# QR-DECOMPOSITION

HOUSEHOLDER TRANSFORMATIONS

# QR-decomposition

- A mxk matrix with columns $w_1, w_2, \ldots, w_k$. m-vectors.
- We find an orthonomal basis $q_1, q_2, \ldots, q_k, q_{k+1}, \cdots, q_m$ of $R^m$.
- Then since $q_i$ is orthogonal to $w_1, \ldots, w_{k-1}$.
  - $w_1 = (w_1 . q_1) q_1$
  - $w_2 = (w_2 . q_1) q_1 + (w_2 . q_2) . q_2$.
  - ….
  - $w_k = (w_k . q_1) q_1 + (w_k . q_2) q_2 + \ldots + (w_k . q_k) q_k$.

- Use Theorem 3.1.8, A=QR

$$\begin{bmatrix} w_1 & w_2 & \cdots & w_k \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_k \end{bmatrix} \begin{bmatrix} (w_1 \cdot q_1) & (w_2 \cdot q_1) & \cdots & (w_1 \cdot q_k) \\ 0 & (w_2 \cdot q_2) & \cdots & (w_2 \cdot q_k) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (w_k \cdot q_k) \end{bmatrix}$$

**Theorem 7.10.1** (***QR-Decomposition***) *If $A$ is an $m \times k$ matrix with full column rank, then $A$ can be factored as*

$$A = QR \tag{4}$$

*where $Q$ is an $m \times k$ matrix whose column vectors form an orthonormal basis for the column space of $A$ and $R$ is a $k \times k$ invertible upper triangular matrix.*

- A=QR.
- Since the inverse of Q is $Q^T$, we have $R=Q^TA$.
- See Example 1.

# QR-decompositions and the least square problem

- $Ax=b$. best approximate solution $x'=(A^TA)^{-1}A^Tb$.
- We write $A=QR$. $A^T=R^TQ^T$.
- $A^TAx'=A^tx'$.
- $R^TQ^TQRx=R^TQTb$.
- $R^TRx=R^TQTb$ and $x'=R^{-1}Q^Tx$.

**Theorem 7.10.2** *If $A$ is an $m \times k$ matrix with full column rank, and if $A = QR$ is a $QR$-decomposition of $A$, then the normal system for $A\mathbf{x} = \mathbf{b}$ can be expressed as*

$$R\mathbf{x} = Q^T\mathbf{b} \tag{9}$$

*and the least squares solution can be expressed as*

$$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b} \tag{10}$$

- Example 2.
- We will use Householder reflections to find Q instead since it has advantages in computer calculations.
- We obtain a formula for reflections:
  - Let $a^c$ be the orthogonal hyperplane to span{a}
  - x-refl_$a^c$(x)=2proj_a(x).
  - Thus refl_$a^c$(x)=x-2proj_a(x)=x-2a(x.a)/||a||$^2$.

**Definition 7.10.3** If $\mathbf{a}$ is a nonzero vector in $R^n$, and if $\mathbf{x}$ is any vector in $R^n$, then the ***reflection of* $\mathbf{x}$ *about the hyperplane* $\mathbf{a}^\perp$** is denoted by $\text{refl}_{\mathbf{a}^\perp}\mathbf{x}$ and defined as

$$\text{refl}_{\mathbf{a}^\perp}\mathbf{x} = \mathbf{x} - 2\text{proj}_{\mathbf{a}}\mathbf{x} \tag{11}$$

The operator $T: R^n \to R^n$ defined by $T(\mathbf{x}) = \text{refl}_{\mathbf{a}^\perp}\mathbf{x}$ is called the ***reflection of $R^n$ about the hyperplane* $\mathbf{a}^\perp$**.

- Thus the matrix H_$a^c$ for refl_$a^c$ is H_$a^c$=I-2a$a^T$/$a^T$a.
- If a is a unit vector u, then $u^Tu=||u||^2=1$.
- refl_$a^c$=x-2(x.u)u and H_$u^c$=I-u$u^T$.
- See Example 3 and 4.

**Definition 7.10.4** An $n \times n$ matrix of the form

$$H = I - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T \tag{16}$$

in which $\mathbf{a}$ is a nonzero vector in $R^n$ is called a ***Householder matrix***. Geometrically, $H$ is the standard matrix for the Householder reflection about the hyperplane $\mathbf{a}^\perp$.

**Theorem 7.10.5** *Householder matrices are symmetric and orthogonal.*

- Proof: $H^T = I - (2/a^Ta)(aa^T)^T = H$.
  $HH = (I - 2aa^T/a^Ta)(I - 2aa^T/a^Ta) = I - 4aa^T/a^Ta + (2aa^T/(a^Ta))(2aa^T/(a^Ta)) = I$ (since $4(1/(a^Ta)^{2)}(aa^Taa^T) = 4(1/(a^Ta)^{2)}((a^Ta)aa^T) = 4(1/(a^Ta))(aa^T)$.)

**Theorem 7.10.6** *If* **v** *and* **w** *are distinct vectors in $R^n$ with the same length, then the Householder reflection about the hyperplane* $(\mathbf{v} - \mathbf{w})^\perp$ *maps* **v** *into* **w**, *and conversely.*

# QR-decomposition using householder reflections

- The steps given A nxn matrix
- We apply a Householder matix Q_1 so that Q_1A has (1,0,…,0) as the first column by choosing Q_1 so that Q_1a_1=[1,0,…,0]$^T$.
- Now choose Q_2 which is 1 at (1,1)-entry and zero on elsewhere in the 1-st row and the 1-st column.
- Now concentrate on (n-1)x(n-1)-matrix in Q_1A removing 1st column and the 1st-row
- Q_2 sends the 2nd column a'_2 of Q_1A to [*,1,0,..,0]$^T$.
- We keep doing this… Q_n-1…Q_2Q_1A is upper triangular. We let it be R and Q=Q_n…Q_2Q_1.

- Example 7: