Transfer of energy frequencies in the cubic defocusing nonlinear Schrödinger equation I

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Equation and Conservation laws

• The periodic defocusing cubic nonlinear Schrödinger equation (NLS)

\[
\begin{cases}
- i \partial_t u + \Delta u = |u|^2 u, \\
    u (0, x) := u_0 (x),
\end{cases}
\]

(1)

where \( u (t, x) \) is a complex-valued function and \( x \in \mathbb{T}^2 \) and the initial data is smooth for convenience.

• Hamiltonian (Energy conservation laws)

\[
E [u] (t) := \int_{\mathbb{T}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4 \, dx (t) = E [u] (0).
\]

• Mass conservation laws

\[
M [u] (t) := \int_{\mathbb{T}^2} |u|^2 \, dx (t) = M [u] (0).
\]
• Local well-posedness for $s > 0$ (Bourgain, 1993).
• By conservation laws, we can get global smooth solution to (1) from smooth initial data.
• Later, we introduce the toy model that is the completely integrable.

We construct a solution to (1) that energy of move on to higher Fourier modes. In other words, we construct a solution to (1) with arbitrarily large growth in higher Sobolev norms.
Theorem 1.1 (Main theorem)
Let $s > 1$, $K \gg 1$, and $0 < \delta \ll 1$ be given parameters. Then there exists a global smooth solution $u(t, x)$ to (1) and a time $T > 0$ with

$$
\|u(0)\|_{H^s} \leq \delta
$$

and

$$
\|u(T)\|_{H^s} \geq K
$$

Corollary 1.1 ($H^2$ instability of zero solution)
The global-in-time solution map taking the initial data $u_0$ to the associated solution $u$ of (1) is strongly unstable in $H^s$ near zero for all $s > 1$:

$$
\inf_{\delta > 0} \left( \limsup_{|t| \to \infty} \left[ \sup_{\|u_0\|_{H^2} \leq \delta} \|u(t)\|_{H^s} \right] \right) = 0
$$
Previous result

• High Sobolev norms of solution can grow no faster than exponential-in-time. (Bourgain, 1993)


• Small dispersion NLS

\[ i\partial_t \omega + \delta \Delta \omega = |\omega|^2 \omega \]  

(2)

Smooth norms of solution of (2) evolving from relatively generic data with unit $L^2$ norm eventually grow larger than a negative power of $\delta$. (Kuksin, 1997)
Figure: Outline
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Equation (1) has gauge freedom. Hence we can let

\[ v(t, x) = e^{iGt}u(t, x), \quad G \in \mathbb{R} \]  \hspace{1cm} (3)

then NLS equation (1) is the following equation for \( v \)

\[ (-\partial_t + \Delta) v = \left( G + |v|^2 \right) v \]  \hspace{1cm} (4)

with the same initial data. We write a solution of (2) as following

\[ v(t, x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(n\cdot x + |n|^2t)}. \]  \hspace{1cm} (5)
Substituting (5) into (4) and comparing both sides gives the following infinite system of equations for \( a_n(t) \),

\[
-i\partial_t a_n = G a_n + \sum_{n_1,n_2,n_3 \in \mathbb{Z}^2 \atop n_1 - n_2 + n_3 = n} a_{n_1} \overline{a_{n_2}} a_{n_3} e^{i\omega_4 t}
\]  

(6)

where

\[
\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2
\]  

(7)

For the removal of \( Ga_n \) term with appropriate choosing the gauge parameter \( G \), we split the sum on the right hand side of (6) into the following terms,

\[
\sum_{n_1,n_2,n_3 \in \mathbb{Z}^2 \atop n_1 - n_2 + n_3 = n} = \sum_{n_1,n_2,n_3 \in \mathbb{Z}^2 \atop n_1 - n_2 + n_3 = n} + \sum_{n_1,n_2,n_3 \in \mathbb{Z}^2 \atop n_1 - n_2 + n_3 = n} + \sum_{n_1,n_2,n_3 \in \mathbb{Z}^2 \atop n_1 - n_2 + n_3 = n} - \sum_{n_1,n_2,n_3 \in \mathbb{Z}^2 \atop n_1 - n_2 + n_3 = n}
\]

\[
:= \text{Terms I } + \text{Terms II } + \text{Terms III } + \text{Terms IV}
\]
Since we handle the system as term by term, Term IV is
\[ -a_n(t)\vert a_n(t)\vert^2. \] Term II and Term III are single sums which by Plancherel’s Theorem and mass conservation total,

\[
2a_n(t) \cdot \sum_{m \in \mathbb{Z}} \vert a_m(t)\vert^2 = 2a_n(t) \cdot \|u(t)\|_L^2(T^2) \sum_{m \in \mathbb{Z}} \vert a_m(t)\vert^2 = 2a_n(t)^2 M^2,
\]

where \( M := \|u(t)\|^2_{L^2(T^2)}. \)

We can remove the first term of (5) by choosing \( G = -2M. \)

Equation (4) takes then the following useful form which we denote \( \mathcal{FNSL}, \)

\[
-i \partial_t a_n = -a_n \vert a_n\vert^2 + \sum_{n_1,n_2,n_3 \in \Gamma(n)} a_{n_1} \overline{a_{n_2}} a_{n_3} e^{i\omega_4 t} \quad (8)
\]

where

\[
\Gamma(n) = \left\{ (n_1, n_2, n_3) \in (\mathbb{Z}^2)^3 : n_1 - n_2 + n_3 = n, n_1 \neq n, n_3 \neq n \right\}. \quad (9)
\]
Well-posedness - $\mathcal{F}\text{NLS}$

Let

$$(\mathcal{N}(t)(a, b, c))_n = -a_n \overline{b_n} c_n + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \overline{b_{n_2}} c_{n_3} e^{i\omega_4 t} \quad (10)$$

With this notation, we can reexpress $\mathcal{F}\text{NLS}$ as

$-i \partial_t a_n = (\mathcal{N}(t)(a, a, a))_n$

Lemma 2.1

$$\| (\mathcal{N}(t)(a, b, c))_n \|_{l^1(\mathbb{Z}^2)} \lesssim \| a \|_{l^1(\mathbb{Z}^2)} \| b \|_{l^1(\mathbb{Z}^2)} \| c \|_{l^1(\mathbb{Z}^2)} \quad (11)$$
Resonant truncation - $\mathcal{R}\mathcal{F}\mathcal{N}\mathcal{L}\mathcal{S}$

Define the set of all resonant non-self interactions $\Gamma_{\text{res}} (n) \subset \Gamma (n)$ by

$$\Gamma_{\text{res}} (n) = \left\{ (n_1, n_2, n_3) \in \Gamma (n) : \omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0 \right\}$$  \hspace{1cm} (12)

Note that $(n_1, n_2, n_3) \in \Gamma_{\text{res}}$ precisely when $(n_1, n_2, n_3, n)$ form four corners of a non degenerate rectangle with $n_2$ and $n$ opposing each other, and similarly for $n_1$ and $n_3$.

In resonant set, $\mathcal{F}\mathcal{N}\mathcal{L}\mathcal{S}$ is not oscillates in time anymore. Hence we can simply define the resonant truncation $\mathcal{R}\mathcal{F}\mathcal{N}\mathcal{L}\mathcal{S}$ of $\mathcal{F}\mathcal{N}\mathcal{L}\mathcal{S}$ by,

$$-i\partial_t r_n = -r_n |r_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{\text{res}}(n)} r_{n_1} \overline{r_{n_2}} r_{n_3}. \hspace{1cm} (13)$$
Finite truncation - The frequency set $\Lambda$

We introduce some notations and terminologies on set $\Lambda$ before finite truncation of $\mathcal{R\mathcal{F}}\text{NLS}$.

- For some positive integer $N$, the set $\Lambda$ splits into $N$ disjoint generations $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_N$

[Diagram: Nuclear family]

Figure: Nuclear family
Finite truncation - The frequency set $\Lambda$

We require following properties on set $\Lambda$.

- **Property $I_{\Lambda}$ (Initial data):** $r_n(0) = 0$ whenever $n \notin \Lambda$.
- **Property $II_{\Lambda}$ (Closure):**

\[
(n_1, n_2, n_3) \in \Gamma_{\text{res}}(n), n_1, n_2, n_3 \in \Gamma \Rightarrow n \in \Gamma \tag{14}
\]

**Lemma 2.2**
If $\Lambda$ is a finite set satisfying Property $I_{\Lambda}$, Property $II_{\Lambda}$, and $r(0) \mapsto r(t)$ solves $R\mathcal{F}\text{NLS}$ (8) on $[0, T]$ then for all $t \in [0, T]$, $\text{spt} [r(t)] \subset \Lambda$. 
• Property III_Λ (Existence and uniqueness of spouse and children): 1 \leq \forall j \leq N \text{ and } n_1 \in \Lambda_j \ \exists! \text{ nuclear family } (n_1, n_2, n_3, n_4) \text{ such that } n_1 \text{ is a parent. In particular each } n_1 \in \Lambda_j \text{ has a unique spouse } n_3 \in \Lambda_j \text{ and two unique children } n_2, n_4 \in \Lambda_{j+1}.

• Property IV_Λ (Existence and uniqueness of sibling and parents): 1 \leq \forall j \leq N \text{ and } n_2 \in \Lambda_{j+1} \ \exists! \text{ nuclear family } (n_1, n_2, n_3, n_4) \text{ such that } n_2 \text{ is a parent. In particular each } n_2 \in \Lambda_{j+1} \text{ has a unique sibling } n_4 \in \Lambda_{j+1} \text{ and two unique parents } n_1, n_3 \in \Lambda_j.

• Property V_Λ (Non-degeneracy): The sibling of a frequency \( n \) is never equal to its spouse.

• Property VI_Λ (Faithful): Apart from the nuclear families, \( \Lambda \) contains no other rectangles.
In assumption of existence of $\Lambda$,

$$-i\partial_t r_n (t) = - |r_n (t)|^2 r_n (t) + 2r_{n_{\text{child-1}}} (t) r_{n_{\text{child-2}}} (t) r_{n_{\text{spouse}}} (t)$$

$$+ 2r_{n_{\text{parent-1}}} (t) r_{n_{\text{parent-2}}} (t) r_{n_{\text{sibling}}} (t)$$

(15)

where for each $n \in \Lambda_j$, $n_{\text{spouse}} \in \Lambda_j$ is its spouse,
$r_{n_{\text{child-1}}}, r_{n_{\text{child-2}}} \in \Lambda_{j+1}$ are its two children, $n_{\text{sibling}} \in \Lambda_j$ is its sibling, and $r_{n_{\text{parent-1}}}, r_{n_{\text{parent-2}}} \in \Lambda_{j-1}$ are its parents.

For more simplify ODE, we introduce a condition to $\Lambda$.

- **Property VI$_\Lambda$ (Intragenerational equality):** The function $n \mapsto r_n (0)$ is constant on each generation $\Lambda_j$. Thus $1 \leq j \leq N$ and $n, n' \in \Lambda_j$ imply $r_n (0) = r_{n'} (0)$.

By Gronwall argument, if one has intragenerational equality at time 0 then one has intragenerational equality at all later times.
Finite truncation - Toy model system

By Property VI, we may collapse the function $n \mapsto r_n (t)$, which is currently a function on $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$, to the function $j \mapsto b_j (t)$ on $\{1, 2, \ldots, N\}$, where $b_j (t) := r_n (t)$ whenever $n \in \Lambda_j$. Hence the ODE (15) collapse to the following system that we call *Toy Model System*.

$$-i \partial_t b_j (t) = -|b_j (t)|^2 b_j (t) + 2b_{j-1} (t)^2 \overline{b_j (t)} + 2b_{j+1} (t)^2 \overline{b_j (t)},$$

with the convention that $b_0 (t) = b_{N+1} (t) = 0$.  

(16)
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First ingredient: the frequency set $\Lambda$

Proposition 2.1 (First ingredient: the frequency set $\Lambda$)

Given parameters $\delta \ll 1$, $K \gg 1$, we can find an $N \gg 1$ and a set of frequencies $\Lambda \subset \mathbb{Z}^2$ with,

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$$

which satisfies Property $\Pi_\Lambda$ - Property $VI_\Lambda$ and also,

$$\frac{\sum_{n \in \Lambda_N} |n|^{2s}}{\sum_{n \in \Lambda_1} |n|^{2s}} \gg \frac{K^2}{\delta^2}.$$  \hfill (17)

In addition, given any $\mathcal{R} \gg C(K, \delta)$, we can ensure that $\Lambda$ consists of $N \cdot 2^{N-1}$ disjoint frequencies $n$ satisfying $|n| \geq \mathcal{R}$. 
Second ingredient: instability in the toy model

Proposition 2.2 (Second ingredient: instability in the toy model)

Given \( N > 1, \varepsilon \ll 1 \), there is initial data \( b (0) = (b_1 (0), b_2 (0), \ldots, b_N (0)) \in \mathbb{C}^N \) for (16) and there is a time \( T = T (N, \varepsilon) \) so that

\[
|b_3 (0)| \geq 1 - \varepsilon, \quad |b_j (0)| \leq \varepsilon, \quad j \neq 3
\]
\[
|b_{N-2} (T)| \geq 1 - \varepsilon, \quad |b_j (T)| \leq \varepsilon, \quad j \neq N - 2. \quad (18)
\]

In addition, the corresponding solution satisfies \( \|b (t)\|_{l_\infty} \sim 1 \) for all \( 0 \leq t \leq T \).

Theorem 3.1 (Instability in the toy model)

Let \( N \leq 6 \). Given any \( \varepsilon > 0 \), there exists a point \( x_3 \) within \( \varepsilon \) of \( \mathbb{T}_3 \) (using the usual metric on \( \Sigma \)), a point \( x_{N-2} \) within \( \varepsilon \) of \( \mathbb{T}_{N-2} \), and a time \( t \leq 0 \) such that \( S (t) x_3 = x_{N-2} \).
Third ingredient: the approximation lemma

Let $0 < \sigma < 1$ be an absolute constant (all implicit constants in this subsection may depend on $\sigma$). Let $B \gg 1$, and let $T \ll B^2 \log B$. Let

$$g(t) := \{g_n(t)\}_{n \in \mathbb{Z}^2}$$

be a solution to the perturbed equation of

$$-i\partial_t g(t) = (\mathcal{N}(g(t), g(t), g(t))) + \mathcal{E}(t)$$  \hspace{1cm} (19)

for times $0 \leq t \leq T$, where $\mathcal{N}(t)$ is defined in (10), (7) and where the initial data $g(0)$ is compactly supported.
Lemma 2.3 (Third ingredient: the approximation lemma)

Assume that the solution of above equation $g(t)$ and the error term $E(t)$ obey the bounds of the form

$$\|g(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1}$$  \hspace{1cm} (20)

$$\left\| \int_0^t E(s) \, ds \right\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\sigma}$$  \hspace{1cm} (21)

for all $0 \leq t \leq T$.

We conclude that if $a(t)$ denotes the solution to $\mathcal{F}$NLS (8) with initial data $a(0) = g(0)$, then we have

$$\|a(t) - g(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\sigma}$$  \hspace{1cm} (22)

for all $0 \leq t \leq T$. 

Proof of Lemma 2.3

Write

\[ F(t) := -i \int_0^t \mathcal{E}(s) \, ds, \quad \text{and} \quad d(t) := g(t) + F(t). \]

Observe that

\[-id_t = -ig_t - iF_t = (\mathcal{N}(t)(g(t), g(t), g(t))) + \mathcal{E}(t) - \mathcal{E}(t) = \mathcal{N}(t)(d - F, d - F, d - F)\]

(23)

where we have suppressed the explicit \( t \) dependence for brevity.
Proof of Lemma 2.3 - Continued

By hypothesis, \( d(t) = O_{l_1}(B^{-1}) \). By trilinearity and Lemma 2.1,

\[
-id_t = \mathcal{N}(d, d - F, d - F) - \mathcal{N}(F, d - F, d - F) \\
= \mathcal{N}(d, d, d) + O(\|F\|_{l_1}) O(\|g\|_{l_1})^2 \\
= \mathcal{N}(d, d, d) + O_{l_1}(B^{-3-\sigma}). \tag{24}
\]

Let \( e(t) \) is a smooth error function. Now write \( a := d + e \). Since \( a(t) \) is the solution to \( \mathcal{F}_{\text{NLS}} \),

\[
-i (d + e)_t = -id_t - ia_t + id_t \\
= -ia_t = \mathcal{N}(a, a, a) = \mathcal{N}(d + e, d + e, d + e). \tag{25}
\]

Now, we calculate (24) \(-\) (25) and use trilinearity and Lemma 2.1 again,

\[
 ie_t = O_{l_1}(B^{-3-\sigma}) + O_{l_1}(B^{-2} \|e\|_{l_1}) + O_{l_1}(\|e\|_{l_1}^3). \tag{26}
\]
Hence by the differential form of Minkowski’s inequality, we have
\[ \partial_t \|e\|_{l^1} \lesssim B^{-3-\sigma} + B^{-2} \|e\|_{l^1} + \|e\|_{l^1}^3. \]

To finish the proof we use a bootstrap argument. If \( \|e\|_{l^1} = O\left(B^{-1}\right) \) for all \( t \in [0, T] \) then
\[ \partial_t \|e\|_{l^1} \leq C B^{-2} \|e\|_{l^1}. \]

We have the following inequality by using Gronwall’s inequality and (21)
\[ \|e\|_{l^1} \leq B^{-1-\sigma} \exp\left(C B^{-2} t\right) \]
for all \( t \in [0, T] \). Since we have \( T \ll B^2 \log B \), we thus have \( \|e\|_{l^1} \ll B^{-1-\sigma/2} \), and so we can remove the bootstrap assumption.
Proof of Theorem 1.1

Notation for understanding of proof

- \( a(t) \): The solution of \( \mathcal{F} \text{NLS} \).
- \( b(t) \): The solution of the toy model.
- \( b(\lambda)(t) \): The scaled solution of the toy model.
- \( g(t) \): The solution of the perturbed toy model.

From now, we prove Theorem 1.1.
Given \( \delta, K \), construct \( \Lambda \) as in Proposition 2.1. and so we can construct a traveling wave solution \( b(t) \) to the toy model concentrated at scale \( \varepsilon \) according to Proposition 2.2 above.
This proposition also gives us a time $T_0 = T_0 (k, \delta)$ at which the wave has the traversed the $N$ generations of frequencies. For choosing sufficiently large time, we scale the toy model,

$$b(\lambda) := \lambda^{-1} b \left( \frac{t}{\lambda^2} \right).$$

First of all, the aim is to apply Lemma 2.3 with $g (t) = \{g_n (t)\}_{n \in \mathbb{Z}^2}$ defined by,

$$g_n (t) = b_j^{(\lambda)} (t),$$

for $n \in \Lambda_j$, and $g_n (t) = 0$ when $n \neq \Lambda$. Hence we set

$$\mathcal{E} (t) := - \sum_{[\Gamma(n) \setminus \Gamma_{\text{res}}] \cap \Lambda_3} g_{n_1} \overline{g_{n_2}} g_{n_3} e^{i\omega_4 t}$$

where $\omega_4$ is as in (7).
• 1st condition

By considering its support, the fact that $|\lambda| = C(N)$, and the fact that $\|b(t)\|_{l\infty} \sim 1$, we can be sure that, $\|b(t)\|_{l^1(\mathbb{Z})} \sim C(N)$ and therefore

$$\left\| b^{(\lambda)}(t) \right\|_{l^1(\mathbb{Z})}, \|g(t)\|_{l^1(\mathbb{Z})} \leq \lambda^{-1}C(N)$$

(28)

Thus, (20) holds with the choice $B = C(N)\lambda$.

For large enough $\lambda$, we choose $B$ large enough so that

$$B^2 \log B \gg \lambda^2 T_0.$$
2nd condition

Claim)

\[ \left\| \int_0^t \mathcal{E}(s) \, ds \right\|_{l^1} \lesssim C(N) \left( \lambda^{-3} + \lambda^{-5}T \right). \quad (29) \]

Proof of Claim)

\[ \int_0^T g_{n_1} \overline{g_{n_2}} g_{n_3} e^{i\omega_4 t} \, ds = \int_0^T g_{n_1} \overline{g_{n_2}} g_{n_3} \frac{d}{ds} \left[ \frac{e^{i\omega_4 t}}{i\omega_4} \right] \, ds \]

\[ = g_{n_1}(T) \overline{g_{n_2}(T)} g_{n_3}(T) - g_{n_1}(0) \overline{g_{n_2}(0)} g_{n_3}(0) \]

\[ - \int_0^T \frac{d}{ds} \left[ g_{n_1} \overline{g_{n_2}} g_{n_3} \right] \frac{e^{i\omega_4 t}}{i\omega_4} \, ds \]

By (28), (11) and the fact that \(|\lambda| = C(N)\), the boundary terms are bounded \(\lambda^{-3}\) and the integral term is bounded \(\lambda^{-5}T\).
Once $\lambda$ has been chosen, we choose $\mathcal{R}$ sufficiently large so that initial data $g(0) = a(0)$ has the right size:

$$\left( \sum_{n \in \Lambda} |g_n(0)|^2 |n|^{2s} \right)^{\frac{1}{2}} \sim \delta$$  \hspace{1cm} (30)

It remains to show that we can guarantee,

$$\left( \sum_{n \in \Lambda} |a_n(\lambda^2 T_0)|^2 |n|^{2s} \right)^{\frac{1}{2}} \geq K,$$  \hspace{1cm} (31)

where $a(t)$ is the evolution of the data $a(0)$ under the full system (8).
Claim)

- Estimate to perturbed solution

\[
\left( \sum_{n \in \Lambda} |g_n (\lambda^2 T_0)|^2 |n|^{2s} \right)^{\frac{1}{2}} \gtrsim K,
\]

(32)

- Estimate to error

\[
\sum_{n \in \Lambda} |g_n (\lambda^2 T_0) - a_n (\lambda^2 T_0)|^2 |n|^{2s} \lesssim 1.
\]

(33)

Proof of Claim)
As for first estimate, consider the ratio of this norm of the resonant evolution at time \(\lambda^2 T_0\) to the same norm at the time 0,

\[
Q := \frac{\sum_{n \in \mathbb{Z}^2} |g_n (\lambda^2 T_0)|^2 |n|^{2s}}{\sum_{n \in \mathbb{Z}^2} |g_n (0)|^2 |n|^{2s}}
\]

\[
= \frac{\sum_{i=1}^{N} \sum_{n \in \Lambda_i} |b_n (\lambda^2 T_0)|^2 |n|^{2s}}{\sum_{i=1}^{N} \sum_{n \in \Lambda_i} |b_n (0)|^2 |n|^{2s}}
\]

since \(g_n = 0\) when \(n \not\in \Lambda\).
Let \( S_j := \sum_{n \in \Lambda_j} |n|^{2s} \),

\[
Q = \frac{\sum_{i=1}^{N} \sum_{n \in \Lambda_i} |b_n (\lambda^2 T_0)|^2 |n|^{2s}}{\sum_{i=1}^{N} \sum_{n \in \Lambda_i} |b_n (0)|^2 |n|^{2s}}
\geq \frac{S_{N-2} (1 - \varepsilon)}{\varepsilon S_1 + \varepsilon S_2 + (1 - \varepsilon) S_3 + \varepsilon S_4 + \cdots + \varepsilon S_N}
\]

\[
= \frac{S_{N-2}}{S_{N-2} \cdot \left[ \varepsilon \frac{S_1}{S_{N-2}} + \varepsilon \frac{S_2}{S_{N-2}} + (1 - \varepsilon) \frac{S_3}{S_{N-2}} + \cdots + \varepsilon + \varepsilon \frac{S_{N-1}}{S_{N-2}} + \varepsilon \frac{S_N}{S_{N-2}} \right]}
\]

\[
= \frac{(1 - \varepsilon)}{(1 - \varepsilon)} \frac{S_3}{S_{N-2}} + O(\varepsilon) \gtrsim \frac{K^2}{\delta^2},
\]

by Proposition 2.1 and by choosing \( \varepsilon \lesssim C(N, K, \delta) \) sufficiently small.
As for second estimate, using Lemma 2.3 we obtain that

\[
\sum_{n \in \Lambda} |g_n (\lambda^2 T_0) - a_n (\lambda^2 T_0)|^2 |n|^{2s} \lessapprox \lambda^{-2-\sigma} \sum_{n \in \Lambda} |n|^{2s} \leq 1,
\]

by possibly increasing \( \lambda \) and \( \mathcal{R} \), maintaining (30).
Thank You
for Your Attention!!