Deformation spaces of real projective structures on Coxeter 3-orbifolds (with some survey)

Suhyoung Choi
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KAIST
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Abstract

- A real projective structure on a 3-orbifold is given by locally modeling the orbifold by real projective geometry.
- We present some methodology to study Coxeter groups which are fundamental groups of 3-orbifolds with representations in $\mathrm{SL}_{\pm}(4, \mathbb{R})$ and deformation spaces.
- These examples give us nontrivial deformation spaces of projective structures.

A brief introduction to projective structures

A *geometry* is a pair (G, X) where G is a Lie group acting on a space X (analytically, locally effectively, transitively) as defined by Klein. A study of geometry is the study of G-invariant properties on X.

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- an atlas of charts to X
- where the transition maps are in G.

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$$\{(G, x) - \text{structure on } M\} \leftrightarrow$$
$$\{(\mathbf{dev} : \tilde{M} \to X, h : \pi_1(M) \to G)|$$
$$\mathbf{dev} \circ \vartheta = h(\vartheta) \circ \mathbf{dev}, \vartheta \in \pi_1(M)\}/ \sim$$

where $\mathbf{dev}$ is an immersion and h a homomorphism. $\sim$ is given by

$$(\mathbf{dev}, h) \leftrightarrow (g \circ \mathbf{dev}, g \circ h(\cdot) \circ g^{-1}).$$

Some examples:

- Euclidean manifolds: $(R^n, Isom(R^n))$
- hyperbolic manifolds: $(H^n, PSO(n+1, R))$.
- spherical manifolds: $(S^n, O(n+1, R))$.
- conformal manifolds: $(S^n, Mobius(S^n))$.
- projective manifolds: $(\mathbb{R}P^n, PGL(n+1, \mathbb{R}))$.

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  - ⋆ What about flexible types ones: such as
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A flexible type geometry has no G-invariant metrics on X, and the universal cover of (G, X)-manifold M can be complicated and immerses over X. G-representations of $\pi_1(M)$ are poorly understood.

Origins in Geometry

- Cartan defined projectively flat structures on manifolds as:
  - ⋆ "geodesically Euclidean but with no metrics", i.e.,
    - ∗ torsion-free
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  - Gauge theory: Hitchin-Teichmuller components,
  - affine differential geometry (Calabi-Yau manifolds): Loftin, Labourie's work, and
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- ⋆ 3-dimensional Thurston geometries are almost projective. ($\mathbb{R}P^3 \# \mathbb{R}P^3$ does not admit a projective structure (Cooper-Long).)
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- Most examples of projective manifolds are by taking quotients of a domain in $\mathbb{R}P^n$ by a discrete subgroup of $\mathrm{PGL}(n+1,\mathbb{R})$.

  ⋆ The domains are usually convex and we call the quotient *convex projective manifold*. There are of course projective manifolds that are not from domains.

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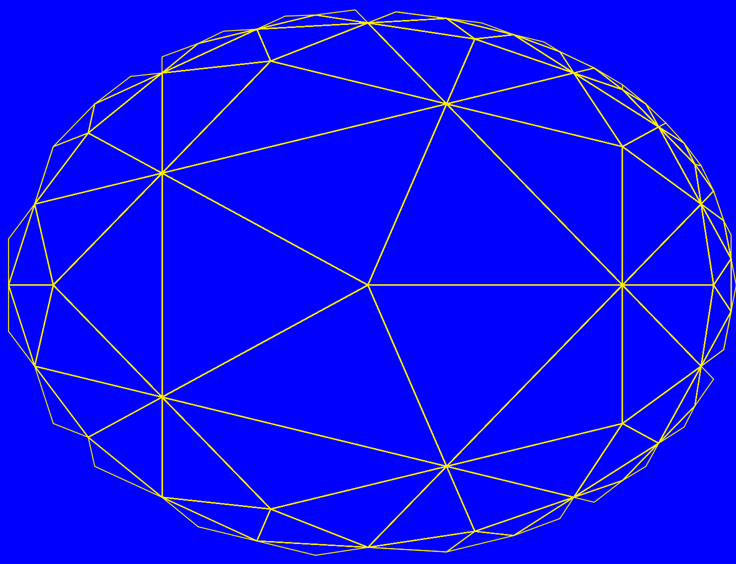
- J.L. Koszul showed that convexity is preserved if one slightly changed the projective structures. (Importance of convex projective manifolds)

- Kac-Vinberg were first to find examples of convex projective surfaces that are not hyperbolic. The examples are based on 2-dimensional Coxeter groups and easy matrix computations. These are related to Kac-Moody algebras.



Kobayashi and convex projective manifolds

- Kobayashi studied metrics on projective manifolds: He considers maps

  $$l \subset \mathbb{RP}^1 \to M$$

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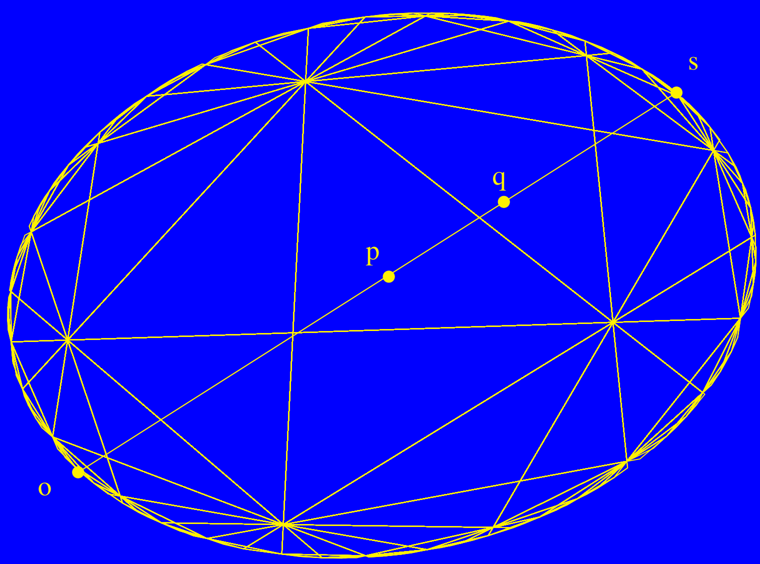
- In this case, Kobayashi metric is Finsler and Hilbert

$$d(p, q) = \log(o, s, q, p).$$

If $\Omega = H^n$, the metric is the standard hyperbolic metric.



Brief history

- Benzecri (and Milnor) showed affine 2-manifold has Euler characteristic $= 0$ (the Chern conjecture for dimension 2).

- Benzecri studied convex domains that arise for convex projective manifolds. He showed that the boundary of Ω is C^1 and if C^2, it is an ellipsoid for 2-dimensional closed convex projective surfaces.

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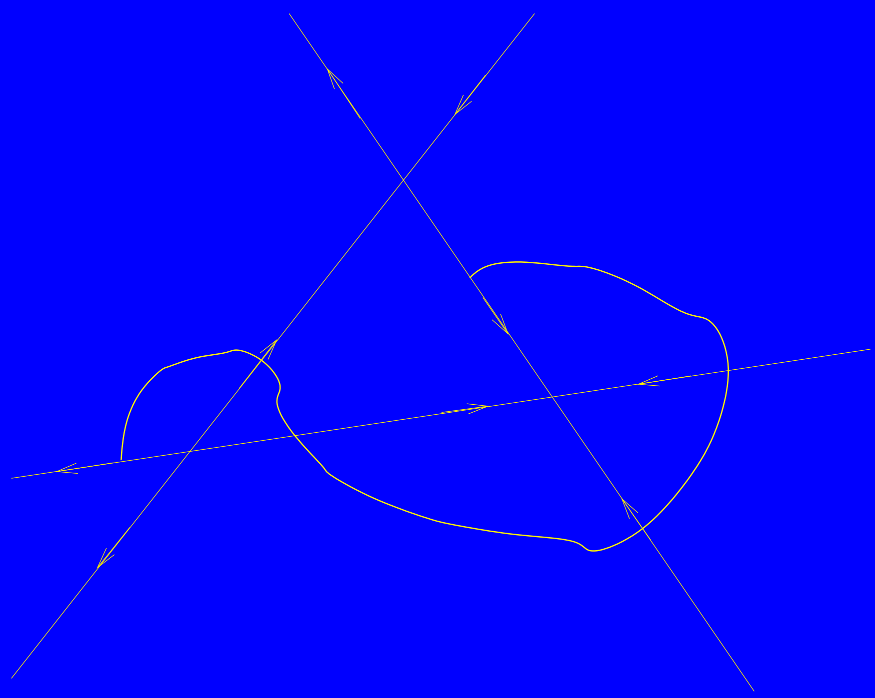
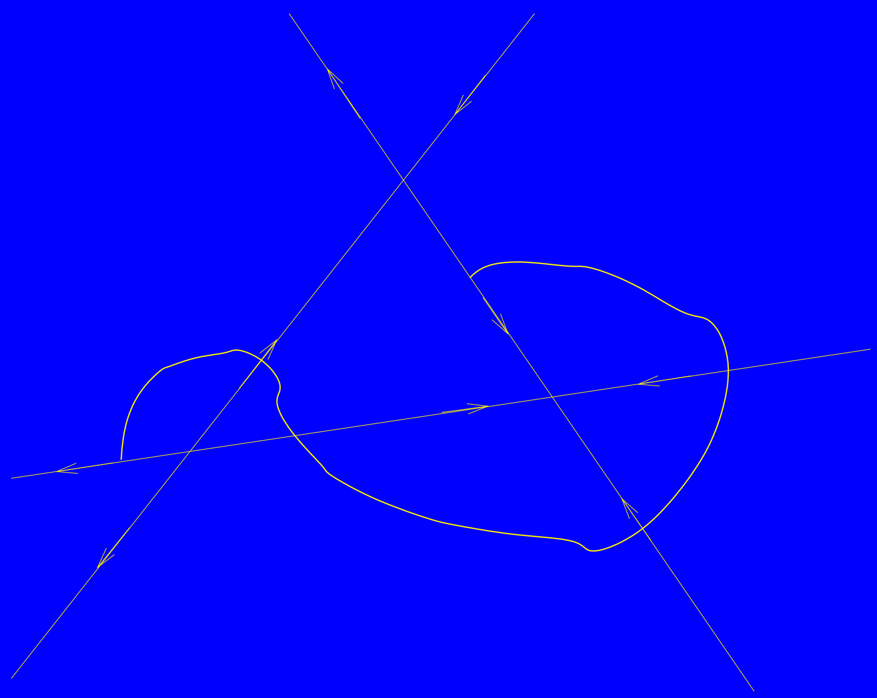
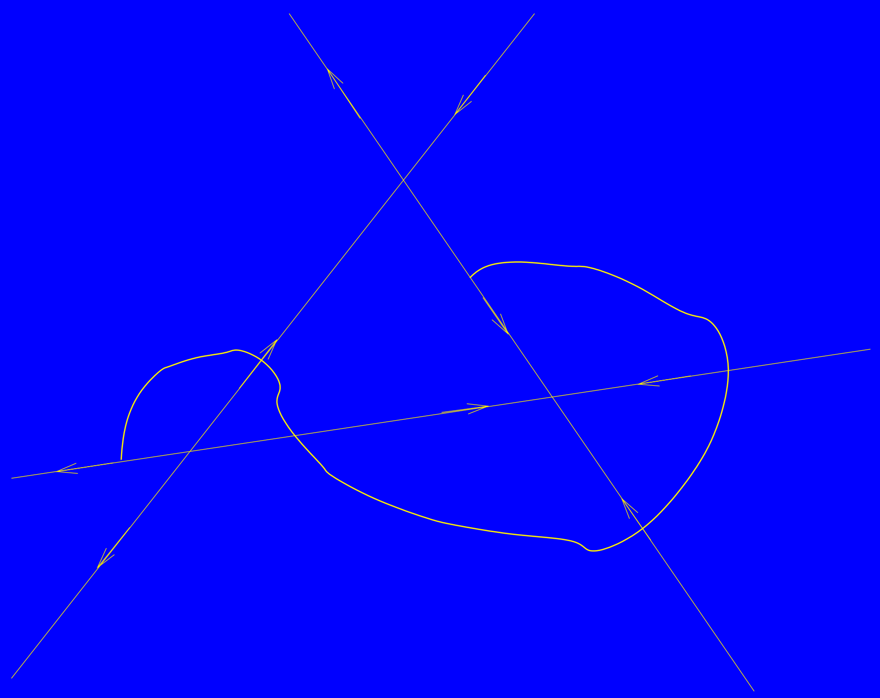




- Grafting: One can insert this type of annuli into a convex projective surface to obtain non-convex projective surfaces.
- Convex decomposition theorem: actually, one can show that this is all that can happen. (—)



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- Goldman's classification of convex projective structures on surfaces:

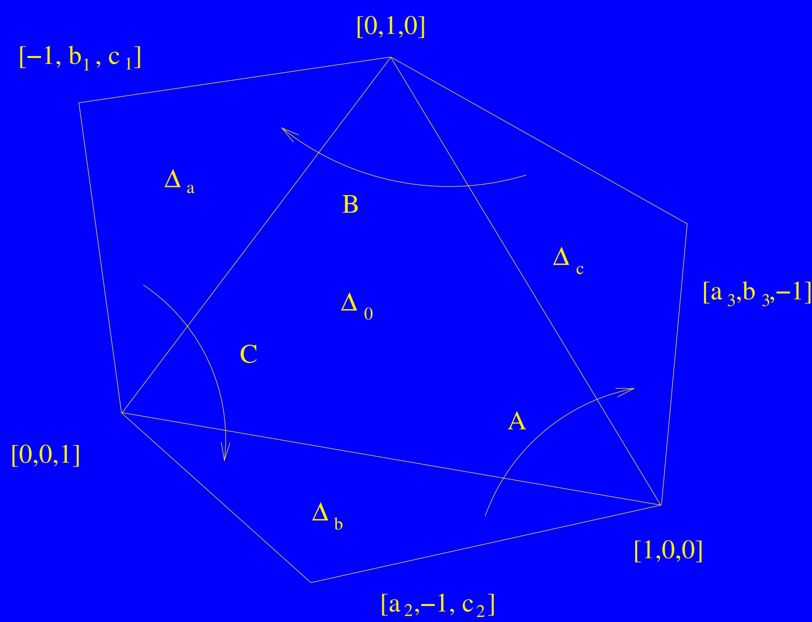
- Goldman's classification of convex projective structures on surfaces: Determining the deformation space $D(\Sigma)$:

  - ⋆ The needed key is that

$$D(P) \to D(\partial P)$$

for a pair of pants P is a principle fibration for a pair of pants P.

  - ⋆ first cut up the surface into pairs of pants.
  - ⋆ Each pair of pants is a union of two open triangles.
  - ⋆ We realize the triangles as geodesic ones.
  - ⋆ We investigate the projective invariants of union of four triangles in $\mathbb{RP}^2$.


[0,1,0]
[−1, b1, c1]
Δa
B
Δc
[a3,b3,−1]
Δ0
C
A
[0,0,1]
Δb
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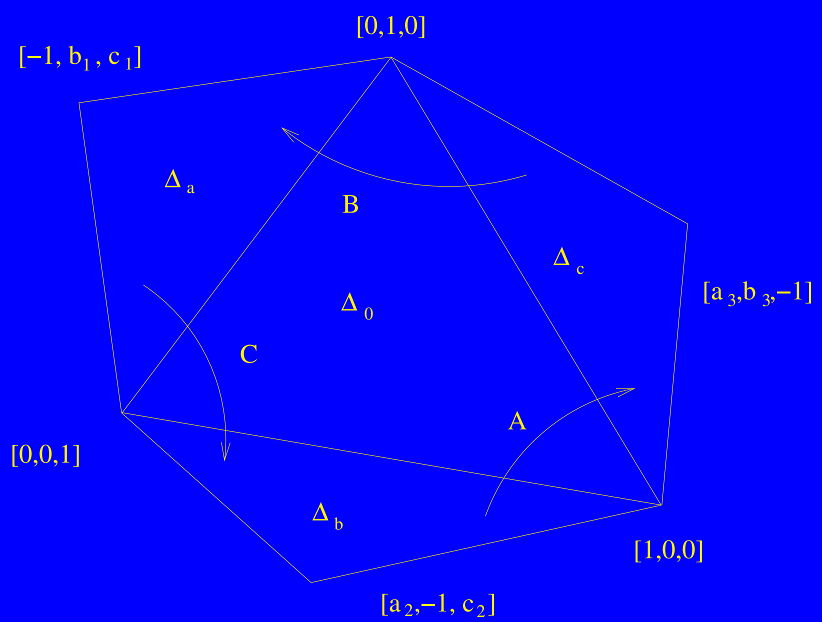

⋆ We generalized this to 2-orbifolds of negative Euler characteristics in a recent paper [CG], showing $D(\Sigma)$ is again homeomorphic to a cell.

Group theory and representations

- One can also consider $D(M)$ the quotient space of all pairs $(\mathbf{dev}, h)$ where $\mathbf{dev} : \tilde{M} \to \mathbb{R}P^n$ is an immersion equivariant with respect to a homomorphism

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- Sending $(\mathbf{dev}, h)$ to h gives us a map

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- As stated earlier, Kac-Vinberg, Koszul started to study the deformations of representations $\Gamma \to \mathrm{PGL}(n, \mathbb{R})$.
- There is a well-known deformation (Apanasov) called bending for projective and conformally flat structures.
- Johnson and Millson found that a certain hyperbolic manifold has a deformation space of projective structures that is singular. (They also worked out this for conformally flat structures also.)

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- Recent work of Benoist (papers "Convex divisibles I-IV"):
  - ⋆ *Theorem*. Γ an irreducible torsion-free subgroup of $GL(m, \mathbb{R})$. Then Γ acts on a proper convex cone C if and only if Γ is positive proximal.
  - ⋆ If C is not a Lorentzian cone, then Γ is Zariski dense in $GL(m, \mathbb{R})$.
  - ⋆ *Theorem*. Let Γ be a discrete torsion-free subgroup of $SL(m, \mathbb{R})$ acting on an open convex domain in $\mathbb{R}P^{m-1}$. Let C be the corresponding cone on $\mathbb{R}^m$. Then one of the following holds.
    - ∗ C is a product

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⋆ Let Γ be as above. Then the following conditions are equivalent:

- Ω is strictly convex.
- $\partial\Omega$ is C^1.
- Γ is Gromov-hyperbolic.
- Geodesic flow on Ω/Γ is Anosov.

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  - ⋆ This representation is unique up to conjugation, but if we consider $G := SO^+(3,1)$ as a subgroup of a larger group Γ, we can search for deformations of the G-representation into the group Γ.
  - ⋆ Out of the first 1000 closed hyperbolic 3-manifolds in the Hodgson-Weeks census, a handful admit non-trivial deformations of their $SO^+(3,1)$-representations into $SL(4,\mathbb{R})$; each resulting representation variety then gives rise to a family of real projective structures on the manifold.

- ⋆ We (J.R.Kim) studied the projective deformation spaces of knot complements numerically and found the dimension to be three. To determine the dimension of the projective deformation spaces, we need three more equations: The dimension is zero usually.
- ⋆ If there is an involution reversing a simple closed geodesic in a hyperbolic 3-manifold, we conjecture that the dimension is one.

Projective structures on 3-dimensional Coxeter orbifolds

- We present some methodology to study Coxeter groups which are fundamental groups of 3-orbifolds with representations in $\mathrm{SL}_{\pm}(4, \mathbb{R})$ and deformation spaces. These examples give us nontrivial deformation spaces of projective structures. (There are related examples by Benoist.)

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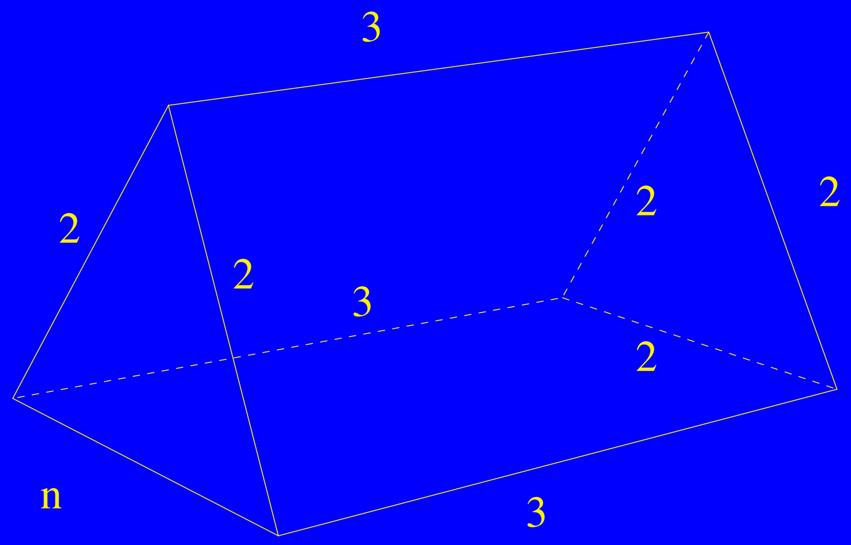
- The fundamental group of the orbifold will be a Coxeter group with a presentation

$$R_i, i = 1, 2, \ldots, f : R_i^2 = 1, (R_i R_j)^{n_{ij}} = 1$$

where R_i is associated with silvered sides and $R_i R_j$ has order n_{ij} associated with an edge.

- Let P be a fixed convex 3-dimensional polyhedron. Let us assign orders at each edge.
  - ⋆ e the number of edges
  - ⋆ e_2 the numbers of edges of order-two among the edges.
  - ⋆ f be the number of sides.



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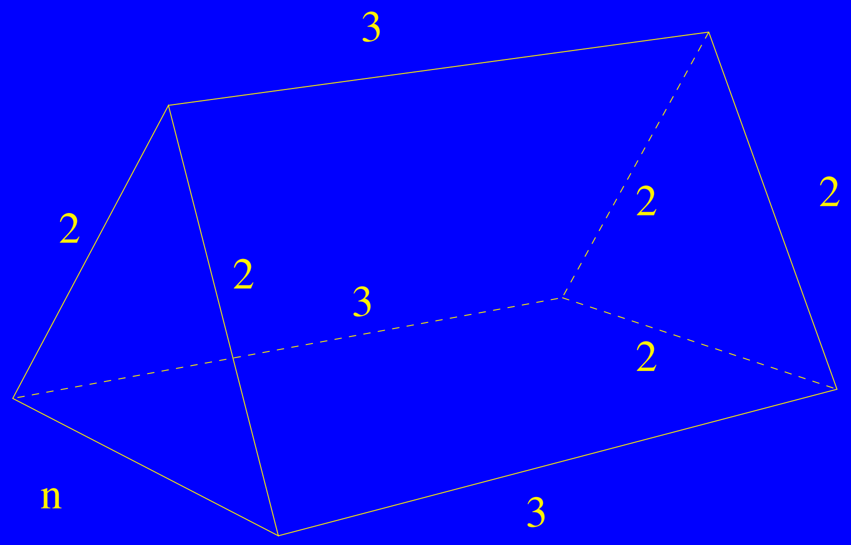
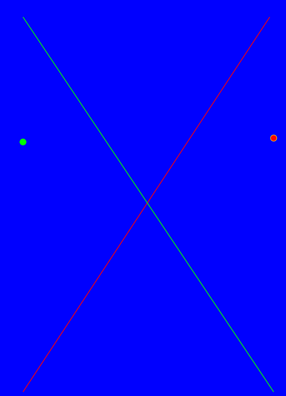


- *Definition*. A *reflection* in S^n is given by a great hypersphere of fixed points and a pair of antipodal points $p, -p$ mapping to each other, which we call *antipodally-fixed points*.

- Given two reflections R_1 and R_2 in S^n $n \geq 1$ with respectively distinct hyperspheres and pairs of antipodally-fixed points we can define a *dihedral angle* between the respective transverse hyperspheres of fixed points P_1 and P_2:
  - ⋆ If $R_1 \circ R_2$ is not order-two and antipodally-fixed points of R_1 is not in the hypersphere of R_2 and vice versa, $R_1 \circ R_2$ will fix $P_1 \cap P_2$ and

$$\sim \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & I^{n-1,n-1} \end{bmatrix}$$

if $n-3 < tr(R_1 \circ R_2) < n+1$. In this case, we define the dihedral angle to be $\theta/2$.



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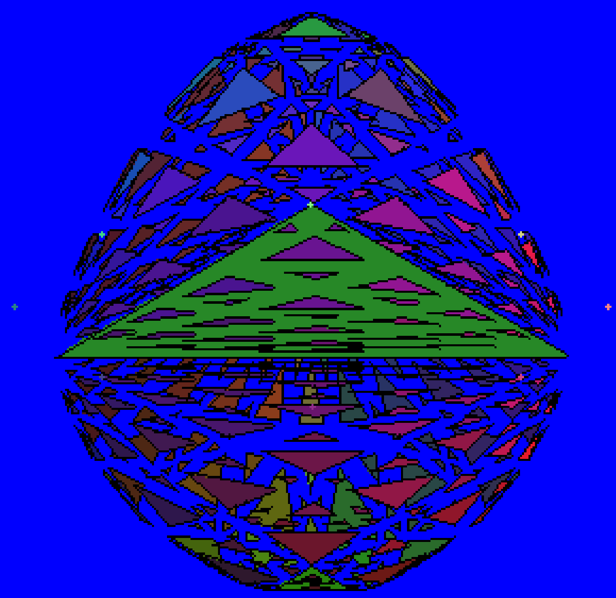
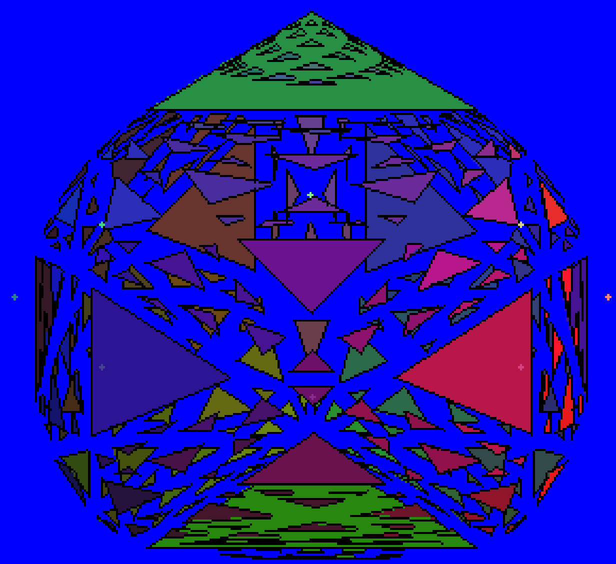
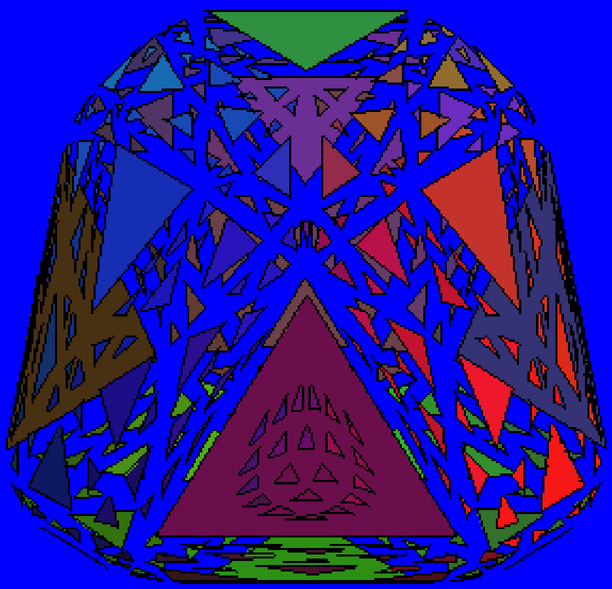
Pictures (due to Yves Benoist)







- One of the main question currently is how to determine if the deformation space is empty or not.
- We also wish to understand about the hyperbolic polyhedrons which we are not necessarily studying in this paper.
- We could approach the Andreev theorem for hyperbolic polyhedra from projective sides if we accomplish all these.

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