8.5. Application of Quadratic forms to optimization

Basically Calculus topic.

Relative extrema of functions of two variables.

- Critical point (x,y): if $f_x(x,y)=0$, $f_y(x,y)=0$.
- If (x_0,y_0) is a critical point, then let $D(x,y)=f(x,y)-f(x_0, y_0)$.
- If $D(x,y) > 0$ for all (x,y) suff close to (x_0, y_0), then (x_0,y_0) relative minimum.
- If $D(x,y) < 0$, -> relative maximum.
- If $D(x,y)$ can have both signs -> saddle point at (x_0, y_0).

Theorem 8.5.1 (***Second Derivative Test***) *Suppose that* (x_0, y_0) *is a critical point of* $f(x, y)$ *and that* f *has continuous second-order partial derivatives in some circular region centered at* (x_0, y_0). *Then:*

(*a*) f *has a relative minimum at* (x_0, y_0) *if*

$$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) > 0 \quad \textit{and} \quad f_{xx}(x_0, y_0) > 0$$

(*b*) f *has a relative maximum at* (x_0, y_0) *if*

$$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) > 0 \quad \textit{and} \quad f_{xx}(x_0, y_0) < 0$$

(*c*) f *has a saddle point at* (x_0, y_0) *if*

$$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) < 0$$

(*d*) *The test is inconclusive if*

$$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) = 0$$

Hessian matrix

- H(x,y) =[[f_xx(x,y), f_xy(x,y)], [f_xy(x,y),f_yy(x,y)]].
- Det[H(x_0, y_0)] = f_xx(x_0,y_0)f_yy(x_0,y_0)-f_xy^2(x_0,y_0).

Theorem 8.5.2 (***Hessian Form of the Second Derivative Test***) *Suppose that* (x_0, y_0) *is a critical point of* $f(x, y)$ *and that* f *has continuous second-order partial derivatives in some circular region centered at* (x_0, y_0). *If* $H = H(x_0, y_0)$ *is the Hessian of* f *at* (x_0, y_0), *then:*

(*a*) f *has a relative minimum at* (x_0, y_0) *if* H *is positive definite.*

(*b*) f *has a relative maximum at* (x_0, y_0) *if* H *is negative definite.*

(*c*) f *has a saddle point at* (x_0, y_0) *if* H *is indefinite.*

(*d*) *The test is inconclusive otherwise.*

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- Example 1. Find critical points -> Take derivatives -> Find Hessian matix at each critical point -> Do the Hessian test.

Constrained extremum problem

- We give a constraint ||x||=1.
- We try to find the maximum and the minimum of $q(x)=x^TAx$

Theorem 8.5.3 (***Constrained Extremum Theorem***) *Let A be a symmetric $n \times n$ matrix whose eigenvalues in order of decreasing size are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then:*

(*a*) *There is a maximum value and a minimum value for $\mathbf{x}^T A\mathbf{x}$ on the unit sphere $\|\mathbf{x}\| = 1$.*

(*b*) *The maximum value is λ_1 (the largest eigenvalue), and this maximum occurs if $\mathbf{x}$ is a unit eigenvector of A corresponding to λ_1.*

(*c*) *The minimum value is λ_n (the smallest eigenvalue), and this minimum occurs if $\mathbf{x}$ is a unit eigenvector of A corresponding to λ_n.*

- Example 2. $q(x)=-23/25x^2-2/25y^2+72/25xy$.
  - Eigenvectors ±(3/5,4/5) for 1, ±(-4/5,3/5) for -2 for A are on the unit sphere
  - Thus, q at ±(3/5,4/5) is 1 and q at ±(-4/5,3/5) is -2.
  - maximum =1.
  - Minimum = -2.

Constrained Extrema and level curves

- Level curves of $x^T A x = k$ meet the circle $||x||=1$.
- If k is the maximum or the minimum, the level curve is tangent to the circle.
- Conversely, the level curve tangent to the circle gives us the maximum and the minimum points and values.

- Example 4. $q(x)=-23/25x^2-2/25y^2+72/25xy$.