Transfer of energy frequencies in the cubic defocusing nonlinear Schrödinger equation I

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Contents

Equation and Conservation laws

- The periodic defocusing cubic nonlinear Schrödinger equation(NLS)

$$\begin{cases} -i\partial_t u + \Delta u = |u|^2 u, \\ u(0,x) := u_0(x), \end{cases} \tag{1}$$

  where $u(t,x)$ is a complex-valued function and $x \in \mathbb{T}^2$ and the initial data is smooth for convenience.
- Hamiltonian (Energy conservation laws)

$$E[u](t) := \int_{\mathbb{T}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4 \, dx(t) = E[u](0).$$

- Mass conservation laws

$$M[u](t) := \int_{\mathbb{T}^2} |u|^2 \, dx(t) = M[u](0).$$

- Local well-posedness for $s > 0$ (Bourgain, 1993).
- By conservation laws, we can get global smooth solution to (1) from smooth initial data.
- Later, we introduce the toy model that is the completely integrable.

We construct a solution to (1) that energy of move on to higher Fourier modes. In other words, we construct a solution to (1) with arbitrarily large growth in higher Sobolev norms.

Theorem 1.1 (Main theorem)

Let $s > 1$, $K \gg 1$, and $0 < \delta \ll 1$ be given parameters. Then there exists a global smooth solution $u(t,x)$ to (1) and a time $T > 0$ with

$$\|u(0)\|_{H^s} \leq \delta$$

and

$$\|u(T)\|_{H^s} \geq K$$

Corollary 1.1 (H^2 instability of zero solution)

The global-in-time solution map taking the initial data u_0 to the associated solution u of (1) is strongly unstable in H^s near zero for all $s > 1$:

$$\inf_{\delta > 0}\left(\limsup_{|t|\to\infty}\left[\sup_{\|u_0\|_{H^2}\leq\delta}\|u(t)\|_{H^s}\right]\right) = 0$$

Previous result

- High Sobolev norms of solution can grow no faster than exponential-in-time. (Bourgain, 1993)
- Sobolev norms grow no faster that polynomial-in-time upper bound. (Bourgain, 2004, Collinder et al., 2001, Salem et al., 1999)
- Small dispersion NLS

$$i\partial_t \omega + \delta \Delta \omega = |\omega|^2 \omega \tag{2}$$

Smooth norms of solution of (2) evolving from relatively generic data with unit L^2 norm eventually grow larger than a negative power of δ. (Kuksin, 1997)

Outline

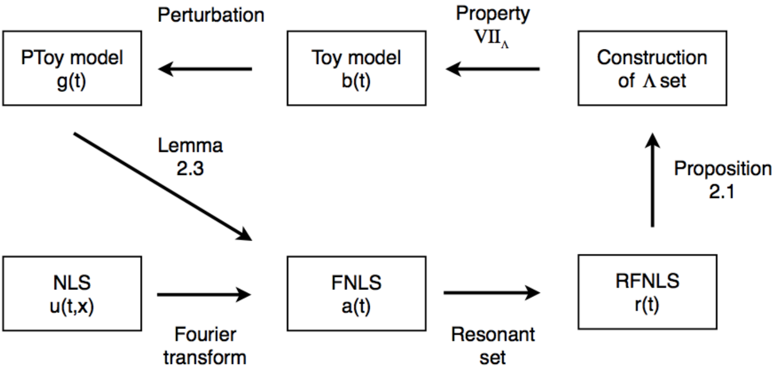

Figure : Outline

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Reductions of Equation - $\mathcal{F}$NLS

Equation (1) has gauge freedom. Hence we can let

$$v(t,x) = e^{iGt} u(t,x), \qquad G \in \mathbb{R} \tag{3}$$

then NLS equation (1) is the following equation for v

$$(-\partial_t + \Delta) v = \left(G + |v|^2\right) v \tag{4}$$

with the same initial data. We write a solution of (2) as following

$$v(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i\left(n \cdot x + |n|^2 t\right)}. \tag{5}$$

Substituting (5) into (4) and comparing both sides gives the following infinite system of equations for $a_n(t)$,

$$-i\partial_t a_n = G a_n + \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a_{n_2}} a_{n_3} e^{i\omega_4 t} \tag{6}$$

where

$$\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 \tag{7}$$

For the removal of $G a_n$ term with appropriate choosing the gauge parameter G, we split the sum on the right hand side of (6) into the following terms,

$$\sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} = \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n_1, n_3 \neq n}} + \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n_1 = n}} + \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n_3 = n}} - \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n \\ n_1 = n_3 = n}}$$

$$:= \text{Terms I } + \text{Terms II } + \text{Terms III } + \text{Terms IV}$$

Since we handle the system as term by term, Term IV is $-a_n(t)|a_n(t)|^2$. Term II and Term III are single sums which by Plancherel's Theorem and mass conservation total,

$$\begin{aligned} 2a_n(t) \cdot \sum_{m \in \mathbb{Z}} |a_m(t)|^2 &= 2a_n(t) \cdot \|u(t)\|^2_{L^2(\mathbb{T}^2)} \\ &= 2a_n(t)^2 M^2, \end{aligned}$$

where $M := \|u(t)\|^2_{L^2(\mathbb{T}^2)}$.
We can remove the first term of (5) by choosing $G = -2M$. Equation (4) takes then the following useful form which we denote $\mathcal{F}$NSL,

$$-i\partial_t a_n = -a_n |a_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \overline{a_{n_2}} a_{n_3} e^{i\omega_4 t} \tag{8}$$

where

$$\Gamma(n) = \left\{ (n_1, n_2, n_3) \in \left(\mathbb{Z}^2\right)^3 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n \right\}. \tag{9}$$

Well-posedness - $\mathcal{F}$NLS

Let

$$\left(\mathcal{N}\left(t\right)\left(a,b,c\right)\right)_n = -a_n\overline{b_n}c_n + \sum_{n_1,n_2,n_3\in\Gamma(n)} a_{n_1}\overline{b_{n_2}}c_{n_3}e^{i\omega_4 t} \quad (10)$$

With this notation, we can reexpress $\mathcal{F}$NLS as
$-i\partial_t a_n = \left(\mathcal{N}\left(t\right)\left(a,a,a\right)\right)_n$

Lemma 2.1

$$\left\|\left(\mathcal{N}\left(t\right)\left(a,b,c\right)\right)_n\right\|_{l^1(\mathbb{Z}^2)} \lesssim \|a\|_{l^1(\mathbb{Z}^2)}\|b\|_{l^1(\mathbb{Z}^2)}\|c\|_{l^1(\mathbb{Z}^2)} \quad (11)$$

Resonant truncation - $R\mathcal{F}$NLS

Define the set of all resonant non-self interactions $\Gamma_{\text{res}}(n) \subset \Gamma(n)$ by

$$\Gamma_{\text{res}}(n) = \left\{ (n_1, n_2, n_3) \in \Gamma(n) : \omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0 \right\} \tag{12}$$

Note that $(n_1, n_2, n_3) \in \Gamma_{\text{res}}$ precisely when (n_1, n_2, n_3, n) form four corners of a non degenerate rectangle with n_2 and n opposing each other, and similarly for n_1 and n_3.
In resonant set, $\mathcal{F}NLS$ is not oscillates in time anymore. Hence we can simply define the *resonant truncation* R$\mathcal{F}$NLS of $\mathcal{F}$NLS by,

$$-i\partial_t r_n = -r_n |r_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n)} r_{n_1} \overline{r_{n_2}} r_{n_3}. \tag{13}$$

Finite truncation - The frequency set Λ

We introduce some notations and terminologies on set Λ before finite truncation of R$\mathcal{F}$NLS.

- For some positive integer N, the set Λ splits into N disjoint generations $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_N$

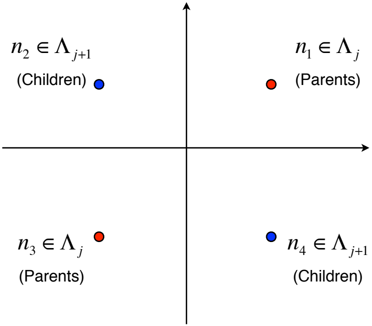

Figure : Nuclear family

Finite truncation - The frequency set Λ

We require following properties on set Λ.

- Property I_Λ(Initial data): $r_n(0) = 0$ whenever $n \notin \Lambda$.
- Property II_Λ(Closure):

$$(n_1, n_2, n_3) \in \Gamma_{\text{res}}(n), n_1, n_2, n_3 \in \Gamma \Rightarrow n \in \Gamma \tag{14}$$

Lemma 2.2

If Λ is a finite set satisfying Property I_Λ, Property II_Λ, and $r(0) \mapsto r(t)$ solves R$\mathcal{F}$NLS (8) on $[0, T]$ then for all $t \in [0, T]$, $\text{spt}[r(t)] \subset \Lambda$.

- Property III_Λ(Existence and uniqueness of spouse and children): $1 \le \forall j \le N$ and $n_1 \in \Lambda_j$ $\exists!$ nuclear family (n_1, n_2, n_3, n_4) such that n_1 is a parent. In particular each $n_1 \in \Lambda_j$ has a unique spouse $n_3 \in \Lambda_j$ and two unique children $n_2, n_4 \in \Lambda_{j+1}$.
- Property IV_Λ(Existence and uniqueness of sibling and parents): $1 \le \forall j \le N$ and $n_2 \in \Lambda_{j+1}$ $\exists!$ nuclear family (n_1, n_2, n_3, n_4) such that n_2 is a parent. In particular each $n_2 \in \Lambda_{j+1}$ has a unique sibling $n_4 \in \Lambda_{j+1}$ and two unique parents $n_1, n_3 \in \Lambda_j$.
- Property V_Λ(Non-degeneracy): The sibling of a frequency n is never equal to its spouse.
- Property VI_Λ(Faithful): Apart from the nuclear families, Λ contains no other rectangles.

In assumption of existence of Λ,

$$-i\partial_t r_n(t) = -|r_n(t)|^2 r_n(t) + 2r_{n_{\text{child-1}}}(t)\, r_{n_{\text{child-2}}}(t)\, \overline{r_{n_{\text{spouse}}}(t)} + 2r_{n_{\text{parent-1}}}(t)\, r_{n_{\text{parent-2}}}(t)\, \overline{r_{n_{\text{sibling}}}(t)} \tag{15}$$

where for each $n \in \Lambda_j$, $n_{\text{spouse}} \in \Lambda_j$ is its spouse, $r_{n_{\text{child-1}}}, r_{n_{\text{child-2}}} \in \Lambda_{j+1}$ are its two children, $n_{\text{sibling}} \in \Lambda_j$ is its sibling, and $r_{n_{\text{parent-1}}}, r_{n_{\text{parent-2}}} \in \Lambda_{j-1}$ are its parents.
For more simplify ODE, we introduce a condition to Λ.

- Property VI_Λ(Intragenerational equality): The function $n \mapsto r_n(0)$ is constant on each generation Λ_j. Thus $1 \le j \le N$ and $n, n' \in \Lambda_j$ imply $r_n(0) = r_{n'}(0)$.

By Gronwall argument, if one has intragenerational equality at time 0 then one has intragenerational equality at all later times.

Finite truncation - Toy model system

By Property VI, we may collapse the function $n \mapsto r_n(t)$, which is currently a function on $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$, to the function $j \mapsto b_j(t)$ on $\{1, 2, \ldots, N\}$, where $b_j(t) := r_n(t)$ whenever $n \in \Lambda_j$. Hence the ODE (15) collapse to the following system that we call *Toy Model System*.

$$-i\partial_t b_j(t) = -|b_j(t)|^2 b_j(t) + 2b_{j-1}(t)^2 \overline{b_j(t)} + 2b_{j+1}(t)^2 \overline{b_j(t)}, \tag{16}$$

with the convention that $b_0(t) = b_{N+1}(t) = 0$.

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First ingredient: the frequency set Λ

Proposition 2.1 (First ingredient: the frequency set Λ)

Given parameters $\delta \ll 1$, $K \gg 1$, we can find an $N \gg 1$ and a set of frequencies $\Lambda \subset \mathbb{Z}^2$ with,

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N \qquad \text{disjoint union}$$

which satisfies Property II_Λ - Property VI_Λ and also,

$$\frac{\sum_{n \in \Lambda_N} |n|^{2s}}{\sum_{n \in \Lambda_1} |n|^{2s}} \gtrsim \frac{K^2}{\delta^2}. \tag{17}$$

In addition, given any $\mathcal{R} \gg C(K, \delta)$, we can ensure that Λ consists of $N \cdot 2^{N-1}$ disjoint frequencies n satisfying $|n| \geq \mathcal{R}$.

Second ingredient: instability in the toy model

Proposition 2.2 (Second ingredient: instability in the toy model)

Given $N > 1$, $\varepsilon \ll 1$, there is initial data $b(0) = (b_1(0), b_2(0), \ldots, b_N(0)) \in \mathbb{C}^N$ for (16) and there is a time $T = T(N, \varepsilon)$ so that

$$\begin{aligned} |b_3(0)| &\geq 1 - \varepsilon, \qquad |b_j(0)| \leq \varepsilon, \qquad j \neq 3 \\ |b_{N-2}(T)| &\geq 1 - \varepsilon, \qquad |b_j(T)| \leq \varepsilon, \qquad j \neq N-2. \end{aligned} \tag{18}$$

In addition, the corresponding solution satisfies $\|b(t)\|_{l^\infty} \sim 1$ for all $0 \leq t \leq T$.

Theorem 3.1 (Instability in the toy model)

Let $N \leq 6$. Given any $\varepsilon > 0$, there exists a point x_3 within ε of $\mathbb{T}_3$ (using the usual metric on Σ), a point x_{N-2} within ε of $\mathbb{T}_{N-2}$, and a time $t \leq 0$ such that $S(t)x_3 = x_{N-2}$.

Third ingredient: the approximation lemma

Let $0 < \sigma < 1$ be an absolute constant (all implicit constants in this subsection may depend on σ). Let $B \gg 1$, and let $T \ll B^2 \log B$. Let

$$g(t) := \{g_n(t)\}_{n \in \mathbb{Z}^2}$$

be a solution to the perturbed equation of

$$-i\partial_t g(t) = (\mathcal{N}(g(t), g(t), g(t))) + \mathcal{E}(t) \tag{19}$$

for times $0 \leq t \leq T$, where $\mathcal{N}(t)$ is defined in (10), (7) and where the initial data $g(0)$ is compactly supported.

Lemma 2.3 (Third ingredient: the approximation lemma)

Assume that the solution of above equation $g(t)$ and the error term $\mathcal{E}(t)$ obey the bounds of the form

$$\|g(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1} \tag{20}$$

$$\left\| \int_0^t \mathcal{E}(t)\, ds \right\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\sigma} \tag{21}$$

for all $0 \leq t \leq T$.

We conclude that if $a(t)$ denotes the solution to $\mathcal{F}$NLS (8) with initial data $a(0) = g(0)$, then we have

$$\|a(t) - g(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\sigma} \tag{22}$$

for all $0 \leq t \leq T$.

Proof of Lemma 2.3

Write

$$F(t) := -i\int_0^t \mathcal{E}(s)\,ds, \quad \text{and} \quad d(t) := g(t) + F(t).$$

Observe that

$$\begin{aligned} -id_t = -ig_t - iF_t &= (\mathcal{N}(t)(g(t), g(t), g(t))) + \mathcal{E}(t) - \mathcal{E}(t) \\ &= \mathcal{N}(t)(d-F, d-F, d-F) \end{aligned} \tag{23}$$

where we have suppressed the explicit t dependence for brevity.

Proof of Lemma 2.3 - Continued

By hypothesis, $d(t) = O_{l^1}\left(B^{-1}\right)$. By trilinearity and Lemma 2.1,

$$\begin{aligned} -id_t &= \mathcal{N}(d, d-F, d-F) - \mathcal{N}(F, d-F, d-F) \\ &= \mathcal{N}(d,d,d) + O\left(\|F\|_{l^1}\right) O\left(\|g\|_{l^1}\right)^2 \\ &= \mathcal{N}(d,d,d) + O_{l^1}\left(B^{-3-\sigma}\right). \end{aligned} \tag{24}$$

Let $e(t)$ is a smooth error function. Now write $a := d + e$. Since $a(t)$ is the solution to $\mathcal{F}$NLS,

$$\begin{aligned} -i(d+e)_t &= -id_t - ia_t + id_t \\ &= -ia_t = \mathcal{N}(a,a,a) = \mathcal{N}(d+e, d+e, d+e). \end{aligned} \tag{25}$$

Now, we calculate (24) − (25) and use trilinearity and Lemma 2.1 again,

$$ie_t = O_{l^1}\left(B^{-3-\sigma}\right) + O_{l^1}\left(B^{-2}\|e\|_{l^1}\right) + O_{l^1}\left(\|e\|_{l^1}^3\right). \tag{26}$$

Hence by the differential form of Minkowski's inequality, we have

$$\partial_t \|e\|_{l^1} \lesssim B^{-3-\sigma} + B^{-2} \|e\|_{l^1} + \|e\|_{l^1}^3 .$$

To finish the proof we use a bootstrap argument.
If $\|e\|_{l^1} = O\left(B^{-1}\right)$ for all $t \in [0, T]$ then

$$\partial_t \|e\|_{l^1} \leq CB^{-2} \|e\|_{l^1} .$$

We have the following inequality by using Gronwall's inequality and (21)

$$\|e\|_{l^1} \leq B^{-1-\sigma} \exp\left(CB^{-2}t\right)$$

for all $t \in [0, T]$. Since we have $T \ll B^2 \log B$, we thus have $\|e\|_{l^1} \ll B^{-1-\sigma/2}$, and so we can remove the bootstrap assumption.

Proof of Theorem 1.1

Notation for understanding of proof

- $a(t)$: The solution of $\mathcal{F}$NLS.
- $b(t)$: The solution of the toy model.
- $b^{(\lambda)}(t)$: The scaled solution of the toy model.
- $g(t)$: The solution of the perturbed toy model.

From now, we prove Theorem 1.1.
Given δ, K, construct Λ as in Proposition 2.1. and so we can construct a traveling wave solution $b(t)$ to the toy model concentrated at scale ε according to Proposition 2.2 above.

This proposition also gives us a time $T_0 = T_0(k, \delta)$ at which the wave has the traversed the N generations of frequencies.
For choosing sufficiently large time, we scale the toy model,

$$b^{(\lambda)} := \lambda^{-1} b\left(\frac{t}{\lambda^2}\right).$$

First of all, the aim is to apply Lemma 2.3 with
$g(t) = \{g_n(t)\}_{n \in \mathbb{Z}^2}$ defined by,

$$g_n(t) = b_j^{(\lambda)}(t),$$

for $n \in \Lambda_j$, and $g_n(t) = 0$ when $n \neq \Lambda$. Hence we set

$$\mathcal{E}(t) := -\sum_{[\Gamma(n)\backslash\Gamma_{\mathrm{res}}]\cap\Lambda_3} g_{n_1}\overline{g_{n_2}}g_{n_3}e^{i\omega_4 t} \tag{27}$$

where ω_4 is as in (7).

- 1st condition

By considering its support, the fact that $|\lambda| = C(N)$, and the fact that $\|b(t)\|_{l^\infty} \sim 1$, we can be sure that, $\|b(t)\|_{l^1(\mathbb{Z})} \sim C(N)$ and therefore

$$\left\|b^{(\lambda)}(t)\right\|_{l^1(\mathbb{Z})}, \|g(t)\|_{l^1(\mathbb{Z})} \leq \lambda^{-1} C(N) \tag{28}$$

Thus, (20) holds with the choice $B = C(N)\lambda$.
For large enough λ, we choose B large enough so that

$$B^2 \log B \gg \lambda^2 T_0.$$

- 2nd condition

Claim)

$$\left\| \int_0^t \mathcal{E}(s)\, ds \right\|_{l^1} \lesssim C(N)\left(\lambda^{-3} + \lambda^{-5} T\right). \tag{29}$$

Proof of Claim)

$$\begin{aligned}\int_0^T g_{n_1} \overline{g_{n_2}} g_{n_3} e^{i\omega_4 t} ds &= \int_0^T g_{n_1} \overline{g_{n_2}} g_{n_3} \frac{d}{ds}\left[\frac{e^{i\omega_4 t}}{i\omega_4}\right] ds \\ &= g_{n_1}(T)\, \overline{g_{n_2}(T)} g_{n_3}(T) - g_{n_1}(0)\, \overline{g_{n_2}(0)} g_{n_3}(0) \\ &- \int_0^T \frac{d}{ds}\left[g_{n_1} \overline{g_{n_2}} g_{n_3}\right] \frac{e^{i\omega_4 t}}{i\omega_4} ds\end{aligned}$$

By (28), (11) and the fact that $|\lambda| = C(N)$, the boundary terms are bounded λ^{-3} and the integral term is bounded $\lambda^{-5} T$.

Once λ has been chosen, we choose $\mathcal{R}$ sufficiently large so that initial data $g(0) = a(0)$ has the right size:

$$\left(\sum_{n\in\Lambda} |g_n(0)|^2 |n|^{2s}\right)^{\frac{1}{2}} \sim \delta \tag{30}$$

It remains to show that we can guarantee,

$$\left(\sum_{n\in\Lambda} \left|a_n\left(\lambda^2 T_0\right)\right|^2 |n|^{2s}\right)^{\frac{1}{2}} \geq K, \tag{31}$$

where $a(t)$ is the evolution of the data $a(0)$ under the full system (8).

Claim)

- Estimate to perturbed solution

$$\left(\sum_{n\in\Lambda}\left|g_n\left(\lambda^2T_0\right)\right|^2|n|^{2s}\right)^{\frac{1}{2}}\gtrsim K, \tag{32}$$

- Estimate to error

$$\sum_{n\in\Lambda}\left|g_n\left(\lambda^2T_0\right)-a_n\left(\lambda^2T_0\right)\right|^2|n|^{2s}\lesssim 1. \tag{33}$$

Proof of Claim)
As for first estimate, consider the ratio of this norm of the resonant evolution at time λ^2T_0 to the same norm at the time 0,

$$Q:=\frac{\sum_{n\in\mathbb{Z}^2}\left|g_n\left(\lambda^2T_0\right)\right|^2|n|^{2s}}{\sum_{n\in\mathbb{Z}^2}\left|g_n\left(0\right)\right|^2|n|^{2s}}$$

$$=\frac{\sum_{i=1}^N\sum_{n\in\Lambda_i}\left|b_n\left(\lambda^2T_0\right)\right|^2|n|^{2s}}{\sum_{i=1}^N\sum_{n\in\Lambda_i}\left|b_n\left(0\right)\right|^2|n|^{2s}}$$

since $g_n=0$ when $n\notin\Lambda$.

Let $S_j := \sum_{n \in \Lambda_j} |n|^{2s}$,

$$Q = \frac{\sum_{i=1}^{N} \sum_{n \in \Lambda_i} \left| b_n \left(\lambda^2 T_0\right)\right|^2 |n|^{2s}}{\sum_{i=1}^{N} \sum_{n \in \Lambda_i} |b_n (0)|^2 |n|^{2s}}$$
$$\gtrsim \frac{S_{N-2}(1-\varepsilon)}{\varepsilon S_1 + \varepsilon S_2 + (1-\varepsilon) S_3 + \varepsilon S_4 + \cdots + \varepsilon S_N}$$
$$= \frac{S_{N-2}(1-\varepsilon)}{S_{N-2} \cdot \left[\varepsilon \frac{S_1}{S_{N-2}} + \varepsilon \frac{S_2}{S_{N-2}} + (1-\varepsilon) \frac{S_3}{S_{N-2}} + \cdots + \varepsilon + \varepsilon \frac{S_{N-1}}{S_{N-2}} + \varepsilon \frac{S_N}{S_{N-2}}\right]}$$
$$= \frac{(1-\varepsilon)}{(1-\varepsilon)} \frac{S_3}{S_{N-2}} + O(\varepsilon) \gtrsim \frac{K^2}{\delta^2},$$

by Proposition 2.1 and by choosing $\varepsilon \lesssim C(N, K, \delta)$ sufficiently small.

As for second estimate, using Lemma 2.3 we obtain that

$$\sum_{n\in\Lambda} \left|g_n\left(\lambda^2 T_0\right) - a_n\left(\lambda^2 T_0\right)\right|^2 |n|^{2s} \lesssim \lambda^{-2-\sigma} \sum_{n\in\Lambda} |n|^{2s} \leq 1,$$

by possibly increasing λ and $\mathcal{R}$, maintaining (30).

Thank You
for Your Attention!!