi_2(\lambda - k^2) e^{i(kx + k^2 t)} \frac{e^{i(\lambda - k^2)t} - 1}{\lambda - k^2} \hat{u}(k_1, \lambda_1)\overline{\hat{u}(k_2, \lambda_2)}\hat{u}(k_3, \lambda_3)$$

$$(1 + |\lambda - \widehat{k^2|)}^{\frac{1}{2} + \frac{\epsilon}{2}} a(k, \lambda) dx dt$$

Let $G(x,t) := (1 + |\lambda - \widehat{k^2|)}^{\frac{1}{2} + \frac{\epsilon}{2}} a(k, \lambda)$.

By Taylor series expansion, we have $\frac{e^{i(\lambda - k^2)t} - 1}{\lambda - k^2} = \sum_{j=1}^{\infty} \frac{1}{j!}(\lambda - k^2)^{j-1}(it)^j$.

Then $\int\int e^{ikx} e^{ik^2 t} (it)^j G(x,t) = (it)^j \widehat{G(x}, t)(k, k^2)$.

Since u is defined on $\mathbf{T} \times [0, T]$, we may assume that $G(x,t)$ is same.

$$\begin{aligned}(it)^j \widehat{G(x}, t)(k, k^2) &= \int \widetilde{(it)^j \psi_1}(\tau) \widehat{G(x,t)}(k, k^2 - \tau) d\tau \\ &= \int \widetilde{(it)^j \psi_1}(\tau) (1 + |\tau|)^{\frac{1}{2} + \frac{\epsilon}{2}} a(k, k^2 - \tau) d\tau,\end{aligned}$$

where ψ_1 is 1 on $[0, 1]$ and 0 on $\mathbb{R} - [0, 2]$.

$$\|\widehat{(it)^j G(x,t)}(k,k^2)\|_{l^2} \leq \|a(k,k^2-\tau)\|_{l^2} \leq \|e^{-ix\tau}\hat{a}(x,t)\|_{L^{4/3}_{x,t}} \leq \|e^{-ix\tau}\hat{a}(x,t)\|_{L^2_{x,t}}.$$

Hence

$$\|\sum\int \psi_2(\lambda-k^2)e^{ikx}\frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)\|$$

$$\leq \sup_{\|b\|_{l^2}\leq 1}\sum\int \frac{c(k_1,\lambda_1)}{\langle\lambda_1-k_1^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_2,\lambda_2)}{\langle\lambda_2-k_2^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_3,\lambda_3)}{\langle\lambda_3-k_3^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{b(k)}{\langle\lambda-k^2\rangle}$$

$$
\begin{aligned}
&\|\sum\int e^{ikx}\frac{e^{i\lambda t}-e^{ik^2 t}}{\lambda-k^2}\hat{u}(k_1,\lambda_1)\overline{\hat{u}(k_2,\lambda_2)}\hat{u}(k_3,\lambda_3)\| \\
&\leq \sup_{\|a\|_{l^2L^2}\leq 1}\sum\int\frac{c(k_1,\lambda_1)}{\langle\lambda_1-k_1^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_2,\lambda_2)}{\langle\lambda_2-k_2^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_3,\lambda_3)}{\langle\lambda_3-k_3^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{a(k,\lambda)}{\langle\lambda-k^2\rangle^{\frac{1}{2}-\frac{\epsilon}{2}}} \\
&+\sup_{\|b\|_{l^2}\leq 1}\sum\int\frac{c(k_1,\lambda_1)}{\langle\lambda_1-k_1^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_2,\lambda_2)}{\langle\lambda_2-k_2^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{c(k_3,\lambda_3)}{\langle\lambda_3-k_3^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}\frac{b(k)}{\langle\lambda-k^2\rangle}
\end{aligned}
$$

Define $a(k,\lambda)=\frac{b(k)}{\langle\lambda-k^2\rangle^{\frac{1}{2}-\frac{\epsilon}{2}}}$.

$$\sum \int \frac{c(k_1,\lambda_1)}{\langle \lambda_1 - k_1^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}} \frac{c(k_2,\lambda_2)}{\langle \lambda_2 - k_2^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}} \frac{c(k_3,\lambda_3)}{\langle \lambda_3 - k_3^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}} \frac{a(k,\lambda)}{\langle \lambda - k^2\rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}$$
$$\leq \int\int F_1^3 F_2 dxdt$$

where $\widehat{F_1}(k,\lambda) = \frac{c(k,\lambda)}{(1+|\lambda-k^2|)^{\frac{3}{8}}}, \widehat{F_2}(k,\lambda) = \frac{a(k,\lambda)}{(1+|\lambda-k^2|)^{\frac{3}{8}}}$

Hence

$$\int\int F_1^3 F_2 dxdt \leq \|F_1\|_4^3 \|F_2\|_4 \leq c \cdot \|c\|^3_{l_k^2 L_\lambda^2} = c\|u\|^3.$$

$$-k^2 + k_1^2 - k_2^2 + k_3^2 = -2k_2^2 + 2k_1k_2 + 2k_3k_2 - 2k_1k_3 = 2(k_1 - k_2)(k_2 - k_3).$$

Hence

$$\max(|\lambda_1 - k_1^2|, |\lambda_2 - k_2^2|, |\lambda_3 - k_3^2|, |\lambda - k^2|) \geq \frac{1}{2}|k_2 - k_1||k_2 - k_3|.$$

Actually

$$\begin{aligned} &|\lambda_1 - k_1^2| + |\lambda_2 - k_2^2| + |\lambda_3 - k_3^2| + |\lambda - k^2| \\ &\geq |(\lambda_1 - k_1^2) - (\lambda_2 - k_2^2) + (\lambda_3 - k_3^2) - (\lambda - k^2)| \\ &= |-k^2 + k_1^2 - k_2^2 + k_3^2|. \end{aligned}$$

And $|k - k_i| \leq |k_1 - k_2| + |k_3 - k_2|$. Assumption $|k| \leq K, \max|k_i| > K + \Delta$ and $k_2 \neq k_1, k_3 (|k_2 - k_1|, |k_3 - k_2| \geq 1)$ gives

$$\max(|\lambda_1 - k_1^2|, |\lambda_2 - k_2^2|, |\lambda_3 - k_3^2|, |\lambda - k^2|) \geq \Delta$$

if $-\frac{1}{2} < b' \leq b < \frac{1}{2}$, then for any $0 < T < 1$ we have

$$\|(1+|\lambda-k^2|)^{b'}\widehat{\psi_1(t/T)u}\|_{l^2_k L^2_\lambda} \leq T^{b-b'}\|(1+|\lambda-k^2|)^b\widehat{u}\|_{l^2_k L^2_\lambda}$$

By interpolating with the trivial case $b' = b$, we may assume that $b' = 0$

$$\|\widehat{\psi_1(t/T)u}\|_{l^2_k L^2_\lambda} \leq T^b\|(1+|\lambda-k^2|)^b\widehat{u}\|_{l^2_k L^2_\lambda} \text{ for } 0 < b < \frac{1}{2}.$$

Two cases : $\langle\tau - k^2\rangle \geq \frac{1}{T}$, $\langle\tau - k^2\rangle \leq \frac{1}{T}$.
1. $\langle\tau - k^2\rangle \geq \frac{1}{T}$:

$$\begin{aligned}\|\widehat{\psi_1(t/T)u}\|_{l^2_k L^2_\lambda} &= \|(1+|\lambda-k^2|)^{-b}(1+|\lambda-k^2|)^b\widehat{\psi_1(t/T)u}\|_{l^2_k L^2_\lambda} \\ &\leq T^b\|(1+|\lambda-k^2|)^b\widehat{u}\|_{l^2_k L^2_\lambda}.\end{aligned}$$

2. $\langle \tau - k^2 \rangle \leq \frac{1}{T}$:

$$\begin{aligned}
\|u(t)\|_{L^2_x} &= \|\mathcal{F}u(t)\|_{l^2_k} \\
&\leq \| \int_{\langle \tau - k^2 \rangle \leq \frac{1}{T}} |\hat{u}(\tau, k)| d\tau \|_{l^2_k} \\
&= \| \int_{\langle \tau - k^2 \rangle \leq \frac{1}{T}} \langle \tau - k^2 \rangle^{-b} \langle \tau - k^2 \rangle^{b} |\hat{u}(\tau, k)| d\tau \|_{l^2_k} \\
&\leq T^{b-\frac{1}{2}} \| (\int_{\langle \tau - k^2 \rangle \leq \frac{1}{T}} \langle \tau - k^2 \rangle^{2b} |\hat{u}(\tau, k)|^2 d\tau)^{\frac{1}{2}} \|_{l^2_k} \\
&= T^{b-\frac{1}{2}} \| (1 + |\lambda - k^2|)^b \widehat{u} \|_{l^2_k L^2_\lambda}
\end{aligned}$$

Hence

$$\begin{aligned}
\| \widehat{\psi_1(t/T) u} \|_{l^2_k L^2_\lambda} &= \| \psi_1(t/T) u \|_{L^2_{x,t}} \\
&\leq (\int_0^T (T^{b-\frac{1}{2}} \| (1 + |\lambda - k^2|)^b \widehat{u} \|_{l^2_k L^2_\lambda})^2 dt)^{\frac{1}{2}} \\
&\leq T^b \| (1 + |\lambda - k^2|)^b \widehat{u} \|_{l^2_k L^2_\lambda}
\end{aligned}$$

$$\sum \int \frac{c(k_1,\lambda_1)}{\langle \lambda_1 - k_1^2 \rangle^{\frac{1}{2}+\frac{\epsilon}{2}}} \frac{c(k_2,\lambda_2)}{\langle \lambda_2 - k_2^2 \rangle^{\frac{1}{2}+\frac{\epsilon}{2}}} \frac{c(k_3,\lambda_3)}{\langle \lambda_3 - k_3^2 \rangle^{\frac{1}{2}+\frac{\epsilon}{2}}} \frac{a(k,\lambda)}{\langle \lambda - k^2 \rangle^{\frac{1}{2}+\frac{\epsilon}{2}}}$$
$$\leq \int \int F_1^3 F_2 dxdt$$

where $\widehat{F_1}(k,\lambda) = \frac{c(k,\lambda)}{(1+|\lambda-k^2|)^{\frac{3}{8}+\tilde{\epsilon}}}, \widehat{F_2}(k,\lambda) = \frac{a(k,\lambda)}{(1+|\lambda-k^2|)^{\frac{3}{8}+\tilde{\epsilon}}}$

Hence
$$\int \int F_1^3 F_2 dxdt \leq \|F_1\|_4^3 \|F_2\|_4 \leq T^{c_3} \Delta^{-c_3} \|u\|^3.$$

$$\int_0^t U(t-\tau)(P_K w)(u,u,u)(\tau)d\tau$$
$$-\int_0^t U(t-\tau)\left[P_K w(u_{K+\Delta},u_{K+\Delta},u_{K+\Delta})(\tau)d\tau + 2\left(\int (|\phi|^2 - |u_{K+\Delta}|^2)dx\right)u_K\right](\tau)d\tau$$

$$|||"||| \leq T^{c_3}\Delta^{-c_3}|||u|||^3$$

And

$$|||\int_0^t U(t-\tau)\left(\int (|\phi|^2 - |u_{K+\Delta}|^2)dx\right)u_K(\tau)d\tau|||$$
$$\leq (|||u_{K+\Delta}|||^2 + \|\phi\|^2)|||\int_0^t U(t-\tau)u_K(\tau)d\tau|||$$
$$\leq T^{c_3}|||u_{K+\Delta}|||(|||u_{K+\Delta}|||^2 + \|\phi\|^2)$$

$$u_k(t) = U(t)\phi_K + i\int_0^t U(t-\tau)(P_K w)(\tau)d\tau$$

$$|||u_K||| \le \|\phi_k\|_2 + T^{c_3}(|||u|||^2 + \|\phi\|_2^2)|||u_{K+\Delta}||| + T^{c_3}\Delta^{-c_3}|||u|||^3.$$

Choosing T sufficiently small, depending on $|||u|||$, $\|\phi\|_2$, and hence $\|\phi\|_2$, we get

$$|||u_K||| \le c_4\|\phi_k\|_2 + \delta|||u_{K+\Delta}||| + \Delta^{-c_3}.$$

where $\delta > 0$ is a sufficiently small constant.
A straightforward iteration r times yields

$$\begin{aligned}
&|||u_K||| \\
&\le (c_4\|\phi_k\|_2 + \delta\|\phi_{k+\Delta}\|_2 + \cdots + \delta^r\|\phi_{k+r\Delta}\|_2) + \delta^r|||u_{K+r\Delta}||| \\
&+ (1+\delta+\cdots+\delta^{r-1})\Delta^{-c_3} \\
&\lesssim \|\phi_{k+r\Delta}\|_2 + \delta^r\|\phi\|_2 + \Delta^{-c_3}
\end{aligned}$$

$$\|u_K\| \lesssim \|\phi_{K+r\Delta}\|_2 + \delta^r \|\phi\|_2 + \Delta^{-c_3}$$

Assuming $N_1 > N_0$ (N_1 is sufficiently large), for an appropriate choice of r, Δ (Take Δ such that $N_1 - N_0 = \frac{-c_3 \ln \Delta - \ln \|\phi\|_2}{\ln \delta} \cdot \Delta$ and let r be $\frac{-c_3 \ln \Delta - \ln \|\phi\|_2}{\ln \delta}$).
Then $\delta^r \|\phi\|_2 = \Delta^{-c_3} \leq (N_1 - N_0)^{-c_5}$ for some $c_5 > c_3$.
Hence

$$\|u_{N_0}\| \lesssim \|\phi_{N_1}\|_2 + (N_1 - N_0)^{-c_5}.$$

And

$$\|u_{N_0}(t)\|_2 \lesssim \|\phi_{N_1}\|_2 + (N_1 - N_0)^{-c_5}.$$

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Lemma 3

Consider the solution u, v to the Cauchy problems

$$\begin{cases} iu_t + u_{xx} + u|u|^2 = 0, \\ u(x,0) = \phi(x) \end{cases}$$

$$\begin{cases} iv_t + v_{xx} + v|v|^2 = 0, \\ v(x,0) = \psi(x) \end{cases}$$

and assume $\|\phi\|_2 = \|\psi\|_2$. Then for $|t| < T(\|\phi\|_2)$ one has

$$\|P_{N_0}(u(t) - v(t))\|_2 \leq \|P_{N_1}(\phi - \psi)\|_2 + \epsilon$$

provided $N_1 - N_0 > C_1 \epsilon^{-C_1}$.

$$(u-v)(t) = U(t)(\phi-\psi) + i\int_0^t u(t-\tau)[w(u,u,u)-w(v,v,v)](\tau)d\tau$$

The difference expression for u and v are both bounded by $T^{c_3}\Delta^{-c_3}(\|u\|^3+\|v\|^3)$

$$\|\int_0^t U(t-\tau)(P_K w)(u,u,u)(\tau)d\tau$$
$$-\int_0^t U(t-\tau)\left[P_K w(u_{K+\Delta},u_{K+\Delta},u_{K+\Delta})(\tau)d\tau + 2(\int(|\phi|^2-|u_{K+\Delta}|^2)dx)u_K\right](\tau)d\tau\|$$
$$\lesssim T^{c_3}\Delta^{-c_3}(\|u\|^3+\|v\|^3)$$
$$\|\int_0^t U(t-\tau)(P_K w)(v,v,v)(\tau)d\tau$$
$$-\int_0^t U(t-\tau)\left[P_K w(v_{K+\Delta},v_{K+\Delta},v_{K+\Delta})(\tau)d\tau + 2(\int(|\psi|^2-|v_{K+\Delta}|^2)dx)u_K\right](\tau)d\tau\|$$
$$\lesssim T^{c_3}\Delta^{-c_3}(\|u\|^3+\|v\|^3)$$

Since

$$\int |\phi|^2 dx = \int |\psi|^2 dx$$

we obtain

$$\int_0^t U(t-\tau)\big[P_K w(u_{K+\Delta}, u_{K+\Delta}, u_{K+\Delta})(\tau)d\tau - P_K w(v_{K+\Delta}, u_{K+\Delta}, v_{K+\Delta})(\tau)$$
$$+2\int (|\phi|^2)(u_K - v_K) - (\int |u_{K+\Delta}|^2 dx)u_K + (\int |v_{K+\Delta}|^2 dx)v_K\big](\tau)d\tau$$

$$\begin{aligned} w(u,u,u) - w(v,v,v) &= |u|^2 u - |u|^2 v + |u|^2 v - u|v|^2 + u|v|^2 - v|v|^2 \\ &= |u|^2(u-v) + uv(\bar{u}-\bar{v}) + |v|^2(u-v) \end{aligned}$$

gives

$$\|\int_0^t U(t-\tau)\big[P_K w(u_{K+\Delta}, u_{K+\Delta}, u_{K+\Delta})(\tau)d\tau - P_K w(v_{K+\Delta}, v_{K+\Delta}, v_{K+\Delta})(\tau)$$
$$+2\int (|\phi|^2)(u_K - v_K) - (\int |u_{K+\Delta}|^2 dx)u_K + (\int |v_{K+\Delta}|^2 dx)v_K\big](\tau)d\tau\|$$
$$\lesssim T^{c_3}(\|u_{K+\Delta}\|^2 + \|v_{K+\Delta}\|^2 + \|\phi\|_2)\|u_{K+\Delta} - v_{K+\Delta}\|$$

For sufficiently small T, one gets

$$\|u_K - v_K\| \leq \|\phi_K - \psi_K\|_2 + \delta\|u_{K+\Delta} - v_{K+\Delta}\| + \Delta^{-c_3}$$

and also

$$\|u_K(t) - v_K(t)\|_2 \leq \|\phi_K - \psi_K\|_2 + \delta\|u_{K+\Delta} - v_{K+\Delta}\| + \Delta^{-c_3}$$

Iterate and gets for $N_1 > N_0$

$$\|u_{N_0} - v_{N_1}\| \leq \|\phi_{N_1} - \psi_{N_1}\|_2 + (N_1 - N_0)^{-c_5}$$

and

$$\|u_{N_0}(t) - v_{N_1}(t)\|_2 \leq \|\phi_{N_1} - \psi_{N_1}\|_2 + (N_1 - N_0)^{-c_5}$$

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Lemma 4

Consider the solution u, v to the Cauchy problems

$$\begin{cases} iu_t + u_{xx} + u|u|^2 = 0, \\ u(x,0) = \phi(x) \end{cases}$$

$$\begin{cases} iv_t + v_{xx} + P_N(v|v|^2) = 0, \\ v(x,0) = \phi(x) \end{cases}$$

where $\phi = P_N\phi$. Then for $|t| < T(\|\phi\|_2)$ one has

$$\|P_{N_0}(u(t) - v(t))\|_2 \leq \epsilon$$

provided $N - N_0 > C_1 \epsilon^{-C_1}$.

$$(u-v)(t) = i\int_0^t u(t-\tau)[w(u,u,u)-w(v,v,v)](\tau)d\tau$$

Hence

$$(u_K - v_K)(t) = i\int_0^t u(t-\tau)[P_K w(u,u,u) - P_K w(v,v,v)](\tau)d\tau$$

provided K is kept less than N. As in the proof of Lemma 3, one obtains

$$|||u_K - v_K||| \leq \delta|||u_{K+\Delta} - v_{K+\Delta}||| + \Delta^{-c_3}$$

and

$$\|u_K(t) - v_K(t)\|_2 \leq \delta|||u_{K+\Delta} - v_{K+\Delta}||| + \Delta^{-c_3}$$

Iterating and keeping $K + r\Delta < N$ yields

$$\|u_K - v_K\| \leq \delta^r(\|u\| + \|v\|) + \Delta^{-c_3}.$$

Hence for $N_0 < N$

$$\|u_{N_0} - v_{N_0}\| \leq (N - N_0)^{-c_5}$$

and

$$\|u_{N_0}(t) - v_{N_0}(t)\|_2 \leq (N - N_0)^{-c_5}.$$