8.2 Similarity and diagonalizability

Coordinate change for diagonalization

Similar matrices

Definition 8.2.1 If A and C are square matrices with the same size, then we say that ***C is similar to A*** if there is an invertible matrix P such that $C = P^{-1}AP$.

- A≈B,B≈C->A≈C. A≈A. A≈B->B≈A

Theorem 8.2.2 *Two matrices are similar if and only if there exist bases with respect to which the matrices represent the same linear operator.*

- Proof: $C=P^{-1}AP$. If P=[v_1,⋯,v_n], then by equation (20) Sec.8.2, we have $[T]\_B=P^{-1}[T]P$ where [T]=A.

Similarity Invariants

- Coordinate change are superficial change.
- Many essential properties remain.
- $\det(P^{-1}AP)=\det(P)^{-1}\det(A)\det(P)=\det(A)$.

Theorem 8.2.3

(*a*) *Similar matrices have the same determinant.*

(*b*) *Similar matrices have the same rank.*

(*c*) *Similar matrices have the same nullity.*

(*d*) *Similar matrices have the same trace.*

(*e*) *Similar matrices have the same characteristic polynomial and hence have the same eigenvalues with the same algebraic multiplicities.*

- Example 1.

Eigenvectors and eigenvalues of similar matrices

- The algebraic multiplicity of an eigenvalue is the multiplicity as a root of the characteristic polynomial.
- The geometric multiplicity of an eigenvalue is the dimension of $(LI-A)x=0$.
- geom mult. $\leq$ alg. mult.
- Example 2.

Theorem 8.2.4 *Similar matrices have the same eigenvalues and those eigenvalues have the same algebraic and geometric multiplicities for both matrices.*

- Proof: $C=P^{-1}AP$. Then
  - $LI-C=LI-P^{-1}AP = P^{-1}(LI-A)P$.
  - $\det(LI-C)=\det(LI-A)$.
  - $(LI-C)x=0 \leftrightarrow P^{-1}(LI-A)Px=0. \leftrightarrow (LI-A)y=0$ for $y=Px$. (substitute variable)
  - Thus dim sol $(LI-C)x=0$ is the same as dim sol$(LI-A)x=0$.

Theorem 8.2.5 *Suppose that $C = P^{-1}AP$ and that λ is an eigenvalue of A and C.*

(a) *If $\mathbf{x}$ is an eigenvector of C corresponding to λ, then $P\mathbf{x}$ is an eigenvector of A corresponding to λ.*

(b) *If $\mathbf{x}$ is an eigenvector of A corresponding to λ, then $P^{-1}\mathbf{x}$ is an eigenvector of C corresponding to λ.*

- Proof (b): Ax=Lx. -> P^{-1}Ax=LP^{-1}x -> P^{-1}AP(P^{-1}x) = L(P^{-1}x) -> C(P^{-1}x)=L(P^{-1}x).

Diagonalization

- We wish to change coordinates so that the matrix is diagonal.
- This is not always possible.

The Diagonalization Problem Given a square matrix A, does there exist an invertible matrix P for which $P^{-1}AP$ is a diagonal matrix, and if so, how does one find such a P? If such a matrix P exists, then A is said to be ***diagonalizable***, and P is said to ***diagonalize*** A.

Theorem 8.2.6 *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.*

- Proof ->: $A=P^{-1}DP$ for a diagonal matrix D with diagonals $L_1,L_2,\cdots,L_n$.
  - $AP=PD$. $P=[p_1,p_2,\cdots,p_n]$
  - $AP=[Ap_1,Ap_2,\cdots,Ap_n]$
  - $PD=[L_1p_1,L_2p_2,\cdots,L_np_n]$
  - Thus $Ap_i=L_ip_i$.
- Proof <−: $p_1,p_2,\ldots,p_n$ linearly independent, eigenvectors.
  - $Ap_i=L_ip_i$.
  - Let $P=[p_1,p_2,\cdots,p_n]$.
  - The same computations show $AP=PD$.
  - Since P is invertible, $P^{-1}AP=D$.

A method for diagonalizing a matrix.

Diagonalizing an $n \times n$ Matrix with n Linearly Independent Eigenvectors

Step 1. Find n linearly independent eigenvectors of A, say $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$.

Step 2. Form the matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$.

Step 3. The matrix $P^{-1}AP$ will be diagonal and will have the eigenvalues corresponding to $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, respectively, as its successive diagonal entries.

- Example 4.

Linear independence of eigenvectors

Theorem 8.2.7 *If* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *are eigenvectors of a matrix A that correspond to distinct eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_k$, *then the set* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is linearly independent.*

- Thus, if some eigenvalues coincide, the corresponding eigenvectors may be depedent. (This is unless they are fundamental solutions.)

Some facts

Theorem 8.2.8 *An $n \times n$ matrix with n distinct real eigenvalues is diagonalizable.*

Theorem 8.2.9 *An $n \times n$ matrix A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is n.*

Theorem 8.2.10 *If A is a square matrix, then:*

(*a*) *The geometric multiplicity of an eigenvalue of A is less than or equal to its algebraic multiplicity.*

(*b*) *A is diagonalizable if and only if the geometric multiplicity of each eigenvalue of A is the same as its algebraic multiplicity.*

- Unifying theorem:

Theorem 8.2.11 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(a) *A is diagonalizable.*

(b) *A has n linearly independent eigenvectors.*

(c) *R^n has a basis consisting of eigenvectors of A.*

(d) *The sum of the geometric multiplicities of the eigenvalues of A is n.*

(e) *The geometric multiplicity of each eigenvalue of A is the same as the algebraic multiplicity.*