QUALIFYING EXAM: COMPLEX ANALYSIS

Scope of the subject

The exam will cover the following topics:

- Graphing of elementary functions
- Logarithmic function, power function
- Möbius transform
- Analytic functions, Cauchy-Riemann equations
- Power series including Laurant series, Identity theorem
- Cauchy theorem, Morera's Theorem
- Cauchy integral formula, Liouville's Theorem, Fundamental theorem of algebra
- Harmonic functions, harmonic conjugate, Poisson integral formula
- Mean value property, maximum principle
- Three types of isolated singularities
- Residue calculus, Rouche's theorem
- Evaluation of real integrals
- Argument principle
- Schwartz lemma, Pick's lemma
- Meromorphic functions, infinite product, Weierstrauss product theorem
- Conformal mapping and Riemann mapping theorem

DO NOT include the followings:

- Analytic continuation
- Picard Theorem, Montel theorem
- Special functions, e.g. Zeta function, Gamma functions.
- Riemann surface

Example Problems

Problem 1. *Find all values of* $(1+2i)^i$ *in* $a+bi$ *form.*

Problem 2. *Sketch the image of* $Log\, z$ *in the following domains.*

(1) *the right half plane,* $Re\, z > 0$
(2) *the half disk,* $|z| < 1, Re\, z > 0$
(3) *the unit circle,* $|z| = 1$
(4) *the horizontal line,* $y = e$
(5) *the vertical line,* $x = e$

Problem 3. *Find all roots of the equation* $\log z = i\pi/2$.

Problem 4. *Find the radius of convergence of the power series expansion of* $(z^2-1)/(z^3-1)$ *at* $z=2$.

Problem 5. *Show that* $\sum \frac{z^k}{k^2}$ *converges uniformly for* $|z| < 1$. *Show that* $\sum \frac{z^k}{k}$ *does not converge uniformly for* $|z| < 1$.

Problem 6. *Let f be an entire function such that, for all z, $|f(z)| = |\sin z|$. Prove that there is a contant c of modulus 1 such that $f(z) = C \sin z$.*

Problem 7. *Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a positive constant. Show that $f(z) = a_1 z$, where a_1 is a complex constant.*

Problem 8. *Show that if z_0 is an isolated singularity of $f(z)$, and if $(z - z_0)^N f(z)$ is bounded near z_0, then z_0 is either removable or a pole of order at most N.*

Problem 9. *Determine the order of the pole at $z = 0$ for*

$$\frac{z}{\sin z - z + z^3/3!}.$$

Problem 10. *Show that $f(x, y) = x^2 - y^2 + 3xy + x + 2y$ is harmonic. Find a harmonic conjugate.*

Problem 11. *Show that $f(z)$ has no singularities in the extended plane other than poles if and only if $f(z)$ is a rational function.*

Problem 12. *Evaluate the following integrals.*

(1)
$$\oint_{|z|=3} \frac{2z^2 - z - 2}{z - 2} dz$$

(2)
$$\oint_{|z-1|=2} \frac{dz}{z(z^2 - 4)e^z}$$

(3)
$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x + x^2} dx$$

(4)
$$\int_0^{\infty} \frac{x^2 + 1}{x^4 + 1} dx$$

(5)
$$\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta}$$

(6)
$$\int_0^{\infty} \frac{x^{-a}}{1 + x} dx$$

(7)
$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} \, dx$$

(8)
$$\int_0^{2\pi} \frac{\cos \theta}{5 + 4 \cos \theta} \, d\theta$$

Problem 13. *Show that $2z^5 + 6z - 1$ has one root in the interval $0 < x < 1$ and four roots in the annulus $\{1 < |z| < 2\}$.*

Problem 14. *How many roots does $z^9 + z^5 - 8z^3 + 2z + 1$ have between the circles $\{|z| = 1\}$ and $\{|z| = 2\}$?*

Problem 15. *Let $f(z)$ be a holomorphic function on a domain D. Prove that $\overline{f(\overline{z})}$ is a holomorphic function on $\{\overline{z} : z \in D\}$.*

Problem 16. *Establish the identity*

$$\int_{-\infty}^{\infty} e^{\alpha x^2} e^{i\beta x}\, dx = \sqrt{\frac{\pi}{\alpha}} e^{-\beta^2/4\alpha}$$

where α, β are real numbers with $\alpha > 0$.

Problem 17. *Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n.*

(1) *Prove that for any $M > 0$, there exists a constant $R > 0$ such that $|p(z)| > M$ on $\{z \in \mathbb{C} : |z| > R\}$.*
(2) *Prove that if $|p(z)|$ attains a local minimum at z_0, then $|p(z_0)| = 0$.*
(3) *Deduce the fundamental theorem of algebra, saying that $p(z)$ has at least a zero in $\mathbb{C}$.*

Problem 18. *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function with compact support. Show that if the Fourier transform of f,*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$$

also has a compact support, then $f \equiv 0$. (Hint. Use the identity theorem)

Problem 19. *Let $f(z)$ be a nonconstant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.*

Problem 20. *Let $f(z)$ be a holomorphic function on $\mathbb{C}$. Show tha if $Re\, f \geq 0$, then f is a constant.*

Problem 21. *Let $f(z)$ be continuous on a domain D. Show $f(z)$ is analytic in D, assuming that*

(1) *$f(z)$ is analytic except on a point $z_0 \in D$*
(2) *$f(z)$ is analytic except on a line segment C in D.*

Problem 22. *Show the equation $e^z = 3z^2$ has two roots inside the unit circle. Show that two roots are real.*

Problem 23. *Show that*

$$\frac{1}{2\pi} \int_0^{2\pi} \log|a - e^{i\theta}| d\theta = \begin{cases} 0, & |a| \leq 1 \\ \log|a|, & |a| > 1 \end{cases}$$

Problem 24. *Let $\{u_n(z)\}$ be a sequence of harmonic functions uniformly converging to $u(z)$ on any compact subset of D. Show that $u(z)$ is harmonic.*

Problem 25. *Suppose that $f(z)$ is analytic in a neighborhood of z_0. Show that $f(z)$ has a zero of degree n at z_0 if and only if $1/f(z)$ has a pole of degree n at z_0.*

Problem 26. *Show the Gamma function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ is analytic on $\{Re\, z > 0\}$ by following steps. ($t^{z-1} = e^{(z-1) Log\, t}$)*

- *Show the integral $\int_0^\infty e^{-t} t^{z-1}\, dt$ converges absolutely for $\{Re\, z > 0\}$. Thus, $\Gamma(z)$ is well-defined on $\{Re\, z > 0\}$.*
- *Show $\Gamma_n(z) = \int_{\frac{1}{n}}^n e^{-t} t^{z-1}\, dt$ is analytic in $\{Re\, z > 0\}$.*
- *Show $\Gamma_n(z)$ converges uniformly to $\Gamma(z)$ on any compact subset of $\{Re\, z > 0\}$. Conclude that $\Gamma(z)$ is analytic on $\{Re\, z > 0\}$.*

Problem 27. *Suppose that $f(z)$ is a meromorphic function in $\mathbb{C}$. Show that if $|f(z)|$ is bounded for $|z| > R$ for some R, then $f(z)$ is a rational function.*

Problem 28. *Let $\{z_n\}$ be a convergent sequence in $\mathbb{C}$. Let $f(z)$ be analytic in $D \subset \mathbb{C}$. Prove or disprove (by giving a counter example) that if $f(z)$ vanishes at z_n, then $f(z) \equiv 0$ in D.*

Problem 29. *Let $f(z)$ be an entire function. Assume that there exists $R > 0$ such that $|f(z)| \leq C|z|^n$ whenever $|z| > R$. Show that $f(z)$ is a polynomial of at most degree n.*

Problem 30. *Show that*

$$\prod_{n=1}^{\infty} (1 + \frac{1}{n^2}) = \frac{e^{\pi} - e^{-\pi}}{2\pi}$$

Problem 31. *Show that $\prod_{n=1}^{\infty} \left(1 + a_j\right)$ converges if and only if $\prod_{j=m}^{n}(1 + a_j) \to 1$ as $m, n \to \infty$.*

Problem 32. *Let $f(z)$ be analytic and satisfy $|f(z)| \leq M$ for $|z - z_0| < R$. Show that if $f(z)$ has a zero of order m at z_0, then*

$$|f(z) \leq \frac{M}{R^m}|z - z_0|^m, \qquad |z - z_0| < R.$$

Show that equality holds at some point $z \neq z_0$ only when $f(z)$ is a constant multiple of $(z - z_0)^m$.

Problem 33. *Show that the conformal maps of the upper half-plane onto the open unit disk are of the form*

$$f(z) = e^{i\phi}\frac{z - a}{z - \overline{a}}, \qquad Im\, a > 0,\ 0 \leq \phi \leq 2\pi.$$

Show that a and $e^{i\phi}$ are uniquely determined by the conformal map.

Problem 34. *Show that amy conformal self-map of the punctured complex plane$\{0 < |z| < \infty\}$ is either a multiplication $z \mapsto az$ or such multiplication followed by the inversion $z \mapsto 1/z$.*