PRELIMENARY EXAM: LEBESGUE INTEGRALS

Scope of the subject

The exam will cover the following topics:

- Riemann integrals
- Measurable sets, $\sigma-$algebra, Borel sets.
- Measurable functions, Lebesgue integration
- Convergence theorems
- L^p spaces and related inequalities.
- Product measure and Fubini theorem on $\mathbb{R}^n$.
- Differentiation and integration e.g. Hardy-Littlewood maximal inequality, Lebesgue differentiation theorem, Fundamental theorem of calculus.
- Function of bounded variation and differentiability of functions on $\mathbb{R}$.

The exam will NOT include the following topics:

- Hahn-Banach theorem and its applications
- Baire category
- Weak topology
- Fourier transform

These topics will be included in Ph.D. qualifying exams.

Sample Problems

Problem 1.

(1) *Let $\{E_k\} \subset \mathbb{R}^d$ be an decreasing sequence of measurable sets. (i.e. $E_k \subset E_{k-1}$) Suppose $\lambda(E_1) < \infty$. Show that*

$$\lambda\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \lambda(E_k).$$

Find an example of a decreasing sequence $\{E_k\}$ with $\lambda(E_1) = \infty$ that the equality does not hold.

(2) *Let $\{B_k\}$ be a sequence of measurable sets. Show that if $\lambda\left(\bigcup_{k=1}^{\infty} B_k\right) < \infty$, then*

$$\limsup_k \lambda(B_k) \le \lambda\left(\bigcap_{j=1}^{\infty} \bigcup_{k\ge j} B_k\right).$$

Problem 2. *Let X be a nonempty set. State the definition of $\sigma-$algebra of X. Let $\{\mathcal{F}_t\}_{t\in I}$ be a collection of $\sigma-$algebras of X. Show that $\bigcap_{t\in I} \mathcal{F}_t$ is a $\sigma-$algebra.*

Problem 3. *Let $\mathcal{M}$ be a collection of subsets of $\mathbb{R}^d$ satisfying $E \in \mathcal{M}$ if and only if for any $\epsilon > 0$, there exists an open set $G \supset E$ and a closed set $F \subset E$ with $\lambda^*(E - F) + \lambda^*(G - E) < \epsilon$. Show that $\mathcal{M}$ is a $\sigma-$algebra.*

Problem 4. *Prove the following and state what convergence theorem you used.*

- $\int_0^1 \log \frac{1}{1-x} dx = \int_0^1 \sum_{n=1}^{\infty} \frac{x^n}{n} dx = 1$
- $\int_0^{\infty} \frac{\sin x}{e^x - 1}\, dx = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nx} \sin x\, dx = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Problem 5. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space.(i.e $\mu(X) < \infty$) Suppose that $\{f_n : X \to \mathbb{R}\}$ are integrable. Show that if $f_n \to f$ uniformly in X, then f is integrable and*

$$\lim_n \int_X f_n\, d\mu = \int_X f\, d\mu.$$

Problem 6. *Let A be a subset of $[0, 1]$ which consists of all numbers which do not have the digit 5 appearing in the decimal expansion. Find its Lebesgue measure $\lambda(A)$.*

Problem 7. *Let $(X, \mathcal{M}, \mu)$ be a measure space where $\mathcal{M}$ is a σ-algebra and $\mu : \mathcal{M} \to [0, \infty]$ satisfying the countable additivity:*

For disjoint $A_k \in \mathcal{M}$, $k = 1, 2, \cdots$, $\sum_{k=1}^{\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$.

Prove that

(1) If $\{A_k \in \mathcal{M} : k = 1, 2, \cdots\}$, then $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k)$.

(2) If $A_k \subset A_{k+1}$ for $k = 1, 2, \cdots$, then $\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$.

Problem 8. *(1) Let $\{f_k : \mathbb{R}^n \to [-\infty, \infty]\}_{k=1}^{\infty}$ be a sequence of measurable functions. Assume the following:*

- *$\lim_{k\to\infty} f_k(x)$ exists a.e. in $\mathbb{R}^n$.*
- *There are nonnegative integrable functions $\{g_k\}_{k=1}^{\infty}$ such that*

$$|f_k(x)| \leq g_k(x) \text{ a.e.} \quad \text{and} \quad \lim_{k\to\infty} g_k(x) = g(x) \text{ a.e.}$$

- *$g \in L^1$ and*

$$\int g\, d\lambda = \lim_{k\to\infty} \int g_k\, d\lambda.$$

Prove that

$$\int \lim_{k\to\infty} f_k\, d\lambda = \lim_{k\to\infty} \int f_k\, d\lambda.$$

(2) Let $\{f_k\}_{k=1}^{\infty}$, $f \in L^1$ and $f_k \to f$ a.e.. Prove that $\lim_{k\to\infty} \int |f_k - f|\, d\lambda = 0$ if and only if $\lim_{k\to\infty} \int |f_k|\, d\lambda = \int |f|\, d\lambda$.

(Hint: Use (1). You are allowed to use (1), even if you cannot prove it.)

Problem 9. *Prove the following identity for $p, q > 0$.*

$$\int_0^1 \frac{x^{p-1}}{1+x^q} dx = \frac{1}{p} - \frac{1}{p+q} + \frac{1}{p+2q} - \frac{1}{p+3q} + \cdots .$$

(You need to justify the convergence.)

Problem 10. *(1) Prove the* Borel-Cantelli lemma, *that is, if $\{A_k\}_{k=1}^\infty$ is a collection of measurable set with*

$$\sum_{k=1}^\infty \lambda(A_k) < \infty,$$

then $\lambda\left(\bigcap_{k=1}^\infty \bigcup_{j\geq k} A_j\right) = 0$.

(2) Prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions $\frac{p}{q}$, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^3}$$

is a set of meausre zero.

Problem 11. *Give an example of a sequence of measurable functions $\{f_k : \mathbb{R} \to \mathbb{R}\}_{k=1}^\infty$ satisfying*

- $\int |f_k| \, d\lambda \leq M < \infty$ *for some $M > 0$,*
- $\lim_{k\to\infty} f_k$ *exists a.e.,*
- $\lim_{k\to\infty} \int f_k \, d\lambda \neq \int \lim_{k\to\infty} f_k \, d\lambda$.

Problem 12. *Let $\{f_n\}$ be a sequence of functions in $L^p, 1 \leq p < \infty.$, which converge a.e. to $f \in L^p$. Show f_n converges to f in L^p if and only if $\|f_n\|_p \to \|f\|_p$.*

Problem 13. *Let $E \in \mathbb{R}^d$ be measurable with $\lambda(E) < \infty$. Let $\mathcal{F}$ be the collection of balls centered at each point $x \in E$. Then for any $\epsilon > 0$, there exists a finite subcollection of balls $\{B_1, \cdots, B_N\}$, which are disjoint and there exist $F \subset E$ with $\lambda(E - F) < \epsilon$ and*

$$F \subset \bigcup_{j=1}^N (2+\epsilon) B_j.$$

(This is a variant of Vitali covering lemma. Please do not use it directly.)

Problem 14. *(Scaling of inequality) We have learned the log-convexity inequality:*

$$\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}, \qquad \text{where} \quad \frac{1}{r} = \theta\frac{1}{p} + (1-\theta)\frac{1}{q}. \tag{1}$$

Show that if (1) *is true for f with $\|f\|_p = 1 = \|f\|_q$, then* (1) *is true for any f. Moreover, show that the scaling condition is necessary.*

Problem 15. *(Rearrangement) Let f be measurable and nonnegative in $\mathbb{R}^d$.*

- *Show for* $\infty > p \geq 1$,

$$\|f\|_p^p = \int_0^\infty p t^{p-1} \lambda\left(\{x : f(x) > t\}\right) \, dt. \tag{2}$$

(Hint: Fubini's theorem)

- *Define a radial function* $f^*(|x|) = \inf\{t : \lambda\left(\{x : f(x) > t\}\right) \leq |x|\}$. *Show that*

$$\lambda\left(\{x : f(x) > t\}\right) = \lambda\left(\{x : f^*(|x|) > t\}\right)$$

and conclude that $\|f\|_p = \|f^*\|_p$.

Problem 16. *Let* $\{f_n\}, f \in L^2(\mathbb{R}^d)$. *Suppose that* $\int f_n g \, dx \to \int f g \, dx, \quad \forall g \in L^2$, *and* $\|f_n\|_2 \to \|f\|_2$ *as* $n \to \infty$. *Then* f_n *converges to* f *in* L^2*- norm.*

Problem 17. *Suppose that* E *is a Borel set in* $\mathbb{R}^2$. *Show for every* y, *the slice* E_y *is a Borel set in* $\mathbb{R}$.

Problem 18. *Let* $1 < p < \infty$ *and* $f \in L^p(\mathbb{R}^d)$.

- *Prove a refined Hardy-Littlewood maximal theorem: There is* $C > 0$ *such that*

$$\lambda\left(\{x : Mf(x) > t\}\right) \leq \frac{C}{t} \int_{\{|f(x)>t/2\}} |f(x)| \, dx.$$

- *Prove that* $Mf \in L^p$ *with* $\|Mf\|_p \leq C\|f\|_p$ *for some* $C > 0$.
(You may use (2), if needed, without proof.)

Problem 19. *Let* $f \in L^1_{loc}(\mathbb{R}^d)$ *and* ϕ_t *is an approximation of identity.*[1] *In a lecture, we have shown that for* $f \in L^p$, $\phi_t * f$ *converges to* f *in* L^p.
Prove that for $f \in L^1_{loc}$, *we have* $\phi_t * f \to f$ *almost everywhere.*

Problem 20. *Let* $f, g : [a, b] \to \mathbb{R}$ *be integrable. Define* $F(x) = \int_a^x f(y) \, dy$ *and* $G(x) = \int_a^x g(y) \, dy$.

(1) *Show* $F(x)G(x)$ *is absolutely continuous.*
(2) *Show that* $(F(x)G(x))' = F(x)g(x) + f(x)G(x)$ *a.e. and conclude that for* $x \in [a, b]$,

$$F(x)G(x) = \int_a^x Fg + \int_a^x fG.$$

Problem 21. *For any continuous increasing function* $F : [a, b] \to \mathbb{R}$, *we can decompose* F *into* $F(x) = \int_a^x f(y) \, dy + h(x)$ *where* f *is a nonnegative integrable function and* $h(x)$ *is continuous increasing function satisfying* $h' = 0$ *a.e. Show that this decomposition is unique.*

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[1] Consider a smooth function $\phi(x)$ satisfying

- $\phi(x) \geq 0$,
- $\phi \in C_c^\infty$ with supp$\phi \subset B_1$,
- $\int \phi(x) \, dx = 1$.

We define a rescaled function $\phi_t(x) = \frac{1}{t^d} \phi(\frac{x}{t})$.

Problem 22. *Let* $1 \leq p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *For given* $g \in L^q(\mu)$, *we define a linear functional* $L(f) = \int fg\,d\mu$ *for locally integrable function* f. *Show that* $L : L^p \to \mathbb{R}$ *is a bounded linear functional with* $\|L\| = \|g\|_q$.

Problem 23. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Assume* $1 \leq p < q \leq \infty$.

(1) *For the Lebesgue measure space* $(\mathbb{R}^n, \mathcal{L}, \lambda)$, *find examples which are in* L^p *but not in* L^q, *and* L^q *but not in* L^p.
(2) *For a finite measure space, show that* $L^p \subset L^q$.
(3) *For* $(\mathbb{Z}, \mathcal{P}, c)$, *show that* $L^q \subset L^p$.

Problem 24. *Let* $Mf(x)$ *be the Hardy-Littlewood maximal function for* $f \in L^1_{loc}$.

(1) *Prove that for any measurable function* f *defined on* $\mathbb{R}^n$,
$$\lambda\left(\{x | Mf(x) > t\}\right) \leq \frac{C_n}{t} \int_{\{|f(x)| > t/2\}} |f(x)|\,dx \qquad \text{for some constant } C_n.$$
(2) *Let* $1 < p < \infty$, *and suppose* $f \in L^p(\mathbb{R}^n)$. *Prove that also* $Mf \in L^p(\mathbb{R}^n)$ *and*
$$\|Mf\|_p \leq D_n \|f\|_p, \qquad \text{for some constant } D_n.$$

Problem 25. *Find an example of function* $f : [0,1] \to \mathbb{R}$, *which is uniformly continuous but not absolutely continuous. Moreover, find an example* f, *which is absolutely continuous but not Lipschitz continuous. (You need to prove your examples satisfy the conditions.)*

Problem 26. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a nonnegative measurable function. For* $1 \leq p < \infty$, *show the followings are equivalent:*

(1) $f \in L^p(\mathbb{R}^n)$,
(2)
$$\sum_{k=-\infty}^{\infty} 2^{kp} \lambda\left(\{x \in \mathbb{R}^n : 2^k \leq f(x) < 2^{k+1}\}\right) < \infty,$$
(3)
$$\sum_{k=-\infty}^{\infty} 2^{kp} \lambda\left(\{x \in \mathbb{R}^n : 2^k \leq f(x)\}\right) < \infty.$$

Problem 27. *Let* $1 \leq p < q \leq \infty$. *Prove that* $L^p([0,1]) \subset L^q([0,1])$, *while* $l^q \subset l^p$. *Here* $l^p = L^p(\mathbb{Z}, \mathfrak{c})$ *where* $\mathfrak{c}$ *is a counting measure. Provide counterexamples disproving the opposite inclusions.*

Problem 28. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of measurable functions on* $\mathbb{R}$. *Show that if* $\{f_n\}$ *converges to* f *in* $L^1(\mathbb{R})$, *then there is a subsequence* $\{f_{n_j}\}$ *converging to* f *almost everywhere.*

Problem 29.

(1) *Let* $1 \leq p < q < r \leq \infty$. *Show if* $f(x) \in L^p(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$, *then* $f \in L^q(\mathbb{R}^d)$.

(2) *Assume that* $f(x) \in L^2(\mathbb{R}^d)$ *and* $|x|^{\frac{d+1}{2}} f(x) \in L^2(\mathbb{R}^d)$. *Show that* $f(x) \in L^1(\mathbb{R}^d)$.

Problem 30. *Let* $W : \mathbb{R} \to \mathbb{R}$ *be the function*

$$W(x) := \sum_{n=1}^{\infty} 4^{-n} \sin(16^n \pi x).$$

$W(x)$ *is well-defined as the series converges absolutely.*

(1) *Show that* $W(x)$ *is a bounded continuous function.*
(2) *Show that for any* $x \in \mathbb{R}$, W *is not differentiable.*
(Hint: Verify that for $j \in \mathbb{Z}, n \in \mathbb{Z}_+$, $|W(\frac{j+\frac{1}{2}}{16^n}) - W(\frac{j-\frac{1}{2}}{16^n})| \geq c4^{-n}$ *for some constant* $c > 0$.*)*

Problem 31. *Let* $f, g : \mathbb{R}^n \to \mathbb{R}$ *be integrable functions. For* $1 \leq p < \infty$, *prove the Young's inequality:*

$$\Big(\int |\int f(x-y)g(y)\,dy|^p\,dx\Big)^{1/p} \leq \big(\int |f(x)|^p\,dx\big)^{1/p} \int |g(x)|\,dx.$$

Problem 32. *For a given nonnegative measurable function* $f : \mathbb{R}^n \to \mathbb{R}_+$, *we have defined its integral by*

$$\int f\,d\mu = \sup\{\int s\,d\mu : 0 \leq s \leq f \quad s \text{ is simple } \}.$$

Prove that if f *and* g *are nonnegative measurable functions, then*

$$\int f\,d\mu + \int g\,d\mu = \int (f+g)\,d\mu.$$

Problem 33. *Show that the class*

$$\mathcal{R} = \{A \subset [0,1] : \chi_A \text{ is Riemann integrable } \}$$

is an algebra. Moreover, show that $\mathcal{R}$ *is not a* σ*-algebra. Note that* χ_A *is the characteristic function, i.e. (* $\chi_A(x) = \begin{cases} 1, x \in A \\ 0, x \in A^c \end{cases}$ *).*

Problem 34. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be an integrable function. Show that for* $\epsilon > 0$, *there exists* $\delta > 0$ *so that*

$$\int_A |f(x)|\,dx < \epsilon \qquad \text{whenever} \quad |A| \leq \delta.$$

Problem 35. *Recall the Hardy-Littlewood maximal function*

$$Mf(x) = \sup_{0<r<\infty} \frac{1}{B(x,r)} \int_{B(x,r} |f(y)|\,dy.$$

Show that if $Mf \in L^1$, *then* $f = 0$.