Study for "On the Regularization Mechanism for the Periodic Korteweg-de Vries Equation ([BIT11])"

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KdV equation

We consider

Periodic KdV equation

$$(\mathrm{KdV}) \begin{cases} \partial_t u = u\partial_x u + \partial_x^3 u, & t > 0,\ x \in \mathbb{T} \\ u(0,x) = u^0(x), & x \in \mathbb{T} \end{cases} .$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and u is a real valued.

Goal

Our goal is to obtain the well-posedness of (KdV) in Sobolev space

Conservation law

If u is a solution of (KdV), then

Conservation law

$$\int_0^{2\pi} u(t,x)dx = \int_0^{2\pi} u^0(x)dx$$
$$\int_0^{2\pi} u(t,x)^2 dx = \int_0^{2\pi} u^0(x)^2 dx$$
$$\vdots$$

We can assume $\int_0^{2\pi} u(t,x)dx = 0$ by the transform $u \mapsto u - \mu$, $x \mapsto x - \mu t$ for $\mu = \frac{1}{2\pi}\int_0^{2\pi} u^0(x)dx$.

Initial condition

Definition

For $s \in \mathbb{R}$,

$$\dot{H}^s := \overline{\left\{ f \in C^\infty(\mathbb{T}) \,\middle|\, \int_0^{2\pi} f(x)dx = 0,\ ||f||_{\dot{H}^s} < \infty \right\}}^{||\cdot||_{\dot{H}^s}},$$

$$||f||_{\dot{H}^s} := \left(\sum_{k \in \mathbb{Z}_0} |k|^{2s} |f_k|^2 \right)^{1/2}, \ \mathbb{Z}_0 := \mathbb{Z} \backslash \{0\},$$

$$f_k := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx.$$

※ For $u \in C([0,T]; \dot{H}^s)$, $u_k(t) = (u(t))_k := \frac{1}{2\pi} \int_0^{2\pi} u(t,x) e^{-ikx} dx$.

Integral form

(KdV) can be rewritten

$$u(t) = e^{t\partial_x^3}u^0 + \frac{1}{2}\int_0^t e^{(t-t')\partial_x^3}\partial_x\left(u(t')^2\right)dt'.$$

We put

$$v(t) := e^{-t\partial_x^3}u(t),\ \ v^0 := u^0 \ \ (\text{i.e. } v_k(t) = e^{ik^3t}u_k(t)).$$

Equation for v is following:

Integral form of KdV

$$(K0)\ \ v(t) = v^0 + \int_0^t B_1(v,v)(t')dt'.$$

Where

$$B_1(v,w)_k := \frac{1}{2}ik \sum_{\substack{k_1+k_2=k \\ k_1,k_2\in\mathbb{Z}_0}} e^{i3kk_1k_2t}v_{k_1}w_{k_2}.$$

Main Theorem

Theorem (BIT11, Thm 4.6, 5.5, 6.5, 6.6, 6.7)

1. *Let $s \geq 0$. (KdV) is unconditionally globally well-posed in $\dot{H}^s$. More precisely, for any $T > 0$ and $v^0 \in \dot{H}^s$, there exists an unique solution $v \in C([0,T]; \dot{H}^s)$ of (K0) on $t \in [0,T]$. Furthermore, the data-to-solution map $S : \dot{H}^s \ni v^0 \mapsto v \in C([0,T]; \dot{H}^s)$ is Lipschitz continuous.*
2. *Let $-1 < \theta \leq 0$ and $T > 0$. If*

$$||v^0||_{\dot{H}^0} + ||w^0||_{\dot{H}^0} \leq M$$

holds for some $M > 0$, then $v = S(v^0)$ and $w = S(w^0)$ satisfy

$$||v - w||_{L^\infty([0,T];\dot{H}^\theta)} \leq C_0(M,\theta)C_1(M,\theta)^T ||v^0 - w^0||_{\dot{H}^\theta}.$$

Main tools are "Normal form method" and "Time averaging induced squeezing".

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Difficulty

Since $B_1(v, v)$ contains a derivative (i.e. there is a derivative loss), we cannot close the iteration argument in $C([0, T]; \dot{H}^s)$ for (K0).

Point

To recover the derivative loss, we will use the Normal form method.

We use the following symbols:

$$\sum_{k_1+k_2=k}^{*} := \sum_{\substack{k_1+k_2=k \\ k_1, k_2 \in \mathbb{Z}_0}}, \quad \sum_{k_1+k_2+k_3=k}^{*} := \sum_{\substack{k_1+k_2+k_3=k \\ k_1, k_2, k_3 \in \mathbb{Z}_0}}$$

First reduction

For the solution of (K0), we have

$$
\begin{aligned}
(\partial_t v)_k &= B_1(v,v)_k \\
&= \frac{1}{2} ik \sum_{k_1+k_2=k}^{*} e^{i3kk_1k_2t} v_{k_1} v_{k_2} \\
&= \frac{1}{6}\partial_t \left(\sum_{k_1+k_2=k}^{*} \frac{e^{i3kk_1k_2t}}{k_1k_2} v_{k_1} v_{k_2} \right) \\
&\quad - \frac{1}{6} \sum_{k_1+k_2=k}^{*} \frac{e^{i3kk_1k_2t}}{k_1k_2} (v_{k_1}\partial_t v_{k_2} + (\partial_t v_{k_1}) v_{k_2}) \\
&=: \frac{1}{6}\partial_t (B_2(v,v)_k) - I
\end{aligned}
$$

by using the differentiation by parts in time.

First reduction

By the symmetry (for k_1 and k_2) and the equation $\partial_t v = B_1(v,v)$,

$$\begin{aligned} I &= \frac{1}{3} \sum_{k_1+k_2=k}^{*} \frac{e^{i3kk_1k_2t}}{k_1k_2} v_{k_1} \partial_t v_{k_2} \\ &= \frac{1}{3} \sum_{k_1+k_2=k}^{*} \frac{e^{i3kk_1k_2t}}{k_1k_2} v_{k_1} B_1(v,v)_{k_2} \\ &= \frac{i}{6} \sum_{k_1+k_2=k}^{*} \sum_{\alpha+\beta=k_2}^{*} \frac{e^{i3(kk_1k_2+k_2\alpha\beta)t}}{k_1} v_{k_1} v_\alpha v_\beta \\ &= \frac{i}{6} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}}^{*} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1} v_{k_1} v_{k_2} v_{k_3} \\ &=: \frac{i}{6} R_3(v,v,v)_k. \end{aligned}$$

※ We used the transform $(\alpha, \beta, k_2) \mapsto (k_2, k_3, k_2+k_3)$ in the forth equality.

First form of KdV

(K0) can be rewritten

First form of KdV

$$(K1) \quad v(t) = v^0 + \frac{1}{6}\left(B_2(v,v)(t) - B_2(v,v)(0)\right) - \frac{i}{6}\int_0^t R_3(v,v,v)(t')dt'.$$

where

$$B_2(u,v)_k := \sum_{k_1+k_2=k}^{*} \frac{e^{i3kk_1k_2t}}{k_1k_2} u_{k_1} v_{k_2},$$

$$R_3(u,v,w)_k := \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3\neq 0}}^{*} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1} u_{k_1} v_{k_2} w_{k_3}.$$

Estimate for R_3

Remark

For $s > 1/2$, we have

$$||R_3(u, v, w)||_{\dot{H}^s} \lesssim ||u||_{\dot{H}^s}||v||_{\dot{H}^s}||w||_{\dot{H}^s}.$$

But for $s \leq 1/2$, we cannot obtain the same estimate.

Resonant and Nonresonant

We put

$$\vec{k} := (k_1, k_2, k_3) \in \mathbb{Z}_0,\ \ \Phi(\vec{k}) := (k_1 + k_2)(k_2 + k_3)(k_3 + k_1)$$

$$\sum_{k_1+k_2+k_3=k}^{\text{res}} := \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2,k_3,k_2+k_3\neq 0 \\ \Phi(\vec{k})=0}},\ \ \sum_{k_1+k_2+k_3=k}^{\text{nres}} := \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2,k_3,k_2+k_3\neq 0 \\ \Phi(\vec{k})\neq 0}}$$

and define

$$R_{3\text{res}}(u,v,w) := \sum_{k_1+k_2+k_3=k}^{\text{res}} \frac{u_{k_1} v_{k_2} w_{k_3}}{k_1}$$

$$R_{3\text{nres}}(u,v,w) := \sum_{k_1+k_2+k_3=k}^{\text{nres}} \frac{e^{i3\Phi(\vec{k})t}}{k_1} u_{k_1} v_{k_2} w_{k_3}.$$

Then $R_3(u,v,w) = R_{3\text{res}}(u,v,w) + R_{3\text{nres}}(u,v,w)$.

Calculation for the resonant part

We note that

$$\sum_{k_1+k_2+k_3=k}^{\text{res}} = \sum_{m=1}^{3} \sum_{(k_1,k_2,k_3)\in S_m},$$

$$S_1 : k_1 + k_2 = 0,\ k_2 + k_3 \neq 0,\ k_3 + k_1 = 0 \Leftrightarrow \vec{k} = (-k, k, k)$$

$$S_2 : k_1 + k_2 = 0,\ k_2 + k_3 \neq 0,\ k_3 + k_1 \neq 0 \Leftrightarrow \vec{k} = (j, -j, k),\ |j| \neq k$$

$$S_3 : k_1 + k_2 \neq 0,\ k_2 + k_3 \neq 0,\ k_3 + k_1 = 0 \Leftrightarrow \vec{k} = (j, k, -j),\ |j| \neq k$$

Therefore

$$\begin{aligned} R_{3\text{res}}(v, v, v) &= \sum_{m=1}^{3} \sum_{(k_1,k_2,k_3)\in S_m} \frac{v_{k_1} v_{k_2} v_{k_3}}{k_1} \\ &= \frac{v_{-k} v_k v_k}{-k} + \sum_{j\in\mathbb{Z}_0, |j|\neq k} \frac{v_j v_{-j} v_k}{j} + \sum_{j\in\mathbb{Z}_0, |j|\neq k} \frac{v_j v_k v_{-j}}{j} \\ &= -\frac{|v_k|^2}{k} v_k =: A_{\text{res}}(v)_k. \end{aligned}$$

Second reduction for nonresonant part

For the solution of (K0), we have

$$
\begin{aligned}
&R_{3\mathrm{nres}}(v,v,v)\\
&=\sum_{k_1+k_2+k_3=k}^{\mathrm{nres}} \frac{e^{i3\Phi(\vec{k})t}}{k_1} v_{k_1}v_{k_2}v_{k_3}\\
&=\frac{1}{3i}\partial_t\left(\sum_{k_1+k_2+k_3=k}^{\mathrm{nres}} \frac{e^{i3\Phi(\vec{k})t}}{k_1\Phi(\vec{k})} v_{k_1}v_{k_2}v_{k_3}\right)\\
&\quad-\frac{1}{3i}\sum_{k_1+k_2+k_3=k}^{\mathrm{nres}} \frac{e^{i3\Phi(\vec{k})t}}{k_1\Phi(\vec{k})}((\partial_t v_{k_1})v_{k_2}v_{k_3}+v_{k_1}(\partial_t v_{k_2})v_{k_3}+v_{k_1}v_{k_2}\partial_t v_{k_3})\\
&=: \frac{1}{3i}\partial_t(B_3(v,v,v)_k)-II
\end{aligned}
$$

by using the differentiation by parts in time.

Second reduction for nonresonant part

By the symmetry (for k_2 and k_3) and the equation $\partial_t v = B_1(v, v)$,

$$\begin{aligned} II &= \frac{1}{3i} \sum_{k_1+k_2+k_3=k}^{\text{nres}} \frac{e^{i3\Phi(\vec{k})t}}{k_1 \Phi(\vec{k})} ((\partial_t v_{k_1}) v_{k_2} v_{k_3} + 2 v_{k_1} v_{k_2} \partial_t v_{k_3}) \\ &= \frac{1}{3i} \sum_{k_1+k_2+k_3=k}^{\text{nres}} \frac{e^{i3\Phi(\vec{k})t}}{k_1 \Phi(\vec{k})} (B_1(v,v)_{k_1} v_{k_2} v_{k_3} + 2 v_{k_1} v_{k_2} B_1(v,v)_{k_3}) \\ &=: \frac{1}{6} B_4(v,v,v,v)_k. \end{aligned}$$

(K1) can be rewritten

Second form of KdV

$$(K2)\quad v(t) = v^0 + \frac{1}{6}\left(B_2(v,v)(t) - B_2(v,v)(0)\right) - \frac{1}{18}\left(B_3(v)(t) - B_3(v)(0)\right) - \frac{i}{6}\int_0^t \left(A_{\mathrm{res}}(v)(t') - \frac{1}{6}B_4(v)(t')\right) dt'.$$

Where $B_3(v) := B_3(v,v,v)$, $B_4(v) := B_4(v,v,v,v)$,

$$B_3(u,v,w)_k := \sum_{k_1+k_2+k_3=k}^{\mathrm{nres}} \frac{e^{i3\Phi(\vec{k})t}}{k_1\Phi(\vec{k})} v_{k_1} v_{k_2} v_{k_3}$$

$$B_4(u,v,w,\varphi)_k := \sum_{k_1+k_2+k_3=k}^{\mathrm{nres}} \frac{e^{i3\Phi(\vec{k})t}}{k_1\Phi(\vec{k})} \left(B_1(u,v)_{k_1} w_{k_2} \varphi_{k_3} + 2u_{k_1} v_{k_2} B_1(w,\varphi)_{k_3}\right)$$

Existence and a priori estimate

Theorem (BIT11, Thm 4.6)

Let $s > 0$ and $T > 0$. There exists the solution $v \in L^\infty([0,T]; \dot{H}^s)$ of (K0). Furthermore, v satisfies the equations (K1) and (K2) and a priori estimate

$$||v||_{L^\infty([0,T];\dot{H}^s)} \le M(s, T, ||v^0||_{\dot{H}^s}).$$

Existence of the solution is proved by Galerkin approximation and compactness argument. For the detail of the proof of existence, see proof of Thm 4.6 in [BIT11].

Remark

Existence for $s = 0$ follows from existence for $s > 0$ and Lipschitz estimate for $s = 0$ (See, proof of Thm 6.6).

Key estimates to obtain a priori estimate

A priori estimate is proved by using following estimates:

Proposition (BIT11, Lemm A.2, A.7, A.13, A.16)

For $s \geq 0$, we have

$$(1)\ ||B_2(u,v)||_{\dot{H}^{s+\epsilon}} \lesssim ||u||_{\dot{H}^s}||v||_{\dot{H}^s} \text{ for } 0 \leq \epsilon \leq 1.$$
$$(2)\ ||B_3(u,v,w)||_{\dot{H}^{s+\epsilon}} \lesssim ||u||_{\dot{H}^s}||v||_{\dot{H}^s}||w||_{\dot{H}^s} \text{ for } 0 \leq \epsilon \leq 2.$$
$$(3)\ ||B_4(u,v,w,\varphi)||_{\dot{H}^{s+\epsilon}} \lesssim ||u||_{\dot{H}^s}||v||_{\dot{H}^s}||w||_{\dot{H}^s}||\varphi||_{\dot{H}^s} \text{ for } 0 \leq \epsilon < 1/2.$$
$$(4)\ ||A_{\text{res}}(v)||_{\dot{H}^{s+\epsilon}} \lesssim ||v||^2_{\dot{H}^0}||v||_{\dot{H}^s} \text{ for } 0 \leq \epsilon \leq 1.$$

Remark

The estimate (1) holds also for $-3/4 < s < 0$ and $0 \leq \epsilon < 3/4$.
The estimate (2) holds also for $s > -1/4$.
The estimate (4) holds also for $s < 0$.

For the detail of the proof of a priori estimate, see proof of Thm 4.4 in [BIT11].

Contents

Difficulty

We put $y(t) := v(t) - v^0$. Then (K2) can be written

$$(K2)' \ y(t) - \frac{1}{3}B_2(v^0, y)(t) = \frac{1}{6}B_2(y,y)(t) - \frac{1}{18}\left(B_3(y+v^0)(t) - B_3(v^0)(0)\right) - \frac{i}{6}\int_0^t \left(A_{\mathrm{res}}(y+v^0)(t') - \frac{1}{6}B_4(y+v^0)(t')\right) dt' \left(= \frac{1}{6}J_1 - \frac{1}{18}J_2 - \frac{i}{6}J_3\right).$$

Difficulty

Lipschitz constant in $L^\infty([0,T];\dot{H}^s)$ is not small because of J_2.

※ If $t > 0$ is small enough (thus y is also small in $L^\infty([0,T];\dot{H}^s)$ since y(0)=0), then J_1 and J_3 have a small Lipschitz constant by the estimate for B_2, B_4, A_{res} and the smallness of y.

Time averaging induced squeezing

For $n \in \mathbb{N}$, we define

$$\Pi_n u_k = (\Pi_n u)_k := \begin{cases} u_k & \text{if } |k| \le n \\ 0 & \text{if } |k| > n \end{cases},$$

$$\Pi_{-n} := Id - \Pi_n$$

$$R^{(n)}_{3\mathrm{nres}0}(u,v,w) := R_{3\mathrm{nres}}(u, \Pi_{-n} v, \Pi_{-n} w)$$

$$R^{(n)}_{3\mathrm{nres}1}(u,v,w) := R_{3\mathrm{nres}}(u,v,w) - R^{(n)}_{3\mathrm{nres}0}(u,v,w).$$

Before second reduction, we split $R_{3\mathrm{nres}}$ as $R_{3\mathrm{nres}} = R^{(n)}_{3\mathrm{nres}0} + R^{(n)}_{3\mathrm{nres}1}$ and apply the reduction to only $R^{(n)}_{3\mathrm{nres}0}$.

Third form of KdV

(K2)' can be rewritten

Third form of KdV

$$(K3)\ y(t) - \frac{1}{3}B_2(v^0, y)(t)$$
$$= \frac{1}{6}B_2(y,y)(t) - \frac{1}{18}\left(B_{30}^{(n)}(y+v^0)(t) - B_{30}^{(n)}(v^0)(0)\right)$$
$$-\frac{i}{6}\int_0^t \left(A_{\mathrm{res}}(y+v^0)(t') + R_{3\mathrm{nres}1}^{(n)}(y+v^0)(t') - \frac{1}{6}B_{40}^{(n)}(y+v^0)(t')\right) dt'.$$

Where

$$B_{30}^{(n)}(v) := B_3(v, \Pi_{-n}v, \Pi_{-n}v),$$
$$B_{40}^{(n)}(v) := \sum_{k_1+k_2+k_3=k}^{\mathrm{nres}} \frac{e^{i3\Phi(\vec{k})t}}{k_1\Phi(\vec{k})} \left(B_1(v,v)_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}v_{k_3}\right.$$
$$\left. + 2v_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}B_1(v,v)_{k_3}\right).$$

Key estimates

The uniqueness and Lipschitz dependence in $C([0,T];\dot{H}^s)$ for $s \geq 0$ (also for $s > -1$ with $\dot{H}^0$-bounded initial data) is proved by following estimates.

Proposition (BIT11, Lemm 6.1, A.6, A.12, A.14, A.15)

(1) For any $0 \leq s \leq 1$ *and* $\alpha \geq 0$,

$$||R^{(n)}_{3\mathrm{nres}1}(u,v,w)||_{\dot{H}^s} \lesssim n^{s+1+\alpha}||u||_{\dot{H}^0}||v||_{\dot{H}^{-\alpha}}||w||_{\dot{H}^s}.$$

(2) For any $-7/4 < s \leq 0$,

$$||B_2(u,v)||_{\dot{H}^s} \lesssim ||u||_{\dot{H}^0}||v||_{\dot{H}^s}$$

(3) For any $s \geq -1$, *there exists* $\alpha > 0$ *such that*

$$||B^{(n)}_{30}(v) - B^{(n)}_{30}(w)||_{\dot{H}^s} \lesssim n^{-\alpha}(||v||_{\dot{H}^0} + ||w||_{\dot{H}^0})^2||v-w||_{\dot{H}^s}$$

(4) For any $-3/2 < s < 1/2$,

$$||B^{(n)}_{40}(v) - B^{(n)}_{40}(w)||_{\dot{H}^s} \lesssim (||v||_{\dot{H}^0} + ||w||_{\dot{H}^0})^3||v-w||_{\dot{H}^s}$$

References

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