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Preface

The author wished to write a book that is relatively elementary and intuitive but rigorous without being too technical. This book exposes the connection between the low-dimensional orbifold theory and geometry that was first discovered by Thurston in 1970s providing a key tool in his proof of the hyperbolization of Haken 3-manifolds. Our main aims are to explain most of the topology of orbifolds but to explain geometric structure theory only for the 2-dimensional orbifolds.

The book was intended for the advanced undergraduates and the beginning graduate students who understand some differentiable manifold theory, Riemannian geometry, some manifold topology, algebraic topology, and Lie group actions. But we do include sketches of these theories in the beginning of the book as a review. Unfortunately, some familiarity with the category theory is needed where the author cannot provide a sufficiently good introduction.

This book is intentionally made to be short as there are many extensive writings on the subject already available. Instead of writing every proof down, we try to explain the reasoning behind the proofs and point to where they can be found. This was done in the hope that the readers can follow the reasoning without having to understand the full details of the proof, and can fast-track into this field. The book hopefully is self-sufficient for people who do not wish to delve into technical details but wish to gain a working knowledge of the field.

In this book, we tried to collect the theory of orbifolds scattered in various literatures for our purposes. Here, we set out to write down the traditional approach to orbifolds using charts, and we include the categorical definition using groupoids. We think that understanding both theories is necessary.

Computer experimentation is important in this field for understanding and research. We will also maintain a collection of Mathematica packages at our homepages so that the readers can experiment with them. We will also give addresses where one can find the computational packages. Many links will be gone soon enough but something related will reappear in other places.

This book is based on the courses that the author gave in the fall term of 2008 at Tokyo Institute of Technology as a visiting professor and in the spring term of 2011 at KAIST. I thank very much the hospitality of the Department of

Mathematical and Computing Sciences at Tokyo Institute of Technology. I thank David Fried, William Goldman, Craig Hodgson, Steve Kerckhoff, Hyuk Kim, Sadayoshi Kojima, Walter Neumann, Alan Reid, William Thurston, and many others not mentioned here for introducing me the ideas of orbifolds and geometric structures on them. Some of the graphics were done with Jfig© and Mathematica™ using the PoincareModel package from the Experimental Geometry Lab of the University of Maryland, College Park, written by William Goldman available from `http://egl.math.umd.edu/` and also the Curved Space by Jeffrey Weeks avaliable from `http://geometrygames.org/CurvedSpaces/index.html`. We also thank Gye-Seon Lee and Kanghyun Choi for drawing many of the graphics using these packages. I also thank many of my colleagues for much patience while this book was being written down. We also thank the anonymous referees for suggesting a number of improvements. The future revisions and errata will be available from `http://mathsci.kaist.ac.kr/~schoi/MSJbook2012.html`. Please send any comments to the author. Finally, this author was partially supported by the Mid-career Researcher Program through the NRF grant funded by the MEST under the grant number 2010-0027001.

Daejeon, July 2012 Suhyoung Choi

Contents

Chapter 1

Introduction

One aim of mathematics is to explore many objects purely defined and created out of imaginations in the hope that they would explain many unknown and unsolved phenomena in mathematics and other fields. As one knows, the manifold theory enjoyed a great deal of attention in the 20th century mathematics involving many talented mathematicians. Perhaps, mathematicians should develop more abstract theories that can accommodate many things that we promised to unravel in the earlier part of 20th century. The theory of orbifolds might be a small step in the right direction as orbifolds have all the notions of the manifold theory easily generalized as discovered by Satake and developed by Thurston. In fact, orbifolds have most notions developed from the manifold theory carried over to them although perhaps in an indirect manner, using the language of the category theory. Indeed, to make the orbifold theory most rigorously understood, only the category theory provides the natural settings.

Orbifolds provide a natural setting to study discrete group actions on manifolds, and orbifolds can be more useful than manifolds in many ways involving in the classification of knots, the graph embeddings, theoretical physics and so on. At least in two- or three-dimensions, orbifolds are much easier to produce and classifiable using Thurston's geometrization program. (See for example the program "Orb" by Heard and Hodgson (2007).) In particular, one obtains many examples with ease to experiment with. The subject of higher dimensional orbifolds is still very mysterious where many mathematicians and theoretical physicists are working on. In fact, the common notion that orbifolds are almost always covered by manifolds is not entirely relevant particularly for the higer-dimensional orbifolds. For example, these kinds of orbifolds might exist in abundance and might prove to be very useful. It is thought that orbifolds are integral part of theoretical physics such as the string theory, and they have natural generalizations in algebraic geometry as stacks.

For 2-manifolds, it was known from the classical times that the geometry provides a sharp insight into the topology of surfaces and their groups of automorphisms as observed by Dehn and others. In late 1970s, Thurston proposed a program to generalize these kinds of insights to the 3-manifold theory. This program is now completed by Perelman's proof of the Geometrization conjecture as is well-known.

The computer programs such as Snappea initiated by Thurston and completed mainly by Weeks, Hodgson, and so on, now compute most topological properties of 3-manifolds completely given the 3-manifold topological data.

It seems that the direction of the research in the low-dimensional manifold theory currently is perhaps to complete the understanding of 3-manifolds by volume ordering, arithmetic properties, and group theoretical properties. Perhaps, we should start to move to higher-dimensional manifolds and to more applied areas.

One area which can be of possible interest is to study the projectively flat, affinely flat, or conformally flat structures on 3-manifolds. This will complete the understanding of all classical geometric properties of 3-manifolds. This aspect is related to understanding all representations of the fundamental groups of 3-manifolds into Lie groups where many interesting questions still remain, upon which we mention that we are yet to understand fully the 2-orbifold or surface fundamental group representations into Lie groups.

This book introduces 2-orbifolds and geometric structures on them for senior undergraduates and the beginning graduate students. Some background in topology, the manifold theory, differential geometry, and particularly the category theory would be helpful.

The covering space theory is explained using both the fiber-product approach of Thurston and the path-approach of Haefliger. In fact, these form a very satisfying direct generalization of the classical covering space theory of Poincaré. The main part of the book is the geometric structures on orbifolds. We define the deformation space of geometric structures on orbifolds and state the local homeomorphism theorem that the deformation spaces are locally homeomorphic to the representation spaces of the fundamental groups. The main emphases are on studying geometric structures and ways to cut and paste the geometric structures on 2-orbifolds. These form a main topic of this book and will hopefully aid the reader in studying many possible geometric structures on orbifolds including affine, projective, and so on. Also, these other types of geometries seem to be of use in the Mirror symmetry and so on.

We will learn the orbifold theory and the geometric structures on orbifolds. We will cover some of the background materials such as the Lie group theory, principal bundles, and connections. The theory of orbifolds has much to do with discrete subgroups of Lie groups but has more topological flavors. We discuss the topology of orbifolds including covering spaces and orbifold-fundamental groups. The fundamental groups of orbifolds include many interesting infinite groups. We obtain the understanding of the deformation space of hyperbolic structures on a 2-orbifold, which is the space of conjugacy classes of discrete faithful $\mathbb{PSL}(2,\mathbb{R})$-representations of the 2-orbifold fundamental group. Finally, we survey the deformation spaces of real projective structures on 2-orbifolds, which correspond to the Hitchin-Teichmüller components of the spaces of conjugacy classes of $\mathbb{PGL}(3,\mathbb{R})$-representations of the fundamental groups.

This book has three parts. In the first part consisting of Chapters 2 and 3, we review the manifold theory with $\mathcal{G}$-structures. In the second part consisting of Chapters 4 and 5, we present the topological theory of orbifolds. In the third part, consisting of Chapters 6, 7, and 8, we present the theory of geometric structures of orbifolds.

In Chapter 2, manifolds and $\mathcal{G}$-structures, we review smooth structures on manifolds starting from topological constructions, homotopy groups and covering spaces, and simplicial manifolds including examples of surfaces. Then we move onto pseudogroups and $\mathcal{G}$-structures. Finally, we review Lie groups and the principal bundle theory in terms of the smooth manifold theory.

In Chapter 3, geometry and discrete groups, we first review Euclidean, spherical, affine, projective, and conformal geometry centering on their properties under the Lie group actions. Next, we go over to hyperbolic geometry. We begin from the Lorentzian hyperboloid model and move onto the Beltrami-Klein model, the conformal model and the upper half-space model. Hyperbolic triangle laws are studied also and the isometry group of hyperbolic spaces is introduced. We also discuss the discrete group actions on manifolds using the Poincaré fundamental polyhedron theorem and discuss Coxeter groups, triangle groups, and crystallographic groups.

In Chapter 4, topology of orbifolds, we start reviewing compact group actions on manifolds. We talk about the orbit spaces and tubes, smooth actions, and equivariant triangulations. Next, we introduce orbifolds from the classical definition by Satake using atlases of charts. We define singular sets and suborbifolds. We also present orbifolds as Lie groupoids from the category theory as was initiated by Haefliger. We present differentiable structures on orbifolds, bundles over orbifolds, the Gauss-Bonnet theorem, and smooth triangulations. To find the universal covers of orbifolds, we start from defining covering spaces of orbifolds and discuss how to obtain a fiber-product of many covering orbifolds. This leads us to the universal covering orbifolds and deck transformation groups and their properties such as uniqueness. We also present the path-approach to the universal covering orbifolds of Haefliger. Hence, we define the fundamental groups of orbifolds.

In Chapter 5, topology of 2-orbifolds, we present how to compute the Euler characteristics of 2-orbifolds using the Riemann-Hurwitz formula. We show how to topologically construct 2-orbifolds from other 2-orbifolds using cutting and sewing methods. This is reinterpreted in two other manners.

In Chapter 6, geometry of orbifolds, we define (G, X)-structures on orbifold using the method of atlases of charts, the method of developing maps and holonomy homomorphisms, and the method of cross-sections to bundles. We show that these definitions are equivalent. We also show that orbifolds admitting geometric structures are always good; that is, they are covered by manifolds. Here, we define the deformation spaces of (G, X)-structures on orbifolds and discuss about the local homeomorphism from the deformation space of (G, X)-structures on an orbifold to the space of representations of the fundamental groups of the orbifold to the Lie group G.

In Chapter 7, the deformation spaces of hyperbolic structures on 2-orbifolds, i.e., the Teichmüller space, we first define the Teichmüller space and present geometric cutting and pasting constructions of hyperbolic structures on 2-orbifolds. We show that any 2-orbifolds decompose into elementary orbifolds. We show how to compute the Teichmüller spaces of elementary orbifolds using hyperbolic trigonometry and piece these together to understand the Teichmüller space of the 2-orbifold following Thurston.

Finally in Chapter 8, we introduce the deformation spaces of real projective structures on 2-orbifolds. We first give some examples. Next, we sketch some history on this subject, including the classification result, Hitchin's conjecture and its solution, and the discrete groups and the representation theory aspects. We go over to the deformation spaces where we use the method very similar to the above chapter. We decompose 2-orbifolds into elementary 2-orbifolds and determine the deformation spaces there and reassemble. Here, we merely indicate proofs. In this chapter, we do many computations for elementary 2-orbifolds.

Our principal source for this lecture note is Chapter 5 of the book [Thurston (1977)]. However, we do not go into his generalization of the Andreev theorem. (Also, the book [Thurston (1997)] is a good source of many materials here.)

We shall maintain some computations files related to graphics in this book. Mathematica™files designated $***$.nb are files that we wrote and maintain in our homepages.

There are many standard textbooks giving us preliminary viewpoint and alternative viewpoints of the foundational material for this book. The book [Kobayashi and Nomizu (1997)] provides us a good introduction to connections on principal bundles and the books [Sharpe (1997); Ivey and Landsberg (2003)] give us more differential geometric viewpoint of geometric structures. The book [Bredon (1972)] is a good source for understanding the local orbifold group actions. Finally, the book [Berger (2009)] provides us with the knowledge of geometry that is probably most prevalently used in this book.

Chapter 2

Manifolds and $\mathcal{G}$-structures

In this chapter, we review many notions in the manifold theory that can be generalized to the orbifold theory.

We begin by reviewing manifolds and simplicial manifolds beginning with cell-complexes and the homotopy and covering theory. The following theories for manifolds will be transferred to orbifolds. We briefly mention them here as a “review” and develop them for orbifolds later (mostly for 2-dimensional orbifolds):

- Lie groups and group actions
- Pseudo-groups and $\mathcal{G}$-structures
- Differential geometry: Riemanian manifolds, principal bundles, connections, and flat connections

We follow a coordinate-free approach to differential geometry. We do not need to actually compute curvatures and so on.

Some of these are standard materials in a differentiable manifold course. We will not give proofs in Chapters 2 and 3 but will indicate one when necessary.

2.1 The review of topology

We present a review of the manifold topology. We will find that many of these directly can be generalized into the orbifold theory later.

2.1.1 *Manifolds*

The useful methods of topology come from taking equivalence classes and finding quotient topology. Given a topological space X with an equivalence relation, we give the quotient topology on $X/\sim$ so that for any function $f : X \to Y$ inducing a well-defined function $f' : X/\sim \to Y$, f' is continuous if and only if f is continuous. This translates to the fact that a subset U of $X/\sim$ is open if and only if $p^{-1}(U)$ is open in X for the quotient map $p : X \to X/\sim$.

A *cell* is a topological space homeomorphic to an n-dimensional open convex domain defined in $\mathbb{R}^n$ for $n \geq 0$. We will mostly use cell-complexes (Hatcher, 2002). A *cell-complex* is a topological space that is a union of n-skeletons X^n defined inductively. A 0-skeleton is a discrete set of points. Let $I = \bigcup_{n\geq 0} I_n$ be the collection of cells. An $(n+1)$-skeleton X^{n+1} is obtained from the n-skeleton X^n as a quotient space of $X^n \cup \bigcup_{\alpha \in I_{n+1}} D^{n+1}_\alpha$ for a collection of $(n+1)$-dimensional balls D^{n+1}_α for $\alpha \in I_{n+1}$ with a collection of functions $f_\alpha : \partial D^{n+1}_\alpha \to X^n$ so that the equivalence relation is given by $x \sim f_\alpha(x)$ for $x \in \partial D^{n+1}_\alpha$. To obtain the topology of X, we use the weak topology that a subset U of X is open if and only if $U \cap X^n$ is open for every n. Most of the times, cell-complexes will be *finite* ones, i.e., have finitely many cells.

A topological n-dimensional manifold (n-manifold) is a Hausdorff space with a countable basis and charts to a Euclidean space E^n; e.g curves, surfaces, and 3-manifolds. The charts could also go to a positive half-space H^n defined by $x_0 \geq 0$ in $\mathbb{R}^n$ for a coordinate function x_0 of $\mathbb{R}^n$. Then the set of points mapping to $\{0\} \times \mathbb{R}^{n-1}$ under charts is well-defined and is said to be the boundary of the manifold. By the invariance of domain theorem, we see that this is a well-defined notion.

For example, $\mathbb{R}^n$ and H^n themselves or open subsets of $\mathbb{R}^n$ or H^n are manifolds of dimension n.

The unit sphere $\mathbf{S}^n$ in $\mathbb{R}^{n+1}$ is a standard example. The quotient space $\mathbb{R}^{n+1} - \{O\}$ by the relation $v \sim w$ for $v, w \in \mathbb{R}^{n+1}$ if $v = sw$ for $s \in \mathbb{R} - \{O\}$ is called the real projective space $\mathbb{RP}^n$ and is another example.

An n-ball is a manifold with boundary. The boundary is the unit sphere $\mathbf{S}^{n-1}$.

Given two manifolds M_1 and M_2 of dimensions m and n respectively, we obtain the product space $M_1 \times M_2$ a manifold of dimension $m + n$.

An annulus is a disk removed with the interior of a smaller disk. It is also homeomorphic to a circle times a closed interval.

A manifold M is a smooth manifold if it has an atlas of charts of form (U, ϕ) where U is an open subset of M and ϕ is a homeomorphism from U to an open subset of $\mathbb{R}^n$ or H^n and transition functions between charts are all smooth.

A *smooth map* $f : M \to N$ for two smooth manifolds M and N is a map represented by smooth maps under coordinate systems of M and N; i.e., $\phi \circ f \circ \psi^{-1}$ is a smooth map from an open subset of Euclidean space or a half-space to another Euclidean space for coordinate charts ϕ of N and ψ of M. A *diffeomorphism* $f : M \to N$ of two smooth manifolds M and N is a smooth map with a smooth inverse map f^{-1}.

Example 2.1. The n-dimensional torus T^n is homeomorphic to the product of n circles $\mathbf{S}^1$. (For 2-torus, see `http://en.wikipedia.org/wiki/Torus` for its embeddings in $\mathbb{R}^3$ and so on.)

A group G *acts* on a manifold M if there is a differentiable map $k : G \times M \to M$ so that $k(g, k(h, x)) = k(gh, x)$ and $k(\mathrm{I}, x) = x$ for $x \in M$ and the identity $\mathrm{I} \in G$. Given an action of G on a manifold, one obtains a homomorphism $G \to \mathbf{Diff}(M)$ so that an element $g \in G$ goes to a diffeomorphism $g' : M \to M$ sending x to $k(g, x)$

where $\mathbf{Diff}(M)$ is the group of diffeomorphisms of M.

Given a group G acting on a manifold M, we obtain the quotient space $M/\sim$ where $\sim$ is given by $x \sim y$ if and only if $x = g(y), g \in G$, which is denoted by M/G. Let $e_i, i = 1, 2, \ldots, n$, denote the standard unit vectors in $\mathbb{R}^n$. Let T_n be a group of translations generated by $T_i : x \mapsto x + e_i$ for each $i = 1, 2, \ldots, n$. Then $\mathbb{R}^n/T_n$ is homeomorphic to T^n.

Example 2.2. We define the *connected sum* of two n-manifolds M_1 and M_2. Remove the interior of the union of two disjointly and tamely embedded closed balls from M_i for each i. Then each M_i has a boundary component homeomorphic to $\mathbf{S}^{n-1}$. We identify the spheres.

Take many 2-dimensional tori or projective planes and do connected sums. Also remove the interiors of some disks. We can obtain all compact surfaces in this way. (See `http://en.wikipedia.org/wiki/Surface`.)

2.1.2 *Some homotopy theory*

We will assume that our topological spaces here are path-connected, locally path-connected and semi-locally simply connected unless we mention otherwise. Here, the maps are always assumed to be continuous.

Let X and Y be topological spaces. A *homotopy* is a map $F : X \times I \to Y$ for an interval I. Two maps f and $g : X \to Y$ are *homotopic* by a homotopy F if $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$ for all x. The homotopic property is an equivalence relation on the set of maps $X \to Y$. A *homotopy equivalence* of two spaces X and Y is a map $f : X \to Y$ with a map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to I_Y and I_X respectively. (See the book [Hatcher (2002)] for details of the homotopy theory presented here.)

The fundamental group of a topological space X is defined as follows: A *path* is a map $f : I \to X$ for an interval $I = [a, b]$ in $\mathbb{R}$. We will normally use $I = [0, 1]$. An endpoint of the path is $f(0)$ and $f(1)$.

Any two paths $f, g : I \to \mathbb{R}^n$ are homotopic by a linear homotopy that is given by $F(t, s) = (1 - s)f(t) + sg(t)$ for $(t, s) \in [0, 1]^2$.

A *homotopy class* is an equivalence class of homotopic maps relative to endpoints.

The *fundamental group* $\pi_1(X, x_0)$ at the *base point* x_0 is the set of homotopy class of paths with both endpoints x_0.

The product in the fundamental group is defined by joining. That is, given two paths $f, g : I \to X$ with endpoints x_0, we define a path $f * g$ with endpoints x_0 by setting $f * g(t) = f(2t)$ if $t \in [0, 1/2]$ and $f * g(t) = g(2t - 1)$ if $t \in [1/2, 1]$. This induces a product $[f] * [g] = [f * g]$, which we need to verify to be well-defined with respect to the equivalence relation of homotopy. The constant path c_0 given by setting $c_0(t) = x_0$ for all t satisfies $[c_0] * [f] = [f] = [f] * [c_0]$. We denote $[c_0]$ by I_{x_0}. Given a path f, we define an inverse path $f^{-1} : I \to X$ by setting $f^{-1}(t) = f(1 - t)$. We also obtain $[f^{-1}] * [f] = \mathrm{I}_{x_0} = [f] * [f^{-1}]$. By verifying

$[f] * ([g] * [h]) = ([f] * [g]) * [h]$ for three paths with endpoints x_0, we see that the fundamental group is a group.

If we change the base to another point y_0 which is in the same path-component of X, we obtain an isomorphic fundamental group $\pi_1(X, y_0)$. Let γ be a path from x_0 to y_0. Then we define $\gamma^* : [f] \in \pi_1(X, x_0) \mapsto [\gamma^{-1} * f * \gamma]$ which is an isomorphism. The inverse is given by $\gamma^{-1,*}$. This isomorphism does depend on γ and hence cannot produce a canonical identification.

Theorem 2.1.1. *The fundamental group of a circle is isomorphic to* $\mathbb{Z}$*.*

This has a well-known corollary, the Brouwer-fixed-point theorem, that a self-map of a disk to itself always has a fixed point.

Given a map $f : X \to Y$ with $f(x_0) = y_0$, we define a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $f_*([h]) = [f \circ h]$ for any path h in X with endpoints x_0.

Theorem 2.1.2 (Van Kampen). *We are given a path-connected space X covered by open path-connected subsets A_i, $i \in I$, containing a common point x_0 for an index set I and such that every intersection of any two or three members is a nonempty path-connected set. Then $\pi_1(X, x_0)$ is a quotient group of the free product $*_{i \in I} \pi_1(A_i, x_0)$. The kernel is the normal subgroup generated by $i_j^*(a) i_k^*(a)^{-1}$ for all a in $\pi_1(A_j \cap A_k, x_0)$.*

A *bouquet of n circles* is the quotient space of a union of n circles with one point from each identified with one another. Then the fundamental group at a basepoint x_0 is isomorphic to a free group of rank n.

For cell-complexes, this theorem is useful for computing the fundamental group: If a space Y is obtained from X by attaching the 2-cells, then $\pi_1(Y, y_0)$ is isomorphic to $\pi_1(X, y_0)/N$ where N is the normal subgroup generated by "boundary curves" of the attaching maps where y_0 is a basepoint in Y.

We will later compute the fundamental groups of surfaces using this method.

2.1.3 *Covering spaces and discrete group actions*

Given a manifold M, we define a *covering map* $p : \tilde{M} \to M$ from another manifold $\tilde{M}$ to be a surjective map such that each point of M has a neighborhood O such that $p|p^{-1}(O) : p^{-1}(O) \to O$ is a homeomorphism for each component of $p^{-1}(O)$. Normally $\tilde{M}$ is assumed to be connected. (See Chapter 5 of the book [Massey (1987)].)

Consider $\mathbf{S}^1$ as the set of unit length complex numbers. The coverings of a circle $\mathbf{S}^1$ are given by $f : \mathbf{S}^1 \to \mathbf{S}^1$ defined by $x \mapsto x^n$. These are finite to one covering maps. Define $\mathbb{R} \to \mathbf{S}^1$ by $t \mapsto \exp(2\pi t i)$. Then this is an infinite covering.

Example 2.3 (Standard Example). Consider a closed disk with interiors of a finite number of disjoint smaller disks removed. Then draw mutually disjoint arcs from the boundary of the disk to all the boundary curves of the smaller disks. We remove mutually disjoint open regular neighborhoods of the disjoint arcs. Call these strips. Let $D, I_1, I_2, \ldots, I_n$ denote the closures of the complement of the union of the strips and the strips themselves. Let α_i^+, α_i^- the two boundary arcs of the strip I_i parallel to the arcs in the counter-clock wise direction. We take a product with a discrete countable set F and label them by $D^i, I_1^i, l_2^i, \ldots, I_n^i$ for $i \in F$. Then we select a permutation $k_j : F \to F$ for each $j = 1, 2, \ldots, n$. For each i, we glue D^i with I_j^i over the arc α_i^+ and then we glue $D^{k_j(i)}$ with I_j^i over α_i^-. We do this for all arcs. Suppose that we obtain a connected space. By sending $D^i \to D, I_j^i \to I_j$ by projections, we obtain a covering.

Another good example is the join of two circles: See pages 56-58 of the book [Hatcher (2002)].

An important property of homotopy with respect to the covering space is the homotopy lifting property: Let $\tilde{M}$ be a covering of M. Given two homotopic maps f and g from a space X to M, we find that if f lifts to $\tilde{M}$, then so does g. If we let $F : X \times I \to M$ be the homotopy, the map lifts to $\tilde{F} : X \times I \to \tilde{M}$. This is completely determined if the lift of f is specified.

For example, one can consider a path to be a homotopy for X equal to a point. Any path in M lifts to a unique path in $\tilde{M}$ once the initial point is assigned.

Moreover, if two paths f and g are homotopic relative to endpoints, and their initial point $\tilde{f}(0)$ and $\tilde{g}(0)$ of the lifts $\tilde{f}$ and $\tilde{g}$ are the same, then $\tilde{f}(1) = \tilde{g}(1)$. Using this idea, we prove:

Theorem 2.1.3. *Let M be a manifold. Let $p : \tilde{M} \to M$ be a covering map and x_0 a base point of M. Given a map $f : Y \to M$ with $f(y_0) = x_0$, and a point $\tilde{x}_0 \in p^{-1}(x_0)$, we can uniquely lift f to $\tilde{f} : Y \to \tilde{M}$ so that $\tilde{f}(y_0) = \tilde{x}_0$ if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{M}, \tilde{x}_0))$.*

An *isomorphism* of two covering spaces X_1 with a covering map $p_1 : X_1 \to X$ and X_2 with $p_2 : X_2 \to X$ is a homeomorphism $f : X_1 \to X_2$ so that $p_2 \circ f = p_1$. The *automorphism group* of a covering map $p : M' \to M$ is a group of homeomorphisms $f : M' \to M'$ so that $p \circ f = p$. We also use the term the *deck transformation group*. Each element is a *deck transformation* or a *covering automorphism*.

Let x_0 be a base point of M. Let $p : \tilde{M} \to M$ be a covering map. The fundamental group $\pi_1(M, x_0)$ acts on $\tilde{M}$ on the right by path-liftings: we choose an inverse image $\tilde{x}_0$ in $\tilde{M}$. For a path γ in M with endpoints x_0, we define $\tilde{x}_0 \cdot \gamma = \tilde{\gamma}(1)$ for the lift $\tilde{\gamma}$ of γ with initial point $\tilde{\gamma}(0) = \tilde{x}_0$. This gives us a right-action $\pi_1(M, x_0) \times p^{-1}(x_0) \to p^{-1}(x_0)$, called a *monodromy action*, since we have $\tilde{x}_0 \cdot (\gamma * \delta) = (\tilde{x}_0 \cdot \gamma) \cdot \delta$.

A covering $p : M' \to M$ is *regular* if the covering map $p : M' \to M$ is a quotient map under the action of a discrete group Γ acting properly discontinuously and freely. Here M is homeomorphic to M'/Γ.

Given a covering map $p : \tilde{M} \to M$, we obtain a subgroup $p_*(\pi_1(\tilde{M}, \tilde{x}_0)) \subset \pi_1(M, x_0)$. Conversely, given a subgroup G of $\pi_1(M, x_0)$, we can construct a covering $\tilde{M}$ containing a point $\hat{x}_0$ and a covering map $p : \tilde{M} \to M$ so that $p_*(\pi_1(\tilde{M}, \hat{x}_0)) = G$ and $p(\hat{x}_0) = x_0$.

One classifies covering spaces of M by the subgroups of $\pi_1(M, x_0)$. That is, two coverings M_1 with basepoint m_1 and the covering map p_1 and M_2 with basepoint m_2 and covering map p_2 of M with $p_1(m_1) = p_2(m_2) = x_0$ are isomorphic with a map sending m_1 to m_2 if we have $p_{1*}(\pi_1(M_1, m_1)) = p_{2*}(\pi_1(M_2, m_2))$. Thus, the set of covering spaces is ordered by inclusion relations of the subgroups. If the subgroup is normal, the corresponding covering is regular.

A manifold has a *universal covering*; a covering space whose fundamental group is trivial. A universal cover covers every other covering of a given manifold.

The universal covering $\tilde{M}$ of a manifold M has the covering automorphism group Γ isomorphic to $\pi_1(M, x_0)$. A manifold M is homeomorphic to $\tilde{M}/\Gamma$ for its universal cover $\tilde{M}$ where Γ is the deck transformation group.

For example, let $\tilde{M}$ be $\mathbb{R}^2$ and T^2 be a torus. Then there is a map $p : \mathbb{R}^2 \to T^2$ sending (x, y) to $([x], [y])$ where $[x] = x \mod 2\pi$ and $[y] = y \mod 2\pi$.

Let M be a surface of genus 2. $\tilde{M}$ is homeomorphic to a disk, identified with a hyperbolic plane. The deck transformation group can be realized as isometries of a hyperbolic plane. We will see this in more details later.

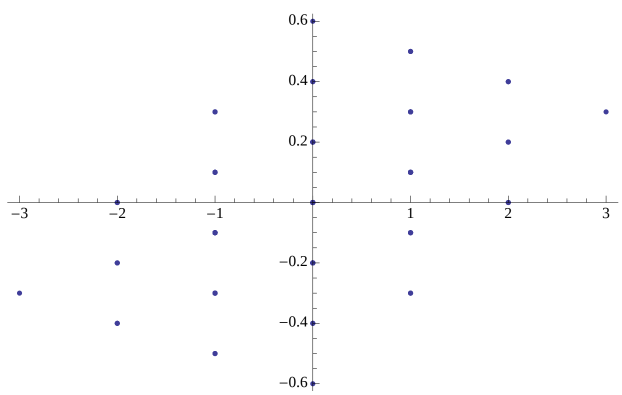

Fig. 2.1 Some orbit points of a translation group of rank two

2.1.4 *Simplicial manifolds*

In this section, we will try to realize manifolds as simplicial sets.

An affine space A^n is a vector space $\mathbb{R}^n$ where we do not remember the origin. More formally, A^n equals $\mathbb{R}^n$ as a set but has an operation $\mathbb{R}^n \times A^n$ given by sending $(a, b) \mapsto a + b$ for $a \in \mathbb{R}^n$ and $b \in A^n$ and satisfies $(a + (b + c)) = (a + b) + c$ for $a, b \in \mathbb{R}^n$ and $c \in A^n$. We can define the difference $b - a$ of two affine vectors a, b

by setting $b - a$ to equal $c \in \mathbb{R}^n$ such that $c + a = b$.

If one takes a point p as the origin, we can make A^n into a vector space $\mathbb{R}^n$ by a map $a \mapsto a - p$ for all $a \in A^n$.

A set of $n+1$ points $v_1, v_2, \ldots, v_{n+1}$ in $\mathbb{R}^n$ is *affinely independent* if the set $v_i - v_1$ for $i = 2, \ldots, n+1$ is linearly independent as vectors. An n-simplex is a convex hull of the set of affinely independent $(n+1)$-points. An n-simplex is homeomorphic to a closed unit ball B^n in $\mathbb{R}^n$.

A *simplicial complex* is a locally finite collection S of simplices so that any face of a simplex is a simplex in S and the intersection of two elements of S, if not empty, is a face of the both. The union is a topological set, which is said to be a *polyhedron*. We can define barycentric subdivisions by taking a barycentric subdivision for each simplex. A link of a simplex σ is the simplicial complex made up of simplicies disjoint from σ in a simplex containing σ.

An n-manifold X can be constructed by gluing n-simplices by face-identifications: Suppose that X is an n-dimensional triangulated space. If the link of every p-simplex is homeomorphic to a sphere of dimension $(n - p - 1)$, then X is an n-manifold. If X is a simplicial n-manifold, we say that X is *orientable* if we can give an orientation on each n-simplex so that over the common faces the orientations extend one another.

2.1.4.1 *Surfaces*

We begin with a construction of a compact surface. Given a polygon with even number of sides, we assign an identification pattern by labeling by $a_1, a_2, \ldots, a_g, a_1^{-1}, a_2^{-1}, \ldots, a_g^{-1}$ so that a_i means an edge labelled by a_i oriented counter-clockwise and a_i^{-1} means an edge labelled by a_i oriented clockwise, and if a pair a_i and a_i or a_i^{-1} occurs, then we identify them respecting the orientations.

- We begin with a bigon. We divide the boundary into two edges and identify by labels a, a^{-1}. Then the result is a surface homeomorphic to a 2-sphere.
- We divide the boundary into two edges and identify by labels a, a. Then the result is homeomorphic to a projective plane.
- Suppose now that we have a quadrilateral with labels a, b, a^{-1}, b^{-1}. We identify the top segment with the bottom one and the right side with the left side. The result is homeomorphic to a 2-torus.

Any closed surface can be represented in this manner.

Let us be given a $4n$-gon. We label edges

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_n, b_n, a_n^{-1}, b_n^{-1}.$$

The result is a connected sum of n tori and is orientable. The genus of such a surface is n.

Suppose that we are given a $2n$-gon. We label edges $a_1 a_1 a_2 a_2 \ldots a_n a_n$. The result is a connected sum of n projective planes and is not orientable. The genus of

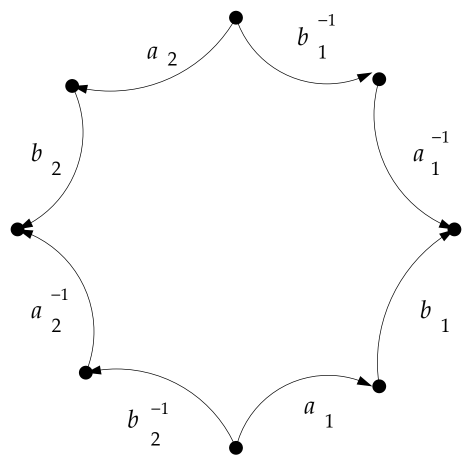

Fig. 2.2 A genus 2 surface as a quotient space of a disk

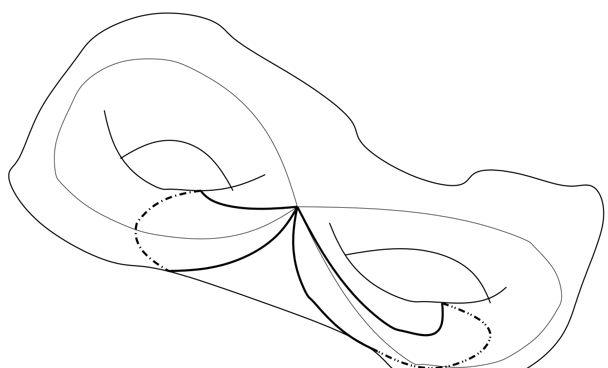

Fig. 2.3 A genus 2-surface patched up

such a surface is n.

We can remove the interiors of disjoint closed balls from the surfaces. The results are surfaces with boundary.

By the Van Kampen theorem, we compute the fundamental group of a surface using this identification. A surface is a cell complex starting from a 1-complex which is a bouquet of circles and attached with a cell. Therefore, we have the fundamental group $\pi_1(S, x)$ for a basepoint x is presented as

$$\langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] \rangle$$

for an orientable surface S of genus g, $g \geq 1$, where the notation implies that the group is isomorphic to a free group generated by $a_1, b_1, \ldots a_g, b_g$ quotient by the normal subgroup generated by the word $[a_1, b_1][a_2, b_2] \ldots [a_g, b_g]$.

The Euler characteristic of a 2-dimensional simplicial complex is given by $F - E + V$ where F denotes the number of 2-dimensional cells, E the number of 1-dimensional cells, and V the number of 0-dimensional cells. This is a topological invariant. We count from the above identification picture that the Euler characteristic of an orientable compact surface of genus g with n boundary components is $2 - 2g - n$.

By a *simple curve* in a surface, we mean an embedded interval. A *simple closed curve* in a surface is an embedded circle. They play important roles in studying surfaces as Dehn and Nielson first discovered.

Let a 2-sphere be given a triangulation. A *pair-of-pants* is the topological space homeomorphic to the complement of the interior of the union of three disjoint closed simplicial 2-cells in the sphere. It has three boundary components homeomorphic to circles. Moreover, a pair-of-pants is obtained by identifying two hexagons in their alternating segments in pairs.

In fact, a closed orientable surface of genus g, $g > 1$, contains $3g - 3$ disjoint simple closed curves so that the complement of its union is a disjoint union of open pairs-of-pants, i.e., spheres with three holes. Hence, the surface can be obtained by identifying boundary components of the pairs-of-pants.

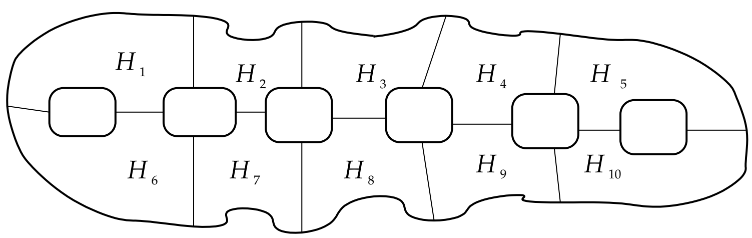

Fig. 2.4 A genus n surface is obtained by doubling a planar surface. That is, we take two copies of this but identify the boundary of the planar surfaces indicated as thick closed arcs. The planar surface is divided into hexagons denoted by H_i by thin lines. Then the doubled hexagons correspond to pairs-of-pants. This process is actually doubling of orbifolds if the boundary is silvered here. (See Section 4.6.1.2 for details on doubling.)

A pair-of-pants P can have a simple closed curve embedded in it but such a circle, if not homotopically trivial, always bounds an annulus containing a boundary component of P. Hence, a pair-of-pants can be built from a pair-of-pants and annuli by identification along circles. One cannot but build a pair-of-pants from a surface other than annuli and a single pair-of-pants. Therefore, a pair-of-pants is an "elementary" surface in that any closed surface can be built from these types of pieces by identifying boundary components where we regard annuli as being trivial elements of the constructions.

2.2 Lie groups

2.2.1 *Manifolds and tangent spaces*

We regard any manifold as being smoothly embedded in some Euclidean space. A *tangent vector* to a manifold M is a vector tangent to a point of M. The *tangent space* $T_x(M)$ at a point x of M is the vector space of vectors tangent to M at x. The *tangent bundle* of M is the space $\{(x,v)|x \in M, v \in T_x(M)\}$ with natural topology. For example, if M is an open subspace of $\mathbb{R}^n$, the tangent vectors are ordinary vectors based at a point of M and the tangent bundle is diffeomorphic to $M \times \mathbb{R}^n$.

At the moment this notion depends on the embedding of M; however, there are definitions showing that these spaces are well-defined.

A smooth map $f : M \to N$ induces a smooth map $Df : T(M) \to T(N)$ restricting to a linear map $D_x f : T_x(M) \to T_{f(x)}(N)$ of the vector spaces at each $x \in M$ defined by

$$\frac{d}{dt}(f \circ \alpha)\Big|_{t=t_0} = Df_{x_0}\left(\frac{d}{dt}\alpha(t)\Big|_{t=t_0}\right)$$

for $x_0 = \alpha(t_0)$. Df is said to be a *differential* of f.

2.2.2 *Lie groups*

A Lie group can be thought of as a space of symmetries of some space. More formally, a *Lie group* is a manifold with a group operation $\circ : G \times G \to G$ that satisfies

- $\circ$ is smooth.
- the inverse $\iota : G \to G$ is smooth also.

From $\circ$, we form a homomorphism $G \to \mathbf{Diff}(G)$ given by $g \mapsto L_g$ and $L_g : G \to G$ is a diffeomorphism given by a left-multiplication $L_g(h) = gh$. Since we have $L_{gh} = L_g \circ L_h$, this is a homomorphism.

As examples, we have:

- The permutation group of a finite set forms a 0-dimensional Lie group, which is a finite set, and a countable infinite group with the discrete topology is a 0-dimensional Lie group.
- $\mathbb{R}$ or $\mathbb{C}$ with $+$ as an operation. ($\mathbb{R}^+$ with $+$ is merely a Lie semigroup.)
- $\mathbb{R} - \{O\}$ or $\mathbb{C} - \{O\}$ with $\times$ as an operation.
- $T^n = \mathbb{R}^n/\Gamma$ with $+$ as an operation and O as the equivalence class of $(0, 0, \ldots, 0)$ and Γ is a group of translations by integral vectors. (The last three are abelian ones.)

We go to the noncommutative groups.

- The general linear group is given by
$$\mathbb{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\} :$$
Here, $\mathbb{GL}(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$. The multiplication is smooth since the coordinate product has a polynomial expression.
- The special linear group is given as
$$\mathbb{SL}(n, \mathbb{R}) = \{A \in \mathbb{GL}(n, \mathbb{R}) | \det(A) = 1\} :$$
The restriction by a system of polynomial equations gives us a smooth submanifold of $\mathbb{GL}(n, \mathbb{R})$. The multiplication is also a restriction.
- The orthogonal group is given by
$$\mathbb{O}(n, \mathbb{R}) = \{A \in \mathbb{GL}(n, \mathbb{R}) | A^T A = \mathrm{I}\}.$$
This is another submanifold formed by a system of polynomial equations.
- The Euclidean isometry group is given by
$$\mathbf{Isom}(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n | T(x) = Ax + b \text{ for } A \in \mathbb{O}(n, \mathbb{R}), b \in \mathbb{R}^n\}.$$

Let us state some needed facts.

- A product of Lie groups forms a Lie group where the product operation is obviously defined.
- A covering space of a connected Lie group forms a Lie group. Here, we need to specify which element of the inverse image of the identity is the identity element.
- A *Lie subgroup* of a Lie group is a closed subgroup that is a Lie group with the induced operation and is a submanifold. For example, consider
  - $\mathbb{SO}(n, \mathbb{R}) \subset \mathbb{SL}(n, \mathbb{R}) \subset \mathbb{GL}(n, \mathbb{R})$.
  - $\mathbb{O}(n, \mathbb{R}) \subset \mathbf{Isom}(\mathbb{R}^n)$.

A *homomorphism* $f : G \to H$ of two Lie groups G and H is a smooth map that is a group homomorphism. The above inclusion maps are homomorphisms. The kernel of a homomorphism is a closed normal subgroup. Hence it is a Lie subgroup also. f induces the unique homomorphism of the Lie algebra of G to that of H which equals the differential $D_e f$ of f at the identity e of G and conversely. (See Subsection 2.2.3 for the definition of the Lie algebras and their homomorphsms.)

2.2.3 *Lie algebras*

A *Lie algebra* is a real or complex vector space V with a bilinear operation $[,] : V \times V \to V$ that satisfies:

- $[x, x] = 0$ for every $x \in V$ (thus, $[x, y] = -[y, x]$),
- and the Jacobi identity: $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in V$.

As examples, we have:

- Sending $V \times V$ to the zero-element O form a Lie algebra. This is defined to be the *abelian Lie algebras*.
- The direct sum of Lie algebras is a Lie algebra.
- A *subalgebra* is a subspace closed under the bracket $[,]$.
- An *ideal* K of V is a subalgebra such that $[x, y] \in K$ for $x \in K$ and $y \in V$.

A homomorphism of a Lie algebra is a linear map preserving $[,]$. The kernel of a homomorphism is an ideal.

2.2.4 *Lie groups and Lie algebras*

Let G be a Lie group. For an element $g \in G$, a *left translation* $L_g : G \to G$ is given by $x \mapsto g(x)$. A *left-invariant vector field* of G is a vector field X so that the left translation leaves it invariant, i.e., $DL_g(X(h)) = X(gh)$ for $g, h \in G$.

- The set of left-invariant vector fields forms a vector space under addition and scalar multiplication and is a vector space isomorphic to the tangent space at I. Moreover, the bracket $[,]$ is defined as vector-fields brackets. This forms a Lie algebra.
- The Lie algebra of G is the Lie algebra of the left-invariant vector fields on G.

A Lie algebra of an abelian Lie group is abelian.

The Lie algebra η of a Lie subgroup H is clearly a Lie subalgebra of the Lie algebra of G: A vector tangent to H at a point h_0 is realizable as a path in H passing h_0. A left-invariant vector field tangent to H at some point is always tangent to H at every point of H since H is closed under left-multiplications by elements of H. The Lie bracket operation is viewed as the derivative of the commutator of two flows generated by two left-invariant vector fields. Therefore, the Lie bracket is a closed operation for any tangent left-invariant vector fields of H.

Let $\mathfrak{gl}(n, \mathbb{R})$ denote the $M_n(\mathbb{R})$ with $[,] : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $[A, B] = AB - BA$ for $A, B \in M_n(\mathbb{R})$. The Lie algebra of $\mathbb{GL}(n, \mathbb{R})$ is isomorphic to $\mathfrak{gl}(n, \mathbb{R})$:

- For X in the Lie algebra of $\mathbb{GL}(n, \mathbb{R})$, we can define a flow generated by X and a path $X(t)$ along it where $X(0) = \mathrm{I}$ for the identity I.
- We obtain an element of $\mathfrak{gl}(n, \mathbb{R})$ by taking the derivative of $X(t)$ at 0 seen as a matrix.
- Now, we show that the brackets are preserved. That is, a vector-field bracket becomes a matrix bracket by the above map. (See the book [Bishop and Crittendon (2002)] for these computations.)

Thus, for any Lie algebra of any finite quotient Lie group of a Lie subgroup of $\mathbb{GL}(n, \mathbb{R})$, the bracket is computed by matrix brackets.

Given X in the Lie algebra $\mathfrak{g}$ of G, we find an integral curve $X(t)$ through I. We define the *exponential map* $\exp : \mathfrak{g} \to G$ by sending X to $X(1)$. The exponential map is defined everywhere, smooth, and is a diffeomorphism near O. With some work, we can show that the matrix exponential defined by

$$A \mapsto e^A = \sum_{i=0}^{\infty} \frac{1}{k!} A^k$$

is the exponential map $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathbb{GL}(n, \mathbb{R})$ from the computation

$$\frac{d}{dt} \left(e^{tA} \right)\big|_{t=1} = e^A A = L_{e^A}(A) = D(L_{e^A})(A)$$

for $A \in \mathfrak{gl}(n, \mathbb{R})$. Hence, this holds for any Lie subgroup of $\mathbb{GL}(n, \mathbb{R})$ and corresponding Lie subalgebra.

(See for example the books [Warner (1983); Bishop and Crittendon (2002)].)

2.2.5 *Lie group actions*

A *left* Lie group G-*action* on a smooth manifold X is given by a smooth map $k : G \times X \to X$ so that $k(e, x) \mapsto x$ and $k(gh, x) = k(g, k(h, x))$. Normally, $k(g, x)$ is simply written $g(x)$. In other words, denoting by $\mathbf{Diff}(X)$ the group of diffeomorphisms of X, k gives us a homomorphism k' $G \to \mathbf{Diff}(X)$ so that $k'(gh)(x) = k'(g)(k'(h)(x))$ and $k'(I) = I_X$. This is said to be the *left-action.* (We will not use notations k and k' in most cases.)

- A *right action* satisfies $(x)(gh) = ((x)g)h$ or more precisely, $(gh)(x) = (h(g(x)))$.
- Define $\chi(X)$ to be the real vector space of vector fields on X. Each Lie algebra element corresponds to a vector field on X by a homomorphism $\chi_{(X,G)} : \mathfrak{g} \to \chi(X)$ defined by $\chi_{(X,G)}(\eta) = \vec{v}$ satisfying

$$\left.\frac{d}{dt}\right|_{t=0} k(\exp(t\eta), x) = \vec{v}(x) \text{ for all } x \in X.$$

- The action is *faithful* if $g(x) = x$ for all x, then g is the identity element of G. This means that only e corresponds to the identity on X. (If this is true in particular, then the correspondence $\chi_{(X,G)}$ is injective.)
- The action is *transitive* if given two points $x, y \in X$, there is $g \in G$ such that $g(x) = y$.

As examples, consider

- $\mathbb{GL}(n, \mathbb{R})$ acting on $\mathbb{R}^n - \{O\}$ faithfully and transitively.
- $\mathbb{PGL}(n+1, \mathbb{R})$ acting on $\mathbb{RP}^n$ faithfully and transitively.

2.3 Pseudo-groups and $\mathcal{G}$-structures

In this section, we introduce pseudo-groups. Topological manifolds and its submanifolds are very wild and complicated objects to study as the topologist in 1950s and 1960s found out. The pseudo-groups will be used to put "calming" structures on manifolds.

Often the structures will be modeled on some geometries. We are mainly interested in classical geometries. We will be concerned with a Lie group G acting on a manifold M. Most obvious ones are Euclidean geometry where G is the group of rigid motions acting on the Euclidean space $\mathbb{R}^n$. The spherical geometry is one where G is the group $\mathbb{O}(n+1)$ of orthogonal transformations acting on the unit sphere $\mathbf{S}^n$.

Topological manifolds form too large a category to understand sufficiently. To restrict the local property, we introduce *pseudo-groups*. A *pseudo-group* $\mathcal{G}$ on a topological space X is the set of homeomorphisms between open sets of X so that the following statements hold:

- The domains of $g \in \mathcal{G}$ cover X.
- The restriction of $g \in \mathcal{G}$ to an open subset of its domain is also in $\mathcal{G}$.
- The composition of two elements of $\mathcal{G}$ when defined is in $\mathcal{G}$.
- The inverse of an element of $\mathcal{G}$ is in $\mathcal{G}$.
- If $g : U \to V$ is a homeomorphism for open subsets U, V of X, and if U is a union of open sets U_α for $\alpha \in I$ for some index set I such that $g|U_\alpha$ is in $\mathcal{G}$ for each α, then g is in $\mathcal{G}$.

Let us give some examples:

- The *trivial pseudo-group* is one where every element is a restriction of the identity $X \to X$ to an open subset.
- Any pseudo-group contains a trivial pseudo-group.
- The maximal pseudo-group of $\mathbb{R}^n$ is TOP formed from the set of all homeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group C^r, $r \geq 1$, is formed from the set of C^r-diffeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group PL is formed from the set of piecewise linear homeomorphisms between open subsets of $\mathbb{R}^n$.
- A (G, X)*-pseudo-group* is defined as follows. Let G be a Lie group acting on a manifold X faithfully and transitively. Then we define the pseudo-group as the set of all pairs $(g|U, U)$ for $g \in G$ where U is an open subset of X.
- The group $\mathbf{Isom}(\mathbb{R}^n)$ of rigid motions acting on $\mathbb{R}^n$ or the orthogonal group $\mathbb{O}(n+1, \mathbb{R})$ acting on $\mathbf{S}^n$ gives examples.

2.3.1 $\mathcal{G}$-*manifolds*

A $\mathcal{G}$-manifold is obtained as a manifold with special type of gluing only in $\mathcal{G}$: Let $\mathcal{G}$ be a pseudo-group on a manifold X. A $\mathcal{G}$-*manifold* is an n-manifold M with a maximal $\mathcal{G}$-atlas.

A $\mathcal{G}$-*atlas* is a collection of charts (embeddings) $\phi : U \to X$ where U is an open subset of M such that whose domains cover M and any two charts are $\mathcal{G}$-compatible.

- Two charts $(U, \phi), (V, \psi)$ are $\mathcal{G}$-*compatible* if the transition map satisfies

$$\gamma = \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \in \mathcal{G}.$$

A set of $\mathcal{G}$-atlases is a partially ordered set under the ordering given by the inclusion relation. Two $\mathcal{G}$-atlases are *compatible* if any two charts in the atlases are $\mathcal{G}$-compatible. In this case, the union is another $\mathcal{G}$-atlas. One can choose a locally finite $\mathcal{G}$-atlas from a given maximal one and conversely. We obtain that the set of compatible $\mathcal{G}$-atlases has a unique maximal $\mathcal{G}$-atlas.

Under the compatibility relation, we obtain that the set of all $\mathcal{G}$-structures is partitioned into equivalence classes. We define the $\mathcal{G}$-*structure* on M as a maximal $\mathcal{G}$-atlas or as an equivalence class in the above partition.

The manifold X is trivially a $\mathcal{G}$-manifold if $\mathcal{G}$ is a pseudo-group on X. A topological manifold has a TOP-structure. A C^r-*manifold* is a manifold with a C^r-structure. A *differentiable manifold* is a manifold with a C^∞-structure. A *PL-manifold* is a manifold with a PL-structure.

A $\mathcal{G}$-map $f : M \to N$ for two $\mathcal{G}$-manifolds is a local homeomorphism or even an immersion so that if f sends a domain of a chart ϕ into a domain of a chart ψ, then

$$\psi \circ f \circ \phi^{-1} \in \mathcal{G}.$$

That is, f is an element of $\mathcal{G}$ locally up to charts.

Given two manifolds M and N, let $f : M \to N$ be a local homeomorphism. If N has a $\mathcal{G}$-structure, then so does M so that the map is a $\mathcal{G}$-map. A $\mathcal{G}$-atlas is given on M by taking open sets so that they map into open sets with charts in N under f and then use the induced charts. This $\mathcal{G}$-structure is said to be the *induced* $\mathcal{G}$-*structure*.

Suppose that M has a $\mathcal{G}$-structure. Let Γ be a discrete group of $\mathcal{G}$-homeomorphisms of M acting properly and freely. Then M/Γ has a $\mathcal{G}$-structure. The charts will be the charts of the lifted open sets. The $\mathcal{G}$-structure here is said to be the *quotient* $\mathcal{G}$-*structure.* (Sullivan and Thurston (1983) explain a class of such examples such as θ-annuli and π-annuli that arise in the study of complex projective and real projective surfaces.)

The torus T^n has a C^r-structure and a PL-structure since so does $\mathbb{R}^n$ and the each element of the group of translations all preserves these structures.

Given a pair (G, X) of Lie group G acting on a manifold X, we define a (G, X)-*structure* as a $\mathcal{G}$-structure and a (G, X)-*manifold* as a $\mathcal{G}$-manifold where $\mathcal{G}$ is the (G, X)-pseudo-group.

A *Euclidean manifold* is an $(\mathbf{Isom}(\mathbb{R}^n), \mathbb{R}^n)$-manifold.
A *spherical manifold* is an $(\mathbb{O}(n+1), \mathbf{S}^n)$-manifold.

2.4 Differential geometry

We wish to understand geometric structures in terms of differential geometry, i.e., methods of bundles, connections, and so on, since such an understanding helps us to see the issues in different ways. Actually, this is not central to the book. However, we should try to relate to the traditional fields where our subject can be studied in another way.

2.4.1 *Riemannian manifolds*

A differentiable manifold has a Riemannian metric, i.e., an inner-product at each tangent space that is smooth with respect coordinate charts. Such a manifold is said to be a Riemannian manifold.

An *isometric immersion* (embedding) of a Riemannian manifold to another one is a (one-to-one) map that preserves the Riemannian metric. We will be concerned with isometric embeddings of M into itself usually. A length of an arc is the value of an integral of the norm of tangent vectors to the arc. This gives us a metric on a manifold. An isometric embedding of M into itself is always an isometry. A geodesic is an arc minimizing length locally.

The *sectional curvature* $K(p)$ of a Riemannian metric along a 2-plane at a point p is given as the rate of area growth of r-balls on a disk $D(p)$ composed of geodesics from p tangent to a 2-plane:

$$K(p) = \lim_{r \to 0_+} 12 \frac{\pi r^2 - A(r)}{\pi r^4}$$

where $A(r)$ is the area of the r-ball centered at p in $D(p)$ with the induced metric. (See Page 133 of the book [Do Carmo (1992)]. This is the Bertrand-Diquet-Puiseux theorem.)

A constant curvature manifold is one whose sectional curvature is identical to a constant in every planar direction at every point.

- A Euclidean space E^n with the standard norm metric of a constant curvature $= 0$.
- A sphere $\mathbf{S}^n$ with the standard induced metric from $\mathbb{R}^{n+1}$ has a constant curvature $= 1$.
- Given a discrete torsion-free subgroup Γ of the isometry group of E^n (resp. $\mathbf{S}^n$. we obtain E^n/Γ (resp. $\mathbf{S}^n/\Gamma$) a manifold with a constant curvature $= 0$ (resp. 1).

2.4.2 *Principal bundles and connections: flat connections*

Let M be a manifold and G a Lie group. A *principal fiber bundle* P over M with a group G is the object satisfying

- P is a manifold.
- G acts freely on P on the right given by a smooth map $P \times G \to P$.
- $M = P/G$ and the map $\pi : P \to M$ is differentiable.
- P is locally trivial. That is, there is a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$ for at least one neighborhood U of any point of M.

We say that P is the *bundle space*, M is the *base space*, and $\pi^{-1}(x)$ is a *fiber* which also equals $\pi^{-1}(x) = \{ug|g \in G\}$ for any choice of $u \in \pi^{-1}(x)$. G is said to be the *structure group*.

As an example, consider $L(M)$: the set of all frames of the tangent bundle $T(M)$. One can give a topology on $L(M)$ so that sending a frame to its base point is the smooth quotient map $L(M) \to M$. $\mathbb{GL}(n, \mathbb{R})$ acts freely on $L(M)$. We can verify that $\pi : L(M) \to M$ is a principal bundle.

Given a collection of open subsets U_α covering M, we construct a bundle by a collection of mappings

$$\{\phi_{\beta,\alpha} : U_\alpha \cap U_\beta \to G\}$$

satisfying

$$\phi_{\gamma,\alpha} = \phi_{\gamma,\beta} \circ \phi_{\beta,\alpha}, \phi_{\alpha,\alpha} = \mathrm{I}$$

for any triple $U_\alpha, U_\beta, U_\gamma$. Then form $U_\alpha \times G$ for each α. For any pair $U_\alpha \times G$ and $U_\beta \times G$, identify by $\tilde{\phi}_{\beta,\alpha} : U_\alpha \times G \to U_\beta \times G$ given by $(x, g) \mapsto (x, \phi_{\beta,\alpha}(x)(g))$. The quotient space is a principal bundle over M.

A principal bundle over M with the structure group G is often denoted by $P(G, M)$. Given two Lie groups G and G', and a monomorphism $f : G' \to G$, we call a map $f : P(G', M) \to P(G, M)$ inducing identity $M \to M$ a *reduction* of the structure group G to G'. There may be many reductions for given G' and G. We say that $P(G, M)$ is reducible to $P(G', M)$ if and only if $\phi_{\alpha,\beta}$ can be taken to be in G'. (See the books [Kobayashi and Nomizu (1997); Bishop and Crittendon (2002)] for details.)

2.4.2.1 *Associated bundles*

Let F be a manifold with a left-action of G. G acts on $P \times F$ on the right by

$$g : (u, x) \to (ug, g^{-1}(x)), g \in G, u \in P, x \in F.$$

Form the quotient space $E = P \times_G F$ with a map $\pi_E : E \to M$ induced from the projection $\pi : P \times F \to M$ and we can verify that $\pi_E^{-1}(U)$ is identifiable with $U \times F$ up to making some choices of sections on U to P. The space E is said to be the *associated bundle* over M with M as the *base space*. The *structure group* is the same

G. The induced quotient map $\pi_E : E \to M$ has a fiber $\pi_E^{-1}(x)$ diffeomorphic to F for any $x \in M$.

Here E can also be built from a cover U_α of M by taking $U_\alpha \times F$ and pasting by appropriate diffeomorphisms of F induced by elements of G as above.

The tangent bundle $T(M)$ is an example. $\mathbb{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ on the left. Let $F = \mathbb{R}^n$. We obtain $T(M)$ as $L(M) \times_{\mathbb{GL}(n,\mathbb{R})} \mathbb{R}^n$. A tensor bundle $T_s^r(M)$ is another example. $\mathbb{GL}(n, \mathbb{R})$ acts on the space of (r, s)-tensors $T_s^r(\mathbb{R}^n)$, and let F be $T_s^r(\mathbb{R})$. Then we obtain $T_s^r(M)$ as $L(M) \times_{\mathbb{GL}(n,\mathbb{R})} T_s^r(\mathbb{R}^n)$.

2.4.2.2 *Connections*

Let $P(M, G)$ be a principal bundle. A *connection* is a decomposition of each $T_u(P)$ for each $u \in P$ so that the following statements hold:

- $T_u(P) = G_u \oplus Q_u$ where G_u is a subspace tangent to the fiber. (G_u is said to be the *vertical space* and Q_u the *horizontal space*.)
- $Q_{ug} = R_g^*(Q_u)$ for each $g \in G$ and $u \in P$.
- Q_u depends smoothly on u.

Let $\mathfrak{g}$ denote the Lie algebra of G. More formally, we define a connection as a $\mathfrak{g}$-valued form ω on P is given as $T_u(P) \to G_u$ obtained by taking the vertical component of each tangent vector of P: We could define a *connection* as a smooth $\mathfrak{g}$-valued form ω.

- $\omega(A^*) = A$ for every $A \in \mathfrak{g}$ and A^* the fundamental vector field on P generated by A, i.e., the vector field tangent to the one parameter group of diffeomorphisms on P generated by the action of $\exp(tA) \in G$ at $t = 0$.
- $(R_g)^*\omega = Ad(g^{-1})\omega$.

A *horizontal* lift of a piecewise-smooth path τ on M is a piecewise-smooth path τ' lifting τ so that the tangent vectors are all horizontal. A horizontal lift is determined once the initial point is given.

- Given a curve on M with two endpoints, we find that the lifts of the curve define a parallel displacement between fibers above the two endpoints (commuting with the right G-actions).
- Fixing a point x_0 on M, these parallelisms along closed loops with endpoints at x_0 form a holonomy group that is identifiable with a subgroup of G acting on the left on the fiber at x_0.
- The curvature of a connection is a measure of how much the horizontal lift of a small loop in M differs from a loop in P. A connection is *flat* if the curvatures are zero identically.
- For the flat connections, we can lift homotopically trivial loops in M to loops in P. Thus, the flatness is equivalent to local lifting of small coordinate charts of M to horizontal sections in P.

- A flat connection on P gives us a smooth foliation of dimension n transversal to the fibers where n is the dimension of M. A *flat bundle* is a bundle with a flat connection.

The associated bundle E also inherits a *connection*, i.e., a splitting of the tangent space of E into vertical space and horizontal space. Here again, the vertical spaces are obtained as tangent spaces to fibers. Again given a curve on M, horizontal liftings and parallel displacements between fibers in E make sense . The flatness is also equivalent to the local lifting property, and the flat connection on E gives us a smooth foliation of dimension n transversal to the fibers.

An *affine frame* in a vector (or affine) space $\mathbb{R}^n$ is a set of $n+1$ points $a_0, a_1, \ldots, a_n$ so that $a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0$ form a linear frame. These assignments give us the canonical map from the space of affine frames $A(\mathbb{R}^n)$ to linear frames $L(\mathbb{R}^n)$. An affine group $\mathbb{A}(n, \mathbb{R})$ acts on $A(\mathbb{R}^n)$ also by sending $(a_0, a_1, \ldots, a_n)$ to $(L(a_0), L(a_1), \ldots, L(a_n))$ for an affine automorphism $L : \mathbb{R}^n \to \mathbb{R}^n$.

An *affine connection* on a manifold M is defined as follows. An affine frame over M is an affine frame on a tangent space of a point of M, treating as an affine space. The set of all affine frames over a manifold forms a manifold of higher dimension. Let $A(M)$ be the space of affine frames over M with the affine group $\mathbb{A}(n, \mathbb{R})$ acting on it fiberwise on the left.

- The Lie algebra $\mathfrak{a}(n, \mathbb{R})$ is a semi-direct product of $M_n(\mathbb{R})$ and $\mathbb{R}^n$.
- There is a natural map $A(M) \to L(M)$ where $L(M)$ is the set of linear frames over M and is given by the natural map $A(\mathbb{R}^n) \to L(\mathbb{R}^n)$.
- An affine connection on M is a linear connection plus the canonical $\mathbb{R}^n$-valued 1-form. The curvature of the affine connection is the sum of the curvature of the linear connection and the torsion.

A nice example is when M is a 1-manifold, say an open interval I. Then P is $I \times G$, and the associated bundle is $I \times X$. A connection is simply given as an infinitesimal way to connect each fiber by a left multiplication by an element of G. In this case, a connection is flat always and $I \times G$ and $I \times X$ are fibered by open intervals transversal to the fibers.

If M is a circle, then P becomes a mapping circle with fiber X and E a mapping circle with fiber E:

$$\begin{array}{c} G \to P \\ \downarrow \\ \mathbf{S}^1. \end{array}$$

Now, such spaces are classified by a map $\pi_1(\mathbf{S}^1) \to G$.

For the affine connections, let M be an interval I, and let $G = \mathbb{A}(1, \mathbb{R})$ and $X = \mathbb{R}$. Then E is now a strip $I \times \mathbb{R}$. An affine connection gives a foliation on the strip transversal to $\mathbb{R}$ and is invariant under translation in the $\mathbb{R}$-direction.

Even for higher-dimensional manifolds, we can think of a connection as the collection of 1-dimensional ones over each path. The local dependence on paths is measured by the curvature.

2.4.2.3 *The principal bundles and (G, X)-structures.*

Given a manifold M of dimension n and a Lie group G acting on a manifold X of dimension n, we form a principal bundle P over a manifold M and the associated bundle E fibered by X with a flat connection. Suppose that we can choose a section $f : M \to E$ which is transverse everywhere to the foliation given by the flat connection. This gives us a (G, X)-structure. The main reason is that the section f sends an open set of M to a transversal submanifold to the foliation. Locally, the foliation gives us a projection to X. The composition gives us charts. The charts are compatible since E has a left-action.

Conversely a (G, X)-structure gives us a principal bundle P, the associated bundle E, the flat connection and a transverse section f.

We will elaborate this later when we are studying orbifolds and geometric structures in Chapter 6.

2.5 Notes

Chapter 0 and 1 of the book [Hatcher (2002)] and the books [Munkres (1991); Warner (1983)] are good source of preliminary knowledges here. The books [Do Carmo (1992); Kobayashi and Nomizu (1997); Bishop and Crittendon (2002)] give us good knowledge of connections, curvature, and Riemannian geometry. Also, the book [Thurston (1997)] is a source for studying (G, X)-structures and pseudo-groups as well as geometry and Lie groups presented here. Goldman's book [Goldman (1988)] treats materials here also in a more abstract manner.

Chapter 3

Geometry and discrete groups

In this section, we will introduce basic materials in the Lie group theory and geometry and discrete group actions on the geometric spaces.

Geometry will be introduced as in the Erlangen program of Klein. We discuss projective geometry in some depth. Hyperbolic geometry will be given an emphasis by detailed descriptions of models. Finally, we discuss the discrete group actions, the Poincaré polyhedron theorem and the crystallographic group theory.

We will not go into details as these are somewhat elementary topics. A good source of the classical geometry is carefully written down in the book [Berger (2009)]. The rest of material is heavily influenced by the books [Ratcliffe (2006); Thurston (1997)]; however, we sketch the material.

3.1 Geometries

We will now describe classical geometries from Lie group action perspectives, as expounded in the Erlangen program of Felix Klein submerging all classical geometries under the theory of Lie group actions: We think of an (G, X)-*geometry* as the invariant properties of a manifold X under a group G acting on it transitively and effectively. Formally, the (G, X)-geometry is simply the pair (G, X) and we should know everything about the (G, X)-geometry from this pair.

Of course, there are many particular hidden treasures under this pair which should surface when we try to study them.

3.1.1 *Euclidean geometry*

The Euclidean space is $\mathbb{R}^n$ (or denoted $\mathbb{E}^n$) and the group $\mathbf{Isom}(\mathbb{R}^n)$ of rigid motions is generated by $\mathbb{O}(n)$ and T_n the translation group. In fact, we have an inner-product giving us a metric.

A system of linear equations gives us a subspace (affine or linear). Hence, we have a notion of points, lines, planes, and angles. Notice that these notions are invariantly defined under the group of rigid motions. These give us the set theoretical model for the axioms of the Euclidean geometry. Very nice elementary introductions can be found in the books [Berger (2009); Ryan (1987)] for example.

3.1.2 *Spherical geometry*

Let us consider the unit sphere $\mathbf{S}^n$ in the Euclidean space $\mathbb{R}^{n+1}$. The transformation group is $\mathbb{O}(n+1, \mathbb{R})$.

Many great spheres exist and they are subspaces as they are given by a homogeneous system of linear equations in $\mathbb{R}^{n+1}$. The lines are replaced by arcs in great circles and lengths and angles are also replaced by arc lengths and angles in the tangent space of $\mathbf{S}^n$.

A *triangle* is a disk bounded by three geodesic arcs meeting transversally in acute angles. Such a triangle up to the action of $\mathbb{O}(n+1, \mathbb{R})$ is classified by their angles $\theta_0, \theta_1, \theta_2$ satisfying

$$0 < \theta_i < \pi \tag{3.1}$$

$$\theta_0 + \theta_1 + \theta_2 > \pi \tag{3.2}$$

$$\theta_i < \theta_{i+1} + \theta_{i+2} - \pi, i \in \mathbb{Z}_3. \tag{3.3}$$

(See Figure 3.2.)

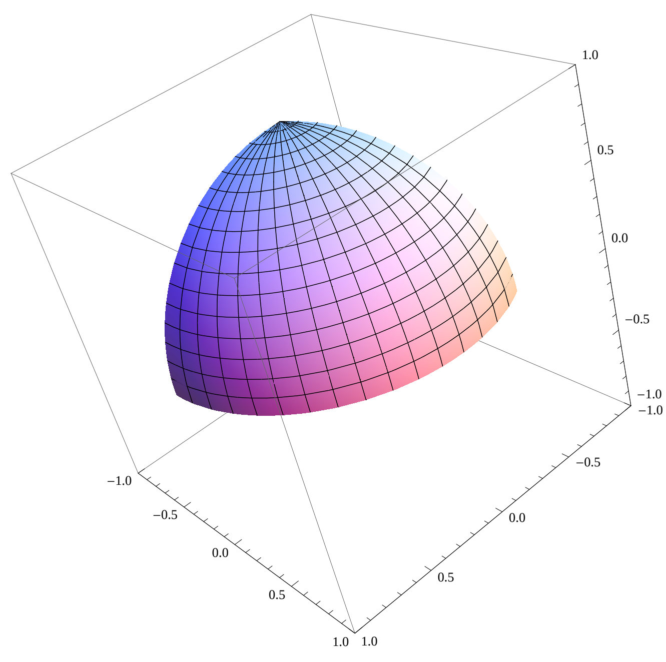

Fig. 3.1 An example of a spherical triangle of angles $2\pi/3, \pi/2, \pi/2$.

Many spherical triangle theorems exist. Given a triangle with angles $\theta_0, \theta_1, \theta_2$

and opposite side lengths l_0, l_1, l_2, we obtain

$$\begin{aligned}\cos l_i &= \cos l_{i+1} \cos l_{i+2} + \sin l_{i+1} \sin l_{i+2} \cos \theta_i,\\ \cos \theta_i &= -\cos \theta_{i+1} \cos \theta_{i+2} + \sin \theta_{i+1} \sin \theta_{i+2} \cos l_i,\\ \frac{\sin \theta_0}{\sin l_0} &= \frac{\sin \theta_1}{\sin l_1} = \frac{\sin \theta_2}{\sin l_2}, i \in \mathbb{Z}_3. \end{aligned} \tag{3.4}$$

(See `http://mathworld.wolfram.com/SphericalTrigonometry.html` for more details and proofs.) This shows for example that a triple of angles detemines the isometry classes of spherical triangles. Also, so does the triples of lengths.

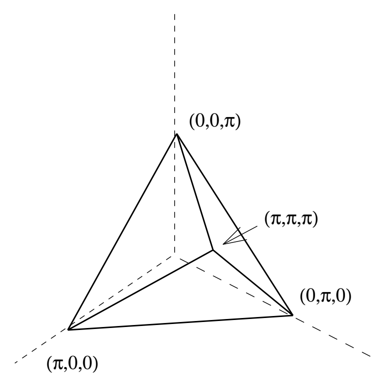

Fig. 3.2 The space of isometric spherical triangles in terms of angle coordinates. See the article [Choi (2011)].

3.1.3 *Affine geometry*

A vector space $\mathbb{R}^n$ becomes an affine space by forgetting about the privileges of the origin. An affine transformation of $\mathbb{R}^n$ is one given by $x \mapsto Ax + b$ for $A \in \mathbb{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. This notion is more general than that of rigid motions.

The Euclidean space $\mathbb{R}^n$ with the group $\mathbb{A}(\mathbb{R}^n) = \mathbb{GL}(n, \mathbb{R}) \cdot \mathbb{R}^n$ of affine transformations forms the affine geometry. Of course, angles and lengths do not make sense. But the notion of lines exists. Also, the affine subspaces that are linear subspaces translated by vectors make sense.

The set of three points in a line has an invariant based on ratios of lengths.

3.1.4 *Projective geometry*

Projective geometry was first considered from fine art. Desargues (and Kepler) first considered points at infinity from the mathematical point of view. Poncelet first added ideal points to the euclidean plane.

A transformation of projecting one plane to another plane by light rays from a point source which may or may not be at infinity is called a *perspectivity*. Projective transformations are compositions of perspectivities. Often, they send finite points to ideal points and vice versa, e.g., perspectivity between two planes that are not parallel. For example, some landscape paintings will have horizons that are from the "infinity" from vertical perspectives. Therefore, we need to add ideal points while the added points are same as ordinary points up to projective transformations.

Lines have well-defined ideal points and are really circles topologically because we added an ideal point at each pair of a direction and its opposite direction. Some notions such as angles and lengths lose meanings. However, many interesting theorems can be proved. Also, theorems always come in dual pairs by switching lines to points and vice versa. Duality of theorems plays an interesting role (Busemann and Kelly, 1953).

A formal definition with topology was given by Felix Klein using homogeneous coordinates. The projective space $\mathbb{RP}^n$ is defined as the quotient space $\mathbb{R}^{n+1} - \{O\}/\sim$ where $\sim$ is given by $v \sim w$ if $v = sw$ for $s \in \mathbb{R} - \{O\}$. Each point is given a homogeneous coordinate: $[v] = [x_0, x_1, \dots, x_n]$ where two homogeneous coordinates are equal if they differ only by a nonzero scalar. That is $[x_0, x_1, \dots, x_n] = [\lambda x_0, \lambda x_1, \dots, \lambda x_n]$ for $\lambda \in \mathbb{R} - \{0\}$. The projective transformation group $\mathbb{PGL}(n+1, \mathbb{R})$ is defined as $\mathbb{GL}(n+1, \mathbb{R})/\sim$ where $g \sim h$ for $g, h \in \mathbb{GL}(n+1, \mathbb{R})$ if $g = ch$ for a nonzero constant c. The group equals the quotient group $\mathbb{SL}_{\pm}(n+1, \mathbb{R})/\{\mathrm{I}, -\mathrm{I}\}$ of the group $\mathbb{SL}_{\pm}(n+1, \mathbb{R})$ of determinant ± 1. Now $\mathbb{PGL}(n+1, \mathbb{R})$ acts on $\mathbb{RP}^n$ where each element sends each ray to a ray by the corresponding general linear map. Each element of g of $\mathbb{PGL}(n+1, \mathbb{R})$ acts by $[v] \mapsto [g'(v)]$ for a representative g' in $\mathbb{GL}(n+1, \mathbb{R})$ of g and is said to be a *projective automorphism*.

Given a basis B of $n+1$ vectors $v_0, v_1, \dots, v_n$ of $\mathbb{R}^{n+1}$, we let $[v]_B = [\lambda_0, \lambda_1, \dots, \lambda_n]_B$ for a point v if we write $v = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n$. Here, $[\lambda_0, \dots, \lambda_n]_B = [c\lambda_0, c\lambda_1, \dots, c\lambda_n]_B$ for $c \in \mathbb{R} - \{0\}$.

Also any homogeneous coordinate change is viewed as induced by a linear map: That is, $[v]_B$ has the same homogeneous coordinate as $[Mv]$ where M is the coordinate change linear map so that $Mv_i = e_i$ for $i = 0, 1, \dots, n$.

- For $n = 1$, $\mathbb{RP}^1$ is homeomorphic to a circle. One considers this as a real line union an infinity.
- A set of points in $\mathbb{RP}^n$ is *independent* if the corresponding vectors in $\mathbb{R}^{n+1}$ are independent. The dimension of a subspace is the maximal cardinality of an independent set minus 1.
- A subspace is the set of points whose representative vectors satisfy a homogeneous system of linear equations. The subspace in $\mathbb{R}^{n+1}$ corresponds to a projective subspace in $\mathbb{RP}^n$ in a one-to-one manner while the dimension drops by 1.
- The affine geometry can be "embedded": $\mathbb{R}^n$ can be identified with the set of points in $\mathbb{RP}^n$ where x_0 is not zero, i.e., the set of points $\{[1, x_1, x_2, \dots, x_n]\}$.

This is called an affine subspace. The subgroup of $\mathbb{PGL}(n+1,\mathbb{R})$ fixing $\mathbb{R}^n$ is precisely $\mathbb{A}(\mathbb{R}^n) = \mathbb{GL}(n,\mathbb{R}) \cdot \mathbb{R}^n$ as can be seen by computations.

- The subspace of points $\{[0, x_1, x_2, \dots, x_n]\}$ is the complement homeomorphic to $\mathbb{RP}^{n-1}$. This is the set of ideal points, i.e., directions in the affine space $\mathbb{R}^n$.
- From affine geometry, one could construct a unique projective geometry and conversely using this idea. (See the book [Berger (2009)] for the complete abstract approach.)
- A hyperspace is given by a single linear equation. The complement of a hyperspace can be identified with an affine space since we can put this into the subspace in the third item.
- A line is the set of points $[v]$ where $v = sv_1 + tv_2$ for $s, t \in \mathbb{R}$ for the independent pair v_1, v_2. Actually a line is $\mathbb{RP}^1$ or a line $\mathbb{R}^1$ with a unique infinity. A point on a line is given a homogeneous coordinate $[s, t]$ where $[s, t] \sim [\lambda s, \lambda t]$ for $\lambda \in \mathbb{R} - \{O\}$.
- $\mathbb{RP}^i$ can be identified to the subspace of points given by $x_0 = 0, \dots, x_{n-i-1} = 0$.
- A subspace is always diffeomorphic to $\mathbb{RP}^i$ for some i, $i = 0, 1, \dots, n$, by a projective automorphism.

The projective geometry has well-known invariants called cross ratios even though lengths of immersed geodesics and angles between smooth arcs are not invariants. (However, we do note that the properties of angles or lengths being $< \pi, = \pi$, or $> \pi$ are invariant properties.)

A *line* is either a subspace of dimension one or a connected subset of it. A *complete affine line* is a complement of a point in a subspace of dimension-one or sometimes we say it is a line of spherical length π. Since they are subsets of a subspace isomorphic to $\mathbb{RP}^1$, we can give it a homogeneous coordinate system $[x_0, x_1]$ regarding it as quotient space of $\mathbb{R}^2 - \{O\}$.

- The cross ratio of four points x, y, z, and t on a one-dimensional subspace $\mathbb{RP}^1$ is defined as follows. There is a unique coordinate system so that $x = [1, 0], y = [0, 1], z = [1, 1], t = [b, 1]$. $b = b(x, y, z, t)$ is defined as the *cross-ratio*. Thus, it is necessary that at least three points x, y, z are mutually distinct.
- If the four points are on a complete affine line, the cross ratio is given as
$$[x, y; z, t] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$
if we find a two-variable coordinate system where
$$x = [1, z_1], y = [1, z_2], z = [1, z_3], t = [1, z_4]$$
by some coordinate change. That is, if x, y, z, and t have coordinates z_1, z_2, z_3, and z_4 respectively in some affine coordinate system of an affine subspace of dimension 1, then the above expression is valid.

- One can define cross ratios of four hyperplanes meeting in a projective subspace of codimension 2. By duality, they correspond to four points on a line.

3.1.4.1 *The $\mathbb{RP}^2$-geometry*

Let us consider $\mathbb{RP}^2$ as an example. We proceed with basic definitions and facts, which can be found in the book [Coxeter (1994)]: We recall that the plane projective geometry is a geometry based on the pair consisting of the projective plane $\mathbb{RP}^2$, the space of lines passing through the origin in $\mathbb{R}^3$ with the group $\mathbb{PGL}(3, \mathbb{R})$, the projectivized general linear group acting on it. $\mathbb{RP}^2$ is considered as the quotient space of $\mathbb{R}^3 - \{O\}$ by the equivalence relation $v \sim w$ iff $v = sw$ for a scalar s.

Here we have a familiar projective plane as topological type of $\mathbb{RP}^2$, which is a Mobiüs band with a disk filled in at the boundary. See `http://www.geom.uiuc.edu/zoo/toptype/pplane/cap/`.

A *point* is an element of $\mathbb{RP}^2$ and a *line* is a codimension-one subspace of $\mathbb{RP}^2$, i.e., the image of a two-dimensional vector subspace of $\mathbb{R}^3$ with the origin removed under the quotient map. Two points are contained in a unique line, and two lines meet at a unique point. Points are *collinear* if they lie on a common line. Lines are *concurrent* if they pass through a common point. A pair of points and/or lines are *incident* if the elements meet with each other.

The dual projective plane $\mathbb{RP}^{2\dagger}$ is given as the space of lines in $\mathbb{RP}^2$. We can identify it as the quotient of the dual vector space $\mathbb{R}^{3,\dagger}$ of $\mathbb{R}^3$ with the origin removed by the scalar equivalence relations as above:

$$\alpha \sim \beta \text{ if } \alpha = s\beta, s \in \mathbb{R} - \{0\}, \alpha, \beta \in \mathbb{R}^{3,\dagger} - \{O\}.$$

An element of $\mathbb{PGL}(3, \mathbb{R})$ acting on $\mathbb{RP}^2$ is said to be a *collineation* or *projective automorphism*. The elements are uniquely represented by matrices of determinant equal to 1. The set of their conjugacy classes is in a one-to-one correspondence with the set of topological conjugacy classes of their actions on $\mathbb{RP}^2$. (Sometimes, we will use matrices of determinant -1 for convenience.)

Among collineations, an order-two element is said to be a *reflection*. It has a unique line of fixed points and an isolated fixed point. Actually, any pair of reflections are conjugate to each other, and given a line and a point not on the line, we can find a unique reflection with these fixed point sets. A reflection will often be represented by a matrix of determinant equal to -1 and the isolated fixed point corresponds to the eigenvector of eigenvalue -1.

Given two lines, we say that a map between the points in one line l_1 to the other l_2 is a *projectivity* or *projective isomorphism* if the map is induced from a rank-two linear map from the vector subspace corresponding to l_1 to that corresponding to l_2.

By *duality*, we mean the one-to-one correspondence between the set of lines in $\mathbb{RP}^2$ with the set of points in $\mathbb{RP}^{2\dagger}$ and one between the points in $\mathbb{RP}^2$ with the lines in $\mathbb{RP}^{2\dagger}$. The correspondence preserves the incidence relationships.

Under duality, a line in $\mathbb{RP}^{2\dagger}$ corresponds to the set of all lines through a point in $\mathbb{RP}^2$, so-called a pencil of lines, and vice-versa.

By duality, given the pencil of lines through a point p and the pencil of lines through another point q, we define that a *projectivity* between the two pencils is a one-to-one correspondence that is the projectivity from the dual line of p to that of q.

Let l_1 and l_2 be two lines and let p_1^1, p_2^1, p_3^1 be three distinct points of l_1 and let p_1^2, p_2^2, p_3^2 be three distinct points in l_2. Then there is a projectivity sending p_i^1 to p_i^2 for $i = 1, 2, 3$.

A quadruple of points in $\mathbb{RP}^2$ in a general position are always equivalent by a collineation. (By a general position, we mean that no three of them are in a line.)

A nonzero vector v in $\mathbb{R}^3$ *represents* a point p of $\mathbb{RP}^2$ if v is in the equivalence class of p or in the ray p. We often label a point of $\mathbb{RP}^2$ by a vector representing it and vice versa by an abuse of notation.

We have another definition.

Definition 3.1. Let y, z, u, v be four distinct collinear points in $\mathbb{RP}^n$ with $u = \lambda_1 y + \lambda_2 z$ and $v = \mu_1 y + \mu_2 z$. The cross-ratio $[y, z; u, v]$ is defined to be $\lambda_2\mu_1/\lambda_1\mu_2$.

Given a set of four mutually distinct points $p_1^1, p_2^1, p_3^1, p_4^1$ on a line l_1 and another such set $p_1^2, p_2^2, p_3^2, p_4^2$ on a line l_2, we obtain a projectivity $l_1 \to l_2$ sending p_i^1 to p_i^2 iff

$$[p_1^1, p_2^1; p_3^1, p_4^1] = [p_1^2, p_2^2; p_3^2, p_4^2].$$

For example, if the coordinates y, z, u, v of four points are $y = 1, z = 0$, and $1 > u > v > 0$ in some affine coordinate system of an affine line, then the cross ratio $[1, 0, u, v]$ equals

$$\frac{1-u}{u}\frac{v}{1-v}$$

which is positive and realizes any values in the open interval $(0, 1)$.

The cross-ratio of four concurrent lines in $\mathbb{RP}^2$ is also defined similarly (see the book [Busemann and Kelly (1953)]) using the dual projective plane where they become four collinear points.

Given a notation $[y, z; u, v]$ with four points y, z, u, v, we usually assume that they are to be on an image of a segment under a projective map where y, z the endpoints and y, v separates u from z. This is the standard position of the four points in this paper.

However, if we exchange y, z or u, v, we obtain a reciprocal. If we exchange y, z and u, v at the same time, we do not change the cross ratios. The symmetry properties of cross ratios are well-known and we skip the discussion here.

3.1.4.2 *Oriented projective geometry*

Note that $\mathbf{S}^n$ double-covers $\mathbb{RP}^n$. Moreover, the group $\mathbb{SL}_{\pm}(n+1,\mathbb{R})$, i.e., linear maps of $\mathbb{R}^{n+1}$ with determinant ± 1, maps to $\mathbb{PGL}(n+1,\mathbb{R})$ with discrete kernels in the center. Then $(\mathbf{S}^n, \mathbb{SL}_{\pm}(n+1,\mathbb{R}))$ defines a geometry called an *oriented projective geometry.*

This is an old idea actually, and there are several advantages working in this space.

Each point is given a homogeneous coordinate: $[v] = [x_0, x_1, \dots, x_n]$ where two homogeneous coordinates are equal if they differ only by a positive scalar; i.e., $[x_0, x_1, \dots, x_n] = [\lambda x_0, \lambda x_1, \dots, \lambda x_n]$ for $\lambda \in \mathbb{R}, \lambda > 0$.

Two points are *antipodal* if their homogeneous coordinates are negatives of the other.

Subspaces are defined by linear equations as above. A great circle is a subspace of dimension 1. A set of a point is not a subspace. A pair of antipodal points is a subspace. The independence is defined as above.

Again a great circle has a homogeneous coordinate system: A great circle is the set of points $[v]$ where $v = sv_1 + tv_2$ for $s, t \in \mathbb{R}$ for the independent pair v_1, v_2. A point on a great circle is given a homogeneous coordinate $[s,t]$ where $[s,t] \sim [\lambda s, \lambda t]$ for $\lambda \in \mathbb{R}, \lambda > 0$. Cross ratios can be defined on four distinct points (x, y, z, t) on a great circle with the first homogeneous coordinates positive.

A *hemisphere* is a subset defined by

$$\{[x_0, x_1, \dots, x_n] | f(x_0, x_1, \dots, x_n) \geq 0\}$$

for a linear function f on $\mathbb{R}^{n+1}$. A *convex subset* of $\mathbf{S}^n$ is a subset such that any two points can be connected by a segment in the subset of length $\leq \pi$. A convex subset is always a subset of a hemisphere of dimension n or $\mathbf{S}^n$ itself. (Under this definition, the intersection of two convex subsets may not be convex. However, if they intersect in their interiors, this problem does not happen.) See the article [Choi (1994a)] for this point of view.

3.1.5 *Conformal geometry*

We can introduce two classes of symmetries of $\mathbb{R}^n$. The first class is the set of reflections of $\mathbb{R}^n$. Let the hyperplane $P(a,t)$ given by $a \cdot x = t$ for a unit vector a. Then the reflection about $P(a,t)$ is given by $\rho(x) = x + 2(t - a \cdot x)a$. The second class is the set of inversions. Let the hypersphere $S(a,r)$ be given by $|x - a| = r$. Then the inversion about $S(a,r)$ is given by $\sigma(x) = a + (\frac{r}{|x-a|})^2(x - a)$.

We compactify $\mathbb{R}^n$ to $\hat{\mathbb{R}}^n = \mathbf{S}^n$ by adding infinity. This is to be accomplished as follows: Let $\mathbf{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and identify $\mathbb{R}^n$ with the subspace $x_{n+1} = -1$. Consider the stereographic projection from the point $(0, 0, \dots, 1)$. Taking the inverse image of $\mathbb{R}^n$ in $\mathbf{S}^n$, we obtain a copy of $\mathbb{R}^n$ in $\mathbf{S}^n$. The usual differentiable structure of $\mathbf{S}^n$ extends that of embedded $\mathbb{R}^n$. Since the stereographic map preserves angles, the angles of $\mathbb{R}^n$ agree with those of the copy in $\mathbf{S}^n$ with the standard metric. The reflections and inversions of $\mathbb{R}^n$ become diffeomorphisms of the copy in $\mathbf{S}^n$, which extend uniquely to real analytic diffeomorphisms of $\mathbf{S}^n$ respectively; that is, their Jacobians are nowhere zero. Since the maps preserve angles almost everywhere, they do so everywhere by a limiting argument. Thus, these reflections and inversions induce conformal homeomorphisms of $\hat{\mathbb{R}}^n = \mathbf{S}^n$; that is, they preserve angles.

- The group of transformations generated by these homeomorphisms is called the *Mobiüs transformation group.*
- They form the conformal transformation group of $\hat{\mathbb{R}}^n = \mathbf{S}^n$.
- For $n = 2$, $\hat{\mathbb{R}}^2$ is the Riemann sphere $\hat{\mathbb{C}}$ and a Mobiüs transformation is a either a fractional linear transformation of form
$$z \mapsto \frac{az+b}{cz+d}, ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$
or a fractional linear transformation pre-composed with the conjugation map $z \mapsto \bar{z}$.
- In higher-dimensions, a description as a sphere of positive null-lines and the special Lorentzian group exists in the Lorentzian space $\mathbb{R}^{1,n+1}$.

3.1.5.1 *Poincaré extensions*

We can identify $\mathbb{E}^{n-1}$ with $\mathbb{E}^{n-1} \times \{O\}$ in $\mathbb{E}^n$ and extend each Mobiüs transformation of $\hat{\mathbb{E}}^{n-1}$ to one of $\hat{\mathbb{E}}^n$ that preserves the upper half space U^n. That is, we extend reflections and inversions in the obvious way: by extending a reflection in $\mathbb{E}^{n-1}$ about a hyperplane to a reflection in $\mathbb{E}^n$ about a hyperplane containing the hyperplane and perpendicular to $\mathbb{E}^{n-1}$, and extending the inversion in $\mathbb{E}^{n-1}$ about a sphere of radius r with center $x \in \mathbb{E}^{n-1}$ to the inversion in $\mathbb{E}^n$ with the same radius and center.

Each Mobiüs transformation m of $\hat{\mathbb{E}}^{n-1}$ is a composition of reflections and inversions, say $r_1 r_2 \dots r_n$. Denoting $\hat{r}_i$ the extension, we let the extension $\hat{m}$ of m be given by $\hat{r}_1 \hat{r}_2 \dots \hat{r}_n$.

- The Mobiüs transformations of $\hat{\mathbb{E}}^n$ that preserve the open upper half space are exactly the extensions of the Mobiüs transformations of $\hat{\mathbb{E}}^{n-1}$. Therefore, $M(U^n)$ is identical with $M(\hat{\mathbb{E}}^{n-1})$.
- We put the pair $(U^n, \hat{\mathbb{E}}^{n-1})$ to $(B^n, \mathbf{S}^{n-1})$ by a Mobiüs transformation η of $\hat{\mathbb{E}}^n$. Thus, $M(U^n)$ is isomorphic to $M(\mathbf{S}^{n-1})$ for the boundary sphere by a conjugation by η.
- By a similar reason to the above, $M(B^n)$ is identical with $M(\mathbf{S}^{n-1})$ by considering the Poincaré extension of reflections and inversions on hyperplanes and spheres orthogonal to $\mathbf{S}^{n-1}$.

3.1.6 *Hyperbolic geometry*

A hyperbolic space $\mathbb{H}^n$ is defined as a complete Riemannian manifold of constant curvature equal to -1. Such a space cannot be realized as a submanifold in a Euclidean space of even very large dimensions. But it is realized as a "sphere" in a Lorentzian space as we will see soon. A *Lorentzian space* is the vector space $\mathbb{R}^{1+n}$ with an inner product

$$x \cdot y = -x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n y_n.$$

We will denote it by $\mathbb{R}^{1,n}$.

- A Lorentzian norm $||x|| = (x \cdot x)^{1/2}$ is a positive number, a positive imaginary number, or zero. The vector is said to be *space-like*, *null*, or *time-like* depending on its norm being positive, zero, or a positive imaginary number.
- The *null* vectors form a light cone divided into a cone of positive null vectors, a cone of negative null vectors, and $\{O\}$.
- The subspace of time-like vectors also has two components where $x_0 > 0$ and $x_0 < 0$ respectively. A time-like vector is also *positive* or *negative* depending on which component it lies in.
- Ordinary notions such as orthogonality can be defined by the Lorentzian inner product. A basis is *orthonormal* if its vectors have norms of 1 or i and they mutually orthogonal. The Gram-Schmidt orthogonalization is possible also for a set of vectors starting with a positive time-like vector.
- A subspace of $\mathbb{R}^{1,n}$ is either *space-like* where all vectors in it are space-like, is *null* where at least one nonzero-vector is null and no vector is time-like, or finally *time-like* where at least one vector is time-like: This can be seen by looking at the restriction of the Lorentzian inner product on the subspace where it could be either positive-definite, semi-definite, or definite with at least one vector with an imaginary norm.
- A pair of space-like vectors v and w spanning a space-like subspace have an angle between them given by the formula $\cos\theta = v \cdot w / ||v|| ||w||$. This can be generalized to the situations where they do not span a space-like subspace and span a null subspace or a time-like subspace. (For details, see the book [Ratcliffe (2006)]).

3.1.6.1 *The Lorentz group*

A *Lorentzian transformation* is a linear map preserving the inner-product. A *Lorentzian matrix* is a matrix corresponding to a Lorentzian transformation under a standard coordinate system. For the diagonal matrix J with entries $-1, 1, \ldots, 1$, $A^t J A = J$ if and only if A is a Lorentzian matrix.

The set of Lorentzian transformations forms a Lie group $\mathbb{O}(1, n)$ given by

$$\{A \in \mathbb{GL}(n+1, \mathbb{R}) | A^t J A = J\},$$

which is a subgroup of $\mathbb{GL}(n+1,\mathbb{R})$. A Lorentzian transformation sends time-like vectors to time-like vectors. Thus, by continuity, it either preserves both components of the subspace of positive time-like vectors or switches the components. It is either *positive* or *negative* if it sends positive time-like vectors to positive time-like ones or negative time-like ones. The set of positive Lorentzian transformations forms a Lie subgroup $\mathbb{PO}(1,n)$.

The quotient map

$$\mathbb{GL}(n+1,\mathbb{R}) \to \mathbb{PGL}(n+1,\mathbb{R})$$

maps the subgroup diffeomorphic to its image subgroup. Hence, there is an inclusion map

$$\mathbb{PO}(1,n) \to \mathbb{PGL}(n+1,\mathbb{R}).$$

We regard the first group as the subgroup of the next.

3.1.6.2 *The hyperbolic space*

For two positive time-like vectors, the subspace spanned by them is time-like and the Lorentzian inner product restricts to an inner product of signature $-1, 1$. In a new coordinate system with coordinate functions s, t, the inner product becomes $-s^2 + t^2$. Since the vectors are positive time-like, the absolute values of second components of the two vectors are smaller than those of the first components. Thus, the Lorentzian inner-product of the two vectors is a negative number. Their norms are positive imaginary numbers, and the absolute value of the inner-product is greater than the product of the absolute values of their norms as can be verified by simple computations. Given $(s_1, t_1), (s_2, t_2), s_i > 0, s_i > t_i$, we can show

$$(-s_1 s_2 + t_1 t_2)^2 > (-s_1^2 + t_1^2)(-s_2^2 + t_2^2),$$

which follows from

$$2 s_1 s_2 t_1 t_2 < s_1^2 t_2^2 + s_2^2 t_1^2.$$

Therefore, there is a *time-like angle* $\eta(x,y)$ for two time-like vectors x and y defined by

$$x \cdot y = |||x||||||y||| \cosh \eta(x,y)$$

where $|||v|||$ for a vector v denotes the absolute value of the norm $||v||$ of v.

A hyperbolic space $\mathbb{H}^n$ is an upper component of the submanifold defined by $||x||^2 = -1$ or $x_0^2 = 1 + x_1^2 + \cdots + x_n^2$. This is a subset of a positive cone, the upper sheet of a hyperboloid. Topologically, it is homeomorphic to $\mathbb{R}^n$ since one realizes it as a graph of the function. Sometimes, this object is called a *hyperboloid model* of the hyperbolic space. (See also `http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/mnkw/`.)

One induces a metric from the Lorentzian space: for two tangent vectors x, y to the hyperboloid, we define $x \cdot y$ by the Lorentzian inner product. Since the tangent vectors at a point u of the hyperboloid is orthogonal to u, the tangent space is space-like and the norms are always positive. This gives us a Riemannian metric of constant curvature -1. (The computation of curvature is very similar to the computations for the sphere.)

A *hyperbolic line* is an intersection of $\mathbb{H}^n$ with a time-like two-dimensional vector subspace. A *triangle* is given by three segments meeting at three vertices. Denote the vertices by A, B, and C and the opposite segments by a, b, and c. By denoting their angles and lengths again by A, B, C, a, b, and c respectively, we obtain

- Hyperbolic law of sines:
$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}.$$
- Hyperbolic law of cosines:
$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C, \tag{3.5}$$
$$\cosh c = \frac{\cosh A \cosh B + \cos C}{\sin A \sin B}. \tag{3.6}$$

One can assign any interior angles to a hyperbolic triangle as long as the sum is less than π. One can assign any lengths to a hyperbolic triangle as long as the lengths satisfy the triangle inequality.

We note that the triangle formula can be generalized to formulas for quadrilaterals, pentagons, hexagons with some right angles. Basic philosophy here is that one can push the vertex outside and the angles become distances between lines. (See the book [Ratcliffe (2006)] or `http://online.redwoods.cc.ca.us/instruct/darnold/staffdev/Assignments/sinandcos.pdf`)

Since $\mathbb{PO}(1, n)$ includes $\mathbb{O}(n, \mathbb{R})$ acting on the subspace given by $x_0 = 0$ and $\mathbb{PO}(1, 1)$ acting transitively on the hyperbolic line through e_0 and $\sqrt{2}e_0 + e_1$, it follows that $\mathbb{PO}(1, n)$ acts transitively on $\mathbb{H}^n$. Given any isometry k, we can find an element $g \in \mathbb{PO}(1, n)$ so that $g \circ k$ fixes e_0 and every vector at the tangent space at e_0. By analyticity of the isometry group, it follows that $k = g^{-1}$. Therefore, the Lie group $\mathbb{PO}(1, n)$ is the isometry group of $\mathbb{H}^n$ and acts faithfully and transitively.

3.1.7 *Models of hyperbolic geometry*

3.1.7.1 *Beltrami-Klein models of hyperbolic geometry*

The hyperboloid model $\mathbb{H}^n$ is a bit complicated in that we have to see a one-dimension higher space to realize its meaning. We will give more intrinsic definitions which are obtainable from the hyperboloid model easily.

The *Klein model* is directly obtained from the hyperboloid model. Recall that an affine patch $\mathbb{R}^n$ in $\mathbb{RP}^n$ is identifiable with a complement of a subspace. A standard one is given by $x_0 \neq 0$. The standard affine patch has coordinate functions $x_1, \ldots, x_n$. There is an embedding from $\mathbb{H}^n$ onto an open unit ball B in the standard affine patch $\mathbb{R}^n$ of $\mathbb{RP}^n$:

$$[x_0, x_1, x_2, \ldots, x_n] \to (x_1/x_0, x_2/x_0, \ldots, x_n/x_0)$$

induced from a standard radial projection $\mathbb{R}^{n+1} - \{O\} \to \mathbb{RP}^n$.

We regard B as a ball of radius 1 with the center at O in $\mathbb{R}^n$. The hyperboloid has a distance metric induced from the Riemannian metric. By the projection, we obtain a distance metric d_k on B. We compute that $d_k(P,Q) = 1/2|\log(ab, PQ)|$ where a, P, Q, b are on a segment with endpoints a, b and

$$[ab, PQ] = \left|\frac{aP}{bP}\frac{bQ}{aQ}\right| \tag{3.7}$$

where aP, bP, bQ, and aQ denote the Euclidean distances between the designated points respectively.

We can verify this formula as follows: The metric is induced on B by the radial projection

$$\pi_{\mathbb{RP}^n} : \mathbb{H}^n \subset \mathbb{R}^{n+1} - \{O\} \to B \subset \mathbb{RP}^n.$$

Since $\lambda(t) = (\cosh t, \sinh t, 0, \ldots, 0)$ define a unit speed geodesic in $\mathbb{H}^n$, we have $d_k([e_1], [(\cosh t, \sinh t, 0, \ldots, 0)]) = t$ for t positive under the Riemannian metric d_k. On the right side of equation 3.7, we compute the same. Since any geodesic segment of same length is congruent under the isometry, we see that the two metrics coincide.

The isometry group $\mathbb{PO}(1,n)$ also maps injectively to a subgroup of $\mathbb{PGL}(n+1, \mathbb{R})$ that preserves B. Since the isometry corresponds to a linear map in $\mathbb{R}^{1+n}$ and it preserves $\mathbb{H}^n$, it follows that an isometry corresponds to a projective automorphism of B. Conversely, we see that a projective automorphism of B preserves d_k because it preserves the cross-ratios and hence, it must come from the isometry. The projective automorphism group of B is precisely $\mathbb{PO}(1,n)$.

- The Beltrami-Klein model is "nice" because you can see outside in $\mathbb{RP}^n$. The outside has the natural structure of the anti-de Sitter space. (See `http://en.wikipedia.org/wiki/Anti_de_Sitter_space`.) We can treat points outside and inside together.
- Each hyperplane in the model is dual (i.e., orthogonal by the Lorentzian inner-product) to a point outside. A point in the model is dual to a hyperplane outside. In fact, any subspace of dimension i is dual to a subspace of dimension $n - i - 1$ by orthogonality.
- For $n = 2$, the dual of a line is given by taking tangent lines to the disk at the endpoints and taking the intersection.
- The distance between two hyperplanes can be obtained by two dual points. The two dual points span a 2-dimensional orthogonal subspace to the both hyperperplanes and hence provide the shortest geodesic.

3.1.7.2 *The conformal ball model (Poincaré ball model)*

We consider a stereo-graphic projection $\mathbb{H}^n$ to the subspace P in $\mathbb{R}^{1+n}$ given by $x_0 = 0$ from the point $(-1, 0, \ldots, 0)$. The formula for the map $\kappa : \mathbb{H}^n \to B_P$ is given by

$$\kappa(y) = \left(\frac{y_1}{1+y_0}, \ldots, \frac{y_n}{1+y_0} \right),$$

where the image is the open ball B_P of radius 1 with the center O in P. The inverse is given by

$$\zeta(x) = \left(\frac{1+|x|^2}{1-|x|^2}, \frac{2x_1}{1-|x|^2}, \ldots, \frac{2x_n}{1-|x|^2} \right).$$

Since this is a diffeomorphism, B_P has an induced Riemannian metric of constant curvature -1. We show by computations

$$\cosh d_{B_P}(x, y) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}.$$

This formula shows that all inversions acting on B_P preserve the metric, and so does the group $M(B_P)$ of Mobiüs transformations of B_P. The corresponding Riemannian metric is $g_{ij} = 2\delta_{ij}/(1-|x|^2)^2$. Note that for two points x, y of B_P, there exists a circle perpendicular to the topological boundary sphere $\mathrm{bd}B_P$ of B_P containing x and y. We can choose a hypersphere passing the midpoint of the segment between x and y. Also, a stabilizer of a point x of B_P is generated by reflections about hyperspheres containing x. Since $M(B_P)$ is generated by reflections about spheres orthogonal to $\mathrm{bd}B_P$, it follows that $M(B_P)$ is transitive on B_P and the stabilizer of a point is easily seen to be isomorphic to $\mathbb{O}(n)$. Since the isometry group of B_P has the same property, it follows that the group of Mobiüs transformations acting on B_P is precisely the isometry group of B_P.

Moreover, $\mathbf{Isom}(B_P)$ can be identified with $M(\mathbf{S}^{n-1})$ where $\mathbf{S}^{n-1}$ is the boundary sphere of B_P (see Section 3.1.5.1).

Geodesics would be lines through O or would be arcs on circles perpendicular to the sphere of radius 1. A *sphere* in $\mathbf{S}^n$ is a sphere in $\mathbb{R}^n$ or the closure of an affine subspace of $\mathbb{R}^n$ in the sphere $\hat{\mathbb{R}}^n$ compactified at ∞. A *horosphere* in B_P is a sphere S in $\mathrm{Cl}(B_P)$ tangent to a point x in $\mathrm{bd}B_P$ with the point $\{x\} = S \cap \mathrm{bd}B_P$ removed. Given a point x of $\mathrm{bd}B_P$, we obtain a one parameter family of horospheres whose closures meet x.

3.1.7.3 *The upper-half space model.*

Let U be the upper half-space in $\mathbb{R}^n$. Then U is homeomorphic to an open ball in the compactification $\hat{\mathbb{R}}^n = \mathbf{S}^n$. Since B_P is an open ball, we can find a Mobiüs transformation sending B_P to U by a composition of two reflections. Now we put B_P to U by the Mobiüs transformation. This gives a Riemannian metric of constant curvature -1 on U.

We have by computations that $\cosh d_U(x, y) = 1 + |x - y|^2/2x_n y_n$ and that the Riemannian metric is given by $g_{ij} = \delta_{ij}/x_n^2$. Then $I(U) = M(U) = M(\mathbb{E}^{n-1})$. Geodesics would be arcs on lines or circles perpendicular to $\mathbb{E}^{n-1}$.

3.1.7.4 *The classification of isometries*

Let U^2 denote the 2-dimensional upper-half space model of the hyperbolic plane and U^3 the 3-dimensional one of the hyperbolic space. The topological boundary $\mathrm{bd}U^2$ in $\mathbf{S}^2$ can be identified with the compactification $\hat{\mathbb{E}}^1$ of the Euclidean line $\mathbb{E}^1$ and $\mathrm{bd}U^3$ in $\mathbf{S}^3$ can be done with the compactification $\hat{\mathbb{E}}^2$ of the Euclidean plane $\mathbb{E}^2$. Since $\hat{\mathbb{E}}^1$ is a circle and $\hat{\mathbb{E}}^2$ equals the complex sphere $\hat{\mathbb{C}}$, we obtain $\mathbf{Isom}^+(U^2) = \mathbb{PSL}(2, \mathbb{R})$ and $\mathbf{Isom}^+(U^3) = \mathbb{PSL}(2, \mathbb{C})$ respectively. In this model, it is easier to classify isometries.

- Apart from the identity, orientation-preserving isometries of hyperbolic plane U^2 can have at most one fixed point. An *elliptic* isometry is one fixing a unique point. A *hyperbolic* isometry is one preserving a unique line. The remaining type one is a *parabolic* isometry. The elliptic, hyperbolic, and parabolic isometries are ones conjugate to
$$\begin{aligned} &z \mapsto \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}, \theta \neq 0 \mod 2\pi, \\ &z \mapsto az, a \neq 1, a \in \mathbb{R}^+, \\ &z \mapsto z + 1 \end{aligned}$$
in $M(U^2)$ respectively.
- Orientation-preserving isometries of a hyperbolic space are classified as loxodromic, hyperbolic, elliptic, or parabolic. A *loxodromic* isometry is one acting on a geodesic translating and having a nonzero rotation angle about the geodesic and fixes two points in $\mathrm{bd}U^3$ corresponding to the endpoints of the geodesic. A *hyperbolic* isometry is one acting on a geodesic translating and having a zero rotation angle about the geodesic and fixes two points in $\mathrm{bd}U^3$ corresponding to the endpoints of the geodesic. An *elliptic* isometry is one acting on a geodesic fixing each points of it and its closure and having a nonzero rotation angle about the geodesic. Finally, a *parabolic* isometry is one fixing no point and acting on no geodesic in U^3 but fixing a unique point in $\mathrm{bd}U^3$ and acts on each of the horsopheres at this point. Up to conjugations, they are represented as Mobiüs transformations on $\mathrm{bd}U^3$ which have forms
  - $z \mapsto \alpha z, \mathrm{Im}\alpha \neq 0, |\alpha| \neq 1$.
  - $z \mapsto az, a \neq 1, a \in \mathbb{R}^+$.
  - $z \mapsto e^{i\theta} z, \theta \neq 0$.
  - $z \mapsto z + 1$.

The proofs are omitted but can be found in standard textbooks such as [Ratcliffe (2006)].

3.2 Discrete groups and discrete group actions

Here, we let X be generally a manifold with some Lie group G acting on it transitively. In order for most of the developed theory to work, we need that X be a sphere $\mathbf{S}^n$ with Lie groups such as $\mathbb{O}(n+1,\mathbb{R})$, $\mathbb{GL}(n+1,\mathbb{R})$, and the Mobiüs transformation group acting on it; $\mathbb{RP}^n$ with $\mathbb{PGL}(n+1,\mathbb{R})$ acting on it; $\mathbb{R}^n$ with $\mathbb{O}(n,\mathbb{R})\cdot\mathbb{R}^n$ or $\mathbb{A}(\mathbb{R}^n) = \mathbb{GL}(n,\mathbb{R})\cdot\mathbb{R}^n$ acting on it; or $\mathbb{H}^n$ with $\mathbb{PO}(1,n)$ acting on it. Sometimes, we cannot let X be a symmetric space with its isometry group even or a complex hyperbolic space. The reason is that there seems to be no good notion of m-planes, i.e., m-dimensional subspaces with pleasant intersection properties. (See Section 3.2.1 for details) It is a hope of geometric topologists that we can overcome these difficulties.

We will present facts for X that will be useful in many cases with some additional assumptions on X. However, the reader may wish to see X as one of the above. These will be mostly sufficient.

Let X be a manifold. A *discrete group* is a group with a discrete topology. (It is usually a finitely generated subgroup of a Lie group.) Any group can be made into a discrete group. We have many notions of a group action $\Gamma \times X \to X$ which induces a homomorphism $\Gamma \to \mathbf{Diff}(X)$ where $\mathbf{Diff}(X)$ denotes the group of diffeomorphisms of X with the C^r-topology ($r \geq 1$):

- The action is *effective* if an element g of Γ corresponds to I_X if and only if g is the identity in Γ. The action is *free* if an element g fixes a point of X if and only if g is the identity in Γ.
- The action is *discrete* if Γ is discrete in the group of homeomorphisms of X with the compact open topology. (We used the fact that $\mathbf{Diff}(X)$ is a subgroup of the group of homeomorphisms.)
- The action has *discrete orbits* if every x has a neighborhood U so that the number of orbit points in U is finite.
- The action is *wandering* if every x has a neighborhood U so that the set of elements γ of Γ so that $\gamma(U) \cap U \neq \emptyset$ is finite.
- The action is *properly discontinuous* if for every compact subset K the set of γ such that $K \cap \gamma(K) \neq \emptyset$ is finite.

The conditions of discrete action, discrete orbit action, wandering action, and properly discontinuous are strictly stronger according to the order presented here as long as X is a manifold. The proof of this fact without the strictness is not very involved by showing that the later condition implies the given condition (see Section 3.5 of [Thurston (1997)]).

- If the action is wandering and free, then the action gives a manifold quotient which is possibly non-Hausdorff.
- The action of Γ is free and properly discontinuous if and only if X/Γ is a (Hausdorff) manifold quotient and $X \to X/\Gamma$ is a covering map.
- Suppose that Γ is a discrete subgroup of a Lie group G acting on X with a compact stabilizer. Then X has a G-invariant Riemannian metric. Any (G, X)-manifold now has an induced Riemannian metric. Suppose that Γ acts properly discontinuously on X. Let us call this the *standard discrete action.*
- A *complete* (G, X)*-manifold* is one isomorphic to X/Γ where Γ acts freely and properly discontinuously. (The notion of completeness agrees with that of the induced Riemannian metric for G acting with compact stabilizers. Hence, this is a natural generalization.)
- We define the *deformation space of complete* (G, X)*-structures* on M as the set of equivalence classes of diffeomorphisms $f : M \to X/\Gamma$ for a discrete subgroup Γ of G acting freely and properly discontinuously with the equivalence relation that $f_1 : M \to X/\Gamma_1 \sim f_2 : M \to X/\Gamma_2$ if there is an (G, X)-diffeomorphism $g : X/\Gamma_1 \to X/\Gamma_2$ where $g \circ f_1$ is isotopic to f_2.
- Suppose that X is simply-connected. For a manifold M, the deformation space of complete (G, X)-structures on M is in a one-to-one correspondence with the space of the conjugacy classes of discrete faithful representations h of $\pi_1(M) \to G$, each of which giving a diffeomorphism $M \to X/h(\pi_1(M))$.

We remark that if we allow G to act on X without the compact stabilizer condition, then we call this *standard flexible type* action.

As examples, we give:

- $\mathbb{R}^2 - \{O\}$ with the group generated by $g_1 : (x, y) \to (2x, y/2)$. This is a free wondering action but is not properly discontinuous.
- $\mathbb{R}^2 - \{O\}$ with the group generated by $g : (x, y) \to (2x, 2y)$. This is a free and properly discontinuous action.
- The modular group $\mathbb{PSL}(2, \mathbb{Z})$ is the group of Mobiüs transformations or isometries of the hyperbolic plane given by

$$z \mapsto \frac{az + b}{cz + d}, a, b, c, d \in \mathbb{Z}, ad - bc = 1.$$

  This is not a free action but a properly discontinuous action on the upper-half space model U^2 of $\mathbb{H}^2$ as the action is a standard discrete one. (See `http://en.wikipedia.org/wiki/Modular_group`.)

3.2.1 *Convex polyhedrons*

For $\mathbf{S}^n$, a *geodesic* is the arc segment in a 1-plane not containing an antipodal pair except at the endpoints. It could be a singleton. For $\mathbb{RP}^n$, a geodesic is just an arc segment in a 1-plane.

Suppose that X is a space where a Lie group G acts effectively and transitively. Furthermore, suppose X has notions of m-planes. An *m-plane* is an element of a collection of submanifolds of X of dimension m so that given generic $m+1$ points, we have a unique one containing them. We require also that every 1-plane contains geodesic between any two points in it if geodesics are defined for the (G, X)-geometry. Obviously, we assume that elements of G send m-planes to m-planes. (For complex hyperbolic spaces, such notions seem to be absent.)

We also need to assume that X satisfies the increasing property: if we are given an m-plane and every set of generic $m+1$-points in it lies in an n-plane for $n \geq m$, then the entire m-plane lies in the n-plane.

For example, any geometry with models in $\mathbb{RP}^n$ and G a subgroup of $\mathbb{PGL}(n+1, \mathbb{R})$ has a notion of m-planes. Thus, hyperbolic, euclidean, spherical, and projective geometries have notions of m-planes. Conformal geometry may not have such notions since a generic pair of points have infinitely many circles through them.

Suppose that the (G, X)-geometry has notions of geodesics well-defined. A *convex subset* of X is a subset A such that for any pair of points of A, there exists a geodesic segment in A between them. (We caution the readers that the intersection of two convex subsets may not be convex under this definition.)

A *convex hull* of a subset A is a minimal convex subset in X containing A. This is usually a well-defined set.

Assume that X is $\mathbf{S}^n$, $\mathbb{R}^n$, $\mathbb{H}^n$, or $\mathbb{RP}^n$ with Lie groups acting on X. Let us state some facts about convex sets:

- The dimension of a convex set is the least integer m such that C is contained in a unique m-plane $\hat{C}$ in X.
- The interior C^o and the boundary ∂C are defined as the topological interior and the topological boundary in $\hat{C}$ respectively.
- The closure of C is in $\hat{C}$. If C is convex, then the interior and the closure are convex. They are domains with the dimensions equal to that of $\hat{C}$.
- A *side* of C is a nonempty maximal convex subset of ∂C.
- A *convex polyhedron* is a nonempty closed convex subset such that the set of sides is locally finite in X.

3.2.2 *Convex polytopes*

Using the Beltrami-Klein model, the open unit ball B, i.e., the hyperbolic space, is a subset of an affine patch $\mathbb{R}^n$. In $\mathbb{R}^n$, one can talk about convex hulls.

- A *convex polytope* in $B = \mathbb{H}^n$ is a convex polyhedron with finitely many vertices and is the convex hull of its vertices in $B = \mathbb{H}^n$.

- A polyhedron P in $B = \mathbb{H}^n$ is a *generalized convex polytope* if its closure is a polytope in the affine patch. A generalized polytope may have ideal vertices. An *ideal vertex* is a vertex in the boundary of B. The triangle with all three vertices at the boundary of B is said to be the *ideal triangle*.
- For $X = \mathbb{RP}^n$ or $\mathbf{S}^n$, a *convex polytope* is given as a convex polyhedron in an affine patch or an open hemisphere with finitely many vertices and is a convex hull of its vertices.
- In general, for X with notions of m-planes, we define a *convex polytope* as above.

Note here that these definitions do not depend on the model of the hyperbolic space almost by coincidence. Of course, one needs to verify this.

A compact simplex which is a convex hull of $n+1$ points in $B = \mathbb{H}^n$ is an example of a convex polytope.

Take the origin O in B, and its tangent space $T_O B$. (In fact, O could be any point.) Start from the origin O in $T_O B$ and expand the infinitesimal euclidean polytope from an interior point radially in $T_O B$ using linear expansion maps given by scalars. Now map the vertices of the convex polytope by an exponential map to B. The convex hull of the vertices is a convex polytope. Thus for any Euclidean polytope, we obtain a one parameter family of hyperbolic polytopes. (We caution that sometimes the combinatorial structures of the polytope might change. But in many cases, they do not.)

A regular hyperbolic dodecahedron with all dihedral angles $\pi/2$ as seen from inside is pictured in Figure 3.6. This is to be constructed by the above method. Actually, the dihedral angle changes from near 116.565 degrees which is realized by a very small regular hyperbolic dodecahedron, i.e., when s is very small, to 60 degrees which is realized by an ideal dodecahedron, i.e., when $s = +\infty$. Therefore, the regular hyperbolic dodecahedron of 90-degree dihedral angles is achievable. (See also `http://demonstrations.wolfram.com/HyperbolizationOfADodecahedron/`.)

3.2.3 *The fundamental domains of discrete group actions*

Recall $\mathbf{S}^n$ with spherical geometry, $\mathbb{E}^n$ with Euclidean geometry and $\mathbb{H}^n$ with a hyperbolic geometry. Let X be $\mathbf{S}^n, \mathbb{E}^n$ or $\mathbb{H}^n$ or more generally a geometrical space with m-planes. Let Γ be a group acting on X. A *fundamental domain* for Γ is an open domain F so that $\{gF|g \in \Gamma\}$ is a collection of disjoint sets and their closures cover X. The fundamental domain is *locally finite* if the above closures are locally finite.

Suppose that X is either a hyperbolic, euclidean, or spherical space. Then the *Dirichlet domain* $D(u)$ for $u \in X$ is the intersection of all

$$H_g(u) = \{x \in X | d(x,u) < d(x, gu)\}, g \in \Gamma - \{\mathrm{I}\}.$$

Then the closure of $D(u)$ is a convex fundamental polyhedron. If X/Γ is compact, and Γ acts properly discontinuously, then $D(u)$ is a convex polytope. (If X is some other types of geometries, this is somewhat only vaguely understood.)

The regular octagon example of a hyperbolic surface of genus 2 is an example of a Dirichlet domain $D(u)$ with the origin as u. (See Figure 3.3.)

3.2.4 *Side pairings and the Poincaré fundamental polyhedron theorem*

A *tessellation* of X is a locally finite collection of polyhedra covering X with mutually disjoint interiors.

If P is a convex fundamental polyhedron of a discrete group Γ of isometries acting on X, then Γ is generated by

$$\Phi = \{g \in \Gamma | P \cap g(P) \text{ is a side of } P\} :$$

To see this, let g be an element of Γ, and let us choose a point x of P^o and consider its image $g(x)$ in $g(P^o)$. Then we choose a path from the initial point x to the terminal point $g(x)$. We perturb the path so that it meets only the interiors of the sides of the tessellating polyhedrons. Each time the path crosses a side $h(S)$ for a translate $h(P)$ for an element h of Γ, we take the side-pairing g_S obtained as below. Then multiplying all such side-pairings in the reverse order to what occurred, we obtain an element $g' \in \Gamma$ so that $g'(P) = g(P)$ as hg_Sh^{-1} moves $h(P)$ to the image of P adjacent in the side $h(S)$ for every $h \in \Gamma$. Since P is a fundamental domain, $g^{-1}g'$ is the identity element of Γ.

- Given a side S of a convex fundamental domain P, there is a unique element g_S such that $S = P \cap g_S(P)$. And $S' = g_S^{-1}(S)$ is also a side of P.
- $g_{S'} = g_S^{-1}$ since $S' = P \cap g_S^{-1}(P)$.
- The Γ-side-pairing is the set of g_S for sides S of P.
- The equivalence class at P is generated by $x \cong x'$ if there is a side-pairing sending x to x' for $x, x' \in P$.
- The equivalence class $[x]$ is finite for $x \in P$ where $[x]$ equals $P \cap \Gamma(x)$.

- A cycle relation for each side S of P.
  - Let $S_1 = S$ for a given side S. Choose the side R of S_1. Obtain S_1'. Let S_2 be the side adjacent to S_1' so that $g_{S_1}(S_1' \cap S_2) = R$ and so on. We obtain $S_1, S_1', S_2, S_2', \ldots, S_i, S_i'$.
  - Let S_{i+1} be the side of P adjacent to S_i' such that

$$g_{S_i}(S_i' \cap S_{i+1}) = S_{i-1}' \cap S_i.$$

- Then we obtain
  - There is an integer l such that $S_{i+l} = S_i$ for each i.
  - $\sum_{i=1}^{l} \theta(S_i', S_{i+1}) = 2\pi/k$ where θ is the dihedral angle measure on X.
  - $g_{S_1} g_{S_2} \cdots g_{S_l}$ has order k.

  - The period l is the number of sides of codimension one coming into the image of a given side R of codimension two in X/Γ. (Of course l depends on the side R.)

- If X does not have a G-invariant metric, we have instead of the angle condition that for each $x \in R^o$, there exists a neighborhood N_i in P of x_i identified to x by $g_{S_1}g_{S_2}\cdots g_{S_i}$ for each i, $1 \leq i \leq l$ so that we obtain a neighborhood of x in X of form

$$N \cup g(N) \cup \cdots \cup g^{k-1}(N) \text{ where } g := g_{S_1}g_{S_2}\cdots g_{S_l} \text{ and}$$
$$N := g_{S_1}(N_1) \cup g_{S_1}g_{S_2}(N_2) \cup \cdots \cup g_{S_1}g_{S_2}\cdots g_{S_l}(N_l).$$

Also, these are all the relations since we can push any relation disk occurring in the presentation to be transversal to the codimension 2-sides of the images of P under Γ. Thus, any such disk reduces to a union of disks meeting the codimension 2-sides once. Thus, if Γ has a convex fundamental polytope, Γ is finitely presented.

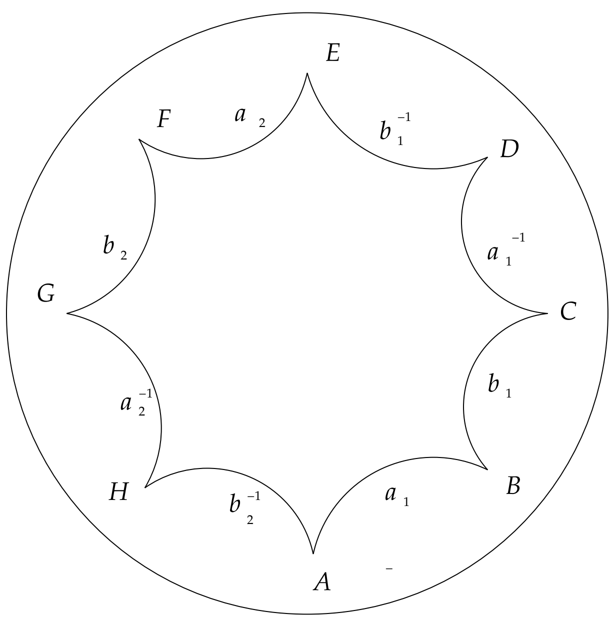

Fig. 3.3 Example: the octahedron in the hyperbolic plane identified to be a genus 2-surface. There is the cycle $(a_1, A), (a_1^{-1}, D), (b_1^{-1}, D), (b_1, C), (a_1^{-1}, C), (a_1, B), (b_1, B), (b_1^{-1}, E), (a_2, E), (a_2^{-1}, H), (b_2^{-1}, H), (b_2, G), (a_2^{-1}, G), (a_2, F), (b_2, F), (b_2^{-1}, A), (a_1, A), (a_1^{-1}, D), \ldots$.

The Poincaré fundamental polyhedron theorem is the converse. We claim that the theorem holds for geometries (G, X) with notions of m-planes. (See pp. 80–84 of the book [Kapovich (2009)].):

Theorem 3.2.1. *Let (G, X) be a geometry with notions of m-planes and geodesics and suppose that X has a G-invariant Riemannian metric. Given a convex polyhe-*

dron P in X with side-pairing automorphisms in G satisfying the above relations, then P is the fundamental domain for the discrete subgroup of G generated by the side-pairing isometries.

If every k equals 1, then the result of the face identification is a manifold. Otherwise, we obtain orbifolds. The results are always complete. (See Jeff Weeks `http://www.geometrygames.org/CurvedSpaces/index.html` for examples of hyperbolic or spherical manifolds as seen from "inside". There are more examples there such as Seifert-Weber manifolds and so on.)

When the side-pairing maps are not isometries or equivalently X has no G-invariant metrics, P is a fundamental domain of a manifold M with an immersion to X. The immersions are often embeddings to open domains. See Chapter 8 for some examples. (See the article [Sullivan and Thurston (1983)] for more details.)

We will be particularly interested in reflection groups. Suppose that X has notions of angles between m-planes. A *discrete reflection group* is a discrete subgroup in G generated by reflections in X about sides of a convex polyhedron. Assume that all the dihedral angles are submultiples of π. Then the side-pairing such that each face is side-paired to itself by a reflection satisfies the Poincaré fundamental theorem.

The reflection group has a presentation $\{S_i : (S_iS_j)^{k_{ij}}\}$ where $k_{ii} = 1$ and $k_{ij} = k_{ji}$ which are examples of Coxeter groups. Notice that k_{ij} is finite if and only if the faces corresponding to S_i and S_j meet in a codimension-two side of P.

The triangle groups are examples of discrete reflection groups.

- Find a triangle in X with angles $\pi/a, \pi/b, \pi/c$ submultiples of π where we assume $2 \leq a \leq b \leq c$. This exists always for $X = \mathbf{S}^2, \mathbb{E}^2$, or $\mathbb{H}^2$.
- We divide into three cases $\frac{\pi}{a} + \frac{\pi}{b} + \frac{\pi}{c} > \pi, = \pi, < \pi$. The triangles are then spherical, euclidean, or hyperbolic ones respectively. They exist and are uniquely determined up to isometry.

  - $> \pi$ cases: $(2, 2, c), (2, 3, 3), (2, 3, 4)$, and $(2, 3, 5)$ respectively corresponding to an index-two-extension of dihedral group of order $2c$, a tetrahedral group, an octahedral group, and an icosahedral group.
  - $= \pi$ cases: $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$. The reflections generate the corresponding wall paper groups.
  - $< \pi$ cases: Any other (p, q, r) gives a hyperbolic tessellation group. Thus, there are infinitely many such groups. (See Proposition 3.2.2.)

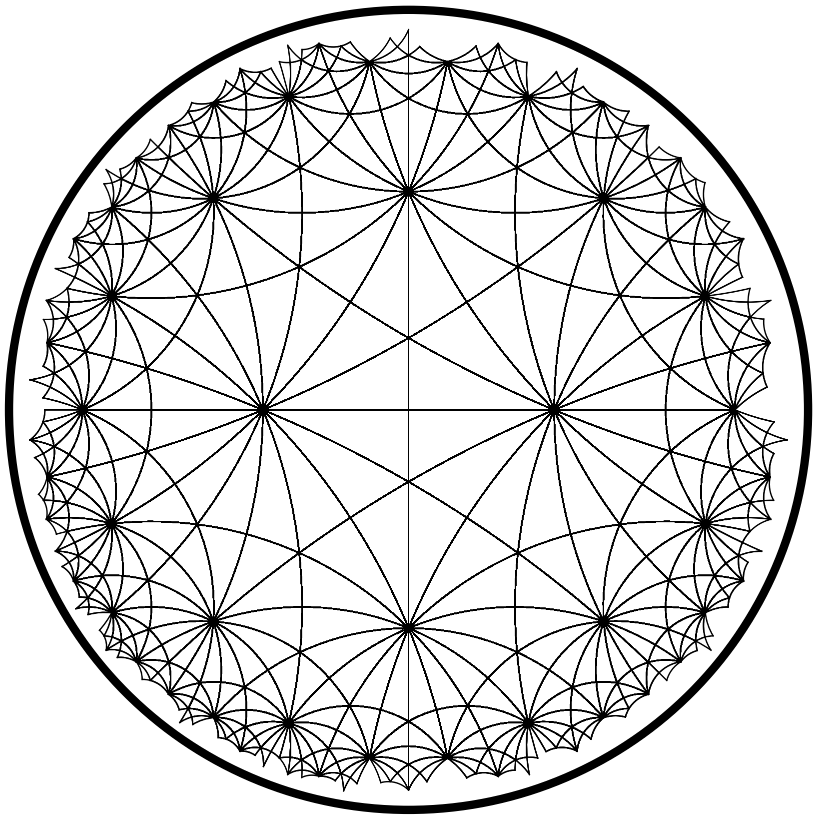

Fig. 3.4 The $(2, 3, 8)$-triangle reflection group in the Poincaré disk model. We used the package "PoincareModel" written by W. Goldman.

Proposition 3.2.2.

- *One can respectively construct a compact geodesic polygon P with angles $\pi/p_1, \pi/p_2, \ldots, \pi/p_n, n \geq 3, p_i \geq 2$ on a two-sphere, a Euclidean plane, or a hyperbolic plane depending on whether the sum of outer angles $\sum_{i=1}^{n} \pi(1 - 1/p_i)$ is smaller than 2π, equal to 2π, or greater than 2π.*
- *This is the necessary and sufficient condition also.*
- *The group generated by the reflection on the sides of P generates a discrete group.*

Proof. One shows that it is possible to construct all triangles in this way. Let us give arbitrary lengths $l, l_1, l_2, \ldots, l_5$.

- We show that a quadrilateral with angles π/p_1, $\pi/p_2, \pi/2, \pi/2$ at respective vertices v_1, v_2, v_3, v_4, and a distinguished edge $\overline{v_3 v_4}$ of length l, and
- a pentagon with angles $\pi/p_1, \pi/2, \pi/2, \pi/2, \pi/2$ at vertices $v_1, v_2, \ldots, v_5$

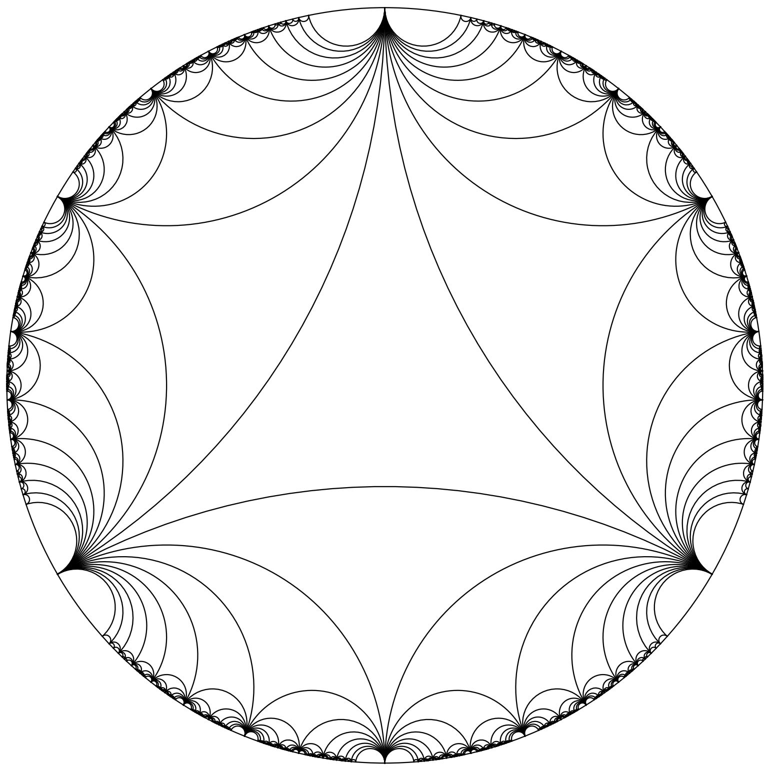

Fig. 3.5 The ideal-triangle reflection group: we use the group generated by the reflections on the sides of an ideal triangle on the hyperbolic plane. We used the package "PoincareModel", written by W. Goldman.

with distinguished edges $\overline{v_2v_3}, \overline{v_4v_5}$ of respective length l_1 and l_2, and
- a hexagon with all angles $\pi/2$ at vertices $v_1, v_2, \ldots, v_6$ with distinguished edges $\overline{v_1v_2}, \overline{v_3v_4}$, and $\overline{v_5v_6}$ of respective lengths l_3, l_4, and l_5 can be constructed.

These are accomplished in Chapter 3 of the book [Ratcliffe (2006)] for example.

Given a topological polygon, we can divide it into quadrilaterals, pentagons, and/or hexagons matching each other on edges of above types up to renaming vertices. Then desired polygon P can be constructed by matching lengths of the distinguished edges.

The necessary part comes from the Gauss-Bonnet theorem. $\square$

3.2.4.1 *Higher-dimensional examples*

To construct a 3-dimensional example, we obtain a Euclidean regular dodecahedron in $T_O B$, put into the hyperbolic space, expand it, and decrease the dihedral angles until we achieve that all dihedral angles are $\pi/3$. (See Section 3.2.2.) There are pictures of these in Geometry Center archives including the Seifert-Weber manifold constructed in such a manner.

One can also achieve a regular octahedron with angles $\pi/2$. These are ideal polytope examples. Heard, Pervova, and Petronio (2008) for example found very many 3-manifolds obtained from an octahedron by side-paring constructions above.

Higher-dimensional examples were analyzed by Vinberg and so on. For example, there is no hyperbolic reflection group of compact type above dimension ≥ 30.

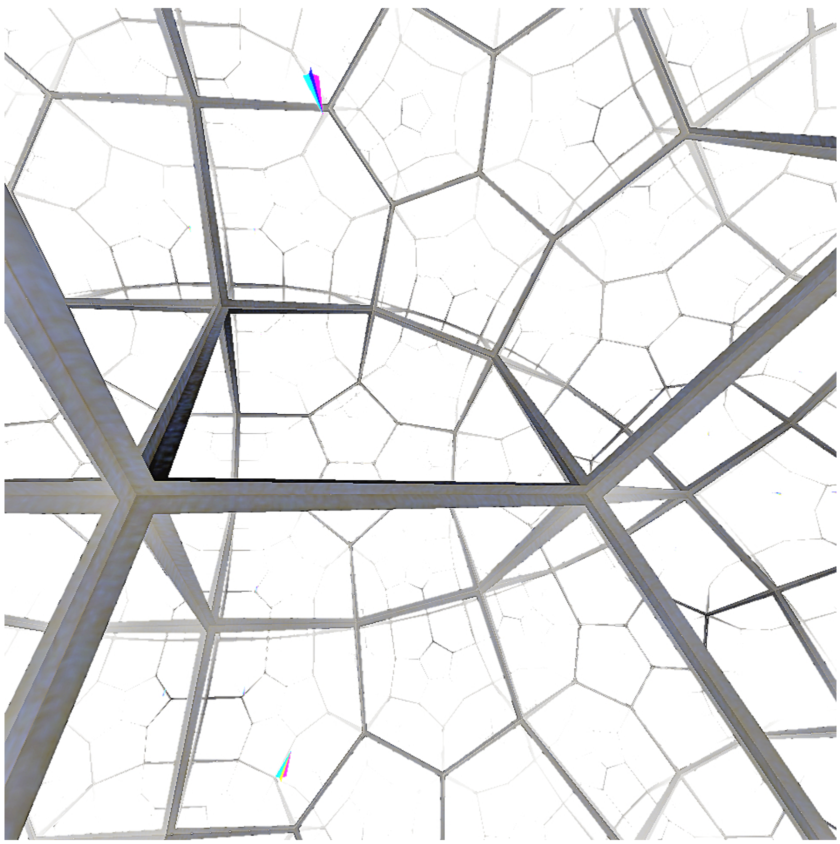

Fig. 3.6 The dodecahedral reflection group as seen by an insider: One has a regular dodecahedron with all edge angles $\pi/2$ and hence it is a fundamental domain of a hyperbolic reflection group. This figure is captured from the program CurvedSpace by J. Weeks.

3.2.5 *Crystallographic groups*

A *crystallographic group* is a discrete group of the rigid motions on $\mathbb{R}^n$ whose quotient space is compact.

The Bieberbach theorem states that

Theorem 3.2.3.

- *A group is isomorphic to a crystallographic group of $\mathbb{R}^n$ if and only if it contains a subgroup of finite index that is free abelian of rank equal to n.*
- *Two crystallographic groups are isomorphic as abstract groups if and only if they are conjugate by an affine transformation.*

Once we have this theorem, the classification of crystallographic groups is reduced to studying the finite group extensions of abelian crystallographic groups, which are lattices. There are only finitely many crystallographic groups for each dimension since once the abelian group action is determined, its symmetry group can be only finitely many. There are 17 wallpaper groups for dimension 2. (See `http://www.clarku.edu/~djoyce/wallpaper/` and see Kali by Weeks `http://www.geometrygames.org/Kali/index.html`.) There are 230 space groups for dimension 3 (Conway, Friedrichs, Huson, and Thurston, 2001). These groups have extensive applications in molecular chemistry. For further informations, see `http://www.ornl.gov/sci/ortep/topology.html`.

3.3 Notes

The figures 3.4 and 3.5 were drawn by packages developed by the Experimental Geometry Laboratory in University of Maryland, College Park. (See `http://egl.math.umd.edu/`.)

A good introduction to Euclidan, affine, and projective geometry can be found in the books [Berger (2009); Rosenbaum (1963)] and some early chapters of books [Thurston (1997); Goldman (1988)]. There are many interactive online courses and materials on projective geometry:

- `http://www.math.poly.edu/courses/projective_geometry/`
- `http://demonstrations.wolfram.com/TheoremeDePappusFrench/`,
- `http://demonstrations.wolfram.com/TheoremeDePascalFrench/`,

In fact, projective geometry is actively researched by engineers working in visions.

The book [Ratcliffe (2006)] gives us extensive descriptions of models of hyperbolic geometry. Discrete group actions and the Poincaré fundamental polyhedron theorems are described well in the books [Ratcliffe (2006); Kapovich (2009)]. In fact, this chapter is heavily influenced by the books [Ratcliffe (2006); Thurston (1997)]. There is also an elementary book [Ryan (1987)].

Chapter 4

Topology of orbifolds

This section begins by reviewing the theory of the compact group actions on manifolds. Then we move on to define orbifold and their maps. We also cover the groupoid definition. We discuss the differentiable structures on orbifolds and the triangulation of orbifolds following the book [Verona (1984)]. We expose the covering theory using the fiber-product approach following Thurston and the path-approach following Haefliger. We make some computations of the fundamental groups. Finally, we relate the fundamental groups with the covering spaces.

We tried to make the abstract definitions into more concrete forms here; however, in many respect, the abstract definitions give us a more accurate sense of what an orbifold is. (For examples, see the article [Lerman (2010)].) This section is somewhat technical but essential to the developments later.

4.1 Compact group actions

Although we need only the result for finite group actions, we will study the situations when G is a compact Lie group. Let X be a space. We are given a group action $G \times X \to X$ with $e(x) = x$ for all x and $gh(x) = g(h(x))$. That is, we have a homomorphism $G \to \mathbf{Diff}(X)$ so that the product operation corresponds to the composition. In this case, X with the action is said to be a G-space.

An *equivariant* map $\phi : X \to Y$ between G-spaces is a map so that $\phi(g(x)) = g(\phi(x))$ for all $x \in X$. An *isotropy subgroup* G_x is defined as $\{g \in G | g(x) = x\}$. We note that $G_{g(x)} = gG_xg^{-1}$ and $G_x \subset G_{\phi(x)}$ for an equivariant map ϕ.

Theorem 4.1.1 (Tietze-Gleason Theorem). *Let G be a compact group acting on a normal space X with a closed invariant set A. Let G also act linearly on $\mathbb{R}^n$. Then any equivariant map $\phi : A \to \mathbb{R}^n$ extends to an equivariant map $\phi : X \to \mathbb{R}^n$.*

An *orbit* of a point x of X is $G(x) = \{g(x)|g \in G\}$. Then we see that $G/G_x \to G(x)$ is one-to-one and onto continuous function. Therefore, the orbit type is given by the conjugacy class of G_x in G. The set of orbit types form a set partially ordered by the reversing the inclusion ordering of the conjugacy classes of subgroups of G.

Denote by X/G the space of orbits with the quotient topology.

For $A \subset X$, define $G(A) = \bigcup_{g \in G} g(A)$ is the *saturation* of A.

- $\pi : X \to X/G$ is an open, closed, and proper map.
- X/G is Hausdorff since G is compact.
- X is compact iff X/G is compact.
- X is locally compact iff X/G is locally compact.

We list some examples:

- Let $X = G \times Y$ and G acts as a product. Then every orbit is homeomorphic to G and the stabilizers are all trivial groups.
- For k, q relatively prime, the action of $\mathbb{Z}_k$ on the unit sphere $\mathbf{S}^3$ in the complex space $\mathbb{C}^2$ is generated by a matrix
$$\begin{bmatrix} e^{2\pi i/k} & 0 \\ 0 & e^{2\pi q i/k} \end{bmatrix}.$$
The quotient space is a Lens space.
- We also consider $\mathbf{S}^1$-actions on $\mathbf{S}^3$ given by
$$\begin{bmatrix} e^{2\pi k i\theta} & 0 \\ 0 & e^{2\pi q i\theta} \end{bmatrix}$$
Then it has three orbit types.
- Consider in general the torus T^n-action on $\mathbb{C}^n$ given by
$$(c_1, \dots, c_n)(y_1, \dots, y_n) = (c_1 y_1, \dots, c_n y_n), |c_i| = 1, y_i \in \mathbb{C}.$$
Then there is a homeomorphism $h : \mathbb{C}^n/T^n \to (\mathbb{R}^+)^n$ given by sending
$$(y_1, \dots, y_n) \mapsto (|y_1|^2, \dots, |y_n|^2)$$
where $\mathbb{R}^+ := \{x \in \mathbb{R} | x \geq 0\}$. (In other words, $(\mathbb{R}^+)^n$ is the closure of the positive 2^n-tant of $\mathbb{R}^n$.) The interiors of sides represent different orbit types.
- Let H be a closed subgroup of Lie group G. Let H act on G by the left action. The left-coset space G/H is the orbit space where G acts on the right also.
- Given a G-action on a space X and $x \in X$, let G_x be the stabilizer of x. A map $G/G_x \to G(x)$ given by $gG_x \mapsto g(x)$ is a homeomorphism if G is compact.
- The twisted product: let X be a right G-space and Y a left G-space. A left action is given by $g(x, y) = (xg^{-1}, gy)$. The twisted product $X \times_G Y$ is the quotient space.
- Let $p : X \to B$ is a principal bundle with G acting on the right. Let F be a left G-space. Now G acts on the right on $X \times F$ by $g(x, f) = (xg, g^{-1}(f))$. Then $X \times_G F$ is the associated bundle. (See Section 2.4.2.1.)

Example 4.1 (Bredon). Let G be the rotation group $\mathbb{SO}(3, \mathbb{R})$, and let X be the vector space of symmetric matrices of trace 0 (hence orthogonally diagonalizable). Suppose that we act by conjugation $G \times X \to X$ given by $g(m) = gmg^{-1}, m \in X$ and each $g \in G$. By linear algebra, we prove that two symmetric matrices are in the same orbit if they have the same eigenvalues with multiplicities. Hence the orbit space is in a one-to-one correspondence with the set of triples (a, b, c) so that $a \geq b \geq c$ and $a+b+c = 0$. The second space is a 2-dimensional cone in $\mathbb{R}^3$. This is homeomorphic to X/G. The isotropy group of a diagonal matrix with three distinct eigenvalues is the group of diagonal matrices with entries ± 1 which is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The isotropy group of a diagonal matrix with exactly two distinct eigenvalues is the group of matrices decomposing into an orthogonal 2×2-matrix and ± 1.

A point x of a space X with a group G acting on it is *stationary* if the stabilizer of x is G.

Example 4.2 (Conner-Floyd). There is an action of $\mathbb{Z}_r$ for $r = pq$, p, q relatively prime, on an Euclidean space of large dimensions without stationary points. This is accomplished in following steps. We sketch the construction here.

- Find a simplicial action $\mathbb{Z}_{pq}$ on $\mathbf{S}^3$ seen as a join $\mathbf{S}^1 \star \mathbf{S}^1$ without fixed points obtained by joining action of $\mathbb{Z}_p$ on the first factor circle and $\mathbb{Z}_q$ on the second factor circle.
- Find an equivariant simplicial map $h : \mathbf{S}^3 \to \mathbf{S}^3$ which is homotopically trivial.
- Build the infinite mapping cylinder using h infinitely many times which is contactible and embed it in an Euclidean space of high-dimensions where $\mathbb{Z}_{pq}$ acts orthogonally.
- Find the contractible neighborhood. Taking the product with the real line makes it into a Euclidean space. Now on this space $\mathbb{Z}_{pq}$ acts orthogonally as well.

4.1.1 *Tubes and slices*

For a compact group action, we need to establish the notion of tubes and slices. These are modeled on twisted product actions: Let G be a compact Lie group, X a right G-space, and S a left G-space. Then $X \times_G S$ is defined as the quotient space of $X \times S$ where $[xg, y] \sim [x, gy]$ for $g \in G$, $x \in X$, and $y \in S$.

Let H be a closed subgroup of G and let A be a left H-space. shThen $G \times_H A$ is a left G-space by the action $g[g', a] = [gg', a]$ where $g, g' \in G, a \in A$ as this sends equivalence classes to themselves. The inclusion $A \to G \times_H A$ induces a homeomorphism $A/H \to (G \times_H A)/G$.

The isotropy subgroup at $[e, a]$ for $a \in A$ and e the identity element of G is computed as follows: $[e, a] = g[e, a] = [g, a] = [h^{-1}, h(a)]$ for $h \in H$. Thus,

$G_{[e,a]} = H_a$ where H_a is the stabilizer of a in H.

As an example, let $G = \mathbf{S}^1$ and A be the unit-disk and $H = \mathbb{Z}_3$ generated by $e^{2\pi i/3}$. G and H act in standard manners in A. Then consider $G \times_H A$. The result is homeomorphic to a solid torus fibered with circles. Each non-central circle is mapped around the quotient solid torus three times and the central circle goes around once.

Let X be a G-space and P an orbit of type G/H. A *tube about* the orbit P is a G-equivariant embedding $G \times_H A \to X$ onto an open neighborhood of P where A is a space where H acts on. We note the following:

- Every orbit passes the image of $e \times A$ where e is the identity of G.
- P equals $G(x)$ for $x = [e, a]$ where a is the stationary point of H in A.
- For general points $x = [e, b]$, not necessarily stationary, we have $G_x = H_b \subset H$.

Let $x \in X$. Suppose that S is a set containing x such that $G_x(S) = S$; i.e., the stabilizer of x acts on S. Then S is said to be a *slice* if $G \times_{G_x} S \to X$ so that $[g, s] \to g(s)$ is a tube about $G(x)$. It is easy to see that S is a slice if and only if S is the image of $e \times A$ for some tube.

Let $x \in S$ and $H = G_x$. Then the following statements are equivalent:

- There is a tube $\phi : G \times_H A \to X$ about $G(x)$ such that $\phi([e, A]) = S$.
- S is a slice at x.
- $G(S)$ is an open neighborhood of $G(x)$ and there is an equivariant retraction $f : G(S) \to G(x)$ with $f^{-1}(x) = S$.

Let X be a completely regular G-space. Let $x_0 \in X$ have an isotropy group H in G. Find an orthogonal representation of G in $\mathbb{R}^n$ with a point v_0 whose isotropy group is H, which always exists by a compact group representation theory. There is an equivalence of orbits $G(x_0)$ and $G(v_0)$. We extend this to a neighborhood by Tietze-Gleason theorem. For $\mathbb{R}^n$, we find the equivariant retraction given by Lemma 5.1 of Chapter 1 of the book [Bredon (1972)]. Transferring this on X, we obtain:

Theorem 4.1.2 (Gleason, Montgomery-Yang). *Let X be a completely regular G-space. There is a tube about any orbit of a completely regular G-space with G compact.*

If G is a finite group acting on a manifold, then a tube is a union of disjoint open sets and a slice is an open subset where G_x acts on.

Theorem 4.1.3 (Path-lifting and the covering homotopy theorem). *Let X be a G-space and G a compact Lie group.*

- *Let $f : I \to X/G$ be any path. Then there exists a lifting $f' : I \to X$ so that $\pi \circ f' = f$.*

- *Assume that every open subspace of X/G is paracompact. Let $f : X \to Y$ be an equivariant map. Let $f' : X/G \to Y/G$ be an induced map. Let $F' : X/G \times I \to Y/G$ be a homotopy preserving orbit types that starts at f'. Then there is an equivariant $F : X \times I \to Y$ lifting F' starting at f. Moreover, any two such liftings of F' differ by composition with a self-equivalence of $X \times I$ covering the identity of $X/G \times I$ and equal to identity on $X \times \{0\}$.*
- *If G is finite and X a smooth manifold with a smooth G-action and if the functions have locally smooth lifts, then the lifts can be chosen to be also smooth. If the derivative of a smooth path with locally smooth lifts is never zero, then the lift is unique up to the action of G.*

4.1.2 *Locally smooth actions*

Let M be a G-space with G a compact Lie group, and let P be an orbit of type G/H. and V a vector space where H acts orthogonally. Then a *linear* tube in M is a tube of the form $\phi : G \times_H V \to M$.

Let S be a slice. S is a *linear slice* if $G \times_{G_x} S \to M$ given by $[g, s] \to g(s)$ is equivalent to a linear tube. In other words, this is the case if the G_x-space S is equivalent to the orthogonal G_x-space.

If there is a linear tube about each orbit, then M is said to be *locally smooth*.

Lemma 4.1.4. *Under the above assumptions, there exists a maximal orbit type G/H for G. (That is, H is conjugate to a subgroup of each isotropy group.)*

Proof. In each tube, there is a maximal orbit type in it and we find the union of maximal orbits in it has to be dense and open. For intersection of two tubes, the union of maximal orbits has to be dense and open in both tubes. Thus, the maximal orbit of a tube is of the maximal orbit type in M. □

The maximal orbits so obtained in a tube are called *principal orbits*.

4.1.3 *Manifolds as quotient spaces.*

Finally, we wish to understand about the quotient spaces. Let M be a smooth manifold (not necessarily connected), and G a compact Lie group acting smoothly on M. We denote by M^* the quotient space M/G. (This is a notation used for this book.) If G is finite, then this is equivalent to the fact that each $i_g : M \to M$ given by $x \mapsto g(x)$ is a diffeomorphism, and the following theorem holds if the dimension of M is ≤ 2.

Theorem 4.1.5. *Let n be the dimension of M and d the dimension of the maximal orbit. Then $M^* = M/G$ is a manifold with boundary if $n - d \leq 2$.*

Proof. Let $k = n - d$ be the codimension of the principal orbits. Consider a linear tube $G \times_K V$ where K is a subgroup of G acting on V. The orbit space $(G \times_K V)/G = (G \times_K V)^*$ is congruent to V^* where $V^* = V/K$. Let S be the unit sphere in V. Then V^* is a cone over S^*. We have that $\dim M^* = \dim V^* = \dim S^* + 1$.

If $k = 0$, then M^* is discrete. If M is a sphere, then M^* is one or two points. (Here, we regard a disconnected 0-sphere as a sphere also.)

If $k = 1$, then M^* is locally a cone over one or two points by the previous steps. Hence M^* is a 1-manifold. If $k = 2$, then M^* is locally a cone over an arc or a circle as S^* is a 1-manifold by the previous step. □

Example 4.1.6. Consider the $\mathbb{Z}_2$-action on $\mathbb{R}^3$ generated by the antipodal map $\vec{x} \mapsto -\vec{x}$. The result is not a manifold.

4.1.4 *Smooth actions are locally smooth*

Recall smooth actions. Let G be a compact Lie group acting smoothly on a manifold M. Then there exists a G-invariant Riemannian metric on M. Then $G(x)$ is a smooth manifold where $G/G_x \to G(x)$ is a diffeomorphism. Recall the exponential map for Riemannian manifolds: For any vector $X \in T_pM$, there is a unique geodesic γ_X with tangent vector at p equal to X. The exponential map $\exp : T_pM \to M$ is defined by $X \mapsto \gamma_X(1)$.

Lemma 4.1.7. *A G-invariant metric on M can always be constructed so that ∂M is totally geodesic.*

Proof. We start with any smooth Riemannian metric μ on M. Next, we integrate to obtain the Riemannian metric $\mu_1 = \int_{g \in G} g^* \mu dg$ on M using the Haar measure on G. Now μ_1 will extend to a Riemannian metric on an open manifold M' containing M. Find a tube T of ∂M in M', i.e., an open neighborhood of ∂M and a submanifold diffeomorphic to $\partial M \times (-\epsilon, \epsilon), \epsilon > 0$. By taking a sufficiently small tube, we assume that μ_1 extends to a metric on T. Here, we assume that the exponential map from the normal bundle of ∂M to T is a diffeomorphism. (See Chapter 4 of the book Hirsch (1976) for details.) Then there exists an antipodal map $\sigma : T \to T$ fixing ∂M by sending a point x of T with a shortest geodesic γ perpendicular to ∂M with $\gamma(\delta) = x$ to $\gamma(-\delta)$ in T again. We may assume that $\sigma(T) = T$. Considering geodesics perpendicular to ∂M, we find that the commutativity $\sigma \circ g = g \circ \sigma$ holds. By comparing distances between two points and their images under σ, we see that $\sigma^* \mu_1$ is also G-invariant in T. We form the G-invariant Riemannian metric $\sigma^* \mu_1 + \mu_1$. Since σ is an isometric involution of this metric, it follows that ∂M is totally geodesic. (For the proof, we followed a note of Francis (2010) here.) Now we use a G-invariant partition of unity to form a metric in M' and hence on M. □

If A is a G-invariant smooth submanifold, then A has an open G-invariant tubular neighborhood. This follows by using the normal bundle to A and the exponential map restricted to the normal bundle N_A. Then this map is a local diffeomorphism in a neighborhood N of A in N_A. By taking the same radius open balls in the normal bundle, we obtain the invariant tubular neighborhood as its image.

Proposition 4.1.8. *Let M be a manifold with boundary ∂M. The smooth action of a compact Lie group is locally smooth.*

Proof. We use the fact that orbits are smooth submanifolds and the above statements and that normal bundles are linear tubes. $\square$

Theorem 4.1.9 (Newman's theorem). *Let M be a connected topological n-manifold. Then there is a finite open covering $\mathcal{U}$ of the one-point compactification of M such that there is no effective action of a compact Lie group with each orbit contained in some member of $\mathcal{U}$.*

The proof follows from algebraic topology.

Corollary 4.1.10. *If G is a compact Lie group acting effectively on M, then the set of fixed points M^G is nowhere dense.*

4.1.5 *Equivariant triangulation*

Illman (1978) proved:

Theorem 4.1.11. *Let G be a finite group. Let M be a smooth G-manifold with or without boundary. Then we have:*

- *There exists an equivariant simplicial complex K and a smooth equivariant triangulation $h : K \to M$.*
- *If $h : K \to M$ and $h_1 : L \to M$ are smooth triangulations of M, there exist equivariant subdivisions K' and L' of K and L, respectively, such that K' and L' are G-isomorphic.*

4.2 The definition of orbifolds

Let X be a Hausdorff second countable topological space. Let n be fixed. Consider a connected open subset $\tilde{U}$ in $\mathbb{R}^n$ with a finite group G acting smoothly on it and a G-invariant map $\phi : \tilde{U} \to U$ for an open subset U of X inducing a homeomorphism $\tilde{U}/G \to U$. ϕ or $(\tilde{U}, \phi)$ is an *orbifold chart*, $\tilde{U}$ or $U = \phi(\tilde{U})$ is a *model neighborhood or model open set*, $(\tilde{U}, G)$ is a *model pair*, and $(\tilde{U}, G, \phi)$ is a *chart* or a *model triple.* An *embedding* $i : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth embedding $i : \tilde{U} \to \tilde{V}$ with $\phi = \psi \circ i$ which induces the inclusion map $U \to V$ for $U = \phi(\tilde{U})$ and $V = \phi(\tilde{V})$.

- Equivalently, i is an embedding inducing the inclusion map $U \to V$ and inducing an injective homomorphism $i^* : G \to H$ so that $i \circ g = i^*(g) \circ i$ for every $g \in G$. $i^*(G)$ will act on the open set that is the image of i.
- Note here i can be changed to $h \circ i$ for any $h \in H$. The images of $h \circ i$ will be disjoint for representatives h for $H/i^*(G)$. Conversely, any embedding $i' : \tilde{U} \to \tilde{V}$ lifting an inclusion $U \to V$ equals $h \circ i$ for $h \in H$. (See Proposition A.1 of the article [Moerdijk and Pronk (1999)].)

Definition 4.2.1. Let $\mathbb{R}_+ := \{x | x \geq 0\}$. Define $\mathbb{R}^n_+$ as the n-fold product of $\mathbb{R}_+$. A *cell* is a nonempty intersection of a convex open set in $\mathbb{R}^n$ with $\mathbb{R}^n_+$.

Two model triples $(\tilde{U}, G, \phi)$ and $(\tilde{V}, H, \psi)$ are *compatible* if for every $x \in U \cap V$ and open sets $U = \phi(\tilde{U})$ and $V = \psi(\tilde{V})$, there is an open neighborhood W of x in $U \cap V$ and the model triple $(\tilde{W}, K, \mu)$ with $\mu(\tilde{W}) = W$ such that there are embeddings to $(\tilde{U}, G, \phi)$ and $(\tilde{V}, H, \psi)$. (One can assume that W is a component of $U \cap V$.)

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible model triples $\{(\tilde{U}, G, \phi)\}$ so that the family of open sets of form $\phi(\tilde{U})$ covers X.
- Two orbifold atlases are *compatible* if model triples in one atlas are compatible with model triples in the other atlas.
- Atlases form a partially ordered set by the inclusion relation. It has a maximal element.
- Given an atlas, we obtain a unique maximal atlas containing it by Zorn's lemma.
- An *orbifold* $\mathcal{O}$ is a topological space X with a maximal orbifold atlas. We say that X is the *underlying space* of $\mathcal{O}$ and write $X = |\mathcal{O}|$ and we say that $\mathcal{O}$ is based on $|\mathcal{O}|$.
- One can obtain an atlas of linear charts only: that is, charts of form $(\tilde{U}, G, \phi)$ where $\tilde{U}$ is an open subset of $\mathbb{R}^n$ and $G \subset \mathbb{O}(n, \mathbb{R})$. For each point $x \in \tilde{U}$, one can find a finite subgroup G_x stabilizing the point and a suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice O_x of coordinate charts since a smooth action is locally smooth, i.e., linear and orthogonal, by Proposition 4.1.8. (Note, if x is in the boundary, then O_x can be identified with an open set intersected with an upper-half space and G_x is acting orthogonally on the half-space.) We call such a chart (O_x, G_x, ϕ) a *linear chart*. Therefore, given an orbifold atlas, there is a compatible orbifold atlas consisting of only linear charts.
- G_x is called a *local group*. If the local group G_x is not trivial, then x is said to be *singular*.
- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.

- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ where $\mathcal{U}$ and $\mathcal{V}$ are maximal atlases is *smooth* if for each point $x \in X$, there is a model triple $(\tilde{U}, G, \phi) \in \mathcal{U}$ with $x \in U = \phi(\tilde{U})$ and a model triple $(\tilde{V}, H, \psi) \in \mathcal{V}$ with $f(x) \in V$ so that $f(U) \subset V = \psi(\tilde{V})$ and f lifts to a smooth map $\tilde{f} : \tilde{U} \to \tilde{V}$. In this case, f is said to be an *orbifold-map*.
- If above f has local lifts $\tilde{f} : \tilde{U} \to \tilde{V}$ that is an immersion for the every pairs of model triples as above, then f is said to be an *orbifold-immersion*.
- Two orbifolds are *diffeomorphic* if there is a smooth orbifold-map with a smooth inverse orbifold-map.

Sometimes, the orbifolds are called effective (or reduced) orbifolds as we defined here (Adem, Leida, and Ruan, 2007). There are ineffective orbifolds, where for a model neighborhood $(\tilde{U}, G, \phi)$ the group G is allowed to be not effective on $\tilde{U}$. This is not a well-studied area.

We also note our convention that an orbifold has certain topological property if the underlying space has that property.

Definition 4.1. A covering of an orbifold is *good* if each model neighborhood is connected, the open set in the triple is homeomorphic to a cell, and the group acts linear orthogonally and the intersection of any finite collection again has such properties.

We will show later that each orbifold has a good cover. (See Proposition 4.4.2.)

Given an orbifold $\mathcal{O}$, if we allow some open sets $\tilde{U}$ in model triples to be open subsets of the closed upper half space $\mathbb{R}^{n-1} \times \mathbb{R}_+$, then the orbifold has *boundary*. A *boundary subset* of an orbifold is the subset of the underlying space orbifolds where each element is so that each of its inverse image points in the model open sets goes to the boundary of $\mathbb{R}^{n-1} \times \mathbb{R}_+$ under charts. The complement of the boundary is the *interior* of the orbifold. If a finite group G acts on a subspace V, we denote by $G|V$ the homomorphism image of G as restrictions $\{g|V | g \in G\}$. The boundary has an orbifold structure also by restricting each model triple $(\tilde{U}, G, \phi)$ to $(\tilde{U} \cap V, G|V, \phi|V)$ for $V = \mathbb{R}^{n-1} \times \{0\}$ whenever $\tilde{U} \cap V \neq \emptyset$ as the model triples are all compatible. The *boundary* of an orbifold is the boundary subset with this orbifold structure. (We will show that the boundary is a suborbifold. See Definition 4.2.2.)

A compact orbifold with empty boundary is said to be a *closed* orbifold.

4.2.1 *Local groups and the singular set*

Let $x \in X$. A *local group* G_x of x is obtained by taking a model triple $(\tilde{U}, G, \phi)$ for x and finding the stabilizer G_y of y for an inverse image point y of x.

- This is independently defined up to conjugacy for any choice of y.
- We reason as follows: Smaller charts will give you the smaller or identical conjugacy class. The stabilizer group eventually does not change under

taking smaller and smaller charts up to conjugations. Thus, one can take a linear chart. Once a linear chart is achieved, the local group is well-defined up to conjugacy (Thus, as an abstract group with an action.)

The *singular set* is a set of points where G_x is not trivial. In each chart, the set of fixed points of each subgroup of G_x is a closed submanifold.

Let (O_x, G_x) and (O_y, G_y) be two charts. Subgroups H of G_x and H' of G_y are *strictly topologically conjugate* if there is a chart (U_z, G_z) with morphisms into (O_x, G_x) and (O_y, G_y) in the orbifold atlas so that H and H' correspond to conjugate subgroups in G_z. H and H' are *topologically conjugate* if there exists a sequence $H_1 = H, H_1, \cdots, H_n = H'$ where H_i and H_{i+1} are strictly topologically conjugate.

The connected maximal subset of the singular set where the topological conjugacy class of the stabilizer G_x of each of its element x is constant is a relatively closed submanifold. Thus X becomes a stratified smooth topological space where each stratum is given by the connected component of the set where the smooth topological conjugacy classes of subgroups of local groups G_x for $x \in X$ is constant. (Here, a *stratified space* is a space that is a union of disjoint relatively closed connected submanifolds. A *stratum* is one of these submanifolds. See Section 4.5.1.)

Because G_y is trivial for y in a dense open subset of O_x, a generic point of an orbifold has a trivial local group. Hence, there exists a dense open subset in the underlying space of an orbifold that is nonsingular. The set of singular points is nowhere dense also.

The singularity of a 1-orbifold is unique: a silvered point. Its neighborhood is modeled on an open interval where $\mathbb{Z}_2$ acts as a reflection group fixing a point. Thus, a connected 1-orbifold has a base space homeomorphic to the circle $\mathbf{S}^1$ or an interval (half-open, open, or closed) and is diffeomorphic to $\mathbf{S}^1$, a closed interval with one or two silvered points, a half-open interval with one or no silvered point, or an open interval.

To classify the singular points of 2-orbifolds, we classify finite groups in $\mathbb{O}(2, \mathbb{R})$ acting on open subsets of $\mathbb{R}^2$ since we are looking at finite subgroups of $\mathbb{GL}(2, \mathbb{R})$: These are as follows: $\mathbb{Z}_2$ acting as a reflection group or a rotation group generated by a rotation of angle π, cyclic groups C_n of order $n \geq 3$ and dihedral groups D_n of order $2n \geq 4$. The singular points of a two-dimensional orbifold fall into three types:

(i) The mirror point: $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by the reflection on the y-axis.
(ii) The cone-points of order n: $\mathbb{R}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acting by rotations by angles $2\pi m/n$ for integers m.
(iii) The corner-reflector of order n: $\mathbb{R}^2/D_n$ where D_n is the dihedral group generated by reflections about two lines meeting at an angle π/n. (Note that D_n is of order $2n$. However, the order of the corner-reflector itself is n.)

From this, we see that the underlying space of a 2-orbifold is a surface with corner since each model neighborhood is diffeomorphic to a surface with corner by above. (This also follows from the proof of Theorem 4.1.5. See the beginning of Section 4.5 for the definition.)

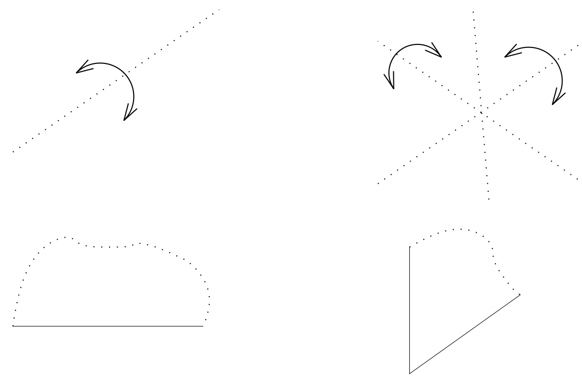

Fig. 4.1 The actions here are isometries on $\mathbb{R}^2$.

Definition 4.2. Given two orbifolds X and Y, we find a natural product orbifold-structure on $|X| \times |Y|$ where $|X|$ and $|Y|$ are the respective underlying spaces. We assume that the boundary of one of X or Y is empty. For a point $(x, y) \in |X| \times |Y|$, an orbifold neighborhood is $U \times V$ for respective model neighborhoods U and V of x and y where $(\tilde{U}, G, \phi)$ is the model triple for x and $(\tilde{V}, H, \psi)$ is one for y with $\phi(\tilde{U}) = U$ and $\psi(\tilde{V}) = V$. The group $G \times H$ acts on $\tilde{U} \times \tilde{V}$, and $(\tilde{U} \times \tilde{V}, G \times H, \phi \times \psi)$ is the model triple for (x, y). Then these charts $\phi \times \psi$ form an atlas of $|X| \times |Y|$ giving us an orbifold structure. We denote the orbifold by $X \times Y$ and call it the product orbifold of X and Y.

If both $\partial X \neq \emptyset$ and $\partial Y \neq \emptyset$, then we can put on $|X| \times |Y|$ an orbifold-structure with corner. (See Section 4.5.2 for detail.)

Definition 4.2.2. A *suborbifold* Y of an orbifold X is an embedded subset such that for each point y in Y and a chart $(\tilde{V}, G, \phi)$ of X for a neighborhood V of y there is a chart for y given by $(P, G|P, \phi)$ where P is a closed submanifold of $\tilde{V}$ where G also acts on and $G|P$ is the image of the restriction homomorphism of G to P. (We caution the readers that $G \to G|P$ is sometimes not injective.)

Clearly, an open subset inherits an orbifold structure to make them into a suborbifold, and the boundary of an orbifold is a suborbifold. (See Remark 4.2.5.)

A suborbifold in our sense is a "suborbifold" in the sense of Definition 2.3 of the book [Adem, Leida, and Ruan (2007)], which is easy to show from the definitions. However, our definition is strictly stronger. Also our definition is strictly weaker than the one in Section 6.1 of the book [Kapovich (2009)]. (Actually, we should say our suborbifolds are "strong" suborbifolds. However, we do not need their definition.) The basic reason for our definition is so that we wish do surgeries along the suborbifolds in later sections.

Let I be the orbifold based on $[0, \epsilon)$ with 0 given the silvered point structure. Then $I \times I$ is a 2-orbifold covered by $(-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ with $\mathbb{Z}_2^2$ acting on it by two reflections about the axes. That is, $I \times I$ has a corner-reflector of order 2 at which two silvered edges meet. The diagonal $\delta \subset [0, \epsilon) \times [0, \epsilon)$ can be given a suborbifold structure in the sense of Definition 2.3 of the book [Adem, Leida, and Ruan (2007)] by Example 2.6 of the same book. However, the inverse image of δ in $(-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ is not an embedded arc, i.e, a union of two transversal arcs, and it cannot be a suborbifold in our sense.

Now consider $J = \{0\} \times I$. Then J is given an orbifold-structure with one-silvered point. Then J is a suborbifold in our sense. However, J is not a suborbifold in the sense of Section 6.1 of the book [Kapovich (2009)]. The reason is that the local groups are required to be mutually isomorphic in the later case.

Clearly, manifolds are orbifolds. But as an orbifold, it might carry more charts. By an abuse of notations, a manifold in this paper will mean a manifold with the extended collection of charts as orbifolds: To explain, in general, let G be a finite group acting on a manifold M smoothly and freely. Then M/G is a manifold with an orbifold structure with an atlas of charts based on some H-invariant open set in M diffeomorphic to an open subset of $\mathbb{R}^n$ and a subgroup H of G as a model. For example, $\mathbb{RP}^n$, $n \geq 2$, will have a chart with the $\mathbb{Z}_2$-action on it.

Conversely, we say that an orbifold is a *manifold* if there is an atlas in the orbifold atlas with model triples with trivial groups only. A submanifold of a manifold has a suborbifold structure when the manifold is considered as an orbifold.

4.2.2 *Examples*: *good orbifolds*

Let $M = T^n$ and $\mathbb{Z}_2$ act on it with generator acting by $-I$. For $n = 2$, $M/\mathbb{Z}_2$ is topologically a sphere and has four singular points. For $n = 4$, we obtain a Kummer surface with sixteen singular points. In general, a regular branched covering of a surface by another surface gives us an orbifold structure.

Theorem 4.2.3. *Let M be an n-orbifold with boundary possibly empty and Γ be a discrete group of orbifold-diffeomorphisms of M acting properly discontinuously but not necessarily freely. Then the quotient space M/Γ has a natural structure of an orbifold.*

Proof. For each point $x \in M$, the stabilizer group Γ_x is a finite group since x has a neighborhood U whose closure is compact. Since G_x is finite, we form an open neighborhood $\bigcup_{g \in G_x} g(U)$ of x. By taking U sufficiently small, we may assume that U has a model triple (V, G, ϕ) for an open subset V in $\mathbb{R}^n$ or in a half-space

$\mathbb{R}^{n-1} \times \mathbb{R}_+$. Now, U has a finite group G_U acting on it. Each element $g : U \to U$ is an embedding and hence lifts to a diffeomorphism $\tilde{g} : V \to V$.

Let G_V be the finite group generated by these lifts and G. Then it follows that $g \in G_V$ for a homeomorphism of V iff $\phi \circ g = h \circ \phi$ for $h \in G_x$. Let $p : M \to M/\Gamma$ be the quotient map. Hence $(V, G_V, p \circ \phi)$ is then a model triple of $p(x) \in M/\Gamma$.

The sets of these types form an atlas of M/Γ and hence give us an orbifold structure. □

We say that M/Γ is a *quotient orbifold* of an orbifold M. In fact, in many cases orbifolds are of this form. If M is a manifold, they are called "good orbifolds". We will talk about these later.

- Consider the Euclidean plane $\mathbb{R}^2$ and the discrete group generated by order-two rotations at $(k+n, l+m)$ for $n, m \in \mathbb{Z}^2$ and fixed real numbers $k, l > 0$.
- Cut a rectangle of height 1 and length 2 containing two fixed points rotations on the top side and two the bottom side respectively. We glue by an isometry given by the composition of the two rotations on the top side, which is identical with that of the two rotations at the bottom side. We obtain an annulus. (See Figure 4.2.)
- Then we crease the top circle and the bottom circle at the cone-points and glue by the order 2 rotations. (This is called "folding". See Section 5.2.1.)
- Thus, the Poincaré polyhedron theorem exactly fits into this situation.
- We can modify this construction easily by taking a nonstandard $\mathbb{Z}_2$-lattice. This might be a good exercise for readers.

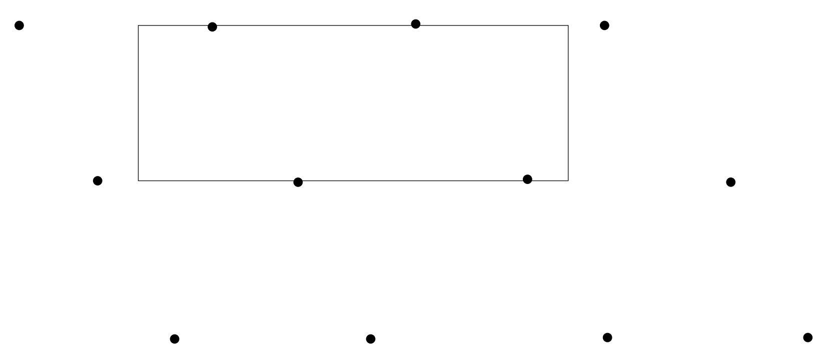

Fig. 4.2 The rectangle and the fixed points

This type of orbifold is an example of an Euclidean orbifold which is a quotient orbifold of the euclidean space by a wall paper group. (We call "pillows" tetrapaks to emphasize the Euclidean structure.) See Figures 4.4 and 4.5.

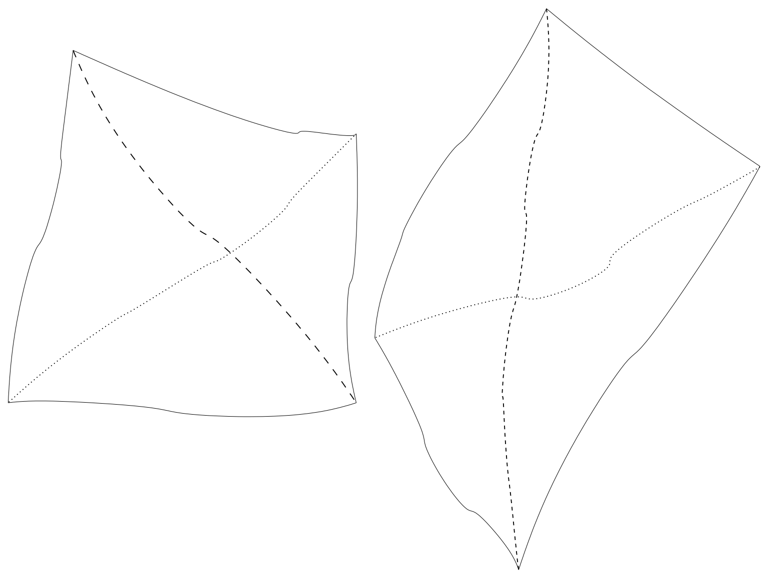

Fig. 4.3 Tetrapaks often called “pillows”.

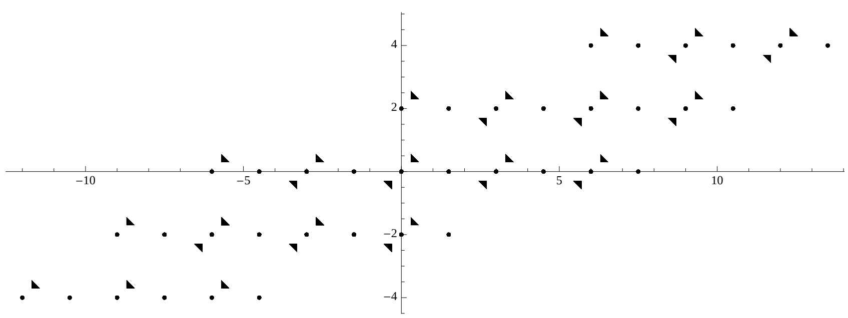

Fig. 4.4 A wall paper group p2: The points are fixed points of some elements of order 2 including generators and a triangle is mapped by various elements of the group. See wall2a.nb

4.2.3 *Examples*: *silvering*

Given a manifold M with boundary, we obtain a *doubled* $\hat{M}$ by taking $M \times \mathbb{Z}_2/\sim$ where $(x,0) \sim (y,1)$ if and only if $x = y \in \partial M$. A $\mathbb{Z}_2$-action $\hat{M}$ is induced by $(x,0) \mapsto (x,1)$ and $(x,1) \mapsto (x,0)$ for $x \in M$. We build a collar neighborhood of ∂M in M diffeomorphic to $\partial M \times [0,\epsilon)$. Then the $\mathbb{Z}_2$-action here can be extended to $\partial M \times (-\epsilon,\epsilon)$ by $(x,t) \to (x,-t)$. This is a smooth action. Hence, we can double M as a smooth manifold $\hat{M}$ and obtain a smooth $\mathbb{Z}_2$-action. Thus, M can be given a smooth orbifold structure modeled on $\mathbb{Z}_2$-invariant open subsets of $\hat{M}$ with $\mathbb{Z}_2$-action or open subsets of M^o with trivial group actions.

Now the boundary of M became now a set of singular points, called *silvered*

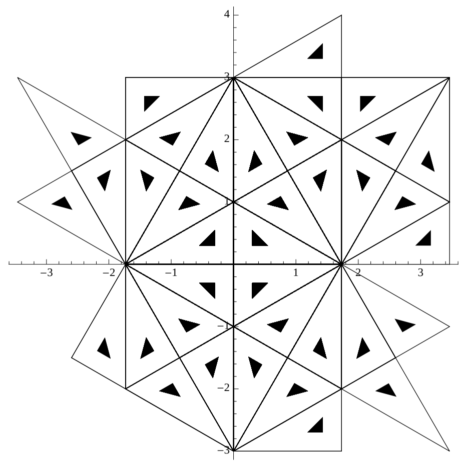

Fig. 4.5 A $(2, 3, 6)$-triangle reflection group. The fundamental domain is one of the bigger triangles and an inside triangle is mapped to many other by various elements of the group. See wall17a.nb

points. Actually, we can do this for the interior U of a properly and smoothly embedded submanfold of ∂M. Define $\hat{M}_U$ as $M \times \mathbb{Z}_2/\sim$ where $(x, 0) \sim (y, 1)$ if and only if $x = y \in U$. Then we can find an orbifold structure on M with U silvered in the above way. (See also Proposition 4.4.3.)

Example 4.2.4. Consider a surface with corner, its boundary that is a union of smooth arcs ending at corner points, and the set of its corner points.

- We choose some collection of these arcs $\alpha_1, \ldots, \alpha_n$ and finite set of points in the interior $q_1, \ldots, q_m$.
- We let the set of points where the the endpoints of half-arcs of the arcs in the collection coincide be called *distinguished corner points*. Denote them by $p_1, \ldots, p_l$. Each p_i is given an order n_i, $n_i \geq 2$. Let each point q_i be given orders m_i, $m_i \geq 2$. If α_i is a loop, then its unique endpoint is a distinguished corner point.
- We give a Riemannian metric on a neighborhood N of the boundary by ϕ-equivariantly immersing the universal cover of the neighborhood into the Euclidean space $\mathbb{E}^2$ so that the boundary arcs are geodesic, the angle at each distinguished corner point p_i is π/n_i and at the non-distinguished corner points the angles are $\pi/2$, where the homomorphism $\phi : \pi_1(N) \to \mathbf{Isom}(\mathbb{R}^2)$ can be chosen.
- Then each point of the arc α_i is silvered by taking as a model open set a small open ball in $\mathbb{E}^2$ containing its image and invariant under the reflection about the image of α_i.
- At each point p_i, we take a model open set as a small open ball in $\mathbb{E}^2$ containing its image and invariant under the two reflections about the images

of α_k and α_l ending there forming an angle π/n_i for some k, l.

- At q_i, we model its open neighborhood by an open ball with a cyclic action by $\mathbb{Z}_{m_i}$. The neighborhood here is chosen to be disjoint from ones of the boundary points.
- For other points, we model an open neighborhood of the point disjoint from boundary or $\{q_1, \ldots, q_m\}$ by an open set in $\mathbb{E}^2$ without any group actions.
- Finally, we see that then these charts are compatible and hence gives rise to an orbifold structure.

Remark 4.2.5. When we say the boundary or interior of an orbifold, we do not mean the boundary or interior of the underlying space. They are different concepts. Of course the boundary of an orbifold is in the boundary of the underlying space but the converse is not necessarily true. For example, supposing that the underlying space is a topological manifold, a silvered $(n-1)$-dimensional open manifold in the boundary of the underlying space is in the interior of the orbifold. The interior of the underlying space is in the interior of the orbifold but the converse is not necessarily true.

4.3 The definition as a groupoid

We will try to avoid the definitions using the category theory as related to the theory of stacks in algebraic geometry as much as possible and use the more concrete set theoretic approach. See for example the articles [Moerdijk (2002); Moerdijk and Pronk (1997); Pohl (2010); Lerman (2010)] and Chapter III$\mathcal{G}$ of the book [Bridson and Haefliger (1999)] and the book [Adem, Leida, and Ruan (2007)]. (See the articles [Haefliger (1990, 1984a); Haefliger and Quach (1984b)] also for the beginning of this.) However, there are many reasons to learn orbifolds as groupoids since this framework provides us with more tools and insights from the category theory and even from the smooth manifold theory in the categorical setting. These definitions are mainly introduced to study sheaf theoretic considerations and bundles and so on. (The main reason we are introducing these definitions is to explain the path approach to covering spaces following Haefliger.)

Here, we will try to minimize the theoretical aspect. In spite of the technical nature, readers somewhat acquainted with the category theory will recognize that these definitions are very concrete. Only the abstract nature of the category theory comes when discussing the equivalences of these structures.

We follow mostly the expositions in the book [Adem, Leida, and Ruan (2007)] and the paper [Moerdijk (2002)].

4.3.1 Groupoids

A *topological groupoid* G consists of a space G_0 of objects and a space G_1 of arrows with five continuous maps:

- a source map $s : G_1 \to G_0$,
- a target map $t : G_1 \to G_0$,
- an associative composition map $m : G_{1s} \times_t G_1 \to G_1$ where

$$G_{1s} \times_t G_1 := \{(h, g) \in G_1 \times G_1 | s(h) = t(g)\}.$$

- a unit map $u : G_0 \to G_1$ so that $su(x) = x = tu(x)$ and $gu(x) = g$ if $s(g) = x$ and $u(x)g = g$ if $t(g) = x$, and
- an inverse map $i : G_1 \to G_1$ so that if $g : x \to y$, then $i(g) : y \to x$ and $i(g)g = u(x)$ and $gi(g) = u(y)$.

It will be convenient to think of these arrows at points as restrictions of maps to the singletons. Given a topological groupoid G, we will denote by G_0 the space of objects and G_1 the space of arrows. The arrow $u(x)$ in G_1 from a point x of G_0 to itself is denoted by I_x.

A *Lie groupoid* is one G where G_0 and G_1 are smooth manifolds and the five maps are smooth and s and t are submersions. (This implies that $G_{1s} \times_t G_1$ is a smooth manifold.)

Let M be a smooth manifold. If $G_0 = G_1 = M$ and every arrow is of form I_x for $x \in G_0$, then this is the *unit groupoid* on M.

As a simple example, let a Lie group K act smoothly on a smooth manifold M. The *action Lie groupoid* L is given by $L_0 = M$ and $L_1 = K \times M$ with s as the projection to the M factor and t as the action $K \times M \to M$. The unit map is the inclusion map $x \mapsto (e, x)$ for the unit element e of K. The inverse map $K \times M \to K \times M$ is given by $(g, x) \mapsto (g^{-1}, g(x))$.

If K is the trivial group, we obtain the unit Lie groupoid.

- Given a groupoid G, we define the *isotropy group* at x to be the set of all arrows from x to itself; i.e.,

$$\begin{aligned} G_x &= \{g \in G_1 | (s,t)(g) = (x, x)\} \\ &= (s,t)^{-1}(x, x) \\ &= s^{-1}(x) \cap t^{-1}(x) \subset G_1. \end{aligned}$$

- A *homomorphism* of Lie groupoids $\phi : H \to G$ is a pair of smooth maps $\phi_0 : H_0 \to G_0$ and $\phi_1 : H_1 \to G_1$ commuting with all structure maps.
- The fiber-product: Given two homomorphisms $\phi : H \to G, \psi : K \to G$ of Lie groupoids, we define the *fiber product* $H \times_G K$ to be the Lie groupoid whose objects are (y, g, z) for $y \in H_0, z \in K_0$, and arrow $g : \phi(y) \to \psi(z)$ and whose arrows $(y, g, z) \to (y', g', z')$ are pairs (h, k) of arrows $h : y \to y', k : z \to z'$ so that $g'\phi(h) = \psi(k)g$.

An *étale* map of a Lie groupoid is a homomorphism $\phi : G \to H$ so that $\phi_0 : G_0 \to H_0$ is a local homeomorphism. A homomorphism of Lie groupoids $\phi : H \to G$ is an *equivalence* if

- $t \circ \pi_1 : G_{1s} \times_\phi H_0 \to G_0$ is a surjective submersion.
- the square

$$\begin{array}{ccc} H_1 & \stackrel{\phi}{\to} & G_1 \\ (s,t)\downarrow & & \downarrow(s,t) \\ H_0 \times H_0 & \stackrel{\phi\times\phi}{\to} & G_0 \times G_0 \end{array}$$

is a fiber product of manifolds.

We can show that two groupoids are equivalent if and only if they are Morita equivalent; i.e., there exists another groupoid and equivalences from it to the two groupoids. This essentially means that there is a larger groupoid containing both.

4.3.1.1 *A nerve of a groupoid and the homotopy groups*

Let G be a Lie groupoid. Define

$$G_n = \{(g_1, \dots, g_n) | g_i \in G_1, s(g_i) = t(g_{i+1})\}$$

as a fiber product. The face operator $d_i : G_n \to G_{n-1}$ is defined by sending $(g_1, \dots, g_n)$ to $(g_1, \dots, g_i g_{i+1}, \dots, g_n)$. This forms an abstract simplicial manifold, said to be the *nerve* of the groupoid G.

The classifying space BG is defined to be the geometric realization as a simplicial complex (Adem, Leida, and Ruan, 2007). We will not give much details here.

4.3.2 *An abstract definition*

- A groupoid G is *proper* if $s \times t : G_1 \to G_0 \times G_0$ is proper.
- A groupoid G is *étale* if s and t are local diffeomorphisms.
- A groupoid G is *foliation* if each isotropy group G_x is discrete.
- An *orbifold groupoid* is a proper étale Lie groupoid.

If G is an étale groupoid, then any arrow $g : x \to y$ in G induces a well-defined germ of a diffeomorphism $\tilde{g} : U_x \to V_y$ for neighborhoods U_x of x and V_y of y in G_0, defined as $\tilde{g} = t \circ \hat{g}$, where $\hat{g} : U_x \to G_1$ is a section of the source map $s : G_1 \to G_0$ with $\hat{g}(x) = g$. (By étale property, such sections exist.) We call an étale groupoid G *effective* (or *reduced*) if the assignment $g \mapsto \tilde{g}$ is faithful; or equivalently, if for each point $x \in G_0$ this map $g \mapsto \tilde{g}$ defines an injective group homomorphism $G_x \to \mathbf{Diff}(U_x)$.

Some authors define proper foliation Lie groupoids to be orbifold groupoids. However, they are equivalent under a Morita equivalence. Orbifold groupoids are usually effective groupoids. Also, G_x is finite for each point $x \in G_0$ if G is a proper foliation groupoid.

The set $ts^{-1}(x) = \{y | \exists z \in G_1, z : x \to y\}$ is called the *orbit* of x. The *orbit space* $|\mathcal{G}|$ of a groupoid $\mathcal{G}$ is the quotient space of its space of objects G_0 under the equivalence relation $x \sim y$ if and only if x and y are in the same orbit.

Theorem 4.3.1.

- *Let $\mathcal{G}$ be a proper effective étale groupoid. Then its orbit space $|\mathcal{G}|$ can be given the structure of an orbifold.*
- *Two effective orbifold groupoids $\mathcal{G}$ and $\mathcal{G}'$ represent the same orbifold up to isomorphism if and only if they are Morita equivalent.*

We do not prove this theorem (Adem, Leida, and Ruan, 2007); however, we show below that an orbifold gives rise to a proper effective étale groupoid.

Example 4.3. Let M be a smooth orbifold with the locally finite covering $\mathcal{U}$ by model neighborhoods in the orbifold atlas and the underlying space X. Each nonempty finite intesection of the members of $\mathcal{U}$ has a model (U, G, ϕ) in the orbifold atlas for some domain $U \subset \mathbb{R}^n$, a finite group G acting on it effectively, and ϕ inducing a homeomorphism U/G to its image. Let M_0 be the disjoint union of the model open sets in $\mathbb{R}^n$ of all finite intersections of members of $\mathcal{U}$, and let M_1 be the set of arrows obtained by restrictions to points in M_0 of all embeddings $U \to V$ for model triples (U, G, ψ) and (V, H, ϕ) lifting the inclusion maps and their compositions and the inverse arrows. (Here, it is possible that $U = V$ and $G = H$.) Also, we include I_x for all $x \in M_0$. Then the space of orbits is homeomorphic to X and M_0 and M_1 contain all the information of the atlas. The fact that this is a proper effective étale groupoid follows by checking the above definitions.

We note the alternative definition:

Definition 4.3. An *orbifold structure* on a paracompact Hausdorff space X consists of orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$. Two orbifold structures $(\mathcal{G}, f)$ and $(\mathcal{H}, g : |\mathcal{H}| \to X)$ are *equivalent* if there is a groupoid equivalence $\phi : \mathcal{H} \to \mathcal{G}$ inducing the homeomorphism $|\phi| : |\mathcal{H}| \to X = |\mathcal{G}|$ so that $f \circ |\phi| = g$.

4.3.2.1 *Examples*

Let a discrete group Γ act on a connected manifold X properly discontinuously. Then (Γ, X) has an orbifold structure. We think of it as a groupoid where X_0 is given as X itself and X_1 as the space of arrows sending $x \to \gamma(x)$ for $x \in X_0$ for $\gamma \in \Gamma$. Hence, there are cardinality of Γ of components of X_1 homeomorphic to X_0. (This is the good orbifold discussed above. See Theorem 4.2.3.)

We obtain a 2-orbifold from a compact orientable Seifert fibered 3-manifold M: We choose X_0 to be the union of finitely open disks that are disjoint and bounded away from one anoother and each flow line meets at least one of them. We choose X_1 to be the space of flow lines with both end points in these disks.

The fiber order of a closed flow curve is the order of the germ of the return map to a transversal disk along the curve.

The orbifold X will be a 2-dimensional orbifold with cone-points whose orders are the fiber-orders of the corresponding closed flow lines.

4.3.2.2 *Actions of a Lie groupoid*

Let G be an orbifold groupoid. A *left G-space* is a manifold E equipped with an action by G: Such an action is given by two maps: an anchor $\pi : E \to G_0$ and an action $\mu : G_1 \times_{G_0} E \to E$.

- This map is defined on (g, f) with $\pi(f) = s(g)$ and written $\mu(g, f) = g.f$ for $f \in E$.
- It satisfies the action identity: $\pi(g.f) = t(g)$, $\mathrm{I}_x.f = f$, and $g.(h.f) = (g.h).f$ for $h : x \to y$ and $g : y \to z$ and $f \in E$ with $\pi(f) = x$.

A *right G-space* is the left G^{op}-space obtained by switching the source and target maps of G_1.

4.4 Differentiable structures on orbifolds

Now, we go back to the original definition of orbifolds using charts.

Let $\mathcal{O}$ be an orbifold. We are given a smooth structure on each $(\tilde{U}, G, \phi)$; i.e., $\tilde{U}$ is given a smooth structure and G is a finite group with a smooth action on it. All embeddings in the atlas are smooth. Then M is given a *smooth* structure under embeddings. Given a chart $(\tilde{U}, G, \phi)$, we define *the space of smooth forms* to be the space of smooth forms in $\tilde{U}$ invariant under the G-action. A *smooth form* on the orbifold is the collection of smooth forms on all model open sets of the charts so that they match under embeddings and the local group actions.

This enables one to define the space $\Lambda^p(\mathcal{O}), p \geq 0$ of smooth p-forms on $\mathcal{O}$ and the boundary operators, which are defined as usual since one can define boundary operators on the model neighborhoods. Let $H^p(\mathcal{O})$ denote the p-th de Rham cohomology of $\mathcal{O}$. Let $H^q_c(\mathcal{O}), q \geq 0$, denote the q-th de Rham cohomology of $\mathcal{O}$ defined from compactly supported smooth forms.

A *smooth simplex* defined from a simplex Δ to an orbifold $\mathcal{O}$ is simply a smooth map. One can define an integral of a differential form with respect to a smooth singular simplex into a model neighborhood by lifting to the model neighborhood by Theorem 4.1.3. A smooth singular simplex may have different lifts to model neighborhoods; however, the integral itself is well-defined. (One needs to look at the currents in the inverse image of the simplex.) This can be extended to any smooth simplex using partition of unity and barycentric subdivisions of the simplex. Given a locally finite covering of $\mathcal{O}$, we can define a smooth partition of unity (in the same way as in the manifold case). (See for example the book [Munkres (1991)].)

- We refine to obtain a cover by open sets whose closures are invariant compact subsets.
- The idea is to find a smooth function on each chart which vanishes outside the invariant compact subsets.
- The images of compact subsets can be chosen to cover $\mathcal{O}$.
- Thus, these functions become functions on $\mathcal{O}$ which sum to a positive valued function.
- We divide by the sum.

An orbifold $\mathcal{O}$ is *orientable* if one can choose an atlas of charts where $\tilde{U}$ is given an orientation with G acting in an orientation-preserving manner and each embedding of charts to another charts is orientation-preserving. For example, a reflection about a hypersurface is excluded and hence silvered boundary is excluded. (However, one can use densities or forms of odd degrees to replace n-forms and can integrate when $\mathcal{O}$ is not orientable. See the book [de Rham (1984)].)

An n-form ω can be integrated on an orientable orbifold $\mathcal{O}$: Let $(\tilde{U}, G, \phi)$ is a model triple for a model neighborhood U of $\mathcal{O}$ and let ω' denote the n-form on $\tilde{U}$ representing ω. Then the integral of ω on U is defined as

$$\int_U \omega = \frac{1}{|G|} \int_{\tilde{U}} \omega'$$

where $|G|$ is the order of G. Then for any n-form, the integral upon $\tilde{\mathcal{O}}$ can be integrated by using a partition of unity.

The Poincaré duality pairing: For an orientable orbifold $\mathcal{O}$,

$$\int : H^p(\mathcal{O}) \otimes H_c^{n-p}(\mathcal{O}) \to \mathbb{R}$$

is given by sending (ω, η) for a closed p-form ω and a closed and compactly supported $(n-p)$-form η to $\int_{\mathcal{O}} \omega \wedge \eta$. This is a nondegenerate bilinear form when $\mathcal{O}$ is a closed orientable orbifold. Adem, Leida, and Ruan (2007) prove this.

4.4.1 *Bundles over orbifolds*

An *orbifold-bundle* (or V-bundle) E over an orbifold $\mathcal{O}$ is given by a smooth orbifold E and a smooth map $\pi : E \to \mathcal{O}$ with the following properties:

- Let F be a smooth manifold with a Lie Group $\mathbf{G}$ acting on it smoothly.
- A pair of defining families $\mathcal{F}$ for $\mathcal{O}$ and $\mathcal{F}'$ for E so that a model triple (U, K, ϕ) of $\mathcal{O}$ corresponds to a model triple (U^*, K^*, ϕ^*) so that $U^* = U \times F$ and $\pi \circ \phi^* = \phi \circ \pi_1$ where $\pi_1 : U^* \to U$ is the projection to the first factor.

- Given $(U, K, \phi), (U^*, K^*, \phi^*)$, and $(U', K', \phi), (U'^{,*}, K'^{,*}, \phi'^{,*})$, we require that there is a one-to-one correspondence of embeddings $\lambda : (U, K, \phi) \to (U', K', \phi)$ and
$$\lambda^* : (U^*, K^*, \phi^*) \to (U'^{,*}, K'^{,*}, \phi'^{,*})$$
where $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ for $(p, q) \in U^* = U \times F$ with $g_\lambda(p) \in \mathbf{G}$.
- We have
$$g_{\mu\circ\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p) \tag{4.1}$$
for embeddings
$$(U, K, \phi) \xrightarrow{\lambda} (U', K', \phi') \xrightarrow{\mu} (U'', K'', \phi'').$$
- If $F = \mathbf{G}$, then this is a principal orbifold bundle (with a right $\mathbf{G}$-action).

Notice that by the one-to-one correspondence property of the third item, there is an isomorphism $K \to K^*$ given by sending $\sigma \in K$ to $\sigma^* \in K^*$ defined by
$$\sigma^*(\tilde{p}, q) = (\sigma(\tilde{p}), g_\sigma(\tilde{p})q), \tilde{p} \in U.$$

Conversely, the above data are enough to construct an orbifold-bundle as we can verify that the quotient space of the collection of sets of form $U \times F$ by the identification map is still Hausdorff and second-countable and hence an orbifold.

4.4.1.1 *Principal bundles using the groupoids language.*

Finally, using the groupoid language, we can define the principal bundles. See the article [Moerdijk (2002)] and the book [Adem, Leida, and Ruan (2007)] for details.

A *principal L-bundle* for a Lie group L over a Lie groupoid G is a G-space P with a right action $P \times L \to P$ which makes $\pi : P \to G_0$ into a principal L-bundle over the manifold G_0 and is compatible with the G-action in the sense that $g.(p.l) = (g.p).l$ for $p \in P, l \in L$ and an arrow $g : x \to y$.

4.4.2 *Tangent bundles and tensor bundles*

Given the orbifold $\mathcal{O}$, we build a *tangent orbifold-bundle* $T(\mathcal{O})$ by taking $F = \mathbb{R}^n$, $\mathbf{G} = \mathbb{GL}(n, \mathbb{R})$, and $g_\lambda(p)$ to be the Jacobian of λ at p for each embedding $\lambda : (U, K, \phi) \to (U', K', \phi)$ as above. We can build any tensor bundles in this way by letting $F = T^r_s(\mathbb{R}^n)$ and $\mathbf{G} = \mathbb{GL}(n, \mathbb{R})$ and $g_\lambda(p)$ be the induced map $T^r_s(\mathbb{R}^n) \to T^r_s(\mathbb{R}^n)$ of λ at p.

A *reduction* of a Lie group $\mathbf{G}$ to a subgroup H means an injective homomorphism $H \to \mathbf{G}$ which induces a bundle morphism of the principal bundle with the Lie group H to the principal bundle with the Lie group $\mathbf{G}$.

A frame bundle is obtained by taking F to be $F_n(\mathbb{R}^n)$ the space of frames in $\mathbb{R}^n$, $\mathbf{G}$ to be $\mathbb{GL}(n, \mathbb{R})$, and $g_\lambda(p)$ to be the induced map $F_n(\mathbb{R}^n) \to F_n(\mathbb{R}^n)$ of λ at p.

An *affine frame bundle* is given by taking $F = A(\mathbb{R}^n)$ the space of affine frames and $\mathbf{G} = \mathbb{A}(\mathbb{R}^n)$, the Lie group of affine autormorphisms. An *affine tangent bundle* is given by taking $F = \mathbb{R}^n$ with the same Lie group.

An orthogonal frame bundle is a reduction of the frame bundle to $\mathbb{O}(n, \mathbb{R})$: Orthogonal frame bundles can be built in this way. We let $F = O_n(\mathbb{R}^n)$ the space of orthonormal frames and let $\mathbf{G} = \mathbb{O}(n, \mathbb{R})$ and choose $g_\lambda(p)$ be a map $O_n(\mathbb{R}^n) \to O_n(\mathbb{R}^n)$ corresponding to each λ at p.

Let $\mathbf{G}$ be a Lie group with a Lie algebra $\mathfrak{g}$. Given a principal bundle P, one defines a *connection* to be an assignment of an equivariant connection on every model triple (U^*, K^*, ϕ^*) corresponding to a model triple (U, K, ϕ) of $\mathcal{O}$ which form a collection that are consistently defined under the embeddings. The *curvature* is also defined as the $\mathfrak{g}$-valued 2-form on $\mathcal{O}$ which comes from the curvature of each orbifold chart.

A *linear connection* is a connection on a frame bundle or a tangent bundle with Lie group $\mathbb{GL}(n, \mathbb{R})$. An *affine connection* is a connection on an affine frame bundle or an affine tangent bundle with the Lie group $\mathbb{A}(\mathbb{R}^n)$. Given an affine connection on an affine tangent bundle, a *geodesic* is defined as a smooth map from an open arc to $\mathcal{O}$ so that in each chart it lifts to a geodesic under the connection.

A *Riemannian metric* on an orbifold is given by an equivariant Riemannian metric on each chart which matches up under embeddings or simply as a smooth section of symmetric covariant tensor bundle $ST^2(\mathcal{O})$ whose image lie in the positive definite forms. A Riemannian metric can be built using a partition of unity again from any given Riemannian metrics on charts.

The group $\mathbb{O}(n, \mathbb{R}) \cdot \mathbb{R}^n$ is the group of rigid motions on $\mathbb{R}^n$. We can also replace the group $\mathbb{A}(\mathbb{R}^n)$ with $\mathbb{O}(n, \mathbb{R}) \cdot \mathbb{R}^n$ by reduction of the group. This corresponds to choosing a section to $ST^2(\mathcal{O})$. Then the connections on the reduced affine bundles are also called *affine connections*. (As usual, an $\mathbb{O}(n, \mathbb{R})$-connection of a tangent bundle or a frame bundle is also considered an affine connection since we can always construct a canonical affine connection from a linear connection by the Levi-Civita constructions. The set of geodesics does not change here. See the reasoning in [Kobayashi and Nomizu (1997)] that can be directly generalized to the orbifold setting.)

Finally, one defines an exponential map $\exp : T(\mathcal{O}) \to \mathcal{O}$: one defines the exponential map using the linear or affine connection on each model neighborhood and then patching up the consistent results.

Lemma 4.4.1. *Given an orbifold $\mathcal{O}$ with boundary, we can give a Riemannian metric on $\mathcal{O}$ so that boundary components are totally geodesic.*

Proof. Let x be a point of $\partial\mathcal{O}$. Then we find a model triple (U, G, ϕ) of x. We obtain a reflection r_F fixing ∂U and we form a finite group L_U isomorphic to $\mathbb{Z}_2$ generated by these. Then let $U' = \bigcup_{g \in L_U} g(U)$ is an invariant open set in $\mathbb{R}^n$ generated by G and L_U. We find an invariant Riemannian metric g_U on U'.

Now, we cover $\mathcal{O}$ by a locally finite covering by model open sets U_i with models $(\tilde{U}_i, G_i, \phi_i)$. Let U_i' be obtained as above by taking the union under the reflections in faces of U_i. Obtain Riemannian metric g_{U_i} for each U_i. We use a partition of unity to obtain a Riemannian metric μ on $\mathcal{O}$. This induces a new Riemannian metric g_{U_i}' on U_i'.

Let O' be an open n-orbifold containing $\mathcal{O}$ and a tubular neighborhood N of $\partial\mathcal{O}$. This can be obtained by taking open model open sets instead of half-open ones in $\mathbb{R}^n$. Extend the metric μ to O'.

For each component F of $\partial\mathcal{O}$, we find a reflection r_F defined on a tubular neighborhood of F in O' given by sending points of distance r on a geodesic perpendicular to F to its opposite point on the geodesic with same distance. Then we form the Riemannian metric $(r_F^*\mu + \mu)/2$. We use a partition of unity so that we have a Riemannian metric, on $\mathcal{O}$, that is invariant under r_F in a smaller tubular neighborhood of F in O'. Then F is totally geodesic this metric as in the note [Francis (2010)]. (See also Lemma 4.1.7.) □

An *isotopy* $F : Y_1 \times I \to Y_2$ for two orbifolds Y_1 and Y_2 is an orbifold-map such that for each $t \in I$ where I is an interval, F restricts to a diffeomorphism of $Y_1 \times \{t\}$ into suborbifolds of Y_2. (We will often consider codimension-zero suborbifolds.)

Let $\mathcal{O}$ be an n-orbifold with boundary. A *neatly embedded suborbifold* is a suborbifold A of $\mathcal{O}$ such that $\partial A = \partial\mathcal{O} \cap A$ or $\partial A = \emptyset$ and $A \cap \partial\mathcal{O} = \emptyset$. (See Section 1.4 of the book [Hirsch (1976)].) In this case, we can make A perpendicular to $\partial\mathcal{O}$ by an isotopy from the inclusion map of A. Basically, we make the inverse image of A in the model open sets be perpendicular to boundary and then we use averaging of the defining functions of A and use partition of unity to build an isotopied suborbifold in $\mathcal{O}$ and the defining functions in the models C^2-close to the original ones. Finally, we show that we can achieve this by an isotopy generated by the vector fields.

A *normal vector* of a suborbifold $\mathcal{O}_1$ at a point x in $\mathcal{O}$ is an equivalence class of a vector v in the tangent space of model neighborhood (U, G, ϕ) with a chart ϕ at a point $\tilde{x}$ corresponding to x and perpendicular to the tangent vectors of the inverse image of $\mathcal{O}_1$ in U under ϕ.

Let Σ be an i-dimensional neat suborbifold of $\mathcal{O}$ for $i < n$. Denote by $N(\Sigma)$ the space of normal vectors of Σ. The exponential map is a diffeomorphism from

$$N_\epsilon(\Sigma) := \{v \in N_x(\Sigma) | \, x \in \Sigma, ||v|| < \epsilon(x)\}$$

to its image provided $\epsilon : \Sigma \to (0, \infty)$ is a sufficiently small valued function. The proof is entirely similar to those in the Riemannian manifold theory and we omit these. (See Sections 4.5 and 4.6 of the book [Hirsch (1976)].) The image is said to be a *tubular-neighborhood* of Σ. (Here we use the total geodesic properties and orthogonality of boundary components of $\mathcal{O}$ meeting the suborbifold.)

Since we understand the normal bundle of Σ, the orbifold structure of a tubular-neighborhood can be understood as an orbifold-bundle over Σ where the fiber over $x \in \Sigma$ can be described as D^{n-i}/G_x for an $(n-i)$-disk D^{n-i} and G_x is a finite group.

If Σ is a boundary component, then we define $N^+(\Sigma)$ to be the set of vectors pointing inside. Each boundary component Σ of an orbifold $\mathcal{O}$ has a *collar*, i.e., a neighborhood diffeomorphic to $\Sigma \times [0, 1)$. Using the exponential map from the normal bundle $N^+(\Sigma)$ to $\mathcal{O}$, and taking the image of vectors of length $< \epsilon(x)$ for some small valued function $\epsilon : \Sigma \to \mathbb{R}^+$, we obtain a *collar*.

4.4.3 *The existence of a locally finite good covering*

Recall Definition 4.1.

Proposition 4.4.2. *Let $\mathcal{O}$ be an orbifold with boundary. Then there exists a good covering.*

Proof. First, give a Riemannian metric on $\mathcal{O}$ where the boundary suborbifolds are totally geodesic. Each point has an orbifold chart with an orthogonal action. Now choose a sufficiently small ball in the model neighborhood centered at the origin so that it has a convexity property. (That is, any path in a model open set can be homotopied into a geodesic.) (See Chapter 3 of the book [Do Carmo (1992)].) Find a locally finite subcollection. Then the intersection set of any finite collection is still convex and hence has cells as finite coverings. □

4.4.4 *Silvering the boundary components*

In fact, we can fully generalize the results in Section 4.2.3:

Proposition 4.4.3. *Let $\mathcal{O}$ be an n-dimensional orbifold with a boundary component Σ. Then we can obtain an orbifold $\mathcal{O}'$ with the same underlying space and every point of Σ is now singular with generic manifold points becoming a silvered point.*

Proof. The proof of Lemma 4.4.1 contains the proof. Basically, we add the reflections to the groups of the model triples with small image open sets. □

4.4.5 *The Gauss-Bonnet theorem*

Let $\mathcal{O}$ be an orbifold with the underlying space X. We will show shortly that X admits a finite smooth triangulation so that the interior of each simplex lies in the connected set of singular points with locally constant local groups in Theorem 4.5.4.

We define the Euler characteristic to be

$$\chi(\mathcal{O}) = \sum_k (-1)^{\dim s_k} \frac{1}{N_{s_k}}$$

where s_k denotes the open kth-cell in the triangulation and N_{s_k} the order of the local group.

Theorem 4.4.4 (Satake). *Let M be a closed orbifold of even dimension m with a Riemannian metric. Then*

$$(2/O_m)\int_M K d\mu = \chi(M),$$

where K is the Pfaffian of the curvature form, $d\mu$ is the volume measure of M, and O_m is the volume of the standard unit m-sphere.

The proof essentially following that of Chern for manifolds is given by Satake (1957). Here Satake's work only allows for codimension ≥ 2 singularities. We see that by doubling M, the theorem holds. (See Section 4.6.1.2 for details on doubling.) Thus, the theorem holds for M by divisions by 2 by Proposition 5.1.3.

4.5 Triangulation of smooth orbifolds

In general, a smooth orbifold has a smooth topological stratification and a smooth triangulation so that each open cell is contained in a single stratum. A smooth topological stratification satisfying certain weak conditions admits a triangulation. We now show that the stratification of an orbifold by orbit types satisfies this condition. We mainly follow pp. 37–38 and pp. 126–127 of the book [Verona (1984)]. (See also the article [Moerdijk and Pronk (1999)].)

We denote by $\mathbb{R}_+$ the subset $\{x \in \mathbb{R} | x \geq 0\}$. A *manifold M with corner* is a topological manifold with boundary with atlas of charts to $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n)|x_1 \geq 0, \ldots, x_n \geq 0\}$ with smooth transition maps. Each point of M has a neighborhood with a chart to an open subset diffeomorphic to $\mathbb{R}^i_+ \times \mathbb{R}^{n-i}$ for a minimal i, $0 \leq i \leq n$. Such a point is said to be of *corank i*. A set of points of corank 0 is the set of interior points of M and the set of points of corank ≥ 1 is the set of boundary points of M to be denoted by ∂M.

Let M be a manifold with corners and let ∂M be the boundary of M. A *face* of M is a closure of a component of the set of corank 1 in ∂M. It itself is a cornered manifold B in ∂M with an embedding $F_B : U_B \to B \times \mathbb{R}_+$ for an open neighborhood U_B of B, called a *collar* of B, where $F_B(x) = (x, 0)$ for al $x \in B$.

4.5.1 *Triangulation of the stratified spaces*

Let X, Y be two subsets of a topological space A with $X \cap Y = \emptyset$. If $X \subset \mathrm{Cl}(Y)$, then we write $X < Y$. We say $X \leq Y$ if $X = Y$ or $X < Y$.

A *face* of a topological space A is a closed subset of A with a smooth embedding $F_B : U_B \to B \times \mathbb{R}_+$ for a neighborhood U_B of B sending B to $B \times \{0\}$. F_B and U_B are said to be the *collar* and the *collar neighborhood*. We write $F_B = (p_B, r_B)$ where $p_B : U_B \to B$ and $r_B : U_B \to \mathbb{R}_+$ are smooth functions.

A Hausdorff, locally compact, paracompact space with a countable basis is said to be a *nice* space. Let A be a nice topological space and $X \subset A$ be a locally closed set. A *tube* T_X of X is a neighborhood of X in A with a retraction $\pi_X : T_X \to X$ and a function $\rho_X : T_X \to \mathbb{R}$ such that $\rho_X^{-1}(0) = X$.

Given positive valued functions $\epsilon, \delta : X \to \mathbb{R}$ with $0 \leq \epsilon < \delta$, we define

$$X \times (\epsilon, \delta) = \{(x,t) \in X \times \mathbb{R} | \epsilon(x) < t < \delta(x)\}$$

with obvious extensions to closed interval cases.

Define $T_X^\epsilon = \{a \in T_X | \rho_X(a) < \epsilon(\pi_X(a))\}$ for a function $\epsilon : X \to \mathbb{R}_+$ where $\epsilon > 0$. If $X \subset U \subset A$ for an open U, then $T_X^\epsilon \subset U$ for some ϵ. $(\pi_X, \rho_X)|T_X^\epsilon$ is a proper map into

$$X \times [0, \epsilon) = \{(x,t) | x \in X, 0 \leq t < \epsilon(x)\}$$

by choosing sufficiently small ϵ.

An *abstract stratification* $\mathcal{A}$ consists of

(i) a nice space A and
(ii) a locally finite family $\mathcal{A}$ of locally closed connected subsets A' (strata) of A so that A is a disjoint union of $\mathcal{A}$ whose members are smooth manifolds
(iii) a family of tubes of the strata $\{\tau_X = (T_X, \pi_X, \rho_X) : X \in \mathcal{A}\}$.
(iv) a family of closed subsets $\mathcal{A}^*$ of A called *faces*

satisfying the following properties:

- If $X, Y \in \mathcal{A}$ with $X \cap \mathrm{Cl}(Y) \neq \emptyset$, then we have $X \leq Y$.
- For any face $A_i \in \mathcal{A}^*$, there exists an open neighborhood U_{A_i} and a homeomorphism $F_{A_i} : U_{A_i} \to A_i \times \mathbb{R}_+$ onto an open subset so that
  - $F_{A_i}(a) = (a, 0), a \in A_i$
  - for any $X \in \mathcal{A}$, if $X \cap A_i \neq \emptyset$, then

    $$F_{A_i}(X \cap U_{A_i}) \subset (X \cap A_i) \times \mathbb{R}_+$$

    for a collar F_{A_i} of A_i. We define

    $$p_{A_i} : U_{A_i} \to A_i \text{ and } r_{A_i} : U_{A_i} \to \mathbb{R}_+ \text{ by}$$

    $$F_{A_i}(a) = (p_{A_i}(a), r_{A_i}(a)) \text{ for } a \in U_{A_i}.$$

- Each stratum $X \in \mathcal{A}$ is a manifold with faces $X_i := X \cap A_i, A_i \in \mathcal{A}^*$ with collars

  $$F_{X_i} = F_{A_i} | X \cap U_{A_i} : U_{X_i} = X \cap U_{A_i} \to X_i \times \mathbb{R}_+$$

  whenever $X \cap A_i \neq \emptyset$.
- For $X \in \mathcal{A}$ and $A_i \in \mathcal{A}^*$, we have $\pi_X^{-1}(X_i) = A_i \cap T_X$ and

  $$F_{X_i} \circ \pi_X = ((\pi_X | T_X \cap A_i) \times \mathrm{I}_{\mathbb{R}_+}) \circ F_{A_i}$$

  in an open neighborhood of X_i provided $X_i \neq \emptyset$.

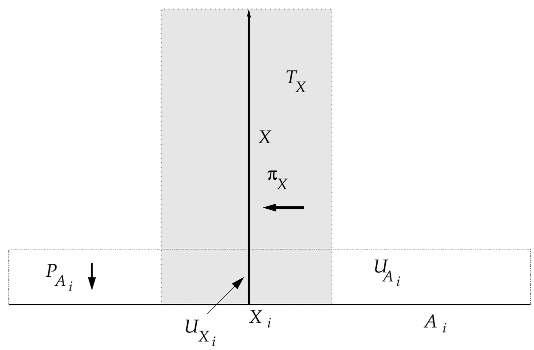

Fig. 4.6 An illustration of tubes and faces and so on.

- For any $X \in \mathcal{A}$, X has $\epsilon_X : X \to \mathbb{R}_+$, $\epsilon_X > 0$, so that $T_X^{\epsilon_X} \cap Y \neq \emptyset$ for $Y \in \mathcal{A}$ implies $X < Y$ and
$$(\pi_X, \rho_X) : T_X^{\epsilon_X} \cap Y \to X \times (0, \epsilon_X)$$
is a smooth submersion.
- For any $X, Y \in \mathcal{A}$, $X \subset \mathrm{Cl}(Y)$, there exist positive functions ϵ_X defined on X and ϵ_Y defined on Y satisfying the statement that
$$a \in T_X^{\epsilon_X} \cap T_Y^{\epsilon_Y} \text{ implies}$$
$$\pi_Y(a) \in T_X, \pi_X(\pi_Y(a)) = \pi_X(a) \text{ and } \rho_X(\pi_Y(a)) = \rho_X(a).$$
- For any $X \in \mathcal{A}$, and $A_i \in \mathcal{A}^*$, we have $\rho_X = \rho_X \circ p_{A_i}$ in a neighborhood of A_i.

The *dimension of a stratum* is the dimension as a manifold. If $X \subset \mathrm{Cl}(Y)$ for strata X and Y, then the dimension of X is strictly less than that of Y. The *depth* of a stratified space is the maximal cardinality of collections of form $\{X_1, \dots, X_n\}$ of strata X_i satisfying $X_i < X_{i+1}$ for $i = 1, \dots, n-1$. Note that the dimensions strictly increase in the chain. The maximal dimensional strata are open manifolds where the tubes are identical with themselves. A stratification has a finite depth if the maximal dimension of the strata is finite.

A *triangulation* of a topological space A consists of a pair (K, ϕ) where K is a countable locally finite simplicial complex and $\phi : |K| \to A$ for a geometric realization $|K|$ of K is a homeomorphism.

A *relative manifold* (with corners) is a pair of topological spaces $(V, \delta V)$ so that δV is a closed subset of V and $V - \delta V$ is a manifold with corners. A triangulation (K, ϕ) of a relative manifold $(V, \delta V)$ is *smooth* if K contains a subcomplex δK so that $\phi(\delta K) = \delta V$ and for any simplex σ of K, the restriction ϕ to $|\sigma| - |\delta K|$ is smooth and for each $x \in |K| - |\delta K|$ the differential $D\phi_x$ of ϕ at x is injective.

A *smooth triangulation* of an abstract stratification $\mathcal{A}$ is a triangulation (K, ϕ) of A satisfying the condition that for each stratum X, there is a subcomplex K_X so that $K_X, \phi|K_X$ is a smooth triangulation of $(\mathrm{Cl}(X), \mathrm{Cl}(X) - X)$.

Theorem 4.5.1 (Verona). *Let X be a nice space, and let $\mathcal{A}$ be an abstract stratification of X of finite depth. Then there exists a smooth triangulation of $\mathcal{A}$.*

4.5.2 *Orbifolds as stratified spaces*

Lemma 4.5.2. *Let V be a Euclidean vector space or $\mathbb{R}^i \times \mathbb{R}_+^{n-i}$ for a fixed $i = 0, 1, \ldots, n$. Let G be a finite group effectively acting on V orthogonally preserving each face of $\mathbb{R}^i \times \mathbb{R}_+^{n-i}$.*

- *The fixed-point set of a linear finite group G action is a closed subspace of V.*
- *The subset $F_{G'}$ of points fixed exactly by a subgroup G' of G is a vector subspace with a finite number of closed subspaces removed. $F_{G'}$ is dense open in the subspace of fixed points of G'.*
- *F_G and $F_{G'}$ are orthogonal to faces of $\mathbb{R}^i \times \mathbb{R}_+^{n-i}$.*
- *For distinct subgroups G' and G'', $F_{G'}$ and $F_{G''}$ are disjoint.*
- *If $G'' \subset G'$ properly, then $F_{G'}$ is in the closure of $F_{G''}$.*

Proof. The first item is clear.

The second item follows from the fact that the fixed-point set of any subgroup is a subspace. One has to remove subspaces fixed by a larger group from inside. The third item and the fourth items are also clear. The final item follows from the second item. □

To prove our result, we will use the results from Section 4.4. (This is strictly for convenience, and we will need simple results in exponential maps.)

First, let G_x be a nontrivial local subgroup of a point x of an orbifold $\mathcal{O}$. Then the set of points with local groups locally conjugate to G_x forms a locally closed connected manifold by the existence of linear charts and Lemma 4.5.2.

Thus, the underlying space X of $\mathcal{O}$ is a disjoint union of connected submanifolds determined by the local topological conjugacy classes of the local groups. Let us call the collection of connected submanifolds $\mathcal{A}$. Since X is a nice topological space, the set $\mathcal{A}$ forms a stratification:

Suppose $X \cap \mathrm{Cl}(Y) \neq \emptyset$ for two strata X, Y. Given the local linear chart U for $x \in X$, we see that the stabilizer G_x corresponding to x is the maximal local group in the chart. Then $X \cap U \subset \mathrm{Cl}(Y) \cap U$ for each linear chart neighborhood U of x. Hence $X \subset \mathrm{Cl}(Y)$.

We need a slight generalization of orbifolds with boundary. Recall $\mathbb{R}_+$ is the space of nonnegative real numbers and $\mathbb{R}_+^n$ the Cartesian product. We define an *orbifold with corners* as an orbifold $\mathcal{O}$ with the following properties:

- Each point has a model (U, G, ψ) where U is an open subset of $\mathbb{R}_+^n$ and G is a finite group acting on it that acts on each face of $\mathbb{R}_+^n$ that U meets.

- In the manifold cases, we define the corank of a point of $\mathcal{O}$ as the corank in the models.
- We define the *face* as a subset of $\partial\mathcal{O}$ as the closure of a component of the set of corank 1 in $\partial\mathcal{O}$ and it is required to a face of the underlying space $|\mathcal{O}|$.

(Recall Remark 4.2.5 also.)

The following is a direct generalization of Lemma 4.4.1 with an almost identical proof.

Lemma 4.5.3. *Given an orbifold $\mathcal{O}$ with corners, we can give a Riemannian metric on $\mathcal{O}$ so that*

- *faces are totally geodesic and they are perpendicular to each other when they meet at codimension-two subspaces.*
- *each stratum X of $\mathcal{O}$ is a totally geodesic manifold with faces in $\partial\mathcal{O}$ and perpendicular to faces in $\partial\mathcal{O}$ and is neatly embedded with a collar about $X \cap F$ for every face F of $\partial\mathcal{O}$.*

Proof. Let x be a point of $\partial\mathcal{O}$. Then we find a model triple (U, G, ϕ) of x. For each face F of U, we obtain a reflection r_F: actually the Euclidean one will do, and we form a finite group L_U generated by these. We require that the reflections always commute with one another. Then let $U' = \bigcup_{g \in L_U} g(U)$ is an invariant open set in $\mathbb{R}^n$ generated by G and L_U. We find an invariant Riemannian metric g_U on U'.

Now, we cover $\mathcal{O}$ by a locally finite covering by model open sets U_i with models $(\tilde{U}_i, G_i, \phi_i)$. Let U'_i be obtained as above by taking the union under the reflections in faces of U_i. Obtain a Riemannian metric g_{U_i} for each U_i. We use a partition of unity to obtain a Riemannian metric μ on $\mathcal{O}$. This induces a new Riemannian metric g'_{U_i} on U'_i. Also, every pair of intersecting faces of $\mathcal{O}$ are orthogonal to each other.

Let O' be an open n-orbifold containing $\mathcal{O}$. Extend the metric μ to O'.

Take a face F of $\partial\mathcal{O}$. We find a reflection r_F defined on a tubular neighborhood of F in O' given by sending points of distance r on a geodesic perpendicular to F to its opposite point on the geodesic with same distance. (We might need to define this on an ambient manifold containing F and extending F slightly.) Then we form the Riemannian metric $(r_F^*\mu + \mu)/2$. We use a partition of unity so that we have a Riemannian metric on $\mathcal{O}$ which is invariant under r_F in a smaller tubular neighborhood of F in O' bounded by some extensions of other faces. Then F is totally geodesic in this metric and still perpendicular to other faces. (See the note [Francis (2010)].)

Using the reflection r_F for the new metric, perhaps a little changed now, we can silver F by taking a small tubular N_F neighborhood of F in O' bounded by some extensions of other faces and define charts by using charts of points of F with images in N_F and adding r_F to the group.

We now do an induction process and we can silver every face of $\partial\mathcal{O}$ since the set of faces is locally finite. Now it is clear that the faces are all totally geodesic and orthogonal when they meet.

Each model neighborhood (U, G) of $\mathcal{O}$ has an invariant Riemannian metric induced from that of $\mathcal{O}$. Then since G acts on U intersected with faces of $\mathbb{R}^i \times \mathbb{R}_+^{n-i}$, it follows that each fixed point set of a subgroup of G is a submanifold A perpendicular to faces of U.

Since a subgroup of G fixes each point of A, it follows that A is totally geodesic. Thus, each stratum of $\mathcal{O}$ is totally geodesic.

Using the exponential map from the normal vector bundle of each face F of U, in this case using normal vectors in one direction, we obtain an ϵ-collar of F for a positive valued function $\epsilon : F \to \mathbb{R}$. We obtain a collar of the image of F in U/G. Since X is totally geodesic, the collar restricts to $X \cap F$ and we obtain an ϵ-collar of $X \cap F$. By patching together, we see that each i-dimensional stratum X has a collar about $X \cap F$ for each face F. $\square$

Now we move to the main theorem of this section.

Theorem 4.5.4. *Let $\mathcal{O}$ be an n-orbifold with corners. Each singularity x of an orbifold $\mathcal{O}$ with a local group G_x always lies in a submanifold of points whose local groups are locally conjugate to G_x. Then the collection of such submanifolds with the nonsingular components forms an abstract stratification of the underlying space of the orbifold $\mathcal{O}$ with corner. Therefore, $\mathcal{O}$ with the stratification is smoothly triangulated.*

Proof.

First, we put a Riemannian metric with totally geodesic faces by Lemma 4.5.3. We let $\mathcal{A}^*$ be the set of totally geodesic faces of $\partial\mathcal{O}$. Cover $\mathcal{O}$ by locally finite linear models.

Suppose that $\mathcal{O}$ has only codimension-one strata. Then the result is clear.

As an induction hypothesis, suppose that we proved the result when orbifolds have only codimension i strata.

Let $\mathcal{O}$ have a mutually disjoint collection of codimension-$(i+1)$ strata $Y_1, Y_2, \dots$ but no higher codimension ones. Since Y_i is relatively closed and with no lower dimensional stratum in its closure, it follows that Y_i is a properly embedded manifold with $\partial Y_i \subset \partial\mathcal{O}$. In fact, $Y = \bigcup_{i=1,2,\dots} Y_i$ is a properly embedded manifold. Since Y_i is in a stratum of conjugate local groups, it follows that Y_i is a neat suborbifold of $\mathcal{O}$. Hence, Y_i has a tubular neighborhood. (See Section 4.4.2.)

Define a smooth positive valued function ρ_{Y_i} for each Y_i so that $\rho_{Y_i}^{-1}(0) = Y_i$ and define each tubular neighborhood $T_{Y_i}^j$ as $\rho_{Y_i}^{-1}([0, \epsilon_j^i))$ for some small positive valued functions $\epsilon_1^i, \epsilon_2^i : Y_i \to \mathbb{R}$, $0 < \epsilon_1^i < \epsilon_2^i$ so that the tubular neighborhoods are mutually disjoint for fixed j. We assume that $\epsilon_2^i = 2\epsilon_1^i$. We may assume that $T_{Y_i}^j$ are tubular neighborhoods of Y_i formed by exponential maps of the normal bundles of Y_i. Let $U^j = \bigcup_{i=1,2,\dots} T^j_{Y_i}$. Define $\pi_X : U^2 \to Y$ by the nearest point projection. Define a foliation $\mathcal{F}$ on $U^2 - Y \to Y$ by inverse images of points under π_X. (We need to choose sufficiently small ϵ^i_j.)

We define a map and a graph of $t\epsilon^i_2$, $0 < t < 1$:

$$(\pi_{Y_i}, \rho_{Y_i}) : T^j_{Y_i} \to Y_i \times \mathbb{R}_+ \text{ and } G_{t\epsilon^i_2} := \{(y, t\epsilon^i_2(y)) | y \in Y_i)\}.$$

Define Σ_t be the union of the inverse image of $G_{t\epsilon^i_2}$ for $0 < t < 1$ under (π_{Y_i}, ρ_{Y_i}) for $i = 1, 2, \dots$.

The orbifold $\mathcal{O} - U^1$ is an orbifold with corner and codimension i strata only but with new faces in the boundary. (Note that collars can be obtained by the tubular neighborhoods.) Hence by induction, we can form π_X, T_X, ρ_X for each stratum X in it satisfying the abstract stratification conditions. Let t_0 satisfy $1/2 = \epsilon^i_1/\epsilon^i_2 < t_0 < 1$. Let U' be the open submanifold of U^2 containing X bounded by Σ_{t_0}. Consider $\mathcal{O} - U'$. Now, we radially extend the open set T_X, the stratum X itself, and these maps. T_X is extended by taking $T_X \cap \Sigma_{t_0}$ and isotopying them into $\Sigma_{t'}$ for $0 < t' < t_0$ preserving the leaves of $\mathcal{F}$ and similarly for X. For each t, $0 < t < t_0$, we define $\pi_X : \Sigma_t \to X \cap \Sigma_t$ and $\rho_X : \Sigma_t \to \mathbb{R}^+$ by conjugating the map ρ_X and π_X on Σ_{t_0} by a diffeomorphism and so on for strata X of $\mathcal{O}$ other than Y_is. Now the smoothness of π_X and ρ_X is obtained by smoothing operations that preserve Σ_ts. (We use the coordinates where Σ_t are defined by a coordinate function.) For each face A_i meeting X, we can extend the maps p_{A_i} and r_{A_i} similarly.

Hence, it follows that $\mathcal{O}$ has an abstract stratification.

Given an orbifold $\mathcal{O}$, one can remove tubular neighborhoods of the union of singular loci of dimension-zero forming another orbifold $\mathcal{O}_1$ and removing tubular neighborhoods of the union of singular loci of dimension-one and so on. Therefore, we see that we can build $\mathcal{O}$ starting from a manifold and adding tubular neighborhoods of strata of codimension $n-1, n-2, \dots, 2, 1$. At each step, of course, we obtain orbifolds with corners.

The conditions in Section 4.5.1 are satisfied with our choices. This proves that $\mathcal{O}$ has an abstract stratification. Finally, we obtain the smooth triangulation by Theorem 4.5.1. □

4.6 Covering spaces of orbifolds

Let X be an orbifold. Let X' be an orbifold with a smooth map $p : X' \to X$ so that for each point x of X, there is a connected model (U, G, ϕ) and the inverse image of $\phi(U)$ is a union of open sets $U_i, i \in I$ for an index set I with models isomorphic to (U, G'_i, π_i) where π_i is equivalent to the quotient map $q_i : U \to U/G'_i$ and G'_i is a subgroup of G so that the following diagram commutes for each $i \in I$

$$\begin{tikzcd} U \arrow[rrr, two heads, "\pi_i"] \arrow[d, two heads, "q_i"'] & & & U_i \arrow[d, "p"] \\ U/G_i' \arrow[rrru, "\hat{\pi}_i", "\cong"'] \arrow[r, "q_i'"'] & U/G \arrow[rr, "\hat{\phi}", "\cong"'] & & \phi(U) \end{tikzcd} \tag{4.2}$$

where q_i' is the quotient map, $\hat{\phi}$ is the induced map of ϕ and $\hat{\pi}_i$ is the induced map of π_i.

Then we say that $p : X' \to X$ is a *covering* and X' is a *covering orbifold* of X. Usually, we will require the underlying spaces $|X|$ and $|X'|$ to be connected unless we mention otherwise.

We can see it as an orbifold bundle over X with discrete fibers. We can choose the fibers to be acted upon by a discrete group G (usually on the right), and hence a principal G-bundle.

Given two covering orbifolds $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, we define a *covering morphism* to be a smooth orbifold map $f : X_1 \to X_2$ so that $p_2 \circ f = p_1$. A covering automorphism group of a covering $p : X' \to X$ is a group of diffeomorphisms γ satisfying

$$\begin{array}{ccc} X' & \stackrel{\gamma}{\to} & X' \\ p\downarrow & & p\downarrow \\ X & = & X. \end{array}$$

An element is called a *covering automorphism* or a *deck transformation*. A *regular covering* is a covering where the deck transformation group acts transitively on the fibers. Sometimes, this is called a *Galois covering* and the covering automorphism group is called a *Galois group* or a *deck transformation group*.

4.6.1 ***The fiber product construction by Thurston***

Let us first review the fiber product constructions for the ordinary covering space theory.

Let Y be a connected manifold, and $\tilde{Y}$ a regular covering map $\tilde{p}$ with the covering automorphism group Γ. Let $\Gamma_i, i \in I$ for an index set I be a sequence of subgroups of Γ, and let $p_i : \tilde{Y}/\Gamma_i \to Y$ be the sequence of induced covering maps.

- The projection $\tilde{p}_i : \tilde{Y} \times (\Gamma_i\backslash\Gamma) \to \tilde{Y}$ induces a covering
$$\hat{p}_i : (\tilde{Y} \times (\Gamma_i\backslash\Gamma))/\Gamma \to \tilde{Y}/\Gamma = Y$$
where Γ acts by
$$\gamma(\tilde{x}, \Gamma_i\gamma_i) = (\gamma(\tilde{x}), \Gamma_i\gamma_i\gamma^{-1})$$
- This map is equivalent to $p_i : \tilde{Y}/\Gamma_i \to Y$ since Γ acts transitively on the set of components of $\tilde{Y} \times (\Gamma_i\backslash\Gamma)$.

- We now define the *fiber-product* $\tilde{Y} \times (\prod_{i \in I} \Gamma_i \backslash \Gamma) \to \tilde{Y}$ of $\tilde{p}_i$ for $i \in I$. Define the left-action of Γ by
$$\gamma(\tilde{x}, (\Gamma_i \gamma_i)_{i \in I}) = (\gamma(\tilde{x}), (\Gamma_i \gamma_i \gamma^{-1})), \gamma \in \Gamma.$$
By taking quotients of both sides by Γ, we obtain that the *fiber-product* of $p_i : \tilde{Y}/\Gamma_i \to Y$, $i \in I$ is isomorphic to
$$p^f : Y^f := (\tilde{Y} \times \prod_{i \in I} \Gamma_i \backslash \Gamma)/\Gamma \to \tilde{Y}/\Gamma = Y.$$
(This construction gives us coverings with perhaps many components.)
- For each $i \in I$, there is a covering map
$$p_i^f : Y^f \to \tilde{Y}/\Gamma_i \text{ satisfying } p_i \circ p_i^f = p^f \tag{4.3}$$
induced from the projection
$$\tilde{Y} \times \prod_{i \in I} \Gamma_i \backslash \Gamma \to \tilde{Y} \times (\Gamma_i \backslash \Gamma)$$
(This is the "categorical" universal property we need.)

4.6.1.1 *The fiber product of orbifolds*

Let Y be a connected orbifold. We can let Γ be a discrete group acting on an orbifold $\tilde{Y}$ properly discontinuously but maybe not freely. $Y = \tilde{Y}/\Gamma$ is said to be an *orbifold quotient* of $\tilde{Y}$ and Y is said to be *developable* or *good* if $\tilde{Y}$ is a manifold.

In the above example, we can let Γ be a discrete group acting on an orbifold $\tilde{Y}$ properly discontinuously but possibly not freely. Let Γ_i for each $i \in I$ be a subgroup and $p_i : \tilde{Y}/\Gamma_i \to Y$ be the covering map for each $i \in I$ where I is an index set. $p^f : Y^f \to Y$ is again defined to be the *fiber product* of orbifold maps $p_i : \tilde{Y}/\Gamma_i \to Y$. Moreover, p^f has the *universal property* for the collection $p_i, i \in I$ that there is a covering $p_i^f : \tilde{Y} \to \tilde{Y}/\Gamma_i$ for each i so that $p^f = p_i^f \circ p_i$.

4.6.1.2 *The doubling orbifolds*

A *mirror point* or *silvered point* is a singular point with the stabilizer group $\mathbb{Z}_2$ acting as a reflection group. One can double an orbifold M with mirror points so that mirror points disappear.

- Let V_i for $i \in I$ be the neighborhoods of M with charts (U_i, G_i, ϕ_i), where I is an index set.
- Define new charts $(U_i \times \{-1, 1\}, G_i, \phi_i^*)$ where G_i acts by
$$g(x, l) = (g(x), s(g)l)$$
where $s(g)$ is 1 if g is orientation-preserving and -1 if not and ϕ_i^* is the quotient map to $U_i \times \{-1, 1\}/G_i$.

- For each embedding $i : (W, H, \psi) \to (U_i, G_i, \phi_i)$, we define a lift
$$(W \times \{-1, 1\}, H, \psi^*) \to (U_i \times \{-1, 1\}, G_i, \phi_i^*).$$
These define the gluing maps.
- The result of the quotiening by the gluing maps is the doubled orbifold and the local group actions are orientation preserving. (We just need to verify that the topology is second-countable and Hausdorff.)
- The result double-covers the original orbifold with Galois group or the covering automorphism group isomorphic to $\mathbb{Z}_2$.

Proposition 4.6.1. *A doubled orbifold has no reflection with a hypersurface fixed set. Hence the set of regular points is dense open and locally-path-connected and path-connected.*

Proof. Since there is no orientation reversing element in the local group, the first statement is clear. If there is no reflection, then the singularity is of codimension two or greater and hence the set of regular points is dense open and path-connected locally. Thus, the second statement follows. □

For example, if we double a cell with a corner-reflector, it becomes a cell with a cone-point.

4.6.2 *Universal covering orbifolds by fiber-products*

Let Y be a connected orbifold. A *base point* of a covering is a regular point of the cover mapping to a regular base point of the covered orbifold. A *universal cover* of Y is an orbifold $\tilde{Y}$ so that for any covering orbifold Y' of Y and base points y^* of $\tilde{Y}$ and y' of Y' mapping to a base point y of Y, there exists a covering map $p : \tilde{Y} \to Y'$ satisfying $p(y^*) = y'$.

As some examples, we state without justifications:

- Clearly, manifolds are orbifolds. Manifold coverings provide examples.
- A *tear-drop* is a sphere with one cone-point of order n. Let Y be a tear-drop orbifold with a cone-point of order n. Then this cannot be covered by any other type of an orbifold and hence is a universal cover of itself. (See Section 4.7.1.3 and Theorem 4.7.4.)
- A sphere Y with two cone-points of orders p and q which are relatively prime is a universal cover of itself. (See Section 4.7.1.3 and Theorem 4.7.4.)
- Choose a cyclic action of Y of order m fixing the cone-points. Then $Y/\mathbb{Z}_m$ is an orbifold with two cone-points of order pm and qm, and Y is the universal cover of $Y/\mathbb{Z}_m$.

We will now show that the universal covering orbifold exists by using fiber-product constructions. For this, we need to discuss elementary neighborhoods.

An *elementary* neighborhood for a covering $p : Y' \to Y$ is an open subset $\phi(U)$ with a model triple (U, G, ϕ) so that the situation in equation 4.2 is satisfied.

We can take the model open set in the chart to be one so that U in the model triple (U, G, ϕ) is a cell. Then such an open set is elementary as we can see from below.

4.6.2.1 *Fiber-products for D^n/G_i*

Let D^n be a cell, i.e., a contractible manifold homeomorphic to a convex subset of $\mathbb{R}^n_+$, with possibly nonempty boundary. Suppose that V is an orbifold D^n/G for a finite group G acting effectively. We deduce that

- We can show that any covering of D^n/G is equivalent to D^n/G_1 for a subgroup G_1 of G. (See Proposition 7 in the article [Choi (2004)].)
- Given two covering orbifolds D^n/G_1 and D^n/G_2 for subgroups G_1 and G_2 of G, one can induce a covering morphism $D^n/G_1 \to D^n/G_2$ by $g \in G$ so that $gG_1g^{-1} \subset G_2$.
- The covering morphism is in one-to-one correspondence with the double cosets of form G_2gG_1 for g such that $gG_1g^{-1} \subset G_2$.
- The covering automorphism group of D^n/G_1 for a subgroup G_1 of G is given by $N(G_1)/G_1$ where $N(G_1)$ is the normalizer of G_1 in G.

(For the detailed proofs of these elementary facts, see the article [Choi (2004)].)

Given a collection of coverings $p_i : D^n/G_i \to D^n/G$ for $i \in I$ for a collection I, $G_i \subset G$, and an n-cell D^n, we form a fiber-product.

$$V^f = (D^n \times \prod_{i \in I} G_i \backslash G)/G \to D^n/G.$$

If we choose all subgroups G_i of G, then any covering D^n/G_i of D^n/G is covered by V^f induced by projection to G_i-factor by Section 4.6.1.1. This is the universal property we seek.

4.6.2.2 *The construction of the fiber-product of a collection of covering orbifolds*

Let $Y_i, i \in I$ be a collection of the orbifold-coverings of Y. We cover Y by elementary neighborhoods V_j for $j \in J$ for an index set J forming a good cover. Now fix j. We take components of $p_i^{-1}(V_j)$ each of which is equivalent to a disjoint union of V/G_k for some finite group G_k where V is a convex open subset of $\mathbb{R}^n_+$. Fix j. We take one component of $p_i^{-1}(V_j)$ for each i and form one fiber product. Then we are left with a disjoint union of fiber products indexed by the choice of components of $p_i^{-1}(V_j)$ for each i. Over regular points of V_j, this is the ordinary fiber-product. Now, we wish to patch these up using embeddings. Let $U \to V_j \cap V_k$ be an embedding. We can assume $U = V_j \cap V_k$ which has a convex cell as a cover.

- We form the fiber product $p_U : U^f \to U$ of $p_i^{-1}(U)$, $i \in I$ and form the fiber product $p_{V_j} : V_j^f \to V_j$ and $p_{V_k} : V_k^f \to V_k$.
- $U_j = p_{V_j}^{-1}(U)$ in V_j^f is identifiable with U^f since the fiber-product construction of U_j in V_j^f is identical with one in U^f with just different labeling.
- Similarly, $U_k = p_{V_k}^{-1}(U)$ in V_k^f is identifiable with U^f.
- Thus, each component of the fiber-products can be identified with another one by the natural maps of form $U_j \to U_k$.

By patching, we obtain a covering Y^f of Y with the covering map p^f. Note that Y^f is not necessarily connected. But each component of Y^f is Hausdorff and second-countable and hence is an orbifold.

Let $\tilde{Y}$ be a component of Y^f. Also for any cover (Y_i, y_i), there is a covering morphism $q_i : \tilde{Y} \to Y_i$ with $q_i(y^*) = y_i$ and so that $p_i \circ q_i = p^f$: the basic reason is that for each component of $p_i^{-1}(U)$ for an elementary neighborhood U of p_i in Y, there is a map from a component of $p^{f,-1}(U)$ mapping to it by Section 4.6.1.1 and we can patch these maps together: We show the consistent definition of this map by considering chains of intersecting open components of sets of form $p^{f,-1}(U)$ for an elementary neighbrhood U in Y. Basically, if three such open sets intersect, then we can show that the map is consistently defined. This is similar to the way one obtains developing maps for geometric structures (see Section 6.1.2). (See the bottom of page 178 of the article [Choi (2004)] also.)

4.6.2.3 *Thurston's example of a fiber product*

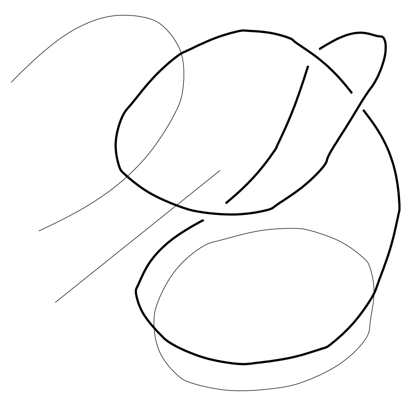

Fig. 4.7 The fiber product of two two-fold covers of the interval I with silvered endpoints by a circle and interval I with silvered endpoints. It is convenient to visualize a cylinder over the bottom circle parallel to the z-axis and the sheet parallel to the y-axis passing the curved arc in the left. The circle is almost on the intersection.

Let I be the unit interval. Make two endpoints into silvered points. Then $I_1 = I$ is double-covered by $\mathbf{S}^1$ with the deck transformation group $\mathbb{Z}_2$. Let p_1 denote the covering map. $I_2 = I$ is also covered by I by a map $x \mapsto 2x$ for $x \in [0, 1/2]$ and $x \mapsto 2-2x$ for $x \in [1/2, 1]$. Let p_2 denote this covering map. Then we determine the fiber product of p_1 and p_2: Cover I by $A_1 = [0, \epsilon), A_2 = (\epsilon/2, 1 - \epsilon/2), A_3 = (\epsilon, 1]$ for $0 < \epsilon < 1/4$.

- $p_1^{-1}(A_1)$ is an open interval and $p_2^{-1}(A_1)$ is a union of two half-open intervals. The fiber-product is a union of two copies of open intervals.
- Over A_2, the fiber product is a union of four copies of open intervals.
- Over A_3, the fiber product is a union of two copies of open intervals.
- By pasting considerations, we obtain a circle mapping 4-1 almost everywhere to I. This could be a long process.

4.6.2.4 *The construction of the universal cover*

Consider the collection Y_i, $i \in I$, of all covers of an orbifold Y. We take each one Y_i with a different choice of a base point y_i over a fixed regular point y of Y. These all are regular points. We take a fiber product of (Y_i, y_i), $i \in I$ and we take a connected component $\tilde{Y}$ containing a base point y^*. Let $\tilde{p}$ denote the restriction of the fiber-product map p^f to $\tilde{Y}$. Hence, $\tilde{Y}$ is a universal cover.

Proposition 4.6.2. *Let Y be a connected orbifold. The universal cover $\tilde{Y}$ of an orbifold Y has an open dense connected set of regular points. Any covering automorphism $\phi : \tilde{Y} \to \tilde{Y}$ that fixes a regular point is the identity map.*

Proof. A universal cover has a morphism to a double Y^2 of the orbifold. Any point mapping to a regular point is also regular. The set of such points is also dense and open and locally path connected. Since the subspace $Y^{2,r}$ of regular points of Y^2 is connected and the set of singular points is at least of codimension 2, the first part follows.

Let $\tilde{Y}^r$ denote the inverse image of the subspace $Y^{2,r}$. Then $\tilde{Y}^r$ is connected and is a covering in the ordinary sense of topology. If ϕ fixes a regular point of $\tilde{Y}$, then it fixes the points of an open model neighborhood. By density, ϕ fixes a point of $\tilde{Y}^r$. If ϕ fixes a point in $\tilde{Y}^r$, then it is the identity on $\tilde{Y}^r$. Since $\tilde{Y}^r$ is dense, ϕ is the identity. □

Theorem 4.6.3. *Let Y be a connected orbifold. The universal cover of an orbifold Y is unique up to covering orbifold-isomorphisms by the universality property.*

Proof. If (Y', y') is another universal cover, then it arises in the list of covers and hence there is a covering morphism $q : \tilde{Y} \to Y'$ with $q(y^*) = y'$. Conversely, we have a morphism $p' : Y' \to \tilde{Y}$ with $p'(y') = y^*$. We obtain a morphism $p' \circ q : \tilde{Y} \to \tilde{Y}$ fixing y^*. By Proposition 4.6.2, $p' \circ q$ is the identity. Similarly, so is $q \circ p'$. □

4.6.2.5 *Properties of the universal cover*

The group of covering automorphisms of the universal cover $\tilde{Y}$ is called a *fundamental group* and is denoted by $\pi_1(Y)$, which is well-defined up to isomorphism by Theorem 4.6.3. (This will be known more accurately as a *Galois-group* in Section 4.7.)

Proposition 4.6.4. *Let $\tilde{Y}$ be a universal cover of an orbifold Y with the covering map $\tilde{p}$.*

- *The deck transformation group $\pi_1(Y)$ of $\tilde{Y}$ acts transitively on fibers of $\tilde{p}^{-1}(x)$ for each x in Y.*
- *$\tilde{p}$ induces a diffeomorphism $\tilde{Y}/\pi_1(Y) \to Y$.*
- *For a subgroup Γ of $\pi_1(Y)$, $\tilde{Y}/\Gamma$ is a covering of Y with the induced covering map from $\tilde{p}$.*
- *Any covering of Y is of form $\tilde{Y}/\Gamma$ for a subgroup Γ of $\pi_1(Y)$.*
- *The set of isomorphism classes of coverings of Y is in one-to-one correspondence with the set of conjugacy classes of subgroups of $\pi_1(Y)$.*

Proof. Let y be a regular base-point of Y. We change the base point of $\tilde{Y}$ to any point z of $\tilde{p}^{-1}(y)$. Then there always is a morphism $q : (\tilde{Y}, y^*) \to (\tilde{Y}, z)$. We find an inverse to q by finding $t = q^{-1}(y^*)$. Then there exists a morphism $q' : (\tilde{Y}, y^*) \to (\tilde{Y}, t)$. Hence, $q \circ q'(y^*) = y^*$. Thus, q' is the inverse and q is a covering automorphism by Proposition 4.6.2. Thus, $\pi_1(Y)$ acts transitively on $\tilde{p}^{-1}(y)$.

Given a point x, we find a path γ in Y with endpoints x and y so that its local lifts to the model neighborhoods have nonzero derivative vectors everywhere. Then each lift to a model open set is unique up to the model group action. Thus, γ lifts to a smooth curve in $\tilde{Y}$ with endpoints a point of $\tilde{p}^{-1}(x)$ and $\tilde{p}^{-1}(y^*)$. In fact the lift is unique up to the choice of the starting point in $\tilde{p}^{-1}(x)$. We see that $\pi_1(Y)$ also acts transitively on the set of lifts. Since we can find a lift starting from any point of $\tilde{p}^{-1}(x)$, we see that $\pi_1(Y)$ acts transitively on $\tilde{p}^{-1}(y)$ for any $y \in Y$.

We see that the quotient orbifold $\tilde{Y}/\pi_1(Y)$ is clearly in a one-to-one correspondence with Y. The charts are also compatible.

We omit the proof of the third item.

For a covering $Y' \to Y$, there is a covering morphism $p' : \tilde{Y} \to Y'$. Now, Y' is actually of form the quotient orbifold $\tilde{Y}/\Gamma$ for a subgroup Γ of $\pi_1(Y)$: Suppose that two regular points p and q of $\tilde{Y}$ go to the same regular point q' of Y' and hence to a point q'' of Y. Then there exists a deck transformation γ so that $\gamma(p) = q$. By considering an elementary neighborhood U of p in Y and the components C_1, $p \in C_1$ and C_2, $q \in C_2$, of its inverse images in $\tilde{Y}$. Consider also the component V of its inverse image in Y' containing q'. Then C_1 and C_2 cover V respectively. This can be seen by a path-lifting argument using curves as above. Clearly, $\gamma(C_1) \subset C_2$ and $\gamma^{-1}(C_2) \subset C_1$ since these are path-components of the inverse image of U. Thus, we have $\gamma(C_1) = C_2$. Since p' sends C_1 and C_2 into V, it follows that every pair of points $(x, \gamma(x))$ of $x \in C_1$ go to a point of V under p'. From this, it follows that every pair of points $(x, \gamma(x))$ for $x \in \tilde{Y}$ go to a point in Y' under p'. If p and q are not regular, then we can find nearby regular points that go to a same point in Y' by lifting a path.

Let Γ be the subset of elements γ of $\pi_1(Y)$ so that the pairs $(x, \gamma(x))$ for all $x \in \tilde{Y}$ are identified under p'. Then Γ is clearly a subgroup. Moreover, if $\gamma \in \pi_1(Y)$ is so that x and $\gamma(x)$ are identified to a point of Y' under p', then $\gamma \in \Gamma$ by the above argument. Hence, it follows that Y' is the quotient orbifold $\tilde{Y}/\Gamma'$.

Given two coverings $Y_1 \to Y$ and $Y_2 \to Y$, we see that an isomorphism $f : Y_1 \to Y_2$ lifts to a diffeomorphism $\tilde{Y} \to \tilde{Y}$. We choose an automorphism fixing y^* by multiplying by an element of $\pi_1(Y)$. By restricting to the regular part, we see that the morphism is the identity map and f is induced by an element of $\pi_1(Y)$. Since Y_1 can be identified with $\tilde{Y}/\Gamma_1$ and Y_2 with $\tilde{Y}/\Gamma_2$, it follows that Γ_1 and Γ_2 are conjugate. The converse is also simple. $\square$

Let Γ be a subgroup of $\pi_1(Y)$. Given the quotient space $\tilde{Y}/\Gamma$, one deduces that an element γ of $\pi_1(Y)$ represents a covering isomorphism $\tilde{Y}/\Gamma \to \tilde{Y}/\Gamma$ if and only if $\gamma\Gamma = \Gamma\gamma$. Thus, γ is in the normalizer $N(\Gamma)$. Conversely, each covering automorphism of $\tilde{Y}/\Gamma \to Y$ lifts to an element $\gamma \in \pi_1(Y)$. Given a covering $\tilde{Y}/\Gamma \to Y$, we determine that the group of covering automorphisms is $N(\Gamma)/\Gamma$. Therefore, a covering is regular or Galois if and only if Γ is a normal subgroup of $\pi_1(Y)$. (These proofs are identical with the ordinary covering-space theory.)

A *good orbifold* is an orbifold with a cover that is a manifold. A *very good orbifold* is an orbifold with a finite cover that is a manifold. A good orbifold has a symply connected manifold as a universal covering space: it has a covering space that is a manifold and the universal covering orbifold must cover this manifold and hence the universal covering space has to be a manifold.

4.6.2.6 *Induced homomorphisms of the fundamental group*

Given two orbifolds Y_1 and Y_2 and an orbifold-diffeomorphism $g : Y_1 \to Y_2$, we obtain that the lift to the universal covers $\tilde{Y}_1$ and $\tilde{Y}_2$ is also an orbifold-diffeomorphism. Furthermore, if the lift value is determined at a point, then the lift is unique.

Proposition 4.6.5. *Let Y_1 and Y_2 be connected orbifolds of same dimension. An isotopy $f_t : Y_1 \to Y_2$ for $t \in [0, 1]$ of orbifold-diffeomorphisms lifts to an isotopy in the universal covering orbifold $\tilde{f}_t : \tilde{Y}_1 \to \tilde{Y}_2$ for each $t \in I$ unique up to a choice of $\tilde{f}_0(y)$.*

Proof. We consider regular parts and model neighborhoods where the lifts clearly exist uniquely for each t. The map $t \mapsto f_t(y)$ for a regular base point y of Y is a path in Y. Then $f_t(y)$ is regular for all $t \in I$. This lifts to a smooth path $\tilde{\gamma} : t \mapsto p^{-1}(f_t(y))$. Since f_t is an orbifold diffeomorphism, there is a lifting diffeomorphism $\tilde{f}_t : \tilde{Y} \to \tilde{Y}$ for each t determined up to post-composing with the deck transformations. By post-composing with elements of $\pi_1(Y)$ if necessary, we can make sure that a lift $\tilde{f}_t : \tilde{Y} \to \tilde{Y}$ satisfies $\tilde{f}_t(y) = \tilde{\gamma}(t)$ for each t. Now, we can verify that $\tilde{f}_t$ forms an isotopy.

□

Given an orbifold-diffeomorphism $f : Y \to Z$ which lifts to a diffeomorphism $\tilde{f} : \tilde{Y} \to \tilde{Z}$, we obtain a homomorphism $\tilde{f}_* : \pi_1(Y) \to \pi_1(Z)$: for each $\gamma \in \pi_1(Y)$, there exists a unique $\delta \in \pi_1(Z)$, so that $\tilde{f} \circ \gamma = \delta \circ \tilde{f}$. If g is isotopic to f and so is its lift $\tilde{g}$ to $\tilde{f}$, then it follows that $\tilde{g}_* = \tilde{f}_*$. (Note that we can define f_* for orbifold-diffeomorphisms only. When f is not a diffeomorphism, we need also the information on the local lifts as well to describe the map using the path-approach below. We will not attempt this in this book.)

Finally notice that if Y_1 is an open suborbifold of Y_2, then we can define a homomorphism $\tilde{\iota}_* : \pi_1(Y_1) \to \pi_1(Y_2)$ where $\tilde{\iota}$ is the lift $\tilde{Y}_1 \to \tilde{Y}_2$ of the inclusion map $\iota : Y_1 \to Y_2$.

Using the path-approach of Haeflger, we obtain a more general result for this. (See Section 4.7.1.2.)

4.7 The path-approach to the universal covering spaces following Haefliger

We will now study the path-approach to the fundamental groups and the universal covering spaces following Bridson and Haefliger (1999). Thus, we see that the ordinary covering theory for topological spaces and the covering theory for orbifolds are very much alike.

4.7.1 *$\mathcal{G}$-paths*

We generalize the notion of paths in the topological spaces to one of those on groupoids: Given an étale groupoid X with the space of arrows $\mathcal{G}$ and the space of objects X_0, we define a $\mathcal{G}$*-path* c to be an object $(g_0, c_1, g_1, \dots, c_k, g_k)$ with a subdivision $a = t_0 \leq t_1 \leq \cdots \leq t_k = b$ of interval $[a, b]$ consisting of

- continuous maps $c_i : [t_{i-1}, t_i] \to X_0$
- elements $g_i \in X_1$ so that $s(g_i) = c_{i+1}(t_i)$ for $i = 0, 1, \dots, k-1$ and $t(g_i) = c_i(t_i)$ for $i = 1, \dots, k$.

The initial point is $t(g_0)$ and the terminal point is $s(g_k)$. We will normally require that the orbit space $|X|$ of X is connected. That is, the underlying space is connected since the orbit space is homeomorphic to the underlying space. (See Example 4.3.)

The three operations define an equivalence relation:

- Subdivision: Add new division point t'_i in $[t_i, t_{i+1}]$ and $g'_i = \mathrm{I}_{c_i(t'_i)}$ and replacing c_i with c'_i, g'_i, c''_i where c'_i, c''_i are restrictions to $[t_i, t'_i]$ and $[t'_i, t_{i+1}]$.
- Adjoining: We reverse the subdivision process.
- Replacement: replace c with $c' = (g'_0, c'_1, g'_1, \dots, c'_k, g'_k)$ as follows. For each i choose continuous map $h_i : [t_{i-1}, t_i] \to X_1$ so that $s(h_i(t)) = c_i(t)$ and define $c'_i(t) = t(h_i(t))$ and $g'_i = h_i(t_i)g_i h^{-1}_{i+1}(t_i)$ for $i = 1, \dots, k-1$ and $g'_0 = g_0 h_1^{-1}(t_0)$ and $g'_k = h_k(t_k)g_k$.

All paths are defined on $[0,1]$ from now on. Given two $\mathcal{G}$-paths $c = (g_0, c_1, \dots, c_k, g_k)$ over $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ and $c' = (g'_0, c'_1, \dots, c'_{k'}, g'_{k'})$ over $0 = t'_0 \leq t'_1 \leq \cdots \leq t'_{k'} = 1$ such that the terminal point of c equals the initial point of c', we define the *composition* $c * c'$ to be the $\mathcal{G}$-path $c'' = (g''_0, c''_1, \dots, c''_{k+k'}, g''_{k+k'})$ so that

- $t''_i = t_i/2$ for $i = 0, \dots, k$ and $t''_i = 1/2 + t'_{i-k}/2$ for $i = k+1, \dots, k+k'$;
- $c''_i(t) = c_i(2t)$ for $i = 1, \dots, k$ and $c''_i(t) = c'_{i-k}(2t-1)$ for $i = k+1, \dots, k+k'$; and
- $g''_i = g_i$ for $i = 0, \dots, k-1$ and $g''_k = g_k g'_0, g''_i = g'_{i-k}$ for $i = k+1, \dots, k+k'$.

The inverse c^{-1} is $(g'_0, c'_1, \dots, c'_k, g'_k)$ over the subdivision where $t'_i = 1 - t_i$ so that $g'_i = g^{-1}_{k-i}$ and $c'_i(t) = c_{k-i+1}(1-t)$.

4.7.1.1 *Homotopies of $\mathcal{G}$-paths*

There are two types of homotopies:

- Equivalences
- An *elementary homotopy* is a family of $\mathcal{G}$-paths $c^s = (g^s_0, c^s_1, \dots, c^s_k, g^s_k)$ over the subdivision $0 = t^s_0 \leq t^s_1 \leq \cdots \leq t^s_k = 1$ so that each of t^s_k, g^s_i, c^s_i depends continuously on s. We require that $t(g^s_0)$ and $s(g^s_k)$ are to be constant independent of s as usual for a homotopy of paths.

Two $\mathcal{G}$-paths a and b are *homotopic* if there is a sequence of $\mathcal{G}$-paths $a = a_1, a_2, \dots, a_n = b$ so that a_i and a_{i+1} are either equivalent or there is an elementary homotopy between them.

A homotopy class of c is denoted $[c]$. $[c * c']$ is well-defined in the homotopy classes $[c]$ and $[c']$. Hence, we define $[c] * [c'] = [c * c']$.

We have the associativity $[c * (c' * c'')] = [(c * c') * c'']$.

The constant path e_x at x is given as $(\mathrm{I}_x, x, \mathrm{I}_x)$. Then $[c * c^{-1}] = [e_x]$ if the initial point of c is x and $[c^{-1} * c] = [e_y]$ if the terminal point of c is y. Thus, $[c]^{-1} = [c^{-1}]$.

We can show easily that the homotopy classes of paths form a *fundamental groupoid*.

4.7.1.2 *The fundamental group $\pi_1(X, x_0)$*

A *loop* is a $\mathcal{G}$-path with the identical initial and terminal points. The *fundamental group* $\pi_1(X, x_0)$ based at $x_0 \in X_0$ is the group of homotopy classes of loops based at x_0. (We will require x_0 to be a regular point.) The associativity, identity and inverse properties are proven above.

Let X be an open suborbifold of Y. Then the inclusion map $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ where $f(x_0)$ is regular also. In fact, f could be any homomorphism of groupoids. Hence f could be an orbifold map $X \to Y$ so that for each point $x \in X$, there exists a model triple (U, G, ϕ) and a model triple (V, H, ψ) of $f(x) \in Y$ so that f lifts to a map $\tilde{f} : U \to V$ unique up to the action

$$\tilde{f} \mapsto h \circ f \circ g \text{ for } h \in H, g \in G.$$

Therefore, a covering map will induce the homomorphism.

Theorem 4.7.1 (Seifert-Van Kampen). *Let X be an orbifold with the space of objects X_0 and the space of arrows $\mathcal{G}$. Assume that the space $|X|$ of orbits is connected. Let $X_0 = U \cup V$ where U and V are open and $U \cap V = W$. Assume that the groupoid restrictions $\mathcal{G}_U$, $\mathcal{G}_V$, $\mathcal{G}_W$ to U, V, W are connected. And let $x_0 \in W$. Then $\pi_1(X, x_0)$ is isomorphic to the quotient group of the free product $\pi_1(\mathcal{G}_U, x_0) * \pi_1(\mathcal{G}_V, x_0)$ by the normal subgroup generated by $j_U(\gamma) j_W(\gamma^{-1})$ for $\gamma \in \pi_1(\mathcal{G}_W, x_0)$ for the induced homomorphism $j_U : \pi_1(\mathcal{G}_W, x_0) \to \pi_1(\mathcal{G}_U, x_0)$ and the induced homomorphism $j_V : \pi_1(\mathcal{G}_W, x_0) \to \pi_1(\mathcal{G}_V, x_0)$.*

Here a groupoid restriction $\mathcal{G}_U$ means restricting the space of objects to U and the space of arrows to those arrows with tails and sources in U.

In a more set theoretic language, this means: Let X be an orbifold so that $|X| = U \cup V$ for two open subsets U and V and let $W = U \cap V$ be a connected open set. Let $x_0 \in W$ and $\hat{U}$, $\hat{V}$, and $\hat{V}$ denote the induced orbifolds. Then $\pi_1(X, x_0)$ is isomorphic to the quotient group of $\pi_1(\hat{U}, x_0) * \pi_1(\hat{V}, x_0)$ by the normal subgroup generated by $j_U(\gamma) j_V(\gamma^{-1})$ for $\gamma \in \pi_1(\hat{W}, x_0)$ for the induced homomorphism $j_U : \pi_1(\hat{W}, x_0) \to \pi_1(\hat{U}, x_0)$ and the one $j_V : \pi_1(\hat{W}, x_0) \to \pi_1(\hat{V}, x_0)$.

The proof is omitted but is remarkably similar to the elementary topology proof using dividing homotopies into small ones mapping into model-neighborhoods. This is an exercise in Chapter III$\mathcal{G}$ in the book [Bridson and Haefliger (1999)].

4.7.1.3 *Examples*

- Consider a tear-drop orbifold. We remove a small disk about the cone-point. The remainder is a disk and has a trivial fundamental group. The disk about the cone-point has the fundamental group isomorphic to the cyclic group of order n by equation 4.4. By the Van-Kampen theorem, a tear-drop has the trivial fundamental group.

- Similarly, we can show that a sphere Y with two cone-points of relatively prime orders p and q has a trivial fundamental group: Here, we remove two disjoint disks around the singularities and the Van-Kampen theorem to prove this.
- Let a discrete group Γ act on a connected manifold X_0 properly discontinuously. Then (Γ, X_0) has an orbifold structure. (See 4.3.2.1.) Let x_0 be a point with trivial stabilizer subgroup. Let g_γ denote the arrow in X_1 with starting point x_0 and the end point $\gamma(x_0)$ for $\gamma \in \Gamma$. Any loop in this groupoid is equivalent to a $\mathcal{G}$-path $(\mathrm{I}_{x_0}, c, g_\gamma)$ so that $\gamma(x_0) = c(1)$ and $c(0) = x_0$ by joining all paths in $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ into a single path, i.e., by changing g_0 to 1 and c_1 to $\gamma_0^{-1} \circ c_1$ where γ_0 is the deck transformation corresponding to g_0, and g_1 to 1 and c_2 to $\gamma_0^{-1} \circ \gamma_1^{-1} \circ c_2$ where γ_1 corresponds to g_1 and so on and joining these paths. Thus, there is an exact sequence for a base point $x_0 \in X_0$:
$$1 \to \pi_1(X_0, x_0) \to \pi_1((\Gamma, X_0), x_0) \to \Gamma \to 1 \tag{4.4}$$
given by sending $[(\mathrm{I}_x, c, g_\gamma)]$ to γ. That is, $\pi_1((\Gamma, X_0), x_0)$ is an extension of Γ by $\pi_1(X_0, x_0)$. (See Example 3.7 in Chapter III.$\mathcal{G}$ in the book [Bridson and Haefliger (1999)].)
- A 2-orbifold that is a disk with an arc silvered has the fundamental group isomorphic to $\mathbb{Z}_2$: A disk with a group action generated by a reflection about an arc covers it. Thus, the result follows from equation 4.4.
- An annulus A with one boundary component silvered has a fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$ since our orbifold is covered by an annulus A_1 by an action of $\mathbb{Z}_2$ which fixes the middle circle of the annulus. There exists a section from $\mathbb{Z}_2$ to $\pi_1(A)$ given by a path γ going to the silvered arc and returning to the base point. Clearly, γ^2 is trivial.
- Consider a 2-orbifold with cone-points which is boundaryless and with no silvered point. One can cover the cone points by sufficiently small disks and we can cut out the disks. Then the Van-Kampen theorem enables one to compute the fundamental group. (See Theorem 5.1.1.)
- Suppose that a two-dimensional orbifold has boundary and silvered points. Then remove open-ball neighborhoods of the cone-points and corner-reflector points. The fundamental group of the remaining part can be computed by the Van-Kampen theorem by considering open neighborhoods of silvered boundary arcs. Finally, adding the open-ball neighborhoods, we compute the fundamental group again using the Van-Kampen theorem.

The last item implies

Corollary 4.7.2. *Let Σ be a compact 2-dimensional orbifold. Then $\pi_1(\Sigma, x_0)$ is finitely presented for any regular point x_0.*

In fact, compact n-orbifolds have finitely presented fundamental groups but we omit the proof that is a higher-dimensional generalization. The fundamental group of a three-dimensional orbifold can be computed similarly using the Van Kampen theorem. However, we need the detailed knowledge of the structure of 3-orbifolds as can be found in the book [Thurston (1977)] and some papers such as [Dunbar (1988)].

4.7.2 *Covering spaces and the fundamental group*

One can build the theory of covering spaces using the fundamental group. We review the relationship of the homotopy group of $\mathcal{G}$-paths to covering spaces first. (Here, we will only consider orbifolds with connected underlying space.)

Let us be given a covering $X' \to X$ for two orbifolds X and X'. For every $\mathcal{G}$-path c in X, there is a lift $\mathcal{G}$-path in X'. If we assign the initial point, the lift is unique. If c' is homotopic to c, then the lift of c' is also homotopic to the lift of c provided the initial points are the same. From this it follows that the induced homomorphism $\pi_1(X', x_0') \to \pi_1(X, x_0)$ is injective.

Moreover, the following familiar proposition holds:

Proposition 4.7.3. *Let $p : X' \to X$ be a covering of an orbifold X with a based point x_0 and let $p'' : X'' \to X$ be another one. Let X' have a base point x_0' going to x_0 under p and X'' has one x_0'' going to x_0 under p''. Then*
$$p''_*(\pi_1(X'', x_0'')) \subset p_*(\pi_1(X', x_0'))$$
if and only if there is a covering map $X'' \to X'$ sending x_0'' to x_0'.

Proof. This is proved using paths as in the covering theory in topology. □

A map from a simply connected orbifold to an orbifold lifts to a cover. The lift is unique if the base-point lift is assigned. Thus, a simply connected cover of an orbifold covers any cover of the given orbifold. From this, we can show that the fiber-product construction is symply connected.

Two simply connected coverings of an orbifold are isomorphic and if base-points are given, we can find an isomorphism preserving the base-points.

Theorem 4.7.4. *A symply connected covering of an orbifold X is a universal cover (Galois-covering) with the Galois-group isomorphic to $\pi_1(X, x_0)$.*

Proof. Consider $p^{-1}(x_0)$. Choose a base-point $\tilde{x}_0$ in it. Given a point of $p^{-1}(x_0)$, we connected it with $\tilde{x}_0$ by a path. Since the paths map to the elements of the fundamental group, the Galois-group acts transitively on $p^{-1}(x)$. Hence the Galois-group is isomorphic to the fundamental group. □

Corollary 4.7.5. *An orbifold-covering $(X', x_0') \to (X, x_0)$ is Galois (regular) if and only if the image of $\pi_1(X', x_0')$ in $\pi_1(X, x_0)$ is normal.*

Proof. Again, Proposition 4.7.3 implies this. □

4.7.2.1 *The existence of the universal cover using the path-approach*

The construction follows that of the ordinary covering space theory. This is included in Exercise 3.20 in Chapter III$\mathcal{G}$ in the book [Bridson and Haefliger (1999)]. Let X be an orbifold with the space of arrows X_1 and the space of objects X_0.

- Let $\hat{X}$ be the set of homotopy classes $[c]$ of $\mathcal{G}$-paths in X with a fixed starting point x_0.
- We define a topology on $\hat{X}$ by open set $U_{[c]}$ that is the set of paths ending at a symply connected open subset U of X_0 with the homotopy class of $c*d$ for a path d in U.
- Define a map $\hat{X} \to X$ sending $[c]$ to its endpoint other than x_0.
- Define a map $\hat{X} \times X_1 \to \hat{X}$ given by $([c], g) \to [c * g]$. This defines a right X_1-action on $\hat{X}$. This makes $\hat{X}$ into a bundle over X.
- Define a left action of $\pi_1(X, x_0)$ on $\hat{X}$ given by $[c] * [c'] = [c * c']$ for $[c'] \in \pi_1(X, x_0)$. This is transitive on fibers.
- We show that $\hat{X}$ is a simply connected orbifold.

4.8 Notes

For compact group actions, see the books [Bredon (1972); Hsiang (1975)]. Good references for triangulation under group actions are articles [Illman (1978, 1983)]. For triangulation of stratified spaces, and hence orbifolds, see the articles [Goresky (1978); Johnson (1983); Verona (1984); Weinberger (1994)]. The work [Verona (1984)] is most self-contained. For general introduction to the orbifold theory, see Chapter 5 of the book [Thurston (1977)] and the article [Matsumoto and Montesinos-Amilibia (1991)]. The original papers [Satake (1956, 1957)] are also very readable. Adem, Leida, and Ruan (2007) and Bridson and Haefliger (1999) treat orbifolds as groupoids. Read the articles [Moerdijk (2002); Moerdijk and Pronk (1997)] for this approach in detail. Haefliger (1990) and Chapter 13 of the book [Ratcliffe (2006)] treat the path approaches to the covering spaces. Chapter 5 in the book [Thurston (1977)] and the article Choi (2004) have contents on the covering space theory using fiber products.

We do not study general maps or morphisms between orbifolds and induced bundles. This is related to defining the notion of suborbifold as well. Perhaps one should view orbifolds as 2-categories as Lerman (2010) has done.

Chapter 5

Topology of 2-orbifolds: 2-orbifold topological constructions

We now wish to concentrate on 2-orbifolds to illustrate more concretely. In many cases, the theory is much easier to understand. Also, we study the topological constructions of 2-orbifolds. We will follow the papers [Choi and Goldman (2005); Scott (1983)].

We first classify smooth 2-orbifolds with possibly empty boundary up to diffeomorphisms. Next 1-dimensional suborbifolds are classified. We discuss the Euler characteristic and the Riemann-Hurwitz formula. We classify the bad orbifolds by discussing about the good, very good, and bad 2-orbifolds. (At present, we can do this for 2-orbifolds only. For higher dimensions, these may not be appropriate terminologies even.)

In the rest of the chapter, we discuss topological cut-and-paste methods applicable to 2-orbifolds.

5.1 The properties of 2-orbifolds

Recall that the singular points of a two-dimensional orbifold fall into three types (See Figure 4.7):

(i) The mirror point: $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflections on the y-axis.
(ii) The cone-points of order n: $\mathbb{R}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acting by rotations by angles $2\pi m/n$ for integers m.
(iii) The corner-reflector of order n: $\mathbb{R}^2/D_n$ where D_n is the dihedral group generated by reflections about two lines meeting at an angle π/n.

From this, we obtain that the underlying space of a 2-orbifold is a surface with corner.

The singular strata associated with conjugate local groups are as follows: a silvered point belongs to a 1-dimensional strata, called a *silvered arc*. The other types have isolated points as strata. Recall that boundary of a 2-orbifold is a suborbifold. The silvered arc may have an end point in the boundary of the 2-orbifold and it may end in a corner-reflector of order ≥ 2 also but not at a cone-point by the local group considerations.

- On the boundary of a surface with a corner, one can choose a collection of mutually disjoint maximal smooth open arcs ending at corners. If two half-arcs in the distinguished arcs end at a corner-point, then the corner-point is a *distinguished one.* If only one of the chosen arc ends at the corner, the corner-point is *ordinary.* The diffeomorphism type of the surface with the collection of chosen arcs will be called the *boundary pattern.*
- Recall Example 4.2.4: given a surface with corner and a collection of discrete points in its interior and the boundary pattern, we can put an orbifold structure on it so that the selected interior points become cone-points and the distinguished corner-points the corner-reflectors of given order and the ordinary end points and points of chosen arcs the silvered points.

Theorem 5.1.1. *Any* 2*-orbifold is obtained from a smooth surface with corner by silvering some arcs and putting cone-points and corner-reflectors. The smooth orbifold topology of* 2*-orbifold is classified by the underlying smooth topology of the surface with corner and the cardinality and orders of cone-points, corner-reflectors, and the boundary pattern of silvered arcs.*

Proof. Given a 2-orbifold, we can forget the orbifold structure and we obtain a smooth surface with corner. Thus, we can obtain the orbifold back by doing the construction as above.

Given two 2-orbifolds Σ_1 and Σ_2 with the same boundary pattern of silvered arcs and corner reflectors and cone-points, we remove the neighborhoods of small corner-reflectors and cone-points diffeomorphic to disks or disks intersected with $\mathbb{R}^2_+$ to obtain Σ_1' and Σ_2'. Then there is a diffeomorphism $f : \Sigma_1' \to \Sigma_2'$ considering Σ_1' and Σ_2' as surfaces with corner. Only silvered arcs remain as the singular sets. f can be isotoped to an orbifold-diffeomorphism.

We can extend f radially at each neighborhood of corner-reflectors and cone-points where we have to smooth the maps radially. We obtain a smooth orbifold diffeomorphism. $\square$

A *full* 1*-orbifold* is an orbifold with the underlying space homeomorphic to an interval and where both endpoints are singular. A boundary full 1-orbifold is a full 1-orbifold in the boundary of 2-orbifold. (Recall Remark 4.2.5.)

Let Σ be a surface with corners. Thurston's notation of a closed 2-orbifold with base space Σ is given by $\Sigma(j_1, j_2, \ldots; k_1, k_2, \ldots)$ where $j_1, j_2, \ldots$ indicate the orders of the cone-points and $k_1, k_2, \ldots$ indicate the orders of the corner-reflectors. This will determine the orbifold diffeomorphism class by additionally specifying on which boundary components of the underlying space Σ the corner-reflectors and the boundary full 1-orbifolds are and in what way by combinatorially ordering these on the boundary components.

From now on, a *special singular point* will mean either a cone-point or a corner-reflector.

5.1.1 *The triangulation of 2-orbifolds*

For 2-orbifolds, the Riemannian metric and triangulation can be approached in much simpler manner. (See Chapter 4 for the full generality.)

Proposition 5.1.2. *One can put a Riemannian metric on a 2-orbifold so that the boundary of the underlying cornered surface is a union of geodesic arcs and each corner-reflector has angles π/n for its order n and each cone-point has angles $2\pi/m$ for its order m. One can give a triangulation by smooth triangles so that slivered arcs and boundary curves are in the union of 1-skeletons and corner-reflectors and cone-points are in 0-skeletons.*

Proof. First construct such a Riemannian metric on the boundary by putting such metrics on the boundary by using a broken geodesic in the Euclidean plane and a singular Euclidean metric around the cone points. (See Example 4.2.4.) We require that the corner-reflector of order n has angle π/n and the cone points of order m have a Euclidean angle $2\pi/m$. Put a Riemannian metric on the surface with neighborhoods of singular points removed so that the boundary is geodesic. We extend the metric to the interior using partition of unity.

By removing open balls around cone-points and corner-reflectors, we obtain a smooth surface with corners and without singular points.

Now we can find a smooth triangulation so that the interior of each edge of a triangle is completely inside the boundary with the corner points removed. Finally we extend the triangulation by cone-construction to the interiors of the removed balls. □

5.1.2 *The classification of 1-dimensional suborbifolds of 2-orbifolds*

A compact 1-orbifold is either a closed arc homeomorphic to a circle, a segment, a segment with one silvered endpoint, or a segment with two silvered endpoint.

Recall that a *neatly* embedded suborbifold is an embedded suborbifold so that its boundary is in the boundary of the ambient orbifold and each point of the boundary has a neighborhood in the orbifold modeled on a half space H^n with the suborbifold neighborhood modeled on another half space H^m embedded in it for $0 \leq m \leq n$. (See Section 4.4.2.) A properly and neatly embedded 1-orbifold in a 2-orbifold with boundary either avoids the singular sets in its topological interior or is entirely contained in a singular set. In the former case we have:

- No silvered-point case: An embedded closed arc avoiding boundary or singular points or an arc segment with two endpoints in the boundary avoiding

singularities.
- One silvered-point case: An arc segment with silvered endpoint at a cone-point of order two or in the interior of a silvered arc and the other endpoint in the boundary.
- Two silvered-points case: An arc segment with silvered endpoints at cone-points of order two or in the interiors of silvered arcs. (This is a full 1-orbifold, which may or may not be the boundary one.)

By a *silvered edge*, we mean a maximal arc whose interior is silvered. (The endpoints must be also singular.)

If the neatly embedded 1-orbifold is in the singular set, we classify them as below:

- a silvered embedded closed arc,
- a maximal segment in a silvered edge with two end points in the boundary of the orbifold,
- a segment in a silvered edge with one endpoint in a corner reflector of order two and the other in the boundary, and
- a segment in a silvered edge with two endpoints in two corner-reflectors of order two. (This is a full 1-orbifold but not the boundary one.)

(It might be useful to recall Remark 4.2.5.)

These are all possible compact neatly embedded 1-orbifolds and we will assume that our 1-orbifolds are of these types only.

5.1.3 *The orbifold Euler-characteristic for orbifolds due to Satake*

Let $\mathcal{O}$ be a compact n-dimensional orbifold with boundary. Then $\mathcal{O}$ admits a finite triangulation by Theorem 4.5.4. Thus, the underlying space of $\mathcal{O}$ has a cell-decomposition. We defined the Euler characteristic to be

$$\chi(\mathcal{O}) = \sum_{c_i} (-1)^{\dim(c_i)} \frac{1}{|\Gamma(c_i)|},$$

where c_i ranges over the open cells and $|\Gamma(c_i)|$ is the order of the group $\Gamma(c_i)$ associated with a point of c_i. (The order is independent of the point in a given open cell.)

Proposition 5.1.3. *Let X be an n-dimensional orbifold for $n \geq 1$. If X is finitely covered by another orbifold X', then $\chi(X') = r\chi(X)$ where r is the number of sheets for regular points.*

Proof. Each point of a strata with a local group G' in X has the inverse image equal to the set of points $p_i, i = 1, \ldots, n$ in a respective stratum S_i in X' with local group G'_i so that

$$r = \sum_{i=1}^{n} \frac{|G'|}{|G'_i|}.$$

This is verified by taking a nearby regular point and counting the inverse images near p_is.

Thus, the inverse image of an open cell c with the local group $\Gamma(c)$ consists of a union of open cells $c_1, \ldots, c_n$ with local groups $\Gamma(c_i)$ of same dimension where we have

$$\frac{r}{|\Gamma(c)|} = \sum_{i=1}^{n} \frac{1}{|\Gamma(c_i)|}.$$

□

The Euler-characteristic of a compact 1-orbifold is as follows: a circle 0, a segment 1, a segment with one silvered-point 1/2, a full 1-orbifold 0.

A *separating* 1-orbifold Y in a 2-orbifold Σ is a 1-dimensional suborbifold in Σ with the topological interior $|Y|^o$ of the underlying space $|Y|$ of Y is in the topological interior $|\Sigma|^o$ of the underlying space $|\Sigma|$ of Σ. so that $|\Sigma| - |Y|$ has two components and moreover, for each point x of Y, the local group H_x of x is included isomorphic to the local group G_x of x in Σ for the model triple embedding $(I, H_x, \phi) \to (U, G_x, \psi)$ where (I, H_x, ϕ) is a model-triple for x in Y and (U, G_x, ψ) is a model-triple for x in Σ.

In fact, this is true for the following two cases only:

- Y is a simple closed curve and has no singularity and lies in the interior of the underlying surface $|\Sigma|$, or
- Y has two silvered points in the interior of silvered arcs in the boundary of $|\Sigma|$ and the interior of $|Y|$ is in the interior of $|\Sigma|$.

In this case, $\Sigma - Y$ then completes with respect to the path-metric into a union of two suborbifolds Σ_1 and Σ_2.

Assuming this, we have the following additivity formula:

$$\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2) - \chi(Y). \tag{5.1}$$

The formula is to be verified by counting open cells with weights since the orders of singular points in the boundary orbifold equal the ambient orders.

5.1.4 *The generalized Riemann-Hurwitz formula*

Suppose that a 2-orbifold Σ with or without boundary has the compact underlying space X_Σ and m cone-points of order q_i and n corner-reflectors of order r_j and n_Σ boundary full 1-orbifolds.

Then the following generalized *Riemann-Hurwitz formula* is very useful:

$$\chi(\Sigma) = \chi(X_\Sigma) - \sum_{i=1}^{m} \left(1 - \frac{1}{q_i}\right) - \frac{1}{2}\sum_{j=1}^{n} \left(1 - \frac{1}{r_j}\right) - \frac{1}{2} n_\Sigma, \tag{5.2}$$

where q_i, $i = 1, \dots, m$, are the orders of cone points and r_j, $j = 1, \dots, n$, are the orders of corner-reflectors and n_Σ is the number of boundary full 1-orbifolds, i.e., the full 1-orbifolds in the boundary of the orbifold Σ.

We prove this formula by a doubling argument and cutting and pasting using equation 5.1. (See [Scott (1983)] for details):

We double the 2-orbifold Σ to Σ'. (See Section 4.6.1.2.) Now we have only closed curve boundary components and cone-points. Then $\chi(\Sigma')$ equals

$$\chi(X_{\Sigma'}) - 2\sum_{i=1}^{m}\left(1 - \frac{1}{q_i}\right) - \sum_{j=1}^{n}\left(1 - \frac{1}{r_j}\right)$$

as can be verified by decomposing Σ' by cutting out disks around the cone-points. We have $\chi(\Sigma) = \chi(\Sigma')/2$ by Proposition 5.1.3, and $\chi(X_{\Sigma'})$ equals $2\chi(X_\Sigma) - n_\Sigma$.

To explain more: while we cannot yet do the full cutting and pasting constructions for 2-orbifolds which we do from Section 5.2.1 to the end of this chapter, we can do this when the 1-orbifolds are separating;

5.1.5 *A geometrization of* 2*-orbifolds : a partial result*

Proposition 5.1.4. *Let S be a 2-orbifold whose underlying space is a disk with at least one special singularity and has nonempty (orbifold) boundary or a disk or 2-sphere with at least three special singular points. Then S is very good and so is regularly covered by a compact surface.*

Proof. First, cover S by a double-cover $\hat{S}$ if S contains silvered points. (See Proposition 4.4.3.) Otherwise let $\hat{S}$ be S. Then $\hat{S}$ has only cone-points. Let n be the number of the cone-points, and let $p_1, p_2, \dots, p_n$ denote their orders. The underlying space is a sphere or a planar surface. Now, the boundary of $\hat{S}$ is a disjoint union of simple closed curves if the underlying space is a planar surface. Let k be the number of boundary components. Then we have either $n \geq 3$ or have $n \geq 1$ and $k \geq 1$.

If $\hat{S}$ has just one singular point that has to be a cone-point, with one boundary component, then $\hat{S}$ is regularly covered by a smooth disk without singularity and we are done. Assume we have either $n \geq 3$, have $n = 2$ and $k \geq 1$, or have $n = 1$ and $k \geq 2$.

We can construct an orbifold structure on a planar subsurface $\mathcal{P}$, diffeomorphic to $|\hat{S}|$, on a 2-sphere $\mathbf{S}^2$ in $\mathbb{R}^3$ so that the cone-points are on the xy-plane, each boundary circle is symmetric with respect to the reflection on the xy-plane, and the reflection on the xy-plane restricts to an orbifold-involution. Denote the resulting orbifold by $\hat{\mathcal{P}}$. By constructing $\hat{\mathcal{P}}$ to have the same number of cone-points as $\hat{S}$, we see that $\hat{S}$ is diffeomorphic to $\hat{\mathcal{P}}$ by Theorem 5.1.1. Now, it follows that the orbifold $\hat{S}$ covers an orbifold Σ with n corner-reflectors on a disk D^2 with orders $p_1, p_2, \dots, p_n$ and with k number of boundary full 1-orbifold.

One can construct a geodesic polygon P with angles $\pi/p_1, \pi/p_2, \dots, \pi/p_n$ and $2k$ angles of $\pi/2$ on the 2-sphere, a Euclidean plane, or a hyperbolic plane depending on whether $\sum_{i=1}^{n} \pi(1-1/p_i) + k\pi$ is smaller than 2π, equal to 2π or greater than 2π. (See Proposition 3.2.2.) (Note that P has at least three vertices.) Then Σ is diffeomorphic to the quotient orbifold of a domain in a 2-sphere, a Euclidean plane, or a hyperbolic plane by the generated reflection group. Thus, $\hat{S}$ admits a spherical, Euclidean, or hyperbolic structure with geodesic boundary. By Selberg's lemma (Corollary 4 in Chapter 7 of the book [Ratcliffe (2006)]), the group $\pi_1(\hat{S})$ has a finite-index normal subgroup that is torsion-free consisting of orientation-preserving isometries. The corresponding covering is an orientable surface $\hat{S}'$ since the group is torsion-free and acts on a sphere, a Euclidean space or a hyperbolic space. Moreover, we can choose the surface $\hat{S}'$ covering regularly $\hat{S}$ by taking the intersection of conjugates of the finite index subgroup $\pi_1(\hat{S}')$ in the fundamental group $\pi_1(\hat{S})$. Finally, $\hat{S}$ covers S regularly. □

5.1.6 *Good, very good, and bad 2-orbifolds*

The purpose of this section is to prove Theorem 5.1.5.

It is fairly easy to distinguish between the good and bad 2-orbifolds as Thurston (1977) shows. We will prove this here.

Since we know the existence of the universal cover of orbifolds from Chapter 4, we can cover any 2-orbifold S with a simply-connected 2-orbifold $\tilde{S}$.

Let S be a compact 2-orbifold with possibly empty boundary. We divide into two cases: $\chi(|S|) \leq 0$ and $\chi(|S|) > 0$ for the underlying space $|S|$, a cornered surface, of S.

If $\chi(|S|) \leq 0$, then the (topological) fundamental group of $|S|$ is infinite, and we obtain a noncompact (topological) cover of $|S|$ which is also an orbifold-cover of S as well. Hence the universal covering $\tilde{S}$ of S is noncompact. Suppose that $\tilde{S}$ has some singular points. Since we can do a doubling-operation otherwise, we see that $\tilde{S}$ has only cone-points. However, $\tilde{S}$ cannot have a cone-point: Otherwise, we can remove a disk-neighborhood D of any cone-point of order say k for an integer $k > 1$. Since $|\tilde{S}|$ is homeomorphic to a disk, and $|\tilde{S}| - D$ has an infinite cyclic fundamental group, we can cover $\tilde{S} - D$ by a k-fold cyclic cover. Hence, by pasting in a disk, we again obtain a nontrivial covering orbifold of $\tilde{S}$, which is absurd. Thus, $\tilde{S}$ is a surface and S is a good orbifold.

Now suppose that $\chi(|S|) > 0$. Thus, $|S|$ is homeomorphic to a sphere, a projective plane, or a disk.

Suppose that S is a disk with at least one cone-point or a corner-reflector with nonempty (orbifold) boundary or a disk or a 2-sphere with at least three cone-points and/or corner-reflectors. By Proposition 5.1.4, S is good.

Suppose that S is a projective plane with at least two cone-points. Then the double-cover of S is good by Proposition 5.1.4 again. Now, suppose that S is a projective plane with one cone-point. Then the double-cover of S is a sphere with two cone-points of identical orders, and is covered by a 2-sphere.

Suppose that S is a disk with empty boundary and has at most two special singular points. Then the double-cover is a sphere with two to four cone-points. If the number of cone points is greater than or equal to three, then S is good. Suppose that we have at most two cone-points. Then S is a disk with one-cone point or a disk with one or two corner-reflectors. In the first case, S is covered by a sphere with two cone-points of identical orders, and S is covered by a 2-sphere. Suppose that S is a disk with two corner-reflectors of identical orders. Then S is covered by a sphere with two cone-points of identical orders. We are left with a disk with one corner-reflector or two corner-reflectors of different orders.

Suppose that S is a 2-sphere with one or two cone-points. If the orders are identical in the second case, then S is regularly covered by a 2-sphere and is good. We are left with the case when S is a 2-sphere with one or two cone-points of distinct orders.

A sphere Σ with cone points of orders p and q with p and q relatively prime is not covered by a manifold since the fundamental group is trivial by the Van Kampen theorem. (See Section 4.7.1.3.) A sphere with one cone point is also not covered by a manifold by the same reason.

The universal cover of a sphere with one cone-point, a sphere with two cone-points of distinct orders, a disk with one cone-point, or a disk with two corner-reflectors of distinct orders is covered by a sphere with one-cone point or two cone-points of relatively prime orders p and q. Hence, we conclude that a sphere with one cone-point, a sphere with two cone-points of distinct orders, a disk with one corner-reflector and a disk with two corner-reflectors of distinct orders are bad, and they are the only bad 2-orbifolds.

We will now continue to show that compact 2-orbifolds are very good except for bad ones.

Theorem 5.1.5. *A sphere with one or two cone-points with orders m and n where $m \neq n$ is a bad orbifold. So is a disk with silvered edges and one or two corner-reflectors of order m and n where $m \neq n$ are bad. Except for these, every other compact 2-orbifold is good. Furthermore, compact good orbifolds are very good. In fact, we can assume that the finite covering is always regular.*

Proof. The first parts were proved in above paragraphs.

We need to show the final statement only by the above discussions. By double-covering, the 2-orbifolds can be assumed to be orientable and have cone-points only as singular points. Let $\mathcal{O}$ denote a 2-orbifold.

If the underlying space is of Euler characteristic ≥ 1, then there is a covering by an orbifold whose underlying space is a sphere or a disk. This was studied above and was shown to be bad or is good. The good ones are very good according to Proposition 5.1.4.

Now suppose that the Euler characteristic of the underlying space is ≤ 0. There exists a disk D containing all the cone-points. Let $\mathcal{D}$ denote the corresponding orbifold. By Proposition 5.1.4, there is a finite regular covering surface S.

The boundary component of D is covered by m boundary components of S and each boundary component of S covers the boundary component of D by n-fold coverings for identical n. The closure of the complement of D is a surface S' of negative Euler characteristic and has infinite homology $H_1(S')$. Suppose that $\partial S'$ has other component than ∂D, then we can find a homomorphism from the group $\pi_1(S')$ of deck transformations of $\tilde{S}'$ to $\mathbb{Z}_n$ so that a simple closed curve in ∂D maps to the generator. Then the kernel of the homomorphism gives us a finite regular covering S'' of the complement S'. We see that S'' has a boundary component mapping to S' in an n-fold way. Hence by attaching copies of S'' for each boundary component of S, we obtain a very good cover of the original orbifold.

Suppose that $\partial S'$ has no other component than ∂D. We can explicitly find a homomorphism $\pi_1(S') \to S_{2n}$ where S_{2n} is a permutation group of $2n$ elements where a simple closed curve in $\partial S'$ is mapped to an order n element: Let the homotopy class of $\partial S'$ is written as $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ for generators $a_i, b_i, i = 1, \dots, g$, $g \geq 1$, of $\pi_1(S')$. Then we find a homomorphism $\phi : \pi_1(S') \to S_{2n}$ by

- $\phi(a_1) = (1, 2, \cdots, n)$,
- $\phi(b_1) = (1, n+1)(2, n+2) \cdots (n, 2n)$, and
- $\phi(a_i) = \phi(b_i) = \mathrm{I}$ for $i \geq 2$.

Again, we can find S'' as above. (We thank the referee for supplying this part.)

By construction, this is a regular covering. (The proof follows the article [Scott (1983)].) □

Remark 5.1.6. The universal cover $\tilde{S}$ of a noncompact 2-orbifold S cannot have singularity: We can assume that $\tilde{S}$ is orientable and hence has only cone-points as singular points. If there is a cone-point, then we can find a cylic branched cover of $\tilde{S}$ by taking a properly embedded arc from the cone-point leaving every compact subset. This is absurd. Thus, noncompact 2-orbifolds are always good.

A noncompact 2-orbifold may not be very good since there is an example with a sequence of cone points of strictly increasing orders. However, noncompact but precompact 2-orbifolds are always very good by Theorem 5.1.5.

5.2 Topological operations on 2-orbifolds: constructions and decompositions

We will now study the question of how to construct and decompose 2-orbifolds:

- Definition of splitting and sewing of 2-orbifolds
- Regular neighborhoods of 1-orbifolds

- Reinterpretation of splitting and sewing.
- Identification interpretations of splitting and sewing

5.2.1 *The definition of splitting and sewing 2-orbifolds*

We will assume that the 2-orbifolds are very good from now on. (Actually, it is sufficient for our purpose that neighborhoods of the involved 1-orbifolds are very good.)

Let S be a very good 2-orbifold so that its underlying space X_S is a pre-compact open surface with a path-metric admitting a compactification to a surface with boundary. Such a metric always exists and the topology of the compactification is unique up to homeomorphism type. Let $\hat{S}$ be a very good cover, that is, a finite regular cover, of S, so that S is orbifold-diffeomorphic to $\hat{S}/F$ where F is a finite group acting on $\hat{S}$.

Since $X_{\hat{S}} = \hat{S}$ is also pre-compact and has a path-metric, we complete it to obtain a compact surface $X'_{\hat{S}}$ with respect to the path metric. The action of the group F extends to $\hat{S}$ by the path-metric. Then $X'_{\hat{S}}/F$ with the quotient orbifold structure is said to be the *orbifold-completion* of S.

- Let S be a 2-orbifold with an embedded circle or a full 1-orbifold l in the interior of S. Obtaining the orbifold-completion $\hat{S}$ of $S - l$ with respect to the path-metric is called *splitting* S along l. Since $S - l$ has an embedded copy in $\hat{S}$, we see that there exists a map $\hat{S} \to S$ sending the copy to $S - l$. Let $\hat{l}$ denote the union of boundary components of $\hat{S}$ corresponding to l under the map.
- Conversely, S is said to be obtained from *sewing* $\hat{S}$ *along* $\hat{l}$.
- If the interior of the underlying space of l lies in the interior of the underlying space of S, then the components of $\hat{S}$ are said to be the *decomposed components of* S *along* l, and we also say that S *decomposes* into $\hat{S}$ along l.
- Of course, if l is a union of disjoint embedded circles or full 1-orbifolds, the same definitions hold.

There are two distinguished classes of splitting and sewing operations:

A simple closed curve boundary component can be made into a set of mirror points and conversely in a unique manner by Proposition 4.4.3.

A boundary full 1-orbifold can be made into a 1-orbifold of mirror points with two corner-reflectors of order two and conversely in a unique manner: (an end point has a neighborhood which is a quotient space of a dihedral group of order four acting on the open ball generated by two reflections.) The forward process is called *silvering* and the reverse process *clarifying.*

5.2.2 *Regular neighborhoods of 1-orbifolds*

5.2.2.1 *The classification of Euler-characteristic zero orbifolds*

Let A be a compact annulus with nonempty boundary. The quotient orbifold of an annulus has a zero Euler characteristic.

Proposition 5.2.1. *The compact* 2*-orbifolds with nonempty boundary and of zero Euler characteristic is as follows:*

(1) *an annulus,*
(2) *a Möbius band,*
(3) *an annulus with one boundary component silvered* (*a* silvered annulus),
(4) *a disk with two cone-points of order two with no mirror points* (*a* $(;2,2)$-disk),
(5) *a disk with two boundary* 1*-orbifolds, two silvered edges* (a silvered strip),
(6) *a disk with one cone-point of order* 2 *and one boundary full* 1*-orbifold* (*a* bigon with a cone-point of order two), *which has only one silvered edge,*
(7) *a disk with two corner-reflectors of order two and one boundary full* 1*-orbifold* (*a* half-square), *which has three silvered edges.*

Proof. The underlying space should have a nonnegative Euler characteristic by the Riemann-Hurwitz formula. If the Euler characteristic of the space is zero, there are none of cone-points, corner-reflectors, and boundary full 1-orbifolds and we obtain cases in (1), (2), or (3).

Suppose now that the underlying space is homeomorphic to a disk. If there is no singular point in the boundary, then (4) holds as there has to be exactly two cone-points of order two by the Riemann-Hurwitz formula. If there are two boundary full 1-orbifolds, then no singular points in the interior and no corner-reflector can exist; thus, (5) holds.

Suppose that exactly one boundary full 1-orbifold exists. If a cone-point exists, then it has to be a unique one and of order two, and (6) holds. If there are no cone-points, but corner-reflectors, then exactly two corner-reflectors of order two and no more. (7) holds. □

5.2.2.2 *Regular neighborhoods of* 1*-orbifolds*

Let l be a circle or a 1-orbifold in the interior of a 2-orbifold S so that $\pi_1(l)$ injects into $\pi_1(S)$. The image is clearly infinite and is not homotopic to a single point. In this case, l is said to be *essential.*

Let l be an essential 1-orbifold. Then l has a deck-transformation-group invariant neighborhood of zero Euler characteristic considering its very good cover and the deck-transformation-group invariant tubular neighborhoods. Since the inverse image of l consists of closed curves which represent generators, we deduce that l is contained in the neighborhood as follows.

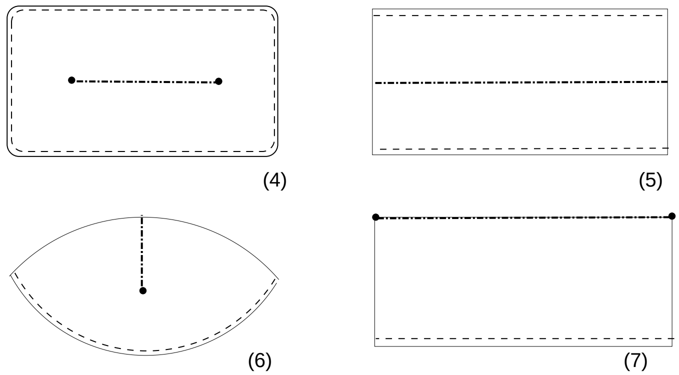

Fig. 5.1 For each orbifold, the arcs with dashed arc nearby are the boundary components and the thicker dotted arc is the 1-orbifold that the 2-orbifold is a regular neighborhood of. The black dot indicates the cone-point of order two or corner-reflectors of order two

- For (1) and (2), l is the closed curve representing the generator of the fundamental group;
- For (3), l is the mirror set that is a boundary component of the underlying space;
- For (4), l is the arc connecting the two cone-points unique up to isotopy.
- For (5), l is an arc connecting two interior points of two silvered edges respectively;
- For (6), l is an arc connecting an interior point of an silvered edge and an cone-point of order two;
- For (7), the silvered edge in the topological boundary connecting the two corner-reflectors of order two.

Given a 1-orbifold l and a neighborhood N of it in some ambient 2-orbifold as in Proposition 5.2.1, we say that N a *regular neighborhood* if the pair (N, l) is diffeomorphic to one of the above.

Proposition 5.2.2. *A* 1*-orbifold embeded as a suborbifold in a good* 2*-orbifold has a regular neighborhood which is unique up to isotopy.*

Proof. The existence is proved above. The uniqueness up to isotopy is proved as follows: In fact, regular neighborhoods are tubular neighborhoods if we use the Riemannian metric on the orbifolds. (See the end of Section 4.4.2 for details.) Each tubular neighborhood fibers over a 1-orbifold with fibers connected 1-orbifolds in the orbifold sense. We can isotopy any tubular neighborhood into any other tubular neighborhood by contracting in the fiber directions. To prove the uniqueness up to isotopy, we can modify the proof of Theorem 5.3 in Chapter 4 of the book [Hirsch (1976)] to be adopted to an annulus with a finite group acting on it and an embedded circle. $\square$

5.2.3 *Splitting and sewing on 2-orbifolds reinterpreted*

Let l be a 1-orbifold embedded in the interior of a 2-orbifold S. If one removes l from the interior of a regular neighborhood, we obtain either a union of one or two open annuli, or a union of one or two open silvered strips. In (2)-(4), an open annulus results. For (1), a union of two open annuli results. For (6)-(7), an open silvered strip results. For (5), we obtain a union of two open silvered strips. The results can be easily path-completed to be unions of one or two compact annuli or unions of one or two silvered strips respectively.

We complete $S - l$ in this manner: We take a closed regular neighborhood N of l in S. We remove $N - l$ to obtain the above types and complete it by the path metric and re-identify with $S - l$ to obtain a compactified 2-orbifold. This process is the splitting of S along l.

Conversely, we describe sewing: Take an open annular 2-orbifold N which is a regular neighborhood of a 1-orbifold l:

- Suppose that l is a circle. We obtain $U = N - l$ which is a union of one or two annuli.
- Take a 2-orbifold S' with a union l' of one or two boundary components which are circles.
- Take an open regular neighborhood of l' and remove l' to obtain V.
- Suppose that U and V are the identical suborbifolds. We identify $S' - l'$ and N along U and V.
- This gives us a 2-orbifold S, and S is obtained from S' by sewing along l'.

- Suppose that l is a closed curve and corresponds to a 1-orbifold l' in S'. We obtain (1),(2),(3)-type neighborhoods of l'' in this way. The operation in case (1) is said to be *pasting*, in case (2) *cross-capping*, and in case (3) *silvering* along simple closed curves.
- Suppose that l is a full 1-orbifold. $U = N - l$ is either an open annulus or a union of one (resp. two) silvered strips.
- The former happens if N is of type (4) and the latter if N is of type (5)-(7).
- In case (4), take a 2-orbifold S' with a boundary component l' a circle. Then we identify U with a regular neighborhood of l' with l' removed to obtain an orbifold S. Then l corresponds a full 1-orbifold l'' in S in a one-to-one manner. l'' has a type-(4) regular neighborhood. The operation is said to be *folding* along a simple closed curve. (See Section 4.2.2.)
- In the remaining cases, take a 2-orbifold S' with a union l' of one (resp. two) boundary full 1-orbifolds. Take a regular neighborhood N of l' and remove l' to obtain V. Identify U with V for $S' - l'$ and $N - l$ to obtain S. Then S is obtained from S' by sewing along l'. Again l corresponds to a full 1-orbifold l'' in S in a one-to-one manner.
- We obtain (5),(6), and (7)-type neighborhoods of l'' in this way, where the operations are said to be *pasting*, *folding*, and *silvering* along full 1-orbifolds

respectively.

- In other words, silvering is the operation of removing a regular neighborhood and replacing by a silvered annulus or a half square. Clarifying is an operation of removing the regular neighborhood and replacing an annulus or a silvered strip.

Proposition 5.2.3. *The Euler characteristic of a* 2*-orbifold before and after splitting or sewing remains unchanged.*

Proof. Form regular neighborhoods of the involved boundary components of the split 2-orbifold and those of the original 2-orbifold. They have zero Euler characteristics. Since their boundary 1-orbifolds have zero Euler characteristics, the lemma follows by the additivity formula (5.1). □

5.2.4 *Identification interpretations of splitting and sewing*

The sewing can be understood as follows: The pasting map f is defined on open neighborhood U of the union of the associated boundary components in an ambient open 2-orbifold S' where f satisfies the equation $\tilde{f} \circ \vartheta = \vartheta' \circ \tilde{f}$ where $\tilde{f}$ is a lift of f defined on $\tilde{U}$ the inverse image and ϑ and ϑ' are corresponding deck transformations acting on components of the inverse images in $\tilde{S}'$ of boundary components of f to be pasted by f.

In the following, we describe the topological identification process of the underlying space involved in these six types of sewings. The orbifold structure on the sewed orbifold should be clear.

Let a 2-orbifold Σ have a boundary component b. (Σ is not necessarily connected.) b is either a simple closed curve or a full 1-orbifold. We find a 2-orbifold Σ'' constructed from Σ by sewing along b or another component of Σ. (We also need the notation here for the later purposes.)

(A) Suppose that b is diffeomorphic to a circle; that is, b is a closed curve. Let Σ' be a component of the 2-orbifold Σ with boundary component b'. Suppose that there is a diffeomorphism $f : b \to b'$. Then we obtain a bigger 2-orbifold Σ'' glued along b and b' topologically.

    (I) The construction so that Σ'' does not create any more singular point results in a 2-orbifold Σ'' so that

$$\Sigma'' - (\Sigma - b \cup b')$$

    is a circle with a neighborhood either diffeomorphic to an annulus or a Möbius band.

    (1) In the first case, we have $b \neq b'$ (pasting).
    (2) In the second case, we have $b = b'$, and $\langle f \rangle$ is of order two without fixed points (cross-capping).

(II) When $b = b'$, the construction so that Σ'' does introduce more singular points to occur in a 2-orbifold Σ'' so that

$$\Sigma'' - (\Sigma - b)$$

is a circle of mirror points or is a full 1-orbifold with endpoints in cone-points of order two depending on whether $f : b \to b$

(1) is the identity map (silvering), or
(2) is of order two and has exactly two fixed points (folding).

(B) Consider when b is a full 1-orbifold with endpoints mirror points.

(I) Let Σ' be a component (possibly the same as one containing b) with boundary full 1-orbifold b' with endpoints mirror points where $b \neq b'$. We obtain a bigger 2-orbifold Σ'' by gluing b and b' by a diffeomorphism $f : b \to b'$. This does not create new singular points (pasting).

(II) Suppose that $b = b'$. Let $f : b \to b$ be the attaching map. Then

(1) if f is the identity, then b is silvered and the end points are changed into corner-reflectors of order two (silvering).
(2) If f is of order two, then Σ'' has a new cone-point of order two and has one-boundary component orbifold removed away. b corresponds to a 1-orbifold in Σ' (folding). This creates just one more singularity of cone-type of order 2.

We can easily put the obvious orbifold structure on Σ'' using the previous descriptions by regular neighborhoods above.

5.3 Notes

Low-dimensional orbifolds were first studied by Thurston (1977) who was to put much emphasis on cut and paste operations, a point of view not attempted before then. There was a nice exposition of the 2-orbifold theory in the article [Scott (1983)], which we followed here. Topological operations with orbifolds are widely used in many papers. They include the papers [Matsumoto and Montesinos-Amilibia (1991); Dunbar (1988); Takeuchi (1989, 1996); Choi and Goldman (2005)]. This topic is not so well treated in groupoid theoretical approach to orbifolds.

Chapter 6

Geometry of orbifolds: geometric structures on orbifolds

In this section, we introduce the geometric structures on orbifolds. The definition is given by the method of atlases of charts, making use of (G, X)-pseudo group structures in Section 2.3. We show that geometric orbifolds are always good by using the foliation theory, an important result due to Thurston (See Chapter 5 of the book [Thurston (1977)].) Then we discuss developing maps, global charts, and associated holonomy homomorphisms. These can also be used as definitions of geometric structures. We also introduce the approach using flat bundles and transverse sections to define the geometric structures. (See Section 2.4.) These observations were first due to Goldman (1987) for manifolds. The article [Goldman (2010)] contains a general introduction to geometric structures on manifolds.

Next, we introduce the deformation spaces of geometric structures on orbifolds using the above three approaches as were done by Goldman for manifolds. We finally mention the local homeomorphism theorem from the deformation space to the representation space.

6.1 The definition of geometric structures on orbifolds

Let (G, X) be a pair defining a geometry. That is, G is a Lie group acting on a manifold effectively and transitively. Let M be a connected n-orbifold with boundary, possibly empty. We have three ways to define a (G, X)-geometric structure on M:

- Atlases of charts.
- A developing map from the universal covering space.
- A cross-section of the flat orbifold X-bundle.

6.1.1 *An atlas of charts approach*

Given an imbedding $f : U \to V$ between two domains U and V in $\mathbb{R}^n$ with groups G_1 and G_2 acting on them respectively, we denote by $f^* : G_1 \to G_2$ the homomorphism determined by sending $\vartheta \in G_1$ to the element of G_2 agreeing with $f \circ \vartheta \circ f^{-1}$ in an open subset provided this is always uniquely determined.

An X-*chart* of a model open set form the triple (U, K, ϕ) of M is simply a h-equivariant homeomorphism from U to an open subset of X where h is an injective homomorphism $K \to G$. Given an atlas of charts for M, for each chart (U, K, ϕ) in the atlas, we suppose that we find an X-chart $\rho : U \to X$ and an injective homomorphism $h : K \to G$ so that ρ is an equivariant map. Let (U, K, ϕ) and (V, H, ψ) be two charts with the inclusion map $\iota : \psi(V) \to \phi(U)$. For an embedding $\tilde{\iota} : (V, H, \psi) \to (U, K, \phi)$ of charts lifting ι, if we have

$$\rho \circ \tilde{\iota} = g \circ \rho', h'(\cdot) = gh(\tilde{\iota}^*(\cdot))g^{-1} \text{ for some } g \in G,$$

then ι or $\tilde{\iota}$ are said to be a (G, X)-*map*. Two X-charts (V, H, ψ) and (U, K, ϕ) in an atlas of X-charts are (G, X)-*compatible* if given any point $x \in \psi(V) \cap \phi(U)$ in M, we have an X-chart (W, K, η) so that $\eta(W)$ is a neighborhood of x in $\psi(V) \cap \phi(U)$ and the embedding of $\eta(W)$ in each of $\phi(U)$ and $\psi(V)$ is a (G, X)-map.

If we simply identify with open subsets of X, the above simplifies greatly and $\tilde{\iota}$ is a restriction of an element of g and $\tilde{\iota}^*$ is a conjugation by g also.

This gives us a way to build an orbifold from open subset pieces of X. A maximal such atlas of compatible X-charts is called *a* (G, X)-*structure* on M.

(Note that this gives no condition on $\partial\mathcal{O}$. Sometimes, it will be necessary to put restrictions to work with deformation spaces. A priori, one does not know what the boundary condition should be.)

An (G, X)-*map* $f : M \to N$ is a smooth map so that for each x and $y = f(x)$, there are charts (U, K, ϕ) and (V, H, ψ) so that f sends $\phi(U)$ into $\psi(V)$ and lifts to an immersion $\tilde{f} : U \to V$ so that

$$\rho' \circ \tilde{f} = g \circ \rho \text{ and } h'(\tilde{f}^*(\cdot)) = gh(\cdot)g^{-1} \text{ for } g \in G.$$

In other words, f is a restriction of an element g of G up to charts with a homomorphism $K \to H$ induced by a conjugation by an element g of G.

Let M be an orbifold. Note that an orbifold-immersion $f : M \to N$ to an orbifold N with a (G, X)-structure μ induces a (G, X)-structure on M so that f becomes a (G, X)-map. M is said to have a (G, X)-*structure induced by* f to be denoted by $f^*(\mu)$. (See Section 2.3.1 also.)

Theorem 6.1.1 (Thurston). *Let M be an n-orbifold with boundary, possibly empty. An (G, X)-orbifold M is always good.*

Proof. Basically we build the space of germs of local (G, X)-maps from M to X which is a principal bundle and is a manifold: M is covered by open sets that can be identified with open sets in X. For a local finite subgroup K of G acting on $U \subset M$ identified with an open subset of X, let K act on $G \times U$ by $k(g, u) = (kg, ku)$ for $u \in U, g \in G, k \in K$. For each (U, K, ϕ), we build $G(U) = (G \times U)/K$ and a projection $G(U) \to U/K$. For any inclusion $V \to U$ for open sets $U, V \subset M$, we obtain $G(V) \to G(U)$ induced by inclusion maps. We paste these together to obtain $G(M)$. Then $G(M)$ is a manifold since K acts on $G \times U$ freely. The foliation given by pasting $g_0 \times U$ in $G(U)$ is a foliation by open manifolds with the same dimension as M. Each leaf of the foliation covers M forming a manifold cover of M.

□

If G is a subgroup of a linear group, then M is very good by Selberg's lemma provided M has finitely generated fundamental group. Thus M is a quotient orbifold $\tilde{M}/\Gamma$ where Γ is finite and contains copies of all of the local group.

6.1.2 *The developing maps and holonomy homomorphisms*

Let a connected orbifold M admit a (G, X)-structure. Let $\tilde{M}$ denote the universal cover of M with a deck transformation group $\pi_1(M)$. Then $\tilde{M}$ is a manifold and we obtain a *developing map* $D : \tilde{M} \to X$ by first finding an initial chart $\rho : U \to X$ and continuing by extending maps by patching. We use a nice cover of $\tilde{M}$ and extend. The map is well-defined independently of which path of charts one took to arrive at a given chart: To show this, we consider a homotopy of paths and consider mutually intersecting three X-charts simultaneously and the map can be consistently defined on their union.

Since we can change the initial chart to $k \circ \rho$ for any $k \in G$, it follows that $k \circ D$ is an another developing map and conversely any developing map is of such a form.

Given a deck transformation $\gamma : \tilde{M} \to \tilde{M}$, we see that $D \circ \gamma$ is a developing map also and hence equals $h(\gamma) \circ D$ for some $h(\gamma) \in G$. Let $\pi_1(M)$ denote the group of deck transformations of $\tilde{M}$. The map $h : \pi_1(M) \to G$ is a homomorphism, so-called the holonomy homomorphism.

The pair (D, h) is said to be the *developing pair*. The development pair is determined up to an action of G given by $(D, h(\cdot)) \to (g \circ D, gh(\cdot)g^{-1})$.

Conversely, a developing map (D, h) gives us X-charts: For each open chart (U, K, ψ), we lift to a component of $p^{-1}(U)$ in $\tilde{M}$ and obtain a restriction of D to the component. This gives us X-charts. A different choice of components gives us the compatible charts. Local group actions and embeddings satisfy the desired properties. Thus, a development pair completely determines the (G, X)-structure on M.

6.1.3 *The definition as flat bundles with transversal sections*

Given a (G, X)-orbifold M with X-charts, we form a G-bundle $G(M)$ as in the proof of Theorem 6.1.1. This is a principal G-bundle. We form an associated X-bundle $X(M)$ using the G-action on X: $X(M) = G(M) \times X/G$ where G acts on the right on $G(M)$ and left on X and G acts on $G(M) \times X$ on the right by

$$g : (u, x) \to (ug, g^{-1}(x)), g \in G, u \in G(M), x \in X.$$

A flat G-bundle is an object obtained by patching open sets $G \times U$ by the left action of G as in the proof of Theorem 6.1.1, and so is a flat X-bundle defined as above.

6.1.3.1 *Flat X-bundles*

One can also define a notion of foliation on n-dimensional orbifolds. Given an n-orbifold M and each model triple (U, K, ϕ), we give a smooth submersion $U \to \mathbb{R}^i$ for some $1 \leq i \leq n-1$ equivariant with respect to a homomorphism $K \to L$ for a finite group acting on $\mathbb{R}^i$ smoothly. The fibers of the maps is said to be leaves. For any embedding $(V, J, \psi) \to (U, K, \phi)$, the leaves of the charts are compatible. A leaf of a foliation is also defined as in the manifold cases as maximal $n-i$-dimensional subset that is a union of images of leaves of model triples.

A foliation in the manifold $G(M)$ with leaves transversal to fibers induces a foliation in $G(M) \times X$ with leaves transversal to fibers and hence a foliation in the orbifold $X(M)$ with leaves transversal to fibers. This corresponds to a flat G-connection. A flat G-connection on $X(M)$ is a way to identify each fiber of $X(M)$ with X locally-consistently. A flat G-connection on $X(M)$ gives us a flat G-connection on $X(\tilde{M})$. Since $\tilde{M}$ is a simply-connected manifold, $X(\tilde{M})$ can be identified with $X \times \tilde{M}$ as an X-bundle where we can regard sets of form $x \times \tilde{M}$ as leaves for the flat connections. $X(\tilde{M})$ covers $X(M)$ and hence

$$X(M) = (X \times \tilde{M})/\pi_1(M)$$

where the connection corresponds to foliations with leaves of type $x \times \tilde{M}$. Hence this gives us a representation $h : \pi_1(M) \to G$ so that for any $\gamma \in \pi_1(M)$, the corresponding action in $X \times \tilde{M}$ is given by $(x, m) \to (h(\gamma)x, \gamma(x))$.

Conversely, given a representation h, we can build $X \times \tilde{M}$ and act by $\gamma(x, m) = (h(\gamma)x, \gamma(m))$ to obtain a flat X-bundle $X(M)$. (This theory is completely analogous to Section 2.4.2.2. See also the books [Kobayashi and Nomizu (1997); Bishop and Crittendon (2002)] for details.)

6.1.3.2 *Flat X-bundles with transversal sections*

A development pair (D, h) of M gives us a flat X-bundle $X(M)$ with a section $s : M \to X(M)$. We obtain a section $D' : \tilde{M} \to X \times \tilde{M}$ transversal to the foliation by taking $D'(x) = (D(x), x)$ for $x \in \tilde{M}$. The transversality of D' to the constant foliation is actually equivalent to the immersive property of D. The left-action of $\pi_1(M)$ gives us a section $s : M \to X(M)$ transversal to the foliation.

On the other hand, given a transversal section $s : M \to X(M)$, we obtain a transversal section $s' : \tilde{M} \to X \times \tilde{M}$. By a projection to X, we obtain an immersion $D : \tilde{M} \to X$ so that $D \circ \gamma = h(\gamma) \circ D$ for some $h(\gamma)$ in G. The map $h : \pi_1(M) \to G$ is a homomorphism. Hence we obtain a development pair.

6.1.4 *The equivalence of three notions.*

Given an atlas of X-charts, i.e., a (G, X)-structure, we determine a development pair (D, h). Given a development pair (D, h), we determine an atlas of X-charts, i.e., a (G, X)-structure. Given a development pair (D, h), we determine a flat X-bundle $X(M)$ with a transversal section $M \to X(M)$. Given a section $s : M \to X(M)$ to a flat X-bundle, we determine a development pair (D, h). Thus, these three classes of definitions are equivalent.

6.2 The definition of the deformation spaces of (G, X)-structures on orbifolds

Consider the set $\mathcal{M}_{(G,X)}(M)$ of all (G, X)-structures on a connected orbifold M. We introduce an equivalence relation: two (G, X)-structures μ_1 and μ_2 are *equivalent* if there is an isotopy $\phi : M \to M$ from the identity map I_M so that $\phi^*(\mu_1) = \mu_2$. The deformation space of (G, X)-structures on M is defined as $\mathcal{M}_{(G,X)}/\sim$. (Currently, we just have a set.)

We reinterpret the space as

- the set of equivalence classes of diffeomorphisms $f : M \to M'$ for M an orbifold and M' a (G, X)-orbifold
- where $f : M \to M' \sim g : M \to M''$ if there exists a (G, X)-diffeomorphism $h : M' \to M''$ so that $h \circ f$ is isotopic to g.

6.2.1 *The isotopy-equivalence space.*

First, we identify $\pi_1(M)$ with $\pi_1(M \times I)$. Consider the set of diffeomorphisms $f : \tilde{M} \to \tilde{M}'$ equivariant with respect to an isomorphism $f_* : \pi_1(M) \to \pi_1(M')$ for a (G, X)-orbifold M'. We introduce an equivalence relation on this set: Given $f : \tilde{M} \to \tilde{M}'$ and $g : \tilde{M} \to \tilde{M}''$, we say that they are *equivalent* if there exists a (G, X)-map $\phi : \tilde{M}' \to \tilde{M}''$ so that $\phi \circ f$ is isotopic to g by an isotopy $\tilde{M} \times I \to \tilde{M}''$ equivariant with respect to both $\phi_* \circ f_*$ and g_* which are equal. Denote this set by $\mathcal{D}_I(M)$.

We claim that $\mathcal{D}_I(M)$ is in one-to-one correspondence with $\mathcal{M}_{(G,X)}/\sim$: Given an element f of the first space, we obtain an induced diffeomorphism $\hat{f} : M \to M'$ for a (G, X)-manifold M'. The equivariant isotopy goes to an isotopy. So this is a well-defined map. The inverse is given by lifting a diffeomorphism $g : M \to M'$ for a (G, X)-manifold M' to the universal covers.

The space $\mathcal{S}(M)$ is defined as follows: Consider the set of triples of form $(D, \tilde{f} : \tilde{M} \to \tilde{M}')$ where $f : M \to M'$ is a diffeomorphism for orbifolds M and M', $\tilde{f} : \tilde{M} \to \tilde{M}'$ is a lift of f, and $D : \tilde{M}' \to X$ is an immersion equivariant with respect to a homomorphism $h : \pi_1(M') \to G$. We define $(D, \tilde{f}) \sim (D', \tilde{f}' : \tilde{M} \to \tilde{M}'')$ if there is a diffeomorphism $\tilde{\phi} : \tilde{M}' \to \tilde{M}''$ so that $D' \circ \tilde{\phi} = D$ and an isotopy

$H : \tilde{M} \times I \to \tilde{M}''$ equivariant with respect to $\tilde{f}'_* : \pi_1(M) \to \pi_1(M'')$ so that $\tilde{\phi} \circ \tilde{f} = H_0$ and $\tilde{f}' = H_1$. We finally give a topology on this space by the C^1-topology on the space of maps $\tilde{M} \to X$ restricting to the space of maps of form $D \circ \tilde{f} : \tilde{M} \to X$. (Here the C^1-topology means the compact C^1-topology.)

There is a G-action on $\mathcal{S}(M)$ given by sending D to $g \circ D$ for $g \in G$.

6.2.2 *The topology of the deformation space*

Theorem 6.2.1. *Let M be a connected orbifold. There is a natural action of G on $\mathcal{S}(M)$ given by $g(D, \tilde{f}) = (g \circ D, \tilde{f}), g \in G$. The quotient space $\mathcal{S}(M)/G$ is in one-to-one correspondence with the deformation space $\mathcal{M}_{(G,X)}/\sim$. This space has the quotient topology from the C^1-topology of $\mathcal{S}(M)$.*

Proof. We show that $\mathcal{D}_I(M)$ is in one-to-one correspondence to $\mathcal{S}(M)/G$.

We first obtain a map $\mathcal{D}_I(M) \to \mathcal{S}(M)/G$: Given an element $\tilde{f} : \tilde{M} \to \tilde{M}'$, we have a developing map $D : \tilde{M}' \to X$ equivariant with respect to $h : \pi_1(M') \to G$. Also, given $\tilde{f}' : \tilde{M} \to \tilde{M}''$, we have a developing map $D' : \tilde{M}'' \to X$ equivariant with respect to $h' : \pi_1(M'') \to G$. If $\tilde{f} : \tilde{M} \to \tilde{M}'$ and $\tilde{f}' : \tilde{M} \to \tilde{M}''$ are equivalent, then there is a (G, X)-diffeomorphism $M' \to M''$ and hence two global charts $D' \circ \tilde{f}$ and $D'' \circ \tilde{f}'$ differ only by an element of G.

Conversely, we obtain a map $\mathcal{S}(M)/G \to \mathcal{D}_I(M)$: given $(D, \tilde{f})$, we obviously obtain a (G, X)-structure on M' If $(D, \tilde{f})$ and $(D', \tilde{f}')$ are equivalent, then there is a diffeomorphism $\phi : M' \to M''$ so that $D' \circ \tilde{\phi} = g \circ D$ for a lift $\tilde{\phi}$ of ϕ. This means $\phi' : M' \to M''$ is a (G, X)-diffeomorphism. The above two maps are clearly inverses of each other. $\square$

We will denote by $\mathcal{D}_{(G,X)}(M)$ the space $\mathcal{S}(M)/G$ with the topology given in the theorem.

6.2.3 *The local homeomorphism theorem*

6.2.3.1 *The representation space*

Suppose that $\pi_1(M)$ is finitely-generated. In particular if M is a compact n-orbifold, this is true. Denote by $g_1, \dots, g_n$ the set of generators and $R_1, \dots, R_m, \dots$ be the set of relations.

The set of homomorphisms $\pi_1(M) \to G$ is to be identified with a subset of G^n by sending a homomorphism h to $(h(g_1), \dots, h(g_n))$. This is clearly an injective map. This image is described as an algebraic subset defined by polynomial relations given by $R_1, \dots, R_m, \dots$; that is, each R_i yields $R_i(h(g_1), \dots, h(g_n)) = \mathrm{I}$, which gives us a system of polynomial equations. (The polynomial relations will always be finitely many.) This follows since if the relations are satisfied, then we can obtain the representation conversely. Denote the space by $\mathbf{Hom}(\pi, G)$, which is an algebraic set.

There is an action of G on $\mathbf{Hom}(\pi_1(M), G)$ given by the conjugation action $(g \star h)(\cdot) = gh(\cdot)g^{-1}$. We denote by $\mathbf{Rep}(\pi_1(M), G)$ the quotient space $\mathbf{Hom}(\pi_1(M), G)/G$.

6.2.3.2 *The map* hol

We define $\mathrm{hol}' : \mathcal{S}(M) \to \mathbf{Hom}(\pi_1(M), G)$ by sending the equivalence class of $(D, \tilde{f})$ to a homomorphism $h \circ \tilde{f}_* : \pi_1(M) \to G$

This induces $\mathrm{hol} : \mathcal{D}_{(G,X)}(M) \to \mathbf{Rep}(\pi_1(M), G)$. We denote by $\mathbf{Hom}(\pi, G)^s$ the subset of $\mathbf{Hom}(\pi_1(M), G)$ where the conjugation action of G given by $h(\cdot) \to gh(\cdot)g^{-1}, g \in G$ is stable; i.e., the orbits are closed and the stabilizers are finite. (See Section 1 of [Johnson and Millson (1987)].) We denote by $\mathcal{S}^s_{(G,X)}(M)$ the inverse image of this set under hol' and a G-invariant set. Denote by $\mathcal{D}^s_{(G,X)}(M)$ the image of this set under the quotient map

$$\mathcal{S}(M) \to \mathcal{S}(M)/G$$

and denote by $\mathbf{Rep}(\pi, G)^s$ the quotient image of $\mathbf{Hom}(\pi, G)^s$.

When M is disconnected as in Chapter 7, the deformation space $\mathcal{D}_{(G,X)}(M)$ is defined as the product space $\prod_{i=1}^n \mathcal{D}_{(G,X)}(M_i)$ for components $M_1, \ldots, M_n$ and $\mathbf{Rep}(\pi_1(M), G)$ is also defined as the product space $\prod_{i=1}^n \mathbf{Rep}(\pi_1(M_i), G)$. Also, similarly, we define

$$\mathcal{D}^s_{(G,X)}(M) := \prod_{i=1}^n \mathcal{D}^s_{(G,X)}(M_i), \ \mathbf{Rep}(\pi_1(M), G)^s := \prod_{i=1}^n \mathbf{Rep}(\pi_1(M_i), G)^s.$$

The main purpose of this section is to state:

Theorem 6.2.2. *Suppose that M is a closed n-orbifold. Then* hol *restricts to a local homeomorphism*

$$\mathcal{D}^s_{(G,X)}(M) \to \mathbf{Rep}(\pi_1(M), G)^s.$$

It is sufficient to prove for the case when M is connected. We just give an informal discussion here since the proof is very complicated (see [Choi (2004)] for details): We send the equivalence class of $(D, \tilde{f})$ to the associated homomorphism $h : \pi_1(M) \to G$. First, it is easy to show that hol' is continuous: If $D' \circ \tilde{f}'$ is sufficiently close to $D \circ f$ in a sufficiently large compact subset of $\tilde{M}$, then the holonomy $h'(g_i)$ of generators g_i are as close to the original $h(g_i)$ as needed.

Conversely, given a geometric structure corresponding to h, if one deforms h by a small amount, then we can change the geometric structure correspondingly by considering local models and changing them slightly and patching up the differences in a consistent way. Finally, we have to show that such a change of a geometric structure is unique up to isotopies.

6.3 Notes

The local homeomorphism result, introduced by Weil (1960, 1962), was a very important and subtle result for the study of deformations of (G, X)-structures on manifolds. For manifolds, Thurston (1977) (and Ehresmann) gave a proof. Later J. Morgan gave a series of lectures on it, which is written up by Walter Lok in Section 1.1 of [Lok (1984)]. Also, Canary, Epsten, and Green gave a short proof of it also (Canary, Epstein, Green, 1987). See also Chapter 7 of [Kapovich (2009)].

Actually, we can find a short transversal section proof given by Goldman (1987) in the manifold cases. It should be possible to modify this proof for the orbifold cases as well. But the proof is conceptually not different.

The main part of this chapter is from the papers [Choi (2004); Choi and Goldman (2005)]. Chapter 6 of the book [Kapovich (2009)] also devotes some pages to geometric orbifolds. The principal bundles, transversal sections, and flat connections are very interconnected and we think that this gives a very pleasant picture of geometric structures and shows that the notion of geometric structures is intrinsic to nature.

Chapter 7

Deformation spaces of hyperbolic structures on 2-orbifolds: Teichmüller spaces of 2-orbifolds

In this section, we define the Teichmüller space of 2-orbifolds as the deformation space of hyperbolic structures. (In some sense, the space should be called a Fricke space when we are talking about hyperbolic structures but not conformal structures, following Goldman.) We discuss the geometric cutting and pasting operations and the relation to the deformation spaces. The decompositions of 2-orbifolds into elementary 2-orbifolds are introduced. Elementary 2-orbifolds are pieces that cannot be decomposed further into negative Euler characteristic 2-orbifolds. We discuss the deformation spaces for elementary 2-orbifolds. (See the beginning of Section 7.3 for definition of elementary 2-orbifolds.) Using the geometric construction, we describe the Teichmüller spaces of 2-orbifolds of negative Euler characteristic. This follows Chapter 5 of the book [Thurston (1977)]. (See also the papers [Matsumoto and Montesinos-Amilibia (1991); Ohshika (1985)].)

Recall that the boundary of an orbifold is a suborbifold. The boundary component of a 2-orbifold is either a boundary full 1-orbifold or a simple closed curve.

Theorem 7.0.1 (Thurston). *Let Σ be a closed 2-orbifold of negative Euler characteristic. The deformation space of hyperbolic structures $\mathcal{T}(\Sigma)$ is homeomorphic to an open cell of dimension*

$$-3\chi(X_\Sigma) + 2k + l + 2n$$

where k is the number of cone-points, l the number of corner-reflectors, and n is the number of boundary full 1-orbifolds of Σ.

7.1 The definition of the Teichmüller space of 2-orbifolds

A *hyperbolic* structure on a 2-orbifold is a geometric structure modeled on $\mathbb{H}^2$ with the isometry group $\mathbb{PSL}(2, \mathbb{R})$. (Or it should be the disk $B^2 \subset \mathbb{RP}^2$ with $\mathbb{PO}(1,2)$ acting on it more closely to our spirit.) The *Teichmüller* space $\mathcal{T}(M)$ of a 2-orbifold M is the deformation space of hyperbolic structures on the 2-orbifold with geodesic boundary. As before, we reinterpret the space as

- the set of equivalence classes of diffeomorphisms $f : M \to M'$ for a 2-orbifold M and a hyperbolic 2-orbifold M' with geodesic boundary where
- $f : M \to M'$ and $g : M \to M''$ for hyperbolic 2-orbifolds with geodesic are equivalent if there exists a hyperbolic isometry $h : M' \to M''$ so that $h \circ f$ is isotopic to g.

A necessary condition for a 2-orbifold to have a hyperbolic structure with geodesic boundary is that the orbifold Euler characteristic is negative: Let the 2-orbifold have a hyperbolic structure with geodesic boundary. The 2-dimensional Gauss-Bonnet theorem states that the integral of a Gaussian curvature times the area form is -2π times the Euler characteristic. (See Theorem 4.4.4 in Chapter 4.)

We can prove the sufficiency by decomposition into elementary 2-orbifolds and finding explicit hyperbolic structures on these and pasting back the results. This process will be clear from the proof of Theorem 7.0.1 in this chapter.

7.2 The geometric cutting and pasting and the deformation spaces

Recall that the interior and boundary of a 2-orbifold in the orbifold sense may be different from the interior and boundary of the underlying surface. (See Remark 4.2.5.) Given a compact hyperbolic 2-orbifold Σ with geodesic boundary, we have that a geodesic segment is either transversal to the boundary components or is contained in it. A compact geodesic 1-suborbifold l without boundary points in Σ either is a closed geodesic in the interior or entirely in the silvered boundary component of $|\Sigma|$ or is a segment with two silvered points as the end points which are either at silvered edges or cone-points of order two. The topological interior l is either in the interior of the topological interior of $|\Sigma|$ or entirely in the boundary of $|\Sigma|$. The geometric isomorphism classes are classified by length and the topological type. Such a geodesic 1-orbifold is covered by a closed geodesic in some cover of the 2-orbifold, which is a surface. (See Section 5.1.2 also.)

Note that geodesic 1-suborbifolds are always essential. (See Section 5.2.2.2)

The *Teichmüller space* $\mathcal{T}(I)$ for a 1-orbifold I is defined as the product of the space of lengths $\mathbb{R}^+$s for all components of I. We technically define $\mathcal{T}(\emptyset)$ as a singleton.

7.2.1 *Geometric constructions.*

Recall from Chapter 5, the topological splitting and pasting constructions. In this chapter, we will do these geometrically.

Recall from Chapter 5: Let Σ be a 2-orbifold with boundary. The pasting map f is defined on open neighborhood U in an ambient open 2-orbifold S' of the union of the associated boundary components in $\partial\Sigma$. Let $\tilde{S}'$ be the universal cover of S'. Now, f satisfies the equation $\tilde{f} \circ \vartheta = \vartheta' \circ \tilde{f}$ where $\tilde{f}$ is a lift of f defined on $\tilde{U}$ the inverse image of U in $\tilde{S}'$ and ϑ and ϑ' are respective deck transformations acting on two components of the inverse images in $\tilde{S}'$ of boundary components of $\partial\Sigma$ to be pasted by f. In the hyperbolic structure case, it is necessary and sufficient that f is an isometry and the boundary components to be glued have the same length.

Recall that a *slide reflection* of $\mathbb{H}^2$ is an isometry acting on a geodesic l as a nontrivial translation but exchanges the two components of $\mathbb{H}^2 - l$.

We will describe how to construct hyperbolic structures on a larger 2-orbifold from smaller ones. Recall the type of topological constructions with 1-orbifolds. Suppose that they are boundary components of 2-orbifolds whose components have negative Euler characteristics. We can do the following operations:

- (A)(I) Pasting or crosscapping along simple closed geodesics.
- (A)(II) Silvering or folding along a simple closed geodesic.
- (B)(I) Pasting along two geodesic full 1-orbifolds.
- (B)(II) Silvering or folding along a geodesic full 1-orbifold.

Now we suppose that the simple closed curves and 1-orbifolds are geodesic and try to obtain geometric versions of the above.

Suppose that the involved 1-orbifolds are geodesic boundary components of a hyperbolic 2-orbifold. We will look at the inverse image of the 1-orbifold in the universal cover. We consider each component of the inverse image. The above operations correspond to reglueing these components with respect to each other.

- (A)(I) For pasting two closed geodesics, it is necessary and sufficient that their lengths match. Also we have an $\mathbb{R}$-amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. The cut and pasting-back constructions are so-called Fenchel-Nielsen twist. (Here the lengths of two closed geodesics have to be the same.) By taking very good covers, the inequivalence reduces to a classical fact. (See [Johnson and Millson (1987)] for example.)
- (A)(I) For cross-capping, we have a unique isometry. The isometry has to be a unique slide reflection of distance equal to the half the length of the closed geodesic. (There is no condition on the boundary component lengths.)
- (A)(II) For folding a closed geodesics, we have an $\mathbb{R}$-amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. The choice depends on the choice of two fixed points of the pasting map. The distance is half of the length of the closed geodesic. (There is no condition on the boundary component.) The inequivalence can be shown as in (A)(I) by double-covering the 2-orbifold so that the folded part lifts to a simple closed curve.
- (A)(II) For silvering, we have a unique isometry to do this; that is, the reflection about the boundary component of the universal cover will do. (There is no

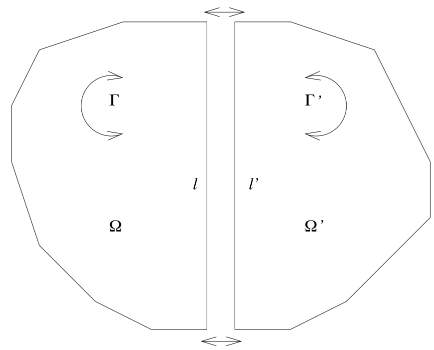

Fig. 7.1 Pasting: The actions here are isometries on the hyperbolic plane seen in the Klein model.

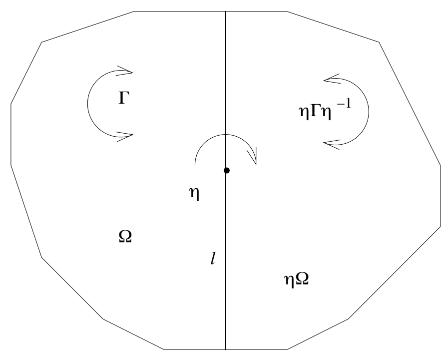

Fig. 7.2 Folding: The actions here are isometries on the hyperbolic plane seen in the Klein model.

condition on the boundary component.)

(B)(I) For pasting along two geodesic full 1-orbifolds, it is necessary and sufficient that their lengths match. We have a unique way to do this. The lengths of the orbifolds have to be the same.

(B)(II) For silvering and folding, we have a unique isometry to do this. (No condition)

7.3 The decomposition of 2-orbifolds into elementary 2-orbifolds.

Suppose that Σ is a compact hyperbolic 2-orbifold with $\chi(\Sigma) < 0$ and geodesic boundary.

Simple closed geodesics and/or simple geodesic segments with endpoints in singular locus in a hyperbolic 2-orbifolds intersect minimally; i.e., they meet the minimal number of times that they can up to isotopies: a disk bounded by two geodesic segments cannot exists in Σ.

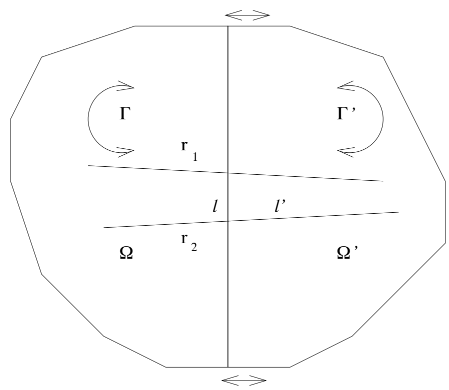

Fig. 7.3 Pasting full 1-orbifolds. The actions here are isometries on the hyperbolic plane seen in the Klein model.

Let $c_1, \dots, c_n$ be a mutually disjoint collection of essential simple closed curves or full 1-orbifolds so that the orbifold Euler characteristic of the completion of each component of $\Sigma - c_1 - \cdots - c_n$ is negative. Then $c_1, \dots, c_n$ are isotopic to simple closed geodesics or geodesic full 1-orbifolds $d_1, \dots, d_n$ respectively where $d_1, \dots, d_n$ are mutually disjoint. Here c_i is isotopic to d_i for each i, and hence c_i is a full 1-orbifold if and only if d_i is one. Also, the isotopy could be chosen simultaneously. See [Choi and Goldman (2005)] for details.

We call such a collection *decomposing* 1-orbifolds.

For example, a 2-orbifold of negative Euler characteristic based on a Möbius band admits a decomposition to an orbifold of negative Euler characteristic based on annulus by decomposing along a simple closed curve in the Möbius band.

Thus, we can decompose Σ into 2-orbifolds of negative Euler characteristic that cannot be applied any more geometric splitting operations; that is, there are no more 1-obifolds decomposing it further into 2-orbifolds with negative Euler characteristic. We call such 2-orbifolds *elementary* 2*-orbifolds*.

A neatly embedded full 1-orbifold in a 2-orbifold is of *mirror-type* if it ends at mirror points only, is of *cone-type* if it ends at cone-points only, and is of *mixed-type* if it ends at a mirror point and a cone-point.

Theorem 7.3.1 (Thurston). *Let Σ be a compact hyperbolic* 2*-orbifold with $\chi(\Sigma) < 0$ and geodesic boundary. Then there exists a mutually disjoint collection of simple closed geodesics and mirror- or cone- or mixed-type geodesic* 1*-orbifolds so that Σ decomposes along their union to a union of elementary* 2*-orbifolds with geodesic boundary or such elementary* 2*-orbifolds with some boundary* 1*-orbifolds silvered additionally.*

For the proof, see Chapter 5 in [Thurston (1977)] and the proof of Theorem 4.3 of [Choi and Goldman (2005)]. The basic strategy is as follows:

- For simplicity assume that Σ is closed and has an orientable surface as the underlying space.
- We can take a disk that contains all the cone-points of Σ unless $|\Sigma|$ is homeomorphic to a 2-sphere. If there are two cone-points of order two, then we take a full 1-orbifold l ending there. Then we decompose Σ along l to obtain a 2-orbifold with a closed geodesic boundary. Thus, we can assume that all cone-points have order > 2 with at most one exception. Unless there is just one cone-point, we can find a closed geodesic bounding all of the cone-points. Then we can decompose the surface further along the closed geodesic to obtain a pair-of-pants, an annulus with a single cone-point, or a disk with two cone-points one of which has order ≥ 3.
- For each boundary component of Σ with corner-reflectors, we can take a closed geodesic homotopic to it bounding a 2-orbifold with negative Euler-characteristic based on an annulus unless Σ is a disk bounded by silvered edges and with corner-reflectors with at most one-cone point.
- The results are much easier to decompose.

7.3.1 *Elementary 2-orbifolds.*

The underlying space of an elementary 2-orbifold has to be homeomorphic to a 2-sphere, a 2-disk, an annulus, or a pair-of-pants since otherwise there is an essential simple closed curve in the interior not freely homotopic to a boundary component just by the topology.

Note that we can also alter some boundary components by silvering it and giving corner-reflector structure of order 2 at the endpoints. The results are still considered to be an elementary 2-orbifold of the same type.

We remark that a Möbius band with some singularities is not elementary as we can use a simple closed geodesic to decompose it further.

We classify elementary 2-orbifolds up to diffeomorphisms by Theorem 5.1.1 and the above decomposition methods.

(P1) A pair-of-pants. ($\chi = -1$.)
(P2) An annulus with one cone-point of order n. ($A(;n)$, $\chi = -1 + 1/n$.)
(P3) A disk with two cone-points of orders p, q, one of which is greater than 2. ($D(;p,q)$, $\chi = -1 + 1/p + 1/q$.)
(P4) A sphere with three cone-points of order p, q, r where $1/p + 1/q + 1/r < 1$. ($\mathbf{S}^2(;p,q,r)$, $\chi = -1 + 1/p + 1/q + 1/r$)

(A1) An annulus with one boundary component a union of a singular segment and one boundary-orbifold. (We call it *two-pronged crown* and denote it by $A(2,2;)$, and we have $\chi = -1/2$. It has two corner-reflectors of order 2 if the boundary components are silvered.)

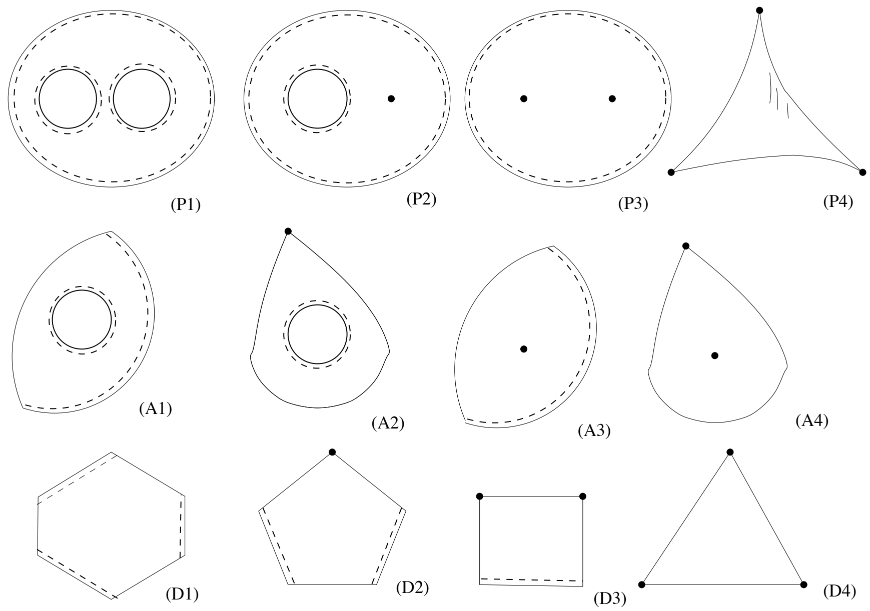

Fig. 7.4 The elementary orbifolds. Arcs with dotted arcs next to them indicate boundary components. Black points indicate singular points.

(A2) An annulus with one boundary component of the underlying space in a singular locus with one corner-reflector of order n, $n \geq 2$. (The other boundary component is a closed curve which is the boundary of the 2-orbifold. We call it a *one-pronged crown* and denote it by $A(n;\,)$, and $\chi = -(n-1)/2n$.)

(A3) A disk with one singular segment and one boundary 1-orbifold and a cone-point of order n greater than or equal to three ($D^2(2,2;\,n)$, $\chi = 1/n - 1/2$.)

(A4) A disk with one corner-reflector of order m and one cone-point of order n so that $1/2m + 1/n < 1/2$ (with no boundary orbifold). (We have $n \geq 3$ necessarily, and denote it by $D^2(m;\,n)$, and we have $\chi = -1/2 + 1/n + 1/2m$.)

(D1) A disk with three silvered edges and three boundary 1-orbifolds. No two boundary 1-orbifolds are adjacent. (*hexagon*, $D^2(2,2,2,2,2,2;\,)$, $\chi = -1/2$)

(D2) A disk with three silvered edges and two boundary 1-orbifolds on the boundary of the underlying space. Two boundary 1-orbifolds are not adjacent, and two silvered edges meet in a corner-reflector of order n, and the remaining silvered one a segment. (*pentagon*, $D^2(2,2,2,2,n;\,)$, $\chi = -1/2(1-1/n)$.)

(D3) A disk with two corner-reflectors of order p, q, one of which is greater than or equal to 3, and one boundary 1-orbifold. The singular locus of the disk is a union of three silvered edges and two corner-reflectors. (*quadrilateral*, $D^2(2,2,p,q;\,)$, $\chi = -1/2 + 1/2p + 1/2q$).)

(D4) A disk with three corner-reflectors of order p, q, r where $1/p + 1/q + 1/r < 1$ and three silvered edges (with no boundary orbifold). (*triangle*, $D^2(p,q,r;\,)$,

$\chi = -1/2 + 1/2p + 1/2q + 1/2r$.)

7.4 The Teichmüller spaces for 2-orbifolds

7.4.1 *The strategy of the proof*

We first prove:

Proposition 7.4.1. *For each elementary 2-orbifold S, $\mathcal{T}(S)$ is homeomorphic to $\mathcal{T}(\partial S)$, where $\mathcal{T}(\partial S)$ is the product of $\mathbb{R}^+$ for each component of ∂S corresponding to the hyperbolic-metric lengths of components of ∂S.*

Note here the rigidity of some closed elementary orbifolds, i.e., elementary orbifolds of type (P4), (A4), and (D4).

Then to obtain the deformation space of a bigger 2-orbifold, we use the above result about the Teichmüller spaces under geometric decompositions.

7.4.2 *The generalized hyperbolic triangle theorem*

A generalized triangle in the hyperbolic plane is one of following:

(a) A hexagon: a disk bounded by six geodesic sides meeting in right angles labeled $A, \beta, C, \alpha, B, \gamma$.
(b) A pentagon: a disk bounded by five geodesic sides labeled A, β, C, α, B where A and B meet in an angle γ, and the rest of the angles are right angles.
(c) A quadrilateral: a disk bounded by four geodesic sides labeled A, C, B, γ where A and C meet in an angle β, C and B meet in an angle α and the two remaining angles are right angles.
(d) A triangle: a disk bounded by three geodesic sides labeled A, B, C where A and B meet in an angle γ and B and C meet in an angle α and C and A meet in an angle β.

For generalized triangles in the hyperbolic plane, we have

$$\begin{aligned}
&\text{(a) } \cosh C = \frac{\cosh\alpha\cosh\beta + \cosh\gamma}{\sinh\alpha\sinh\beta} \\
&\text{(b) } \cosh C = \frac{\cosh\alpha\cosh\beta + \cos\gamma}{\sinh\alpha\sinh\beta} \\
&\text{(c) } \sinh A = \frac{\cosh\gamma\cos\beta + \cos\alpha}{\sin\beta\sinh\gamma} \\
&\text{(d) } \cosh C = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}
\end{aligned} \tag{7.1}$$

In (a), (α, β, γ) can be any positive numbers. In (b), (α, β) can be any positive numbers and γ in $(0, \pi/2]$. In (c), (α, β) can be any positive real numbers in $(0, \pi/2]$ satisfying $\alpha + \beta < \pi$, and γ any real number. In (d), (α, β, γ) can be any

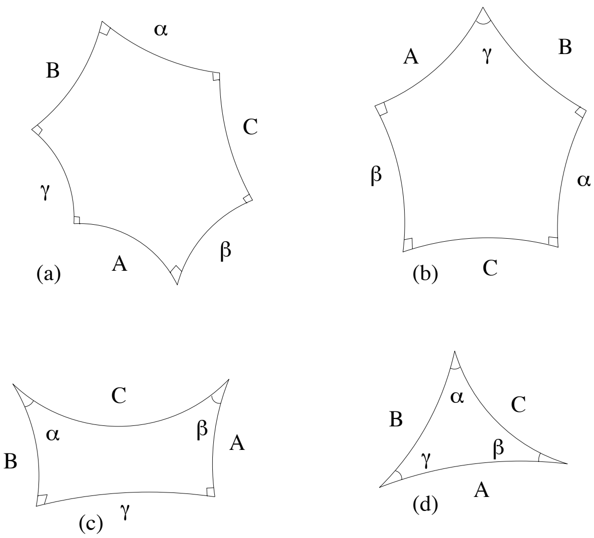

Fig. 7.5 A hexagon, a pentagon, a quadrilateral, and a triangle in the hyperbolic space with our labels.

real numbers in $(0, \pi/2]$ satisfying $\alpha+\beta+\gamma < \pi$. One can use continuity arguments and some geometry to verify these. (These facts are shown in the book [Ratcliffe (2006)] for example.)

7.4.3 *The proof of Proposition 7.4.1.*

The following lemmas imply Proposition 7.4.1.

Lemma 7.4.2. *For elementary* 2*-orbifolds of type* (D1), (D2), (D3), *and* (D4), *silvered edges are labeled by the capital letters* A, B, C. *Assign to each vertex an angle of the form* π/n *where* $n > 1$ *is an integer, for which it is a corner-reflector of that angle. Each edge labeled by Greek letters* α, β, γ *is a boundary full* 1*-orbifold. Then in cases* (D1), (D2), (D3), *and* (D4), $\mathcal{F} : \mathcal{T}(P) \to \mathcal{T}(\partial P)$ *for each of the above orbifolds* P *is a homeomorphism*; *that is,* $\mathcal{T}(P)$ *is homeomorphic to an open cell of dimension* 3*,* 2*,* 1*, or* 0 *respectively.*

Proof. For (D1), we simply notice that we can assign the boundary lengths α, β, γ freely using the equation (a). For (D2), assign $\gamma = \pi/n$. Then α and β can be freely assigned. For (D3), assign $\alpha = \pi/p$ and $\beta = \pi/q$ for $q > 2$. Then γ can be freely assigned with A and B obtained by equation (c). Then the construction of quadrilateral is done. For (D4), we assign $\alpha = \pi/p, \beta = \pi/q, \gamma = \pi/r$ where $1/p + 1/q + 1/r < 1$. Such a triangle always exists uniquely. ☐

For each of hyperbolic elementary orbifolds of type (P1),(P2),(P3), and (P4), there exists an isometric involution acting on each boundary component and the quotient orbifold is of type (D1),(D2),(D3), and (D4): The involution can be constructed explicitly by considering the fundamental domains. That is, we draw shortest geodesics between the appropriate boundary components and/or cone-points to obtain an isometric pair of hexagons, one of pentagons, one of quadrilaterals and one of triangles. Then each involution is given by sending the interior of one domain to the other fixing the geodesics.

Conversely, a hyperbolic orbifold of type (D1)-(D4) is covered by one of type (P1)-(P4) by an orientable double-cover construction of Section 4.6.1.2. The hyperbolic structure is simply obtained by local-lifts of the metrics on ones on (D1)-(D4) or induced by the covering map. (See Sections 6.1 and 2.3.1.) Hence in fact, there is a homeomorphism between the deformation spaces $\mathcal{T}(S) \to \mathcal{T}(S')$ where S double-covers S'. Furthermore $\mathcal{T}(\partial S) \to \mathcal{T}(\partial S')$ is a homeomorphism in these cases.

Hence, $\mathcal{F} : \mathcal{T}(S) \to \mathcal{T}(\partial S)$ is a homeomorphism for the type (P1)-(P4) orbifolds S.

Lemma 7.4.3. *Let S be an elementary 2-orbifold of type* (A1), (A2), (A3), *or* (A4). *Then $\mathcal{F} : \mathcal{T}(S) \to \mathcal{T}(\partial S)$ is a homeomorphism. Thus, $\mathcal{T}(S)$ is an open cell of dimension $2, 1, 1$, or 0 when S is of type* (A1), (A2), (A3) *or* (A4) *respectively. In case* (A4), *$\mathcal{T}(S)$ is a singleton.*

Proof. Here again elementary orbifolds of type (P1), (P2), (P3), and (P4) double-cover orbifolds of type (A1), (A2), (A3), and (A4). Here the involutions are different from the above ones. For (A1), (A3), and (A4), the involutions are about vertical axes and the perpendicular plane containing the vertical axis respectively. (See Figure 7.4.) For (A2) the involutions are about the essential simple closed curve passing the cone-point (See Figure 7.4.) The involutions are realized as isometries uniquely by considering the fundamental domains by drawing shortest geodesics of appropriate relative homotopy classes. This is again sufficient to imply the conclusions here. □

7.4.4 *The steps to prove Theorem 7.0.1.*

We say that a 2-orbifold Σ, each component of which has negative Euler characteristic, is in a *class* $\mathcal{P}$ if the following hold:

(i) The deformation space of hyperbolic structures $\mathcal{T}(\Sigma)$ is homeomorphic to an open cell of dimension

$$-3\chi(X_\Sigma) + 2k + l + 2n$$

where k is the number of cone-points, l the number of corner-reflectors, and n is the number of boundary full 1-orbifolds.

(ii) There exists a fibration

$$\mathcal{F} : \mathcal{T}(\Sigma) \to \mathcal{T}(\partial\Sigma)$$

with fibers homeomorphic to an open cell of dimension $\dim \mathcal{T}(\Sigma) - \dim \mathcal{T}(\partial\Sigma)$. Here $\mathcal{F}$ is the map induced by the restriction of the hyperbolic structures to the metric structures of $\partial\Sigma$.

First of all, elementary orbifolds satisfy these properties.

Let Σ be a compact 2-orbifold whose components are compact orbifolds of negative Euler characteristic, and it splits into an orbifold Σ' in $\mathcal{P}$. We suppose that (i) and (ii) hold for Σ', and show that (i) and (ii) hold for Σ. Since Σ eventually decomposes into a union of elementary 2-orbifolds where (i) and (ii) hold, we would have completed the proof of Theorem 7.0.1 by Proposition 7.4.1.

The proofs of the above statements follow by going through each of the constructions. (For details, see [Choi and Goldman (2005)].) The dimension counting here is easy by knowing that taking diagonal drops dimensions as expected.

(A)(I)(1) Let the 2-orbifold Σ'' be obtained from pasting along two closed curves b, b' in a 2-orbifold Σ'. The map resulting from splitting

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \Delta \subset \mathcal{T}(\Sigma')$$

is a principal $\mathbb{R}$-fibration, where Δ is the subset of $\mathcal{T}(\Sigma')$ where b and b' have equal lengths. Then $\mathbb{R}$ acts by the twisting the gluing of b and b' by isometries. (The operations of cutting along a closed geodesic and re-gluing with nontrivial twists are called Fenchel-Nielsen twists in the hyperbolic surface theory.) Since

$$\mathcal{F} : \mathcal{T}(\Sigma') \to \mathcal{T}(\partial\Sigma')$$

is a fibration, $\mathcal{F}|\Delta$ is a fibration onto Δ' the subset of $\mathcal{T}(\partial\Sigma')$ where b and b' have the same lengths. By forgetting about b and b', we obtain an $\mathbb{R}$-fibration $\Delta' \to \mathcal{T}(\partial\Sigma'')$. Composing with $\mathcal{SP}$, we obtain a fibration

$$\mathcal{F} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\partial\Sigma'')$$

with fibers homeomorphic to an open cell of the desired dimension.

(A)(I)(2) Let Σ'' be obtained from Σ' by cross-capping. The resulting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a homeomorphism. There is an $\mathbb{R}$-fibration $\mathcal{T}(\partial\Sigma') \to \mathcal{T}(\partial\Sigma'')$ by forgetting the boundary component involved in cross-capping. By composing with $\mathcal{SP}$, we obtain the fibration

$$\mathcal{T}(\Sigma'') \to \mathcal{T}(\partial\Sigma'').$$

(A)(II)(1) Let Σ'' be obtained from Σ' by silvering. The clarifying map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a homeomorphism.

(A)(II)(2) Let Σ'' be obtained from Σ' by folding a boundary closed curve l'. The unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a principal $\mathbb{R}$-fibration.

For each of these, the fibration designated by $\mathcal{F}$ can be shown to exist as in (A)(I)(2) above.

(B)(I) Let Σ'' be obtained by pasting along two full 1-orbifolds b and b' in Σ'. The splitting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \Delta \subset \mathcal{T}(\Sigma')$$

is a homeomorphism where Δ is a subset of $\mathcal{T}(\Sigma')$ where the lengths of b and b' are equal. $\mathcal{F}$ is again shown to exist as in (A)(I)(1).

(B)(II) Let Σ'' be obtained by silvering or folding a full 1-orbifold. The clarifying or unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \to \mathcal{T}(\Sigma')$$

is a homeomorphism. $\mathcal{F}$ is again shown to exist as in (A)(I)(2).

7.5 Notes

The Teichmüller theory for 2-orbifolds was created by Thurston in Chapter 5 of [Thurston (1977)] and were written up also in [Matsumoto and Montesinos-Amilibia (1991); Ohshika (1985)]. (See also [Kapovich (2009)].) The materials here are from the papers [Choi (2004); Choi and Goldman (2005)]. We also mention that for examples of the study of 3-dimensional orbifolds and their geometric structures, one could see the books [Cooper, Hodgson, and Kerckhoff (2000); Boileau, Maillot, Porti (2003)].

Chapter 8

Deformation spaces of real projective structures on 2-orbifolds of negative Euler characteristic: An introduction

The main purpose here is to introduce real projective structures on 2-orbifolds to the readers. The theoretical aspects are not completely written here but the readers can find them in articles mentioned. Additionally, we discuss the computational aspect of this theory in a more detailed way.

First, we will give some introduction to real projective structures on orbifolds with relationships to hyperbolic structures. Next, we give some examples of real projective structures on annuli, a torus with one-hole and the orbifolds based on a triangle.

We also give a survey of real projective structures on manifolds (and orbifolds) from a historical point of view: the Hilbert metrics, the topological work of Choi (1994a,b) and Goldman (1990), the gauge theory point of view using Higgs bundles, the Hitchin's conjecture and the group theoretical work of Benoist (2001).

Next, we study real projective structures on 2-orbifolds of negative Euler characteristic. We present Theorem 8.3.1 characterizing the topology of the deformation spaces of convex real projective structures on 2-orbifolds of negative Euler characteristic. Next, we study the relationship between the deformation spaces and the Hitchin-Teichmüller components of the spaces of $\mathbb{PGL}(3,\mathbb{R})$-characters in Section 8.3.1. We try to now understand the deformation spaces of real projective structures on orbifolds. We discuss the geometric constructions available for such structures and the elementary 2-orbifolds and their real projective structures using the work of Goldman (1990). From these, we should be able to prove Theorem 8.3.1 characterizing the topology of the deformation space of real projective structures on 2-orbifolds. However, we do not present the full detail.

8.1 Introduction to real projective orbifolds

Let X be the real projective plane $\mathbb{RP}^n$ and G the group $\mathbb{PGL}(n+1,\mathbb{R})$ of *collineations*, i.e., projective automorphisms of $\mathbb{RP}^n$. An $\mathbb{RP}^n$*-structure* or *real projective structure* on an n-dimensional orbifold Σ is an $(\mathbb{RP}^n, \mathbb{PGL}(n+1,\mathbb{R}))$-structure on Σ. Two $\mathbb{RP}^n$-structures on Σ are equivalent if an isotopy from the identity map

I_Σ of Σ induces one from the other. A *real projective orbifold* or a $\mathbb{RP}^n$*-orbifold* is an orbifold with this structure. The deformation space $\mathbb{RP}^n(\Sigma)$ of $\mathbb{RP}^n$-structures on Σ is the space of equivalence classes of $\mathbb{RP}^n$-structures with appropriate topology.

A hyperbolic space can be represented by the Klein model. We have a standard ellipsoid in $\mathbb{RP}^n$ bounding a convex open domain Ω: This set corresponds to the space of rays in a convex cone in $\mathbb{R}^{n+1}$ given by the equation

$$x_0 > \sqrt{x_1^2 + \cdots + x_n^2}.$$

Then the hyperbolic isometry group is precisely the subgroup of $\mathbb{PGL}(n+1, \mathbb{R})$ acting on Ω, and a discrete group Γ of isometries becomes a discrete group of projective automorphisms. The quotient Ω/Γ has a real projective structure. These are called *hyperbolic real projective structures.* (See Section 3.1.6 for details.)

Given a hyperspace in $\mathbb{RP}^n$, we recall that the complement has the natural affine structure whose geodesic structure extends to projective ones. We call this the *affine subspace.* (See Section 3.1.4 for details.) A domain Ω in $\mathbb{RP}^n$ is *convex* if it forms a convex domain in the affine subspace or equals $\mathbb{RP}^n$ itself. (We can prove this by taking an inverse image in $\mathbb{R}^{n+1}$ with components that are convex cones and we use supporting hyperplanes. See the book [Berger (2009)] for details.) An open domain Ω is *properly convex* if it is contained in some bounded convex closed domain in an affine subspace of $\mathbb{RP}^n$. For a convex domain Ω, this is equivalent to the fact that Ω does not contain a complete 1-dimensional affine space, i.e., a complete affine line. If the boundary of a convex domain Ω does not contain a straight segment, then Ω is said to be *strictly convex.*

In fact, for any convex open domain Ω and Γ acting on Ω cocompactly and properly discontinuously, we obtain a real projective 2-orbifold.

Define $\mathbf{S}^n = (\mathbb{R}^{n+1} - \{O\})/\sim$ where $v \sim w$ iff $v = kw$ for $k > 0$. $\mathbf{S}^n$ has a real projective structure as a double cover of $\mathbb{RP}^n$. A *real projective sphere* $\mathbf{S}^n$ is $\mathbf{S}^n$ with the real projective structure and has a group of projective automorphisms $\mathrm{Aut}(\mathbf{S}^n)$ isomorphic to the group $\mathbb{SL}_\pm(n+1, \mathbb{R})$ of linear maps of determinant ± 1.

A closed real projective orbifold is said to *convex* if any arc in a relative homotopy class can be homotoped to a line relative to the end points. It is *properly convex* if it does not contain a complete affine line, i.e., a subspace projectively isomorphic to a complete real line. A closed real projective orbifold is convex if and only if it is diffeomorphic to Ω/Γ or $\mathbf{S}^n/\Gamma$ for a convex domain Ω in an affine subspace and a real projective automorphism group Γ acting on it or on the real projective sphere $\mathbf{S}^n$ properly discontinuously. It is *properly convex* if and only if it is diffeomorphic to Ω/Γ where Ω is a properly convex domain (Choi, 1994a,b).

There are closed convex real projective orbifolds that are not hyperbolic, which we will state later in detail.

A closed 2-orbifold Σ with $\chi(\Sigma) < 0$ with an $\mathbb{RP}^2$-structure is convex if and only if it is projectively diffeomorphic to the quotient of a properly convex domain in an affine patch by a properly discontinuous action of a group of projective automorphisms.

An arc in Σ that is locally a line is called *geodesic* or *projective geodesic*. If each component $\partial\Sigma$ is locally a line, Σ is said to have geodesic boundary. A closed curve in Σ whose lift develops into a line connecting the unique attracting and repelling fixed points of its holonomy is said to be a *principal* closed geodesic.

When $\partial\Sigma \neq \emptyset$, boundary components are required to be principal geodesic.

Let us discuss for $\mathbb{RP}^2$. A projective automorphism is said to be *positive hyperbolic* if it is diagonalizable and the maximum and minimum modulus eigenvalues are positive and have multiplicity one. Let A be a positive hyperbolic projective automorphism. The conjugation invariants of a positive hyperbolic element A are eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with

$$0 < \lambda_1 < \lambda_2 < \lambda_3, \lambda_1\lambda_2\lambda_3 = 1.$$

Thus, A has three fixed points in $\mathbb{RP}^2$, that are noncollinear, one of which is an attracting fixed point, another one is a repelling one, and the last one is a saddle-type one. There are three A-invariant lines bounding four triangles in $\mathbb{RP}^2$.

The space of invariants for positive hyperbolic matrices is given by $0 < \lambda_1, 0 < \lambda_1 < \lambda_2, \lambda_1\lambda_2^2 < 1$. Here λ_1 and λ_2 completely characterize the conjugacy classes. We denote the region by D, homeomorphic to an open disk. Another way to describe this space is by the *Goldman invariants* of A given by $\lambda = \lambda_1, \tau = \lambda_2 + \lambda_3$. These satisfy

$$0 < \lambda < 1, \frac{2}{\sqrt{\lambda}} < \tau < \lambda + \frac{1}{\lambda^2}.$$

In general, a projective automorphism of $\mathbb{RP}^n$ is represented by a matrix with determinant ± 1 where the largest norm eigenvalue is positive. A projective automorphsim is *positive proximal* if the largest and smallest norm eigenvalues of the corresponding matrix are positive and of multiplicity one.

The following is a summary of the most general results about the geometry of convex real projective manifolds and orbifolds, following [Benoist (2008)]. (Historically, these results were obtained by Kuiper (1954), Benzecri (1960, 1962), Koszul (1965, 1968) and so on.) Recall that group is *hyperbolic* if its Caley graph is Gromov hyperbolic, and a closed curve is *essential* if the fundamental group of the closed curve injects.)

Theorem 8.1.1. *Let Ω/Γ be a closed n-dimensional real projective orbifold M where Ω is a properly convex domain in an affine subspace of $\mathbb{RP}^n$ and Γ is a discrete group of real projective automorphisms acting on Ω and is a hyperbolic group.*

- *Ω is strictly and properly convex.*
- *The holonomy of each essential closed curve is positive proximal with exactly two fixed points in* $\mathrm{bd}\Omega$ *which are an attracting fixed point and a repelling one and acts on the open line in Ω connecting the two fixed points.*
- *Each essential closed curve in M is realized by a closed geodesic.*

- *Suppose that the essential closed curve is homotopic to a simple closed curve. If M is an orientable* 2*-orbifold, then there exists a unique closed geodesic isotopic to it which is a principal closed geodesic or it double covers a segment with two endoints in singularites of order two. If Ω/Γ is not orientable, the closed geodesic is either simple or it double-covers a segment as above or a simple closed geodesic.*
- $\mathrm{bd}\Omega$ *is $C^{1,\alpha}$ and is an ellipsoid if C^2.* (Benzecri)

Note that these hold for each hyperbolic surfaces as well where the corresponding group to Γ is considered a subgroup of $\mathbb{PSO}(1,2)$ and Ω is the interior of a conic.

The following theorem states in the surface case, convex ones are the most important ones. (Choi, 1994a,b).

Theorem 8.1.2. *Let Σ be a compact orientable real projective surface with principal geodesic or empty boundary and $\chi(\Sigma) < 0$. Then Σ has a collection of mutually disjoint simple closed geodesics the components of whose complement have closures that are properly convex real projective surfaces with principal geodesic boundary of negative Euler characteristic or elementary annuli.*

(See Section 8.1.1 for the definition of elementary annuli.)

From this, we obtained later in the paper [Choi and Goldman (1997)].

Theorem 8.1.3. *The deformation space of real projective structures on a closed orientable surface of genus g, $g > 1$, is an infinite countable union of open cells of dimension $16g - 16$.*

8.1.1 ***Examples of real projective 2-orbifolds.***

We recall the terminology and facts in Section 3.1.4:

8.1.1.1 *Elementary annuli*

Let ϑ be a collineation represented by a diagonal matrix with distinct positive eigenvalues. Then it has three fixed points in $\mathbb{RP}^2$: an attracting fixed point of the action of $\langle\vartheta\rangle$, a repelling fixed point, and a saddle-type fixed point. Three lines passing through two of them are ϑ-invariant, as are four open triangles bounded by them. Choosing two open sides of an open triangle ending at an attracting fixed point or a repelling fixed point simultaneously, their union is acted properly and freely upon by $\langle\vartheta\rangle$. The quotient space is diffeomorphic to an annulus. The $\mathbb{RP}^2$-surface projectively diffeomorphic to the quotient space is said to be an *elementary annulus*. (See the left part of Figure 8.1.)

A principal geodesic boundary is one connecting an attracting fixed point of ϑ with a repelling one. This definition is independent of orientation. There is a unique principal geodesic component among the two components. The other component is

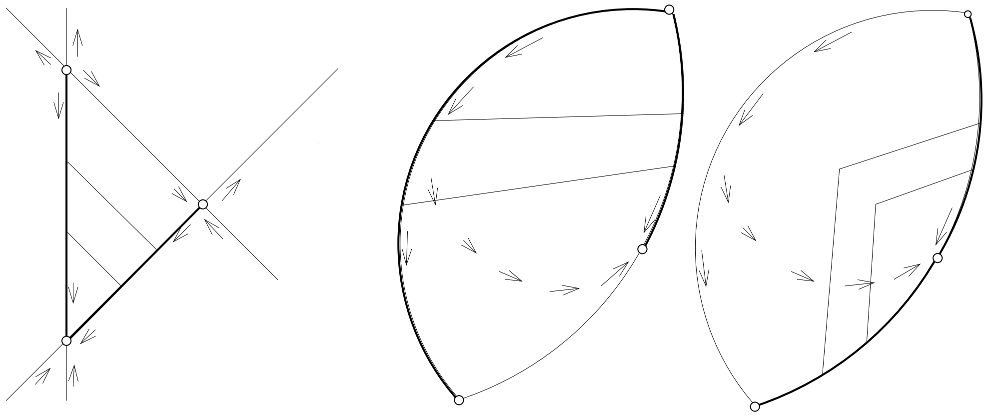

Fig. 8.1 Elementary annuli of hyperbolic and two quasi-hyperbolic types as quotients of domains and actions on them. The thicker lines indicate the included boundary components.

said to be *weak*.

A *pasting* of two boundary components of real projective surfaces with geodesic boundary can be described as attaching and projectively identifying thin regular neighborhoods of the geodesics in some ambient open surface. The necessary condition for pasting to take place is that the holonomy of the generator of the fundamental group of the boundary component is conjugate to the holonomy of the corresponding generator for the other boundary component. This is also the sufficient condition when the boundary components are principal geodesic. (Also, if both boundary components have complete affine lifts, it is also sufficient.)

A real projective annulus with geodesic boundary can be obtained by pasting the above elementary annuli along geodesic boundary of same types.

Goldman showed that each annulus with principal geodesic boundary is obtained by pasting elementary annuli. (See also the article [Sullivan and Thurston (1983)].) In fact, we can draw an arc in $\mathbb{RP}^2$ in a certain manner as in Figure 8.2 and obtain an annulus. This corresponds to the pasting construction.

One can also have an annulus with geodesic boundary where ϑ is *quasi-hyperbolic*, i.e., represented by a non-diagonalizable matrix with two positive eigenvalues. It has two fixed points in $\mathbb{RP}^2$. One is a repelling or attracting fixed point, say x and the other y. A 1-dimensional subspace m passing through x and y is invariant by ϑ. ϑ has an attracting and repelling fixed points x and y on m. There is another ϑ-invariant subspace l of dimension one with unique fixed point y on it. ϑ acts as a translation on $l - \{y\}$ identified with a complete affine line. (See Figure 8.1.)

Let L be a component of $\mathbb{RP}^2 - l - m$. An elementary annulus is the quotient of $L \cup m - \{x, y\}$ or the quotient of $L \cup l' \cup m_1$ for a unique component m_1 of $m - \{x, y\}$ and the component l' of $l - \{y\}$ adjacent to L so that a segment s connecting a

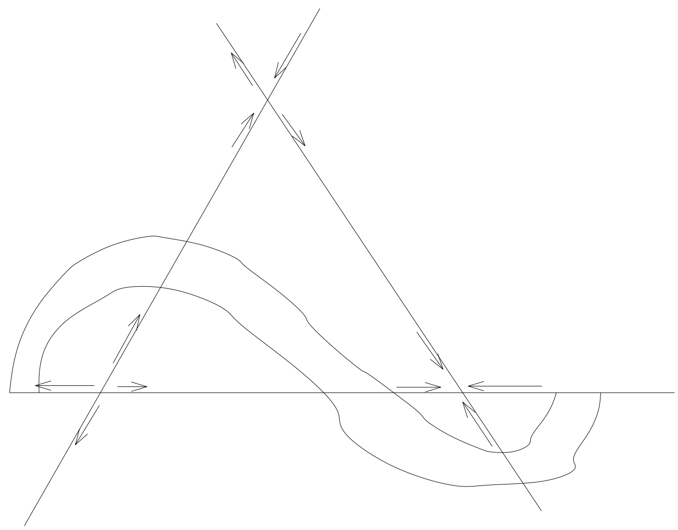

Fig. 8.2 Any immersed arc so that the directions of the action arrows do not change as it crosses the invariant lines corresponds to an annulus with geodesic boundary. To see this, simply act by g and the two arcs will bound a strip glued to an annulus. This was discovered by Goldman (1977). For a fixed holonomy, one can classify them by a free semigroup of rank two.

point of m_1 to l' is disjoint from $\vartheta(l)$. (Note that a wrong choice would give us a non-Hausdorff space.)

We note that the elementary annuli of quasi-hyperbolic type do not occur in convex real projective closed surfaces or 2-orbifolds of negative Euler characteristic. (See [Choi (1994b)].)

8.1.1.2 π-*Annuli*

Let ϑ be a hyperbolic projective automorphism. Take two adjacent ϑ-invariant triangles with three open sides of them all ending in an attracting fixed point or a repelling fixed point. Then the quotient of the union by $\langle\vartheta\rangle$ is diffeomorphic to an annulus. The projectively diffeomorphic surfaces are said to be π-*annuli* (Choi, 1994a,b).

A *reflection* in $\mathbb{RP}^2$ is an involution fixing a line and an isolated point. A reflection in a projective space is determined uniquely by a line of fixed points and a fixed point outside the line with a matrix conjugate to a diagonal matrix with entries $1, 1, -1$.

There is a reflection sending one triangle to the other inducing an order-two group. The quotient map is an orbifold map, and the quotient space carries an orbifold structure so that one boundary component is made of mirror points. Thus, the π-annulus is a double of an elementary annulus with a silvered boundary component.

Now, let ϑ be a quasi-hyperbolic projective automorphism. Then we define one of the two types of annuli to be a π-annulus: that is, an elementary annulus of quasi-hyperbolic type with the lifts of two boundary components ending at a common point.

Also the pasting of two elementary annuli of quasi-hyperbolic type along the boundary components corresponding to the complete affine lines is another type of a π-annulus. (See Figure 8.3.)

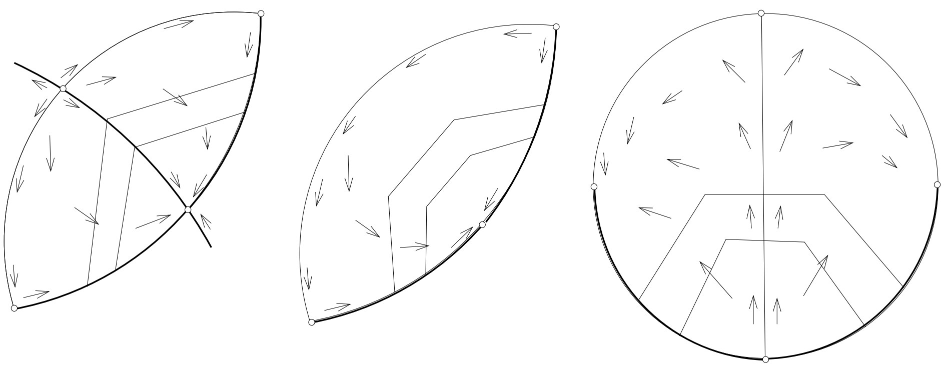

Fig. 8.3 The π-annuli of hyperbolic type and two of quasi-hyperbolic type.

We mention that Nagano and Yagi (1974) and Goldman (1977) essentially classified the real projective structures on annuli, Möbius bands, tori and Klein bottles. To this date, the work was not yet generalized to 2-orbifolds of Euler characteristic zero. The topology of the deformation spaces are still unknown. See [Baues and Goldman (2005)] also.

8.1.1.3 *An example: a bending torus with a disk removed*

Consider $\mathbb{H}^2$ as the inside of a standard ellipse in $\mathbb{RP}^2$ given by the set of null vectors in $\mathbb{R}^3$ with the standard Lorentzian metric from the quadratic form $x_0^2 - x_1^2 - x_2^2$.

Take an orientable hyperbolic 2-orbifold S. Then $S = \mathbb{H}^2/\Gamma$ for a discrete subgroup $\Gamma \subset \mathbf{Isom}(\mathbb{H}^2) = \mathbb{PSO}(1,2) \subset \mathbb{PGL}(3,\mathbb{R})$. Thus, S is identified with a quotient space of a convex open domain in $\mathbb{RP}^2$. Here, $\mathbb{H}^2$ is represented by the Klein model; i.e., it is identified with the standard unit disk in $\mathbb{RP}^2$.

Let S be an orientable hyperbolic closed 2-orbifold or a hyperbolic compact 2-orbifold with geodesic boundary. We can deform this to a parameter of nonhyperbolic real projective surfaces by so-called "bending" first discovered by Thurston (1977). Again denote by $\pi_1(S)$ the group of deck transformations of the universal cover $\tilde{S}$ of S.

An essential simple closed curve is homotopic to a simple closed geodesic by Theorem 8.1.1. Let S contain a simple closed geodesic c.

We have that $\tilde{S}$ is identified with $\mathbb{H}^2$. The inverse image L of c is a disjoint union of straight lines ending in $\mathrm{bd}\mathbb{H}^2$. Take a component l and the other components are of form $g(l)$ for $g \in \pi_1(S)$. Let the cyclic group generated by $\gamma \in \pi_1(S)$ acts on l so that it corresponds to the covering $l \to c$, where l and c are oriented along γ.

We can find an element of $\mathbb{PGL}(3,\mathbb{R})$ namely an element of $\mathbf{Isom}(\mathbb{H}^2)$ that preserves $\mathbb{H}^2$ and sends l to any segment. Therefore, we choose a projective coordinate system so that l has endpoints $[0,1,1]$ and $[0,-1,1]$. Then γ is now represented as a matrix with eigenvalues $\lambda, 1/\lambda$, and 1 at respective points $[0,1,1]$, $[0,-1,1]$, and $[1,0,0]$ for $\lambda > 1$.

Then any projective transformation η with a diagonalizable matrix with eigenvalues $a, 1/(ab)$, and b respectively at the above points commutes with γ. For each component $g(l)$ of L for $g \in \pi_1(S)$, we glue the relative closure of the left adjacent component C of $\mathbb{H}^2 - L$ with a right adjacent component C' by η.

This construction amounts to the following "cut and paste" construction: Cut S by a simple closed curve c and obtain $S - c$. Complete it by the induced path metric to $\bar{S}$ with two boundary components c_1' and c_2'. Find an open ambient real projective 2-orbifold S' containing c_1' and c_2'. Now, η induces a real projective diffeomorphism η' from an open neighborhood N_1 of c_1' in S' to one N_2 of c_2' in S'. Let S_1 be the copy of $S - c$ in S', we take $S_1 \cup N_1 \cup N_2$ in S', and we identify N_1 an N_2 by η'. The resulting 2-orbifold S' is still diffeomorphic to S.

This construction is said to be a *projective bending* of S. For each nonidentity η, we obtain a projective bending. For a parameter of η, we obtain a parameter of bendings. The resulting projective 2-orbifold S' is still properly convex (Goldman, 1990).

In fact, we could have started with any orientable compact properly convex 2-orbifold with geodesic boundary. Each simple closed curve is realized as a simple closed geodesic.

As a specific example, we consider a torus with one hole, i.e., a genus-one orientable hyperbolic surface with one boundary component where S decomposes into one pair-of-pants. We obtain various pictures of deformations and the convex domains that cover the deformed real projective surface.

Let us explain some explicit construction that can be obtained by some computer algebra systems. We did the computation with Mathematica™.

A hyperbolic pair-of-pants with geodesic boundary is first constructed: In $\mathbb{H}^2$ find a geodesic l_1 passing $[0,0,1]$ with endpoints $[1,0,1]$ and $[-1,0,1]$ and another geodesic l_2 passing $[0,0,1]$ with endpoints $[0,1,1]$ and $[0,-1,1]$. We find a matrix A acting on l_1 with eigenvalues $\lambda, 1/\lambda$, and 1 for $\lambda > 1$ with respective fixed points $[-1,0,1]$, $[1,0,1]$, and $[0,1,0]$ and K acting on l_2 with eigenvalues $\mu, 1/\mu, 1$ for $\mu > 1$ with respective eigenvectors $[0,1,1]$, $[0,-1,1]$, and $[1,0,0]$. Let $B = KA^{-1}K^{-1}$.

For λ, μ sufficiently large, one can make $A^{-1}(l_2)$ and $B(l_2)$ are disjoint geodesics. $C := BA$ has an invariant geodesic l_3 meeting $A^{-1}(l_2)$ and $B(l_2)$ at distinct points and containing the shortest segment between them. Then

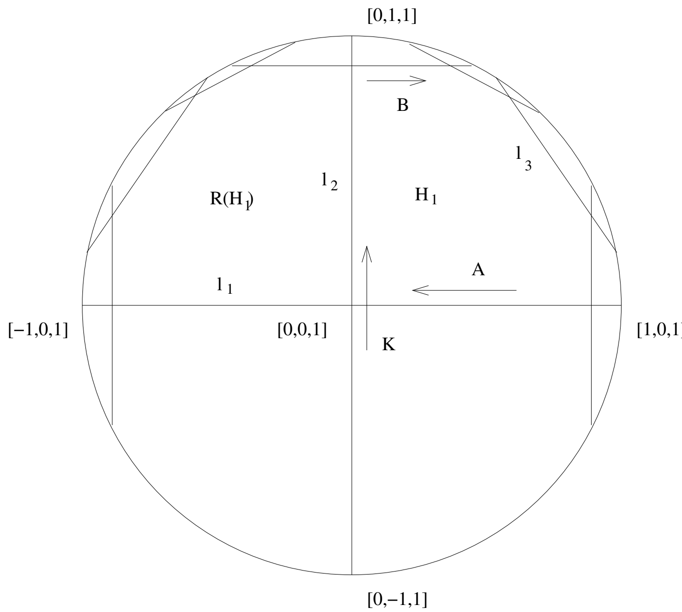

Fig. 8.4 The diagram for a torus with one-hole.

$l_1, l_2, K(l_1), B(l_2), l_3, A^{-1}(l_2)$ bound a hexagon H_1. Reflect H_1 using a reflection R fixing l_2 and $[1, 0, 0]$. The free group $F_2 := \langle A^2, B^2\rangle$ acts freely and properly discontinuously on $\mathbb{H}^2$ and $\bigcup_{g\in F_2} g(H_1 \cup R(H_1))$ forms a universal cover of a pair-of-pants P with geodesic boundary corresponding to A^2, B^2, A^2B^2. This constructs one pair-of-pants. (Actually, this is a double of a hexagonal 2-orbifold with three silvered edges and three boundary components.)

Consider the group generated by A^2, B^2, K. Then this generates a group Γ and $\mathbb{H}^2/\Gamma$ is diffeomorphic to a torus with one hole. (We are attaching the boundary component corresponding to A with that of B by K here.)

Let η be a matrix commuting with A with eigenvalues $\delta, \eta, 1/(\delta\eta)$ and eigenvectors at $[-1, 0, 1]$ $[0, 1, 0]$, and $[1, 0, 1]$. The bending by η corresponds to changing K to $K\eta$. This gives us a two-parameter space of bendings. (See Bending1.nb and Bending2.nb)

Another computations of bending constructions are given by Pat Hooper. See `http://merganser.math.gvsu.edu/~david/~reed03/~projects/hooper/` containing an applet of bendings with parameters. (This was a student project in “Mathematical Graphics: Introduction to Java” in the MSRI Summer School - Reed College, July 13 - July 26, 2003. `http://www.math.ubc.ca/~cass/msri-summer-school/`)

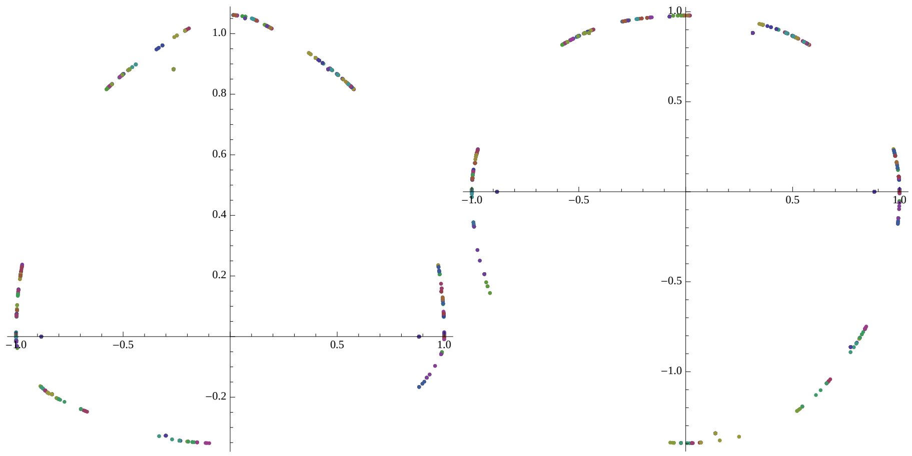

Fig. 8.5 The orbits of bent real projective structures.

8.1.1.4 *Projective triangle reflection groups due to Kac and Vinberg*

Next, we discuss the examples due to Kac and Vinberg (1967). These examples provided the first class of real projective 2-orbifolds and surfaces that are properly convex but not hyperbolic.

Consider a hyperbolic triangle reflection group. $\mathbb{H}^2$ contains a hyperbolic triangle with vertices v_1, v_2, v_3 with respective angles $\pi/p, \pi/q, \pi/r$ satisfying $1/p + 1/q + 1/r < 1$. Let R_1, R_2, and R_3 denote the projective reflections at the edges opposite v_1, v_2, and v_3 respectively. Then we obtain

$$(R_1R_2)^r = \mathrm{I}, (R_2R_3)^p = \mathrm{I}, \text{ and } (R_3R_1)^q = \mathrm{I}. \tag{8.1}$$

A triangle determines the sides of the reflections. We choose the reflection points p_1, p_2, p_3 for the sides e_1, e_2, e_3 respectively. Call the resulting reflections R_1, R_2, and R_3 respectively. They need to satisfy the relations 8.1. Putting the vertices v_1, v_2, and v_3 to $[1, 0, 0], [0, 1, 0]$, and $[0, 0, 1]$ respectively, we obtain the matrices of R_1, R_2, and R_3 as below:

$$R_1 = \begin{pmatrix} -1 & 0 & 0 \\ 2b_1 & 1 & 0 \\ 2c_1 & 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 2a_2 & 0 \\ 0 & -1 & 0 \\ 0 & 2c_2 & 1 \end{pmatrix}, \text{ and } R_3 = \begin{pmatrix} 1 & 0 & 2a_3 \\ 0 & 1 & 2b_3 \\ 0 & 0 & -1 \end{pmatrix} \tag{8.2}$$

where we have $p_1 = (-1, b_1, c_1), p_2 = (a_2, -1, c_2)$, and $p_3 = (a_3, b_3, -1)$.

The necessary and sufficient condition for R_1R_2 to be of order r for $r \geq 2$ is that

$$4a_2b_1 - 1 = \mathrm{tr}(R_1R_2) = \mathrm{tr}(R_1R_2)^{-1} = 1 + 2\cos 2\pi/r \text{ if } r > 2$$

and $a_2 = 0, b_1 = 0$ for $r = 2$. Thus, we obtain

$$4a_2b_1 = 2 + 2\cos 2\pi/r \text{ if } r > 2 \quad \text{or } a_2 = 0, b_1 = 0 \text{ if } r = 2, \tag{8.3}$$

$$4b_3c_2 = 2 + 2\cos 2\pi/p \text{ if } p > 2 \quad \text{or } b_3 = 0, c_2 = 0 \text{ if } q = 2, \text{ and} \tag{8.4}$$

$$4a_3c_1 = 2 + 2\cos 2\pi/q \text{ if } q > 2 \quad \text{or } a_3 = 0, c_1 = 0 \text{ if } r = 2. \tag{8.5}$$

From this, we obtain that if $p, q, r > 2$, then there is a one-parameter space of solutions of the above equations. This gives us a one-parameter space of real projective structures on the disk-orbifold with corner-reflectors of orders p, q, r. We mention that a single parameter value corresponds to the hyperbolic structure (Vinberg, 1971; Kac and Vinberg, 1967).

If any of p, q, r is 2, then there is just one solution. This corresponds to the hyperbolic structure. We computed some examples in TrianglegroupProj.nb and TrianglegroupProj2.nb. See Figure 8.6 for developing images.

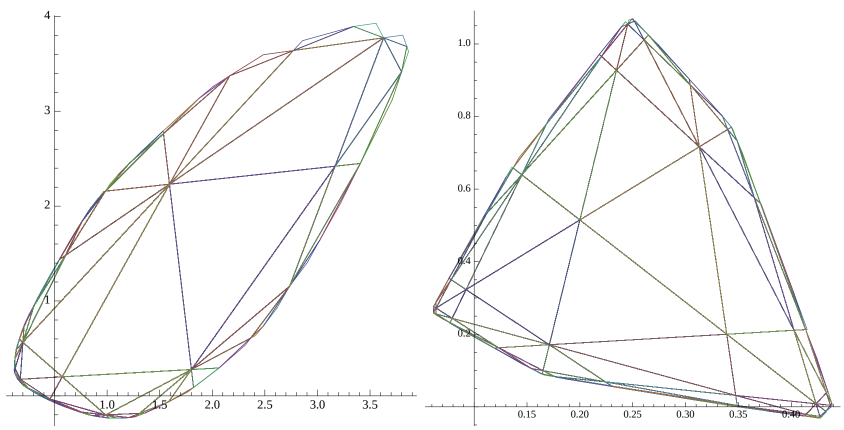

Fig. 8.6 The developing images of two triangle reflection 2-orbifolds of order $(3, 5, 5)$ and $(3, 3, 4)$.

8.2 A survey of real projective structures on surfaces of negative Euler characteristic.

In this section, we sketch some histories of real projective structures.

Historically, Cartan (1924) defined projectively flat structures or real projective structures on manifolds as structures that are "geodesically Euclidean but with no metrics". More precisely, a projectively flat structure on a manifold is given as a torsion-free projectively flat affine connection. "Projectively flat" here means that the connection has same geodesics structures as Euclidean metrics up to reparametrizations.

Later Ehresmann [Pradines (2007)] and Thurston (1977) identified this structure as being a maximal atlas of charts to $\mathbb{RP}^n$ with transition maps in $\mathbb{PGL}(n+1, \mathbb{R})$; that is, it is a geometric structure modeled on $(\mathbb{RP}^n, \mathbb{PGL}(n+1, \mathbb{R}))$. (For an introduction, see the article [Sullivan and Thurston (1983)].)

Kuiper (1954) first studied the convex real projective structures on closed surfaces and showed that they are either a real projective sphere, a real projective plane, a torus or a Klein bottle that is a quotient space of an open triangular domain in $\mathbb{RP}^2$ or is a quotient surface of genus $g, g > 1$ of a properly and strictly convex open domain in $\mathbb{RP}^2$ by a discrete group of projective automorphisms. Benzecri (1960) later generalized this to n-dimensional convex real projective manifolds.

Koszul (1965) showed that convexity is preserved for a closed convex real projective manifold if one deformed the projective structures by a sufficently small amount.

As shown above, Kac and Vinberg (1967) were first to find examples of convex projective surfaces that are not hyperbolic by deforming. The examples are based on Coxeter groups. (See Section 8.1.1.)

Kobayashi (1984) studied metrics on projective manifolds: Given a connected real projective manifold$M$, he considers projective maps

$$l \subset \mathbb{RP}^1 \to M$$

from a bounded interval l and take maximal ones. Using the Hilbert metric of l, he defines the Kobayashi metric. Kobayashi metric is a metric if and only if M has no complete real lines if and only if M is projectively isomorphic to Ω/Γ where Ω is a properly convex domain in $\mathbb{RP}^n$.

In this case, the Kobayashi metric is Finsler and a Hilbert metric given by

$$d(p, q) = |\log[o, s, q, p]|$$

for $p, q \in \Omega$ and o and s are end points of the maximal line containing p, q and o, q separates p, s (See Section 3.1.4.) If $\Omega = \mathbb{H}^n$, the metric is the standard hyperbolic metric. (See Figure 8.7.)

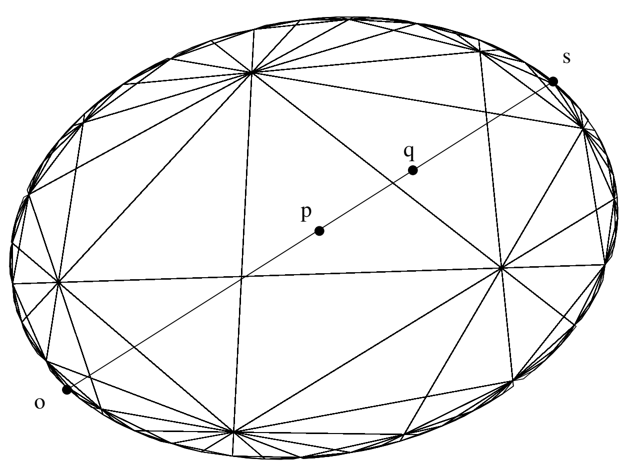

Fig. 8.7 The figure illustrating the cross ratios and the Hilbert metric. The boundary is conic here so that the metric is really a hyperbolic one.

8.2.1 *Topological work*

Nagano and Yagi (1974) classified affine structures on tori, and Goldman (1977) classified projective structures on annuli with geodesic boundary in his senior thesis (Sullivan and Thurston, 1983).

There is a construction called *grafting*: On a closed orientable convex real projective surface of negative Euler characteristic, an essential simple closed curve is homotopic to a simple closed projective geodesic. (See the article [Choi (1994a)] for details.) We cut along the geodesic and complete it to obtain a surface with two new geodesic boundary components. We paste by projective maps the above annuli with principal geodesic boundary to the boundary components with conjugate holonomies. That is, one can insert this type of annuli into a closed convex projective surface to obtain non-convex projective surfaces.

The convex decomposition theorem [Choi (1994a,b)] shows that a closed orientable real projective surface of negative Euler characteristic can be constructed from a closed convex orientable convex real projective surface of negative Euler characteristic by grafting.

Goldman (1990) classified convex projective structures on closed orientable surfaces. Let Σ be a closed orientable surface of genus $g > 1$ and let $\mathcal{CD}(\Sigma)$ denote the deformation space of convex real projective structures on Σ. Then $\mathcal{CD}(\Sigma)$ is homeomorphic to an open cell of dimension $16g - 16 = -8\chi(\Sigma)$. He gave an explicit parameterization to construct back any real projective surface diffeomorphic to Σ. This and Theorem 8.1.2 imply Theorem 8.1.3. Here, the classification is a constructive one.

8.2.2 *The gauge theory and projective structures.*

Atiyah and Bott (1983) studied self-dual connections on surfaces. Corlette (1988) showed that the space of flat connections for manifolds can be realized as the space of harmonic maps to certain symmetric space bundles.

8.2.3 *Hitchin's conjecture and the generalizations.*

Let G be the adjoint group of the split real form of a complex simple group. Hitchin (1992) used the Higgs fields on principal G-bundles over surfaces to obtain parametrizations of flat G-connections over surfaces.

Let Σ be a closed 2-orbifold of negative Euler characteristic. Recall from Chapter 6, the space of homomorphisms

$$\mathbf{Hom}(\pi_1(\Sigma), G)/G.$$

We denote by $\mathbf{Hom}^+(\pi_1(\Sigma), G)$ the subspace of representations which act completely reducibly on Lie algebra of G. It includes the subspace of irreducible representations. (A representation acts *completely reducibly* if every invariant subspace has a complementary invariant subspace. See [Hitchin (1992)] and [Corlette (1988)].)

A Higgs bundle is a pair (V, Φ) where V is a holomorphic vector bundle over a fixed Riemann surface Σ and Φ is a holomorphic section of $\mathrm{End} V \otimes K$ where K is the canonical line bundle. A Teichmüller space $\mathcal{T}(\Sigma)$ is mapped locally homeomorphically by hol to a component of the space

$$\mathbf{Hom}^+(\pi_1(\Sigma), \mathbb{PSL}(2,\mathbb{R}))/\mathbb{PSL}(2,\mathbb{R})$$

of conjugacy classes of Fuchsian discrete faithful irreducible representations by Theorem 6.2.2. A hyperbolic surface naturally corresponds to a conjugacy class of a discrete faithful and irreducible representation $\Gamma \to \mathbb{PSL}(2,\mathbb{R})$ for its fundamental group Γ. Thus, hol is a homeomorphism to the component.

The *Hitchin-Teichmüller component* is a component of

$$\mathbf{Hom}^+(\pi_1(\Sigma), G)/G$$

containing the compositions of form

$$\pi_1(\Sigma) \to \Gamma \to \mathbb{PSL}(2,\mathbb{R}) \to G. \tag{8.6}$$

where the first map is a Fuchsian representation and the second map is the natural irreducible representation $\mathbb{PSL}(2,\mathbb{R}) \to G$ of Kostant. (See Section 4 of [Hitchin (1992)].)

To find a flat connection on a given Higgs bundle, we solve for a unitary connection A

$$F_A + [\Phi, \Phi^*] = 0$$

given a holomorphic section $\Phi \in \mathrm{End} V \otimes K$. The theory of holomorphic sections of holomorphic bundles shows that the Hitchin-Teichmüller component is homeomorphic to an open cell of dimension $(2g-2)\dim G^r$.

Now we restrict our attention to $\mathbb{PGL}(n,\mathbb{R})$. For $n > 2$, Hitchin proved that

$$\mathbf{Hom}^+(\pi_1(\Sigma), \mathbb{PGL}(n,\mathbb{R}))/\mathbb{PGL}(n,\mathbb{R})$$

has three connected components if n is odd and six components if n is even.

A Fuchsian representation is a representation $\pi_1(\Sigma) \to \mathbb{PSL}(2,\mathbb{R})$ with image Γ such that $\mathbb{H}^2/\Gamma$ is homeomorphic to Σ. $\mathbb{PSL}(2,\mathbb{R})$ can be identified as an irreducible subgroup of $\mathbb{PSL}(n,\mathbb{R})$.

A Hitchin representation in $\mathbb{PSL}(n,\mathbb{R})$ is a representation which deforms to a Fuchsian representation, i.e., the ones of form

$$\Gamma \to \mathbb{PSL}(2,\mathbb{R}) \to \mathbb{PSL}(n,\mathbb{R}), \tag{8.7}$$

i.e., those in the Hitchin-Teichmüller component.

A convex projective surface is of form Ω/Γ. Hence, there is a representation $\pi_1(\Sigma) \to \Gamma$ determined only up to conjugation by $\mathbb{PGL}(3,\mathbb{R})$. This gives us a map

$$\mathrm{hol} : \mathcal{CD}(\Sigma) \to \mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3,\mathbb{R}))/\mathbb{PGL}(3,\mathbb{R}).$$

This map was known to be a local-homeomorphism by Ehresmann and Thurston as in Section 6.2.3.2 and is injective to an open subset as shown by Goldman (1990).

Recall that $\mathbb{PGL}(3,\mathbb{R})$, $\mathbb{PSL}(3,\mathbb{R})$, and $\mathbb{SL}(3,\mathbb{R})$ are isomorphic to one another. When Σ is orientable, we obtain a local homeomorphism

$$\mathrm{hol} : \mathcal{CD}(\Sigma) \to \mathbf{Hom}(\pi_1(\Sigma), \mathbb{SL}(3,\mathbb{R}))/\mathbb{SL}(3,\mathbb{R}).$$

The map is in fact a homeomorphism onto the Hitchin-Teichmüller component as shown by Choi and Goldman (1997). (See Section 6.2.3.)

This result was naturally but unexpectedly extended in the early 2000s to the higher-Teichmüller theory developed by Labourie (2006) and Burger, Iozzi, Labourie, Wienhard (2005); however, we will not elaborate on this rather large and rapidly growing topic.

8.2.4 *Group theory and representations*

As stated earlier, Benzecri (1960), Kac and Vinberg (1967), and Koszul (1965) started to study the deformations of representations $\Gamma \to \mathbb{PGL}(n+1,\mathbb{R})$ from the discrete faithful representation $\Gamma \to \mathbb{PSO}(n,1)$ corresponding to hyperbolic manifolds. There is a well-known deformation due to Thurston called *bending* for projective and conformally flat structures: Given a totally-geodesic submanifold S of codimension one in a convex real projective manifold M so that the holonomy homomorphism h restricts in $\pi_1(S)$ to one fixing a point in $\mathbb{RP}^n$, we have a centralizing element η in $\mathbb{PGL}(n+1,\mathbb{R})$ in a one-parameter family of such elements. We can remove S from M and complete it to obtain a manifold with two copies of S as boundary component. Using the centralizing elements, we can re-glue in one-parameter ways. (See Section 8.1.1.3.)

Johnson and Millson (1987) found that certain hyperbolic manifolds have deformation spaces of projective structures that are singular by studying one with many totally geodesic submanifolds codimension one meeting transversally. (They also worked out this for conformally flat structures.)

An element γ of $\mathbb{GL}(m,\mathbb{R})$ is *proximal* if there is an eigenvalue of multiplicity one which is of largest modulus among eigenvalues. Recall that γ is positive proximal if γ is proximal and the largest modulus eigenvalue is positive. A subgroup Γ of $\mathbb{GL}(m,\mathbb{R})$ is *positive proximal* if every proximal element is positive proximal. (This means that it has a pair of an attracting and a repelling fixed point in $\mathbb{RP}^{m-1}$.) We say that Γ *divides* Ω if its image in $\mathbb{PGL}(m,\mathbb{R})$ acts on a properly convex domain $\Omega \subset \mathbb{RP}^{m-1}$ properly discontinuously but not necessarily freely so that the quotient space is compact.

Theorem 8.2.1. *Let* Γ *be an irreducible torsion-free subgroup of* $\mathbb{GL}(m,\mathbb{R})$. *Then* Γ *divides a strictly convex domain* Ω *if and only if* Γ *is positive proximal and discrete. If* Ω *is not a domain bounded by a conic, then* Γ *maps to a Zariski dense subgroup in* $\mathbb{PGL}(m,\mathbb{R})$ *under the projection* $\mathbb{GL}(m,\mathbb{R}) \to \mathbb{PGL}(m,\mathbb{R})$.

This is proved by Benoist (2000).

The recent work of Benoist (papers "Convex divisibles I-IV") proves the following theorem. (See also the survey article [Benoist (2001)].)

Theorem 8.2.2. *Let Γ be a discrete torsion-free subgroup of $\mathbb{GL}(m, \mathbb{R})$ dividing an open convex domain Ω in $\mathbb{RP}^{m-1}$. Let C be the corresponding cone on $\mathbb{R}^m$. The projectivization Γ_0 of Γ is the isomorphic image group in $\mathbb{PGL}(m, \mathbb{R})$. Then the following holds*

- *One of the following is true exclusively:*
  - *C is a product, i.e., a product of irreducible cones in subspaces,*
  - *C is homogeneous; i.e., Γ_0 is Zariski dense in a copy of $\mathbb{PSO}(1, 1-m)$ in $\mathbb{PGL}(m, \mathbb{R})$ acting on Ω transitively,*
  - *Γ_0 is Zariski dense in $\mathbb{PGL}(m, \mathbb{R})$.*
- *If the virtual center of Γ_0 is trivial, i.e., every finite-index subgroup of Γ_0 has a trivial center, then*

  $$E_{\Gamma_0} = \{\rho \in H_{\Gamma_0} | \text{ The image of } \rho \text{ divides a convex open domain in } \mathbb{RP}^{m-1}\}$$

  is closed in

  $$H_{\Gamma_0} := \mathbf{Hom}(\Gamma_0, \mathbb{PGL}(m, \mathbb{R})).$$

  The openness was obtained by Koszul (1965).
- *Let Γ_0 be as above. Then the following conditions are equivalent:*
  - *Ω is strictly convex.*
  - bdΩ *is C^1.*
  - *Γ is a hyperbolic group.*
  - *The geodesic flow on Ω/Γ is Anosov.*

Benzecri (1960) showed that the boundary of Ω is C^1 or is an ellipsoid for closed convex projective manifolds. (See also [Goldman (1988)].)

This completes our survey. However, there were further developments of significance by Cooper, Long, and Thistlethwaite (2007, 2006) which we cannot cover here.

8.3 Real projective structures on 2-orbifolds of negative Euler characteristic.

We begin the study of the deformation spaces of real projective structures on 2-orbifolds.

Recall the orbifold Euler characteristic of orbifolds, a signed sum of the number of open cells with weights given by 1 divided by the orders of groups associated to the open cells. Let Σ be a connected compact 2-orbifold with $\chi(\Sigma) < 0$. The subspace of the deformation space $\mathbb{RP}^2(\Sigma)$ of $\mathbb{RP}^2$-structures on Σ corresponding to convex ones is denoted by $\mathcal{CD}(\Sigma)$ and the closed subspace corresponding to hyperbolic projective structures is denoted by $\mathcal{T}(\Sigma)$, identified as the Teichmüller space of Σ as defined by Thurston (1977). Then we see that $\mathcal{T}(\Sigma)$ is a subspace of $\mathcal{CD}(\Sigma)$, and $\mathcal{CD}(\Sigma)$ is an open subset of $\mathbb{RP}^2(\Sigma)$.

Theorem 8.3.1 (Choi, Goldman). *Let Σ be a compact 2-orbifold with $\chi(\Sigma) < 0$ and $\partial\Sigma = \emptyset$. Then the deformation space $\mathcal{CD}(\Sigma)$ of convex $\mathbb{RP}^2$-structures on Σ is homeomorphic to an open cell of dimension*

$$-8\chi(X_\Sigma) + (6k_c - 2b_c) + (3k_r - b_r)$$

where X_Σ is the underlying space of Σ, k_c is the number of cone-points, k_r the number of corner-reflectors, b_c the number of cone-points of order two, and b_r the number of corner-reflectors of order two.

Let us denote by $C_{\mathcal{T}}(\Sigma)$ the unique component of

$$\mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3,\mathbb{R}))$$

containing the holonomy homomorphisms of hyperbolic $\mathbb{RP}^2$-structures on Σ. Then $C_{\mathcal{T}}(\Sigma)$ is also a component of

$$\mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3,\mathbb{R}))$$

in the part

$$\mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3,\mathbb{R}))^+$$

where $\mathbb{PGL}(3,\mathbb{R})$ acts properly, and $C_{\mathcal{T}}/\mathbb{PGL}(3,\mathbb{R})$ is the *Hitchin-Teichmüller component* as described by Hitchin (1992). We prove:

Theorem 8.3.2. *Let Σ be a closed 2-orbifold with negative Euler characteristic. Then*

$$\mathrm{hol} : \mathcal{CD}(\Sigma) \to C_{\mathcal{T}}(\Sigma)/\mathbb{PGL}(3,\mathbb{R})$$

is a homeomorphism, and $C_{\mathcal{T}}(\Sigma)$ consists of discrete faithful representations of $\pi_1(\Sigma)$.

Corollary 8.3.3. *The Hitchin-Teichmüller component $C_{\mathcal{T}}(\Sigma)/\mathbb{PGL}(3,\mathbb{R})$ is homeomorphic to an open cell of the dimension as above in Theorem 8.3.1.*

We study small 2-orbifolds with rigid hyperbolic structures; i.e., ones with the Teichmüller spaces consisting of singletons.

Corollary 8.3.4.

- *The sphere Σ with cone-points of order p, q, r satisfying $p \leq q \leq r, 1/p + 1/q + 1/r < 1$ has as its Teichmüller space a single point.*
  - *If $p = 2$, then so is $\mathcal{CD}(\Sigma)$.*
  - *If $p > 2$, then $\mathcal{CD}(\Sigma)$ is homeomorphic to $\mathbb{R}^2$.*

- *Let Σ be a 2-orbifold whose underlying space is a disk and with one cone point of order p and a corner-reflector of order q so that $1/p + 1/2q < 1/2$ has as its Teichmüller space a single point.*
  - *If $q = 2$, then so is $\mathcal{CD}(\Sigma)$.*
  - *If $q > 2$, then $\mathcal{CD}(\Sigma)$ is homeomorphic to $\mathbb{R}$.*
- *Let Σ be a 2-orbifold whose underlying space is a disk and with three corner-reflectors of order $p \leq q \leq r$, $1/p + 1/q + 1/r < 1$. Then $\mathcal{T}(\Sigma)$ is a single point.*
  - *If $p = 2$, then so is $\mathcal{CD}(\Sigma)$.*
  - *If $p > 2$, then $\mathcal{CD}(\Sigma)$ is homeomorphic to $\mathbb{R}$.*

8.3.1 *Real projective* 2*-orbifolds and the Hitchin-Teichmüller components*

From now on, we are concerned with explaining the proof of Theorem 8.3.1 but we will not prove it actually.

By an $\mathbb{RP}^2$-*structure* or *projectively flat structure* on a 2-orbifold Σ we mean an $(\mathbb{RP}^2, \mathbb{PGL}(3, \mathbb{R}))$-structure on Σ. From now on, we look at $\mathbb{RP}^2$-orbifolds, that is, 2-orbifolds with $\mathbb{RP}^2$-structures. Here, we require that the boundary components of a surface with a real projective structure are always principal geodesic.

We define the deformation spaces of $\mathbb{RP}^2$-structures on 2-orbifolds, describe local properties, and define convex $\mathbb{RP}^2$-structures (when the 2-orbifolds are boundaryless).

We discuss the relationship between the $\mathbb{RP}^2$-structures and holonomy representations. First, we deduce that the deformation space is Hausdorff from the corresponding property of the holonomy representation variety. Next, we discuss convex $\mathbb{RP}^2$-structures. We show that the deformation space of convex $\mathbb{RP}^2$-structures on a 2-orbifold is an open subset of the full deformation space. We identify the deformation space of convex $\mathbb{RP}^2$-structures on a 2-orbifold with a subset of the space of conjugacy classes of representations of its fundamental group using the above relationship.

8.3.1.1 *Types of Singularities*

Recall that an automorphism of $\mathbb{RP}^2$ is a *reflection* if its matrix is conjugate to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

A reflection has a line of fixed points and an isolated fixed point, which is said to be the *reflection point*. An automorphism of $\mathbb{RP}^2$ is said to be a *rotation of order*

n, $n = 2, 3, \ldots$, if its matrix is conjugate to

$$\begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} & 0 \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A rotation has a unique isolated fixed point, called a *rotation point*, and an invariant line. A one-parameter family of invariant ellipses fills the complement in $\mathbb{RP}^2$ of the rotation point and the invariant line. A rotation of order two is a reflection also and conversely.

For $\mathbb{RP}^2$-orbifolds, the singular points have the neighborhoods with model open sets and finite group actions corresponding to one of the following:

(i) A mirror point: An open disk in $\mathbb{RP}^2$ meeting a line of fixed points of a reflection.
(ii) A cone-point of order n: An open disk in $\mathbb{RP}^2$ containing a rotation point of the rotation of order n.
(iii) A corner-reflector of order n: An open disk in $\mathbb{RP}^2$ containing the intersection point of the lines of fixed points of two reflections g_1 and g_2 generating a dihedral group of order $2n$.

8.3.1.2 *The deformation spaces and holonomy*

We recall some facts from the general (G, X)-structures. (See Chapter 6 for details) We define the deformation space $\mathbb{RP}^2(\Sigma)$ of $\mathbb{RP}^2$-structures on a connected 2-orbifold Σ with principal geodesic boundary as follows (assuming Σ is connected and has empty boundary): Give the C^1-topology to the set $\hat{\mathcal{S}}(\Sigma)$ of all developing pairs $(\mathbf{dev}, h)$ on $\tilde{\Sigma}$. Two pairs $(\mathbf{dev}, h)$ and $(\mathbf{dev}', h')$ are *equivalent under isotopy* if there exists a self-diffeomorphism f of the universal cover $\tilde{\Sigma}$ of Σ commuting with the deck transformations so that $\mathbf{dev}' = \mathbf{dev} \circ f$ and $h' = h$. (We can easily show that $\hat{\mathcal{S}}(\Sigma)$ is homeomorphic to $\mathcal{S}(\Sigma)$ in Section 6.2.1.) We denote by $\mathbb{RP}^{2*}(\Sigma)$ the space of equivalence classes with the quotient topology.

The pairs $(\mathbf{dev}, h)$ and $(\mathbf{dev}', h')$ are equivalent under the $\mathbb{PGL}(3, \mathbb{R})$-action if there exists an element g of $\mathbb{PGL}(3, \mathbb{R})$ so that $\mathbf{dev}' = g \circ \mathbf{dev}$ and $h'(\cdot) = gh(\cdot)g^{-1}$. The quotient space of $\mathbb{RP}^{2*}(\Sigma)$ under the $\mathbb{PGL}(3, \mathbb{R})$-equivalence relation is denote by $\mathbb{RP}^2(\Sigma)$.

Another interpretation of the deformation space is to consider all $\mathbb{RP}^2$-structures on Σ and quotient by the isotopies. One can easily obtain a one-to-one correspondence between the above two spaces.

If two $\mathbb{RP}^2$-structures are distinct up to isotopy, they are *isotopically distinct*. Isotopically distinct $\mathbb{RP}^2$-structures represent different points in the deformation spaces. An example is a pair of $\mathbb{RP}^2$-orbifolds with non-conjugate holonomy homomorphisms (see [Choi (2004)] for details).

By forgetting $\mathbf{dev}$ from the pair $(\mathbf{dev}, h)$, we obtain an induced map

$$\mathrm{hol}' : \mathbb{RP}^{2*}(\Sigma) \to \mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3, \mathbb{R}))$$

to the space of homomorphisms of $\pi_1(\Sigma)$ since the isotopy does not change the holonomy homomorphism.

Since Σ is a compact 2-orbifold, we see that $\pi_1(\Sigma)$ is a finitely presented group by Corollary 4.7.2. From now on, we denote

$$H(\Sigma) = \mathbf{Hom}(\pi_1(\Sigma), \mathbb{PGL}(3, \mathbb{R}))$$

for the $\mathbb{R}$-algebraic subset of $\mathbb{PGL}(3, \mathbb{R})^n$ satisfying the relations corresponding to the relations of the presentation of $\pi_1(\Sigma)$ where n is the number of the generators of $\pi_1(\Sigma)$.

Choi (2004) shows that the map $\mathcal{H}'$ is a local homeomorphism since $\pi_1(\Sigma)$ is finitely presented. (See Section 6.2.3 for detail.)

Let U^n denote the open subset of $\mathbb{PGL}(3, \mathbb{R})^n$ consisting of $(X_1, \ldots, X_n)$ such that no line in $\mathbb{R}^3$ is simultaneously invariant under $X_1, \ldots, X_n$. The $\mathbb{PGL}(3, \mathbb{R})$-action is proper and free on the set

$$U(\Sigma) := H(\Sigma) \cap U^n$$

(Goldman, 1990).

Theorem 8.3.5. *Let Σ be a connected closed 2-orbifold with $\chi(\Sigma) < 0$. Then $\mathbb{RP}^2(\Sigma)$ has the structure of Hausdorff real analytic variety modeled on $U(\Sigma)/\mathbb{PGL}(3, \mathbb{R})$, and the induced map*

$$\mathrm{hol} : \mathbb{RP}^2(\Sigma) \to U(\Sigma)/\mathbb{PGL}(3, \mathbb{R})$$

is a homeomorphism onto an open subset.

8.3.2 *Understanding the deformation space of real projective structures*

8.3.2.1 *The deformation space of 2-orbifolds*

Here, we discuss how to use the above facts to study the deformation space of a given 2-orbifold, in a manner parallel to the Teichmüller space cases. We do not provide the complete proofs here. (See [Choi and Goldman (2005)] for more details.)

Recall that a principal geodesic is a geodesic that lifts to an arc developing to a straight line connecting an attracting fixed point and a repelling fixed point of its holonomy automorphism. A full 1-orbifold is *principal* if an inverse image of it in the universal cover develop into a straight line joining an attracting fixed point and a repelling fixed point of the composition of holonomies of the two reflections.

Recall that the projective invariant of a principal closed geodesic c of a real projective 2-orbifold is given by a point in the domain $D(c)$. The projective invariant of a principal full 1-orbifold c is given as the cross-ratio of the four points in its lift given by the two reflection points and the end points. Hence, we let $D(c)$ be identified with $\mathbb{R}^+$ by taking the absolute values of the logarithms of the cross-ratios.

As in Chapter 7, we can decompose an orientable compact convex real projective 2-orbifold Σ with $\chi(\Sigma) < 0$ and principal geodesic boundary by a mutually disjoint family of essential simple closed principal geodesics or geodesic principal full 1-orbifolds $c_1, \dots, c_n$ so that the orbifold Euler characteristic of the completion of each component of $\Sigma - c_1 - \cdots - c_n$ is negative. The completed 2-orbifolds have all principal geodesic boundary. Moreover, these 2-orbifolds are *elementary* in the sense that we cannot apply the above steps any more.

8.3.2.2 *Geometric constructions.*

To understand this, let S be a 2-orbifold with principal boundary components. The pasting map f is defined on open neighborhood U of the union of the associated boundary components in an ambient open 2-orbifold S' where f satisfies the equation $\tilde{f} \circ \vartheta = \vartheta' \circ \tilde{f}$ where $\tilde{f}$ is a lift of f defined on $\tilde{U}$ the inverse image and ϑ and ϑ' are respective deck transformations acting on components of the inverse images in $\tilde{S}'$ of boundary components of S to be pasted by $\tilde{f}$.

In the real projective structures, it is sufficient that f is a locally projective map in some ambient real projective surface, the boundary components are principal, and ϑ and ϑ' have the same projective invariants described above.

Actually, we can think of the above condition as $f \circ h(c) = h(c') \circ f$ where $h(c)$ and $h(c')$ are holonomy of the closed curves c and c' and the boundary components are principal: The equation is necessary since if the pasting succeeded, then the equation holds. The additional principal geodesic condition is then the sufficient condition.

The geodesics and the full 1-orbifolds are principal always when we are splitting and pasting. (Actually, we need this condition so that the result of pasting is properly convex when the initial real projective 2-orbifolds are properly convex. See [Choi and Goldman (2005)] or [Goldman (1990)])

We describe how to construct convex real projective structures on a larger 2-orbifold from smaller ones. Recall the type of topological constructions with 1-orbifolds from Chapter 7. Suppose that they are boundary components of 2-orbifolds whose components have negative Euler characteristics.

(A)(I) Pasting or crosscapping along a simple closed curve.
(A)(II) Silvering or folding along a simple closed curve.
(B)(I) Pasting along two full 1-orbifolds.
(B)(II) Silvering or folding along a full 1-orbifold.

Now we suppose that the simple closed curves and 1-orbifolds are geodesic and try to obtain geometric versions of the above.

Suppose that the involved 1-orbifolds are geodesic boundary components of a properly convex real projective 2-orbifold with principal geodesic boundary.

(A)(I) For pasting two closed geodesics, we have an $\mathbb{R}^2$-amount of real projective automorphisms to do this. They would create convex real projective structures inequivalent in the deformation spaces. (Here the invariants of two closed geodesics have to be the same.) The possible projective automorphisms B satisfy $AB = BA'$ where A and A' are holonomies of the two closed geodesics. The equation becomes $AB' = B'A$ since we can define $A' = P^{-1}AP$ for $B' = BP^{-1}$ and an invertible P. The solution space of B' is the space of commuting matrices of A and hence is parametrized by $\mathbb{R}^2$.

(A)(I) For cross-capping, we have a unique pasting map. The map must be a real projective automorphism that preserves the orientation of the boundary component but reverses the normal direction and whose second power is the holonomy of the boundary component. The equation is $AB = BA$ and $B^2 = A$ where A is the holonomy of the principal boundary component and B is the pasting map. There is no condition on A other than its positive hyperbolicity. B has eigenvalues that are square roots of those of A and one of middle absolute value has a negative eigenvalue. B is determined since A is positive hyperbolic.

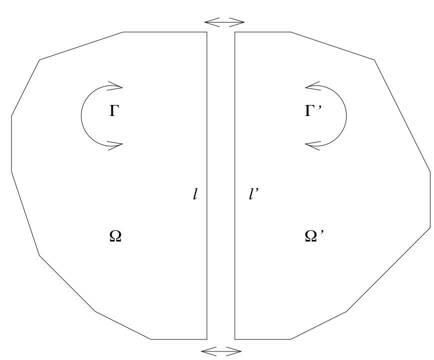

Fig. 8.8 (A)(I) Pasting of two closed principal geodesics

(A)(II)(i) For folding a closed geodesics, we have an $\mathbb{R}$-amount of real projective automorphism f to do this. They would create convex real projective structures inequivalent in the deformation space. The choice depends on the choice of two fixed points of the pasting map. The equation is $AB = BA^{-1}$ and $B^2 = \mathrm{I}$ and B fixes a point p of the principal geodesic l invariant under A. B is uniquely determined by the fixed point p and vice-versa since B switches the two eigenvectors of A and acts on the eigenspace of A of dimension-one as a reflection. Here, $BAB^{-1} = A^{-1}$ and $A \sim A^{-1}$. Therefore, A has eigenvalues $\lambda, 1, \lambda^{-1}$. This is a restriction on the holonomy type of boundary components that we can do folding on.

(A)(II)(ii) For silvering, we have a unique real projective automorphism of order 2 that reverses the normal direction but fixes the points of the boundary component and commutes with the holonomy of the boundary component. The equation is $AB = BA$ and $B^2 = \mathrm{I}$ and B fixes each point of the principal geodesic of A and acts on an eigenspace of dimension one as a reflection. Then B is a unique reflection.

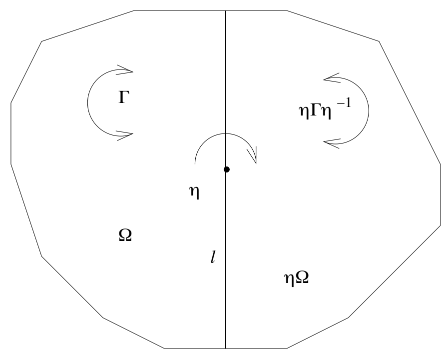

Fig. 8.9 (A)(II)(i) Folding a principal closed geodesic.

(B)(I) For pasting along two geodesic full 1-orbifolds, we have an $\mathbb{R}$-parameter ways to do this, and the invariants of the 2-orbifolds have to be the same: The boundary full 1-orbifolds have holonomy A_1 and A_2 associated with each boundary points, i.e., silvered points where $A_1^2 = A_2^2 = \mathrm{I}$. A_1A_2 represents a closed curve around the full 1-orbifold lifting to a simple closed curve in a double cover of our properly convex real projective 2-orbifold. Hence, it is positive hyperbolic. The equation is $A_iB = BA_i'$ for $i = 1, 2$ and A_i and A_i' are the holonomy elements of the generators of the local groups of the two boundary points of the full 1-orbifolds, acting on the principle geodesics. Moreover, $A_1A_2B = BA_1'A_2'$ since A_1A_2 and $A_1'A_2'$ are corresponding closed paths to become homotopic after pasting. (This corresponds to the cross-ratio invariants of the two full 1-orbifolds being the same.) Since $A_i' = P^{-1}A_iP$ for $i = 1, 2$, the above equation becomes $A_iB' = B'A_i$ and $A_1A_2B' = B'A_1A_2$ for $B' = BP^{-1}$ and $i = 1, 2$. Since A_1A_2 is positive hyperbolic, the solution space is homeomorphic to $\mathbb{R}$ as B' fixes each point of the principal geodesic of A_1A_2 and acts on the eigenspace of A_1A_2 whose corresponding point in $\mathbb{RP}^2$ is not on the geodesic.

(B)(II)(i) For silvering, we have a unique real projective automorphism since there is a unique projective automorphism commuting with the reflections at the end and fixing each point of the boundary component. The equation is $A_iB = BA_i$ and $A_1A_2B = BA_1A_2$ and $B^2 = \mathrm{I}$ and B fixes each point of the principal geodesic fixed by A_1A_2 and acts on an eigenspace of A_1A_2. Here, B is uniquely determined.

(B)(II)(ii) For folding, the full 1-orbifold ends at boundary points. The projective automorphism must send the the full 1-orbifold to itself and make the boundary segments to extend each other where they are sent. There is a unique such automorphism. The equation is $A_1B = BA_2$ and $B^2 = \mathrm{I}$ and B fixes a point of the principal geodesic. Here, B is uniquely determined. (This is similar to (A)(II).)

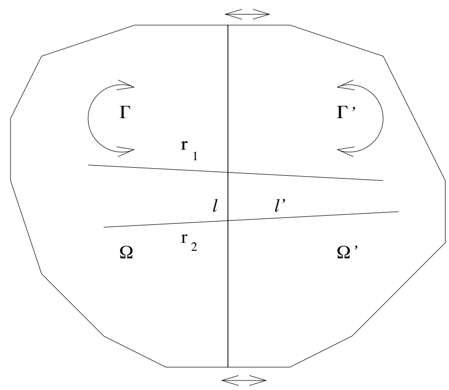

Fig. 8.10 (B)(II)(i) Pasting full 1-orbifolds

8.3.2.3 *Elementary 2-orbifolds and their real projective structures.*

To prove Theorem 8.3.1, we need to study the deformation space of elementary orbifolds and use results in Section 8.3.2.2. The details are in [Choi and Goldman (2005)]. We partially discuss the deformation space of convex real projective structures on elementary orbifolds with principal geodesic boundary. We discuss more about the computational aspects.

8.3.2.4 *A pair-of-pants*

We first discuss a pair-of-pants P. The deformation space was first studied by Goldman (1990). The geodesic boundary components of a convex real projective surface P with principal geodesic boundary are first oriented by a boundary orientation.

Recall that $D(c)$ for a boundary component c of a real projective surface denote the space of invariants (λ, τ) satisfying

$$0 < \lambda < 1 \text{ and } \frac{2}{\sqrt{\lambda}} < \tau < \lambda + \frac{1}{\lambda^2}.$$

Given a hyperbolic automorphism ϑ of $\mathbb{RP}^2$, we have that the invariant for ϑ is given by taking the smallest eigenvalue and the sum of the two other eigenvalues. We define $\mathcal{CD}(\partial P)$ as the product space $\prod_{i=1}^{3} D(c_i)$ where c_i are boundary components of P. Goldman (1990) proved that

$$\mathcal{F} : \mathcal{CD}(P) \to \mathcal{CD}(\partial P)$$

for a pair-of-pants P is a principal $\mathbb{R}^2$-fibration for a pair-of-pants P where $\mathcal{F}$ is given by sending the structure to the invariants of $h(c_i)$ for the boundary components c_1, c_2, c_3 of P.

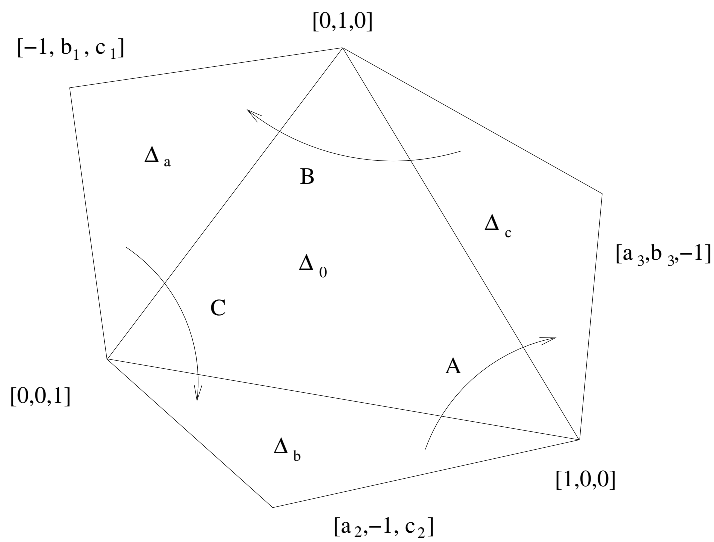

Fig. 8.11 The four adjacent triangles used to understand the convex real projective pair-of-pants

We explain this: Give P an orientation and the induced orientation on ∂P as well. There is a lamination with three leaves that tend to the boundary components in its end and it wraps around each boundary component in the reverse direction to the orientation. We can straighten each leaf so that it is a geodesic. This is accomplished by the fact that P is convex. $P - \partial P$ is a union of two triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ bounded by three lines and vertices removed. In the universal cover $\tilde{P}$, we have a tessellation by these triangles. Under the developing map, each triangle is mapped to a triangle with vertices removed in $\mathbb{RP}^2$. Take one triangle T_0 and adjacent ones T_1, T_2, T_3. Notice that T_0 is in one class of triangles corresponding to $\mathcal{T}_1$ or $\mathcal{T}_2$ and T_1, T_2, and T_3 correspond to the other one.

There exists a deck transformation A sending T_1 to T_2 and B sending T_2 to T_3 and C sending T_3 to T_1. We have $CBA = \mathrm{I}$. In fact, A, B, C correspond to closed curves homotopic to the boundary components in the oriented direction. Since the developing map is a homeomorphism, A, B, and C correspond to elements of $\mathbb{PGL}(3, \mathbb{R})$.

Note the isomorphism $\mathbb{SL}(3, \mathbb{R})$ with $\mathbb{PGL}(3, \mathbb{R})$. We think of A, B, C as matrices of determinant 1 abusing notations.

We can put T_0 to a standard triangle with vertices: $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$ by a projective automorphism and then we obtain:

- T_1 has vertices $[-1, b_1, c_1]$, $[0, 1, 0]$, and $[0, 0, 1]$,
- T_2 has vertices $[1, 0, 0]$, $[a_2, -1, c_2]$, and $[0, 0, 1]$, and
- T_3 has vertices $[1, 0, 0]$, $[0, 1, 0]$, and $[a_3, b_3, -1]$.

Here $b_1, c_1, a_2, c_2, a_3, b_3 > 0$. This position is not canonical. We can still act by transformations with diagonal matrices. Thus, we may assume that $b_1 = 2, c_1 = 2$ without loss of generality. (See Figure 8.11.)

The matrices must be of form

$$A := \begin{bmatrix} \alpha_1 & \alpha_1 a_2 + \gamma_1 c_2 a_3 & \gamma_1 a_3 \\ 0 & -\beta_1 + \gamma_1 b_3 c_2 & \gamma_1 b_3 \\ 0 & -\gamma_1 c_2 & -\gamma_1 \end{bmatrix},$$

$$B := \begin{bmatrix} -\alpha_2 & 0 & -\alpha_2 a_3 \\ -\alpha_2 b1 & \beta_2 & \beta_2 b_3 + \alpha_2 a_3 b_1 \\ \alpha_2 c_1 & 0 & -\gamma_2 + \alpha_2 a_3 c_1 \end{bmatrix}, \text{ and} \tag{8.8}$$

$$C := \begin{bmatrix} -\alpha_3 + \beta_3 a_2 b_1 & \beta_3 a_2 & 0 \\ -\beta_3 b_1 & -\beta_3 & 0 \\ \gamma_3 c_1 + \beta_3 b_1 c_2 & \beta_3 c_2 & \gamma_3 \end{bmatrix}$$

where $\alpha_i, \beta_i, \gamma_i > 0$ are positive real numbers satisfying equation:

$$\alpha_1\alpha_1\alpha_3 = 1, \beta_1\beta_2\beta_3 = 1, \gamma_1\gamma_2\gamma_3 = 1, \alpha_1\beta_1\gamma_1 = 1, \alpha_2\beta_2\gamma_2 = 1, \alpha_3\beta_3\gamma_3 = 1.$$

This follows since the determinants must be 1 and $CBA = \mathrm{I}$.

Solving for $CBA = \mathrm{I}$, we obtain without difficulty: Given $l_1, l_{1,2}, l_2, l_{2,2}, l_3, l_{3,2}$ square roots of smallest positive eigenvalues of A, B, C respectively so that $l_i < l_{i,2}$ for $i = 1, 2, 3$, we have two parameter solutions for $s > 0, t > 0$:

$$\begin{aligned}
&\alpha_1 = l_1^2, \alpha_2 = \frac{l_3}{l_1 l_2 s}, \alpha_3 = s\frac{l_2}{l_3 l_1}, \\
&\beta_1 = s\frac{l_3}{l_1 l_2}, \beta_2 = l_2^2, \beta_3 = \frac{l_1}{l_2 l_3 s}, \\
&\gamma_1 = \frac{l_2}{l_3 l_1 s}, \gamma_2 = s\frac{l_1}{l_2 l_3}, \gamma_3 = l_3^2, \\
&a_2 = t, a_3 = 2, b_1 = \frac{1}{t}\left(1 + \frac{l_2 l_3}{l_1}\tau_3 s + \frac{l_2^2}{l_1^2}s^2\right), b_3 = 2, \\
&c_1 = \frac{1}{2}\left(1 + \frac{l_1 l_2}{l_3}\tau_2 s + \frac{l_1^2}{l_3^2}s^2\right), \text{ and} \\
&c_2 = \frac{1}{2}\left(1 + \frac{l_3 l_1}{l_2}\tau_1 s + \frac{l_3^2}{l_2^2}s^2\right)
\end{aligned} \tag{8.9}$$

where $\tau_1 = l_{1,2}^2 + \dfrac{1}{(l_1^2 l_{1,2})^2}, \tau_2 = l_{2,2}^2 + \dfrac{1}{(l_2^2 l_{2,2})^2}$, and $\tau_3 = l_{3,2}^2 + \dfrac{1}{(l_3^2 l_{3,2})^2}$ hold. The importance of the solution is that we can choose arbitrary boundary invariants $l_i, l_{i,2}$ for $i = 1, 2, 3$, there exists two parameter family of solutions parameterized by $s, t > 0$ proving that $\mathcal{F}$ is a principal $\mathbb{R}^2$-bundle projection. (See Triangle10.nb for computations here.)

8.3.2.5 *Small orbifolds*

Let P be an orbifold that is either an annulus with a singularity $p, p \geq 2$, or a disk with singularity $p, q, p > q \geq 2$ and a sphere with singularity $p, q, r, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. As above, we can divide P into two triangles by segments ending at singular points or winding around components of the boundary in the opposite direction to the boundary orientation. We introduce transformations A, B, and C as above.

In case of the sphere $S_{p,q,r}$ with singularities p, q, r, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, we obtain from the paper [Choi and Goldman (2005)] as the solution space parameterized by $s > 0, t > 0$:

$$\begin{aligned}
\alpha_1 =& 1, \alpha_2 = \frac{1}{s}, \alpha_3 = s, \\
\beta_1 =& s, \beta_2 = 1, \beta_3 = \frac{1}{s}, \\
\gamma_1 =& \frac{1}{s}, \gamma_2 = s, \gamma_3 = 1 \\
a_2 =& t, a_3 = 2, b_1 = \frac{1}{t}(1 + \tau_3 s + s^2), b_3 = 2, \\
c_1 =& \frac{1}{2}(1 + \tau_2 s + s^2), \text{ and } c_2 = \frac{1}{2}(1 + \tau_1 s + s^2) \\
\text{where } \tau_1 =& 2\cos\left(\frac{2\pi}{p}\right), \tau_2 = 2\cos\left(\frac{2\pi}{q}\right), \text{and } \tau_3 = 2\cos\left(\frac{2\pi}{r}\right)
\end{aligned} \tag{8.10}$$

hold.

We do not examin the other cases because of length. The interested readers can download some mathematica files from the authors webpages. See Triangle5graphic.nb,Triangle10graphic.nb, and Triangle10graphicII.nb. We give some examples of the developing images.

8.4 Notes

For computations, one can experiment with various packages that the author and Gye-Seon Lee wrote. Gye-Seon developed from the maple package I wrote. These contain computations where one or more of the cone-point orders are two. One has to be careful about adjusting the coordinates since some points would develop across the line at infinity. This creates problems. But theoretically, a well-chosen affine space would contain the convex domain. These packages will be maintained at `http://mathsci.kaist.ac.kr/~schoi/MSJbook2012.html`.

As a historical note, the closedness of the deformation spaces of real projective structures on closed surfaces of genus ≥ 2 was questioned by Thurston and was given to the author as a doctoral thesis problem in 1985.

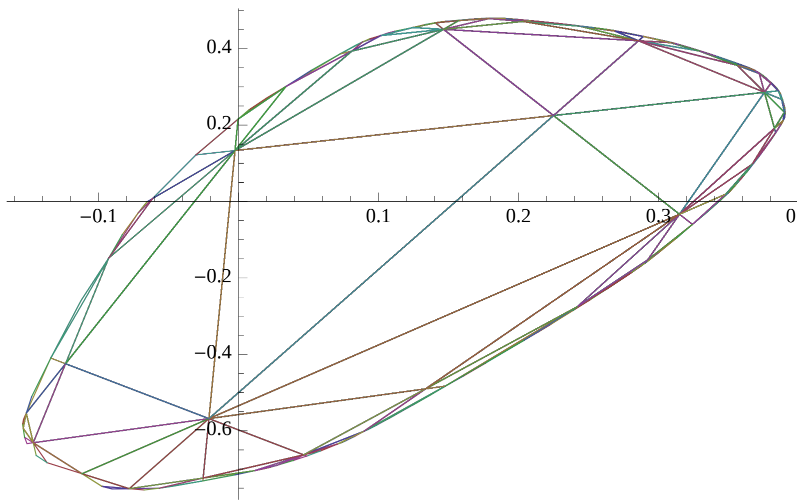

Fig. 8.12 The developing figure of a sphere with cone-points of order $3, 5, 5$. See Triangle5graphic.nb

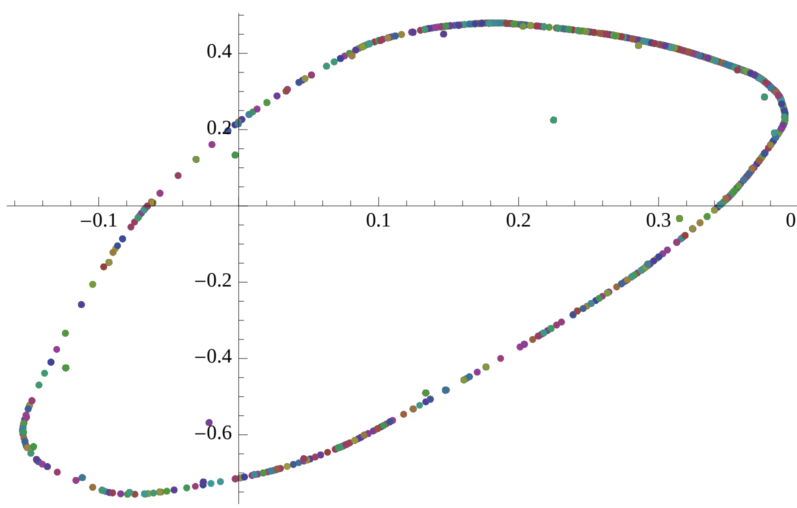

Fig. 8.13 The orbit points of a sphere with cone-points of order $3, 5, 5$. See Triangle5graphic.nb

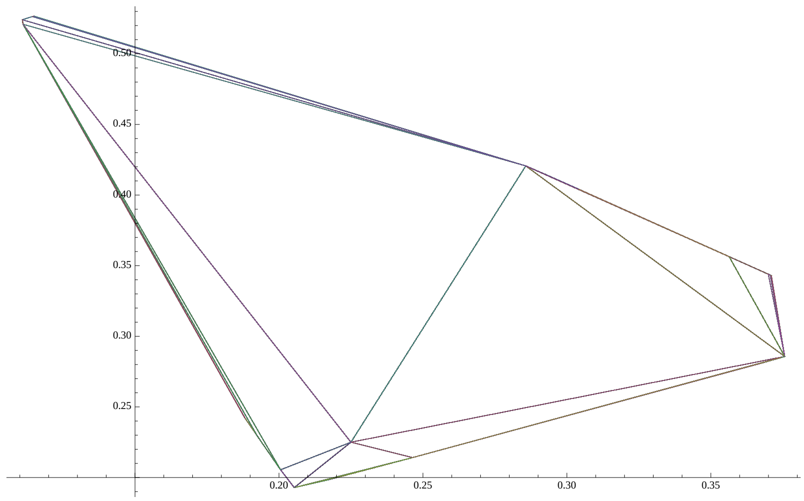

Fig. 8.14 The developing figure of an annulus with cone-points of order 3. See Triangle10graphicII.nb

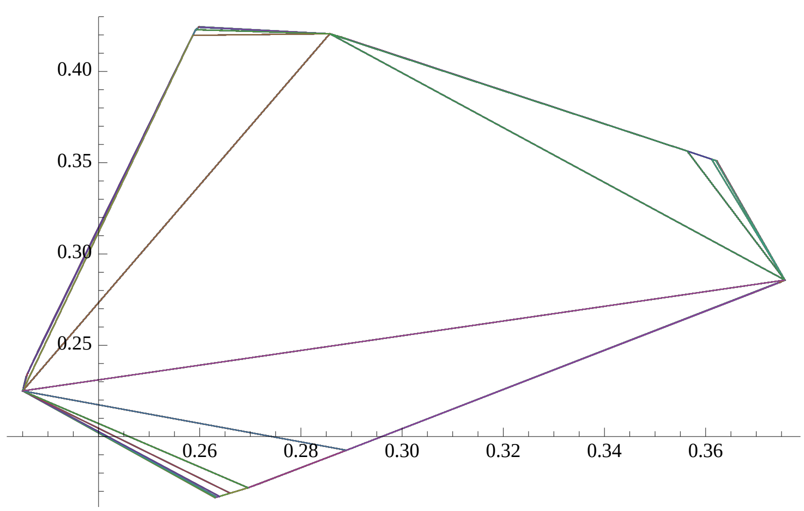

Fig. 8.15 The developing figure of a pair-of-pants. See Triangle10graphic.nb

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