Lyapunov stability of ground states of nonlinear dispersive evolutions equations, (Michael I. Weinstein, CPAM 1986)

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Nonlinear Schrödinger equation

We consider the initial value problem:

$$i\partial_t\phi + \Delta\phi + f(|\phi|^2)\phi = 0, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}_+, \tag{0.1}$$

subject to $\phi(x,0) = \phi_0(x) \in H^1$.

We shall study the stability of solitary waves of (0.1).

- A solitary wave is a localized finite energy solution
- Resulting from a balance of dispersion and "focusing" effect by nonlinearity
- The profile keeps its max. as $t \to \infty$ and decays as $|x| \to \infty$

In this work, we restrict to the following special cases:

- Nonlinearity $f(|\phi|^2) = |\phi|^{2\sigma}$, i.e., $i\partial_t\phi = -\Delta\phi - |\phi|^{2\sigma}\phi$
- Subcritical case, i.e., $\sigma < 2/N$
- $N = 1$ or $N = 3$ (as of 1986)

Ground states and solitary waves

We seek for standing wave solutions of the form, $\psi(x,t) = R(x;E)e^{iEt}$:

$$i\psi_t + \Delta\psi + f(|\psi|^2)\psi = \left(\Delta R - ER + f(R^2)R\right) e^{iEt} = 0$$

To find such a solution, we shall consider the elliptic problem:

$$\Delta u - Eu + f(|u|^2)u = 0 \text{ in } \mathbb{R}^N.$$

$\exists$ a positive, radial, smooth and exp. decaying solution, denoted by R.

- R is called the ground state. (the least energy solution)
- For such R, $\exists$ a one-parameter family of solutions: $\psi(x,t) = R(x;E)e^{iEt}$.
- Note that the scaling invariance: $R(x;E) = E^{1/2\sigma} R(\sqrt{E}x; 1)$
- For any $\lambda, \gamma \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$,

$$\psi_\lambda(x,t) = \lambda^{1/\sigma} e^{i\lambda^2 t} R(\lambda(x + x_0))e^{i\gamma} \text{ is a ground state of NLS}$$

Scale transformation

Consider the power-type nonlinearity, i.e., $f(s^2) = |s|^{2\sigma}$

$$i\phi_t + \Delta\phi + |\phi|^{2\sigma}\phi = 0$$

and the associated elliptic equation:

$$\Delta u - Eu + |u|^{2\sigma}u = 0$$

- Recall that the scaling invariance: $R(x; E) = E^{1/2\sigma}R(\sqrt{E}x; 1)$
- For any $\lambda, \gamma \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$,

$$\psi_\lambda(x,t) = \lambda^{1/\sigma}e^{i\lambda^2 t}R(\lambda(x+x_0))e^{i\gamma} \text{ is a ground state of NLS}$$

- In general,

$$\phi_\lambda(x,t) = \lambda^{1/\sigma}\phi(\lambda x, \lambda^2 t)$$

Notion of stability

- NLS has the phase and translation symmetries:

$$\phi(x,t) \Rightarrow e^{i\gamma}\phi(x+x_0,t) \text{ for } (x_0,\gamma) \in \mathbb{R}^N \times [0,2\pi)$$

- Orbital stability: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\|\psi_0 - \phi_0\|_X < \delta \Rightarrow \|\psi(\cdot + x_0)e^{i\gamma} - \phi\|_X < \varepsilon \text{ for some } (x_0,\gamma) \in \mathbb{R}^N \times [0,2\pi)$$

- Define the orbit of a function ψ by

$$\mathcal{G}_\psi := \{\psi(\cdot + x_0)e^{i\gamma} : (x_0,\gamma) \in \mathbb{R}^N \times [0,2\pi)\}$$

- A metric measuring the deviation of ϕ from the orbit $\mathcal{G}_\psi \sim H^1$ norm:

$$[\rho_E(\phi,\mathcal{G}_\psi)]^2 := \inf_{(x_0,\gamma)\in\mathbb{R}^N\times[0,2\pi)} \{\|\nabla\phi(\cdot+x_0,t)e^{i\gamma} - \nabla\psi\|_{L^2}^2 + E\|\phi(\cdot+x_0,t)e^{i\gamma} - \psi\|_{L^2}^2\}$$

Observations: conserved quantities

- Let the Hamiltonian and the square integral be

$$\mathcal{H}[\phi] := \frac{1}{2}\int |\nabla\phi|^2 - G(|\phi|^2)dx \quad \text{and} \quad \mathcal{N}[\phi] := \frac{1}{2}\int |\phi|^2 dx.$$

Then $\mathcal{H}[\phi(t)] = \mathcal{H}[\phi_0]$ and $\mathcal{L}[\phi(t)] = \mathcal{L}[\phi_0]$ for all $t > 0$.

- A Lyapunov functional $\mathcal{E}[\phi]$ is defined by

$$\mathcal{E}[\phi] = \mathcal{H}[\phi] + E\mathcal{N}[\phi] = \frac{1}{2}\int |\nabla\phi|^2 - G(|\phi|^2) + E|\phi|^2 dx$$

Then $\mathcal{E}[\phi]$ is conserved in time, i.e., $\mathcal{E}[\phi(t)] = \mathcal{E}[\phi_0]$ for all $t > 0$.

- Plugging $\phi(x,t) = u(x)e^{iEt}$ into $\mathcal{E}[\phi]$, we obtain a functional

$$I(u) := \mathcal{E}[u(x)e^{iEt}] = \frac{1}{2}\int |\nabla u|^2 - G(|u|^2) + E|u|^2 dx$$

Existence of the ground states of NLS (Strauss, Berestycki and Lions)

A ground state solution is the least energy solution of the form of $\phi(x,t) = u(x)e^{iEt}$, of which u is a solution to $\Delta u - Eu + f(|u|^2)u = 0$ and, among such solutions, it attains a minimum of the functional:

$$I(u) = \frac{1}{2}\int |\nabla u|^2 - G(|u|^2) + E|u|^2 dx, \quad u \in H^1$$

For the existence of such a solution, we consider the semilinear problem:

$$\Delta u - Eu + g(u) = 0 \text{ in } \mathbb{R}^N$$

Under the conditions (B-L):

- (i) $g(s)/s \to 0$ as $s \to 0$, (superlinear near $s = 0$)
- (ii) $\limsup_{s\to\infty} \frac{|g(s)|}{|s|^p} \le c$ for some $1 < p < \frac{N+2}{N-2}$, i.e., subcritical at ∞
- (iii) $\exists \tau > 0$ s.t. $\int_0^\tau g(t)dt > E\tau^2/2$

Then there is a positive, radial, smooth and exp. decaying ground state.

In our case, let $g(s) = |s|^{2\sigma}s$, i.e., the eq. is $\Delta u - Eu + |u|^{2\sigma}u = 0$.

Variational characterization of the ground state

- For $0 < \sigma < \frac{2}{N-2}$, we consider the minimization problem:

$$J^{\sigma,N}[u] := \frac{\|\nabla u\|^{\sigma N}\|u\|^{2+\sigma(2-N)}}{\|u\|_{2\sigma+2}^{2\sigma+2}} \quad \text{for } u \in H^1$$

Proposition

$\alpha := \inf_{u\in H^1} J^{\sigma,N}[u]$ *is attained at a function R with the properties:*

- $R > 0$ *and radial symmetric,*
- $R \in H^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^N)$,
- $\Delta R - R + R^{2\sigma+1} = 0.$

- Remark: Once existence of the minimizer R is proven, then E-L eq. $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} J^{\sigma,N}[R+\varepsilon\eta] = 0 \;\; \forall \eta \in C_c^\infty$ gives the equations for R

Variational characterization

- This is amount to the best constant problem: Let $\sigma \in (0, \frac{2}{N-2})$. For $f \in H^1$,

$$\|f\|_{2\sigma+2} \leq C_{\sigma,N} \|\nabla f\|_2^{\frac{\sigma N}{2\sigma+2}} \|f\|_2^{1-\frac{\sigma N}{2\sigma+2}}.$$

This is related to the Gagliardo-Nirenberg interpolation estimate:

$$\|f\|_p \leq c_{p,N} \|\nabla f\|_2^{\theta} \|f\|_2^{1-\theta} \quad \text{with } \theta = N(\frac{1}{2} - \frac{1}{p}), \quad 2 < p < \frac{2N}{N-2}$$

- A constrained minimization problem:

$$\inf_{u \in H^1} I(u) = \inf_{u \in H^1} \frac{1}{2} \int |\nabla u|^2 + u^2 dx \quad \text{under } J(u) = \int |u|^{2\sigma+2} dx = C$$

- Key to the existence proof:

Symmetrization: $u(x) = u(|x|)$ with the same L^2 and reduced gradient norm
The embedding $H^1_{\text{rad}}(\mathbb{R}^N) \to L^{2\sigma+2}(\mathbb{R}^N)$ is compact for $\sigma \in (0, \frac{2}{N-2})$

Stability Theorem

Recall our problem:

$$i\partial_t\phi + \Delta\phi + |\phi|^{2\sigma}\phi = 0, \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{0.2a}$$

$$\phi(x,0) = \phi_0(x) \in H^1. \tag{0.2b}$$

Theorem (Orbital Stability)

- *$\sigma < 2/N$ with $N = 1$ or $N = 3$ (as of 1986 CPAM, M. Weinstein)*
- *$\phi \in C\left([0,\infty); H^1\right)$ is a unique solution with initial data $\phi_0 \in H^1$*

Then the ground state R is orbitally stable, i.e., $\forall\varepsilon > 0$, $\exists\delta > 0$:

$$\rho_E(\phi_0, \mathcal{G}_R) < \delta \quad \Rightarrow \quad \rho_E(\phi(t), \mathcal{G}_R) < \varepsilon, \quad \forall t > 0.$$

- Remarks (Global-in-time H^1 solution)
  - (0.2) has a unique global solution in $C\left([0,\infty); H^1\right)$ (e.g. Ginibre and Velo)

Outline of the proof for the stability theorem

- Define the perturbation variable:

$$w(x,t) := \phi(x+x_0,t)e^{i\gamma} - R(x) \text{ and } w(x,t) := u + iv.$$

- Lyapunov functional differentials:

$$\begin{aligned}\triangle\mathcal{E} &:= \mathcal{E}[\phi_0(\cdot)] - \mathcal{E}[R(\cdot)] = \mathcal{E}[\phi(\cdot,t)] - \mathcal{E}[R(\cdot)] = \mathcal{E}[\phi(\cdot + x_0, t)e^{i\gamma}] - \mathcal{E}[R(\cdot)] \\ &= \mathcal{E}[R+w] - \mathcal{E}[R] \\ &\geq (L_+u,u) + (L_-v,v) - C_1\|w\|_{H^1}^{2+\theta} - C_2\|w\|_{H^1}^6 \text{ with } \theta > 0,\end{aligned} \tag{0.3}$$

where $L_+ = -\Delta + 1 - \big(f(R^2) + 2R^2 f'(R^2)\big) = -\Delta + 1 - (2\sigma+1)R^{2\sigma}$,

$$L_- = -\Delta + 1 - f(R^2) = -\Delta + 1 - R^{2\sigma}$$

are the real part and imaginary part, reps., of the NLS linearized operator about the ground state R.

Outline of the proof for the stability theorem

- For x_0, γ chosen for ρ_E metric, we have

$$(L_+ u, u) + (L_- v, v) \geq C_3 \|w\|_{H^1}^2 - C_4 \|w\|_{H^1}^3 - C_5 \|w\|_{H^1}^4. \tag{0.4}$$

- This together with (0.3) yields

$$\triangle \mathcal{E} \geq g(\|w\|_{H^1}) \geq g\left(\rho_E\left(\phi(t), \mathcal{G}_R\right)\right),$$

where $g(t) = ct^2(1 - at^\theta - bt^4), \quad a, b, c, \theta > 0$.

- Properties of g: $g(0) = 0$ and $g(t) > 0$ for $0 < t \ll 1$.
- By the continuity of $\mathcal{E}$ in H^1 near R, we have

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \rho_E(\phi_0, \mathcal{G}_R) < \delta \Rightarrow \triangle \mathcal{E}_0 < g(\varepsilon).$$

- This implies that $g\left(\rho_E\left(\phi(t), \mathcal{G}_R\right)\right) < g(\varepsilon)$, in turn, $\rho_E(\phi(t), \mathcal{G}_R) < \varepsilon \quad \forall t > 0$.

Key Lemmas to be proven

- For the perturbations satisfying $\|\phi\| = \|R\|$, and $x_0 = x_0(t)$ and $\gamma = \gamma(t)$ as in the definition of ρ_E,

$$(L_+u, u) + (L_-v, v) \geq C_3\|w\|_{H^1}^2 - C_4\|w\|_{H^1}^3 - C_5\|w\|_{H^1}^4. \qquad (0.5)$$

Step 1. $(L_-v, v) \geq C''\|v\|_{H^1}^2$

- $\ker L_- = \text{ span } \{R\}$

Step 2. $(L_+u, u) \geq D\|u\|_{H^1}^2 - D'\|\nabla w\|\|w\|^2 - D''\|w\|^4$

(a) $(u, R) = -\frac{1}{2}[(u, u) + (v, v)]$ for $\|\phi\| = \|R\|$

(b) $\inf\limits_{(f,R)=0}(L_+f, f) = 0$ if $\sigma \leq 2/N$

- $\ker L_+ = \text{Span}\{R_{x_j} : j = 1, \cdots, N\}$

Stability with respect to general perturbations

For general perturbations: if $\rho_E(\phi_0, \mathcal{G}_R) < \delta(\varepsilon)$, there is a λ:

- (i) $\psi_\lambda(x,t) = \lambda^{1/\sigma} R(\lambda x) e^{i\lambda^2 t}$ is a ground state of NLS
- (ii) $\|\psi_\lambda\| = \|\phi_0\|$
- (iii) $\|\psi_\lambda - R\|_{H^1} < \frac{\varepsilon}{2}$

Note that

1. $\|\psi_\lambda\|_{L^2} = \lambda^{1/\sigma - N/2} \|R\|_{L^2}$.
2. $h(\lambda) := \|\psi_\lambda - R\|_{H^1}$ is continuous in $\lambda \in (1-\delta, 1+\delta)$.

Therefore, if $\sigma < 2/N$, one can change $\|\psi_\lambda\|_{L^2}$ continuously so that

$$\|\psi_{\lambda_1}\|_{L^2} = \|\phi_0\|_{L^2} \text{ and } \|\psi_{\lambda_1} - R\|_{H^1} < \varepsilon/2.$$

This together with the previous result yields the stability for the general perturbations.

The linearized NLS operator

- Recall the perturbation

$$w(x,t) := \phi(x+x_0,t)e^{i\gamma} - R(x) \text{ and } w(x,t) := u + iv.$$

- By a standard Taylor expansion method, we have

$$\mathcal{E}[R+w] - \mathcal{E}[R]$$
$$= \frac{1}{2}\int |\nabla R + \nabla w|^2 - |\nabla R|^2 - \Big(G(|R+w|^2) - G(R^2)\Big) + |R+w|^2 - R^2 dx$$
$$\geq (L_+u,u) + (L_-v,v) - C_1\|w\|_{H^1}^{2+\theta} - C_2\|w\|_{H^1}^6 \text{ with } \theta > 0,$$

where

$$L_+ = -\Delta + 1 - \Big(f(R^2) + 2R^2 f'(R^2)\Big) \quad \text{and} \quad L_- = -\Delta + 1 - f(R^2)$$

(e.g.)
$$G(|R+w|^2) - G(R^2) = f(R^2)\left(2uR + |w|^2\right) + \tfrac{1}{2}f'(R^2)\left(2uR + |w|^2\right)^2 + \mathcal{O}(|w|^3)$$

Lemma

Under the condition that $\int |\phi(x,t)|^2 dx = \int R^2(x)dx$, *we have*

$$(u, R) = -\frac{1}{2}[(u,u) + (v,v)].$$

Proof.

Remind that

$$u + iv = w(x,t) = \phi(x + x_0, t)e^{i\gamma} - R(x)$$

$\int |\phi(x,t)|^2 dx = \int R^2(x)dx$ and $\|\phi(\cdot)\| = \|\phi(\cdot - x_0)\|$ lead to

$$2\int uRdx = -\int u^2 + v^2 dx.$$

□

Lemma

A minimization of

$$\inf_{(x_0,\gamma)\in\mathbb{R}^N\times[0,2\pi)}\{\|\nabla\phi(\cdot+x_0,t)e^{i\gamma}-\nabla R(\cdot)\|_{L^2}^2+E\|\phi(\cdot+x_0,t)e^{i\gamma}-R(\cdot)\|_{L^2}^2\}$$

over the choice of $x_0 = x_0(t)$ *and* $\gamma = \gamma(t)$ *implies that*

$$\int R^{2\sigma}(x)R_{x_j}(x)u(x,t)dx = 0 \quad \text{for } j = 1, \cdots N$$

and

$$\int R^{2\sigma+1}(x)v(x,t)dx = 0.$$

Proof.

Let $F(x_0,\gamma;t) := \|\nabla\phi(\cdot+x_0,t)e^{i\gamma}-\nabla R(\cdot)\|_{L^2}^2+E\|\phi(\cdot+x_0,t)e^{i\gamma}-R(\cdot)\|_{L^2}^2$.
A straightforward computation of $D_{x_0}F = 0$ and $\partial_\gamma F = 0$ yields the result. □

rmk. $F(x_0,\gamma;t)$ is continuous in x_0,γ and $\lim_{x_0\to\pm\infty} F$ exists

Lemma

For $v \in H^1$ *satisfying* $\int R^{2\sigma+1}(x)v(x,t)dx = 0$, *there exists* $c'' > 0$ *such*

$$(L_- v, v) \geq c'' \|v\|_{H^1}^2 \quad \text{for some } c'' > 0.$$

Proof.

- Note that $L_- R = -\Delta R + R - R^{2\sigma+1} = 0$
- $R > 0$ is the ground state and nondegenerate (unique due to Kwong)
- This implies that L_- is nonnegative, i.e., $(L_- g, g) \geq 0$ for $g \in H^1$
- Consider the minimization problem: $\inf_{(R^{2\sigma+1},v)=0} \dfrac{(L_- v, v)}{(v,v)} = c' > 0$

  if not, i.e., $\inf \frac{(L_- v,v)}{(v,v)} = 0$ then it attains at R. This contradicts to $R > 0$
- $(L_- v, v) \geq c'' \|v\|_{H^1}^2$ for some $c'' > 0$

□

Lemma

Let $\sigma \leq 2/N$. Then

$$\inf_{(f,R)=0} (L_+ f, f) = 0$$

Proof.

- $L_+ R_{x_j} = 0$ since $L_- R = 0$ and $\partial_j (L_- R) = L_+ R_{x_j}$
- $(R_{x_j}, R) = 0$. Thus $\inf_{(f,R)=0} (L_+ f, f) \leq 0$
- Since $J^{\sigma,N}$ attains its min. at R, the second variation $(\delta^2 J^{\sigma,N})(R) \geq 0$
- This leads to

$$(L_+ f, f) \geq c(2 - \sigma N)(\Delta R, f)^2 \quad \text{for some } c > 0$$

- $\sigma \leq 2/N$ implies that $(L_+ f, f) \geq 0$, thus the result is obtained.

□

Non-degeneracy and Uniqueness of the ground state

Lemma

Let $N = 1$ or $N = 3$. Then

$$\ker(L_+) = Span \left\{ \frac{\partial R}{\partial x_j} : j = 1, \cdots, N \right\}$$

• Remark: This is the only place the restriction on the dimensions $N = 1$ and $N = 3$ is imposed. Conjecture: True for all $\sigma \in (0, 2/N - 2)$

Lemma

L_- is a nonnegative self-adjoint in L^2 with

$$\ker(L_-) = \{R\}.$$

Thank you for your attention.