Topology: Midterm Exam (Spring 2007)

Justify your answers fully.

1. Prove or disprove:
   (a) (10pts) An infinite product of path-connected spaces with the product topology is path-connected.
   (b) (10pts) An infinite product of copies of the path-connected space $\mathbf{R}$ with the box topology is path-connected.
   (c) (10pts) If A is a path-connected subset of a topological space, then $\bar{A}$ is path-connected.
   (d) (10pts) Let A be a subspace of a topological space X. Then a retraction $r : X \to A$ is a quotient map.
   (e) (10pts) $\mathbf{R} \times \mathbf{R}$ with the dictionary-order-topology is homeomorphic to $\mathbf{R}_d \times \mathbf{R}$ where $\mathbf{R}_d$ is $\mathbf{R}$ with the discrete topology.

2. Let X be a topological space. Let Y be a compact topological space and let $\pi : X \times Y \to X$ be the projection.
   (a) (20pts) Show that π is a closed map.
   (b) (20pts) Find an example showing that π is not necessarily a closed map when Y is not compact.

3. Let X be a topological space. Let $f, g : X \to \mathbf{R}$ be continuous functions.
   (a) (20pts) Show that $\{x | f(x) \leq g(x)\}$ is closed in X.
   (b) (20pts) Define $h : X \to \mathbf{R}$ by $h(x) = \max\{f(x), g(x)\}$. Show that h is continuous.
   (c) (20pts) Prove or disprove: If $f_1, f_2, \ldots$ be an infinite sequence of bounded continuous functions $X \to \mathbf{R}$ where

$$-N \leq f_i \leq N \text{ for some integer } N \text{ and for all } i,$$

   then $h : X \to \mathbf{R}$ defined by $h(x) = \sup\{f_1(x), f_2(x), \ldots\}$ is continuous.