Deformation spaces of real projective structures on Coxeter 3-orbifolds (with some survey)

Suhyoung Choi
Department of Mathematics
KAIST
305-701 Daejeon, South Korea
schoi@math.kaist.ac.kr
mathx.kaist.ac.kr/schoi

January 10, 2006
Deformation spaces of real projective structures on Coxeter 3-orbifolds (with some survey)

Suhyoung Choi
Department of Mathematics
KAIST
305-701 Daejeon, South Korea
schoi@math.kaist.ac.kr
mathx.kaist.ac.kr/schoi

January 10, 2006
Abstract

- A real projective structure on a 3-orbifold is given by locally modeling the orbifold by real projective geometry.
- We present some methodology to study Coxeter groups which are fundamental groups of 3-orbifolds with representations in $SL_\pm(4, \mathbb{R})$ and deformation spaces.
- These examples give us nontrivial deformation spaces of projective structures.
A brief introduction to projective structures

A \textit{geometry} is a pair \((G, X)\) where \(G\) is a Lie group acting on a space \(X\) (analytically, locally effectively, transitively) as defined by Klein. A study of geometry is the study of \(G\)-invariant properties on \(X\).
A brief introduction to projective structures

A \textit{geometry} is a pair \((G, X)\) where \(G\) is a Lie group acting on a space \(X\) (analytically, locally effectively, transitively) as defined by Klein. A study of geometry is the study of \(G\)-invariant properties on \(X\).

A \((G, X)\)-geometric structure on a manifold \(M\) is given by

- an atlas of charts to \(X\)
- where the transition maps are in \(G\).

This induces “local \((G, X)\)-geometry” on the manifold \(M\) consistently.
A brief introduction to projective structures

A geometry is a pair \((G, X)\) where \(G\) is a Lie group acting on a space \(X\) (analytically, locally effectively, transitively) as defined by Klein. A study of geometry is the study of \(G\)-invariant properties on \(X\).

A \((G, X)\)-geometric structure on a manifold \(M\) is given by

- an atlas of charts to \(X\)
- where the transition maps are in \(G\).

This induces “local \((G, X)\)-geometry” on the manifold \(M\) consistently.

\[
\{(G, x) - \text{structure on } M\} \leftrightarrow \{(\text{dev} : \tilde{M} \to X, h : \pi_1(M) \to G) | \text{dev} \circ \vartheta = h(\vartheta) \circ \text{dev}, \vartheta \in \pi_1(M)\}/\sim
\]
where $\text{dev}$ is an immersion and $h$ a homomorphism. $\sim$ is given by

$$(\text{dev}, h) \leftrightarrow (g \circ \text{dev}, g \circ h(\cdot) \circ g^{-1}).$$

Some examples:

- **Euclidean manifolds:** $(\mathbb{R}^n, \text{Isom}(\mathbb{R}^n))$
- **hyperbolic manifolds:** $(\mathbb{H}^n, \text{PSO}(n + 1, \mathbb{R}))$.
- **spherical manifolds:** $(S^n, \text{O}(n + 1, \mathbb{R}))$.
- **conformal manifolds:** $(S^n, \text{Mobius}(S^n))$.
- **projective manifolds:** $(\mathbb{R}P^n, \text{PGL}(n + 1, \mathbb{R}))$. 
where $\text{dev}$ is an immersion and $h$ a homomorphism. $\sim$ is given by
\[(\text{dev}, h) \leftrightarrow (g \circ \text{dev}, g \circ h(\cdot) \circ g^{-1}).\]

Some examples:

- Euclidean manifolds: $(\mathbb{R}^n, \text{Isom}(\mathbb{R}^n))$
- hyperbolic manifolds: $(H^n, \text{PSO}(n + 1, \mathbb{R}))$.
- spherical manifolds: $(S^n, \text{O}(n + 1, \mathbb{R}))$.
- conformal manifolds: $(S^n, \text{Mobi}us(S^n))$.
- projective manifolds: $(\mathbb{R}P^n, \text{PGL}(n + 1, \mathbb{R}))$.

General remarks:

- What kind of geometric (projective) structures are on a given manifold?
where $\text{dev}$ is an immersion and $h$ a homomorphism. $\sim$ is given by

$$(\text{dev}, h) \leftrightarrow (g \circ \text{dev}, g \circ h(\cdot) \circ g^{-1}) .$$

Some examples:

- Euclidean manifolds: $(\mathbb{R}^n, \text{Isom}(\mathbb{R}^n))$
- hyperbolic manifolds: $(H^n, \text{PSO}(n + 1, \mathbb{R}))$.
- spherical manifolds: $(S^n, O(n + 1, \mathbb{R}))$.
- conformal manifolds: $(S^n, \text{Mobius}(S^n))$.
- projective manifolds: $(\mathbb{R}P^n, \text{PGL}(n + 1, \mathbb{R}))$.

General remarks:

- What kind of geometric (projective) structures are on a given manifold?
Determine the topology and geometry of the deformation space

\[ D(M) = \{\text{geometric (projective) structures on } M\}/\text{isotopies} \]
• Determine the topology and geometry of the deformation space

\[ D(M) = \{ \text{geometric (projective) structures on } M \}/\text{isotopies} \]

• Rigid type geometric structures is
  ★ studied using Margulis type rigidity:
  ★ What about flexible types ones: such as
    ★ conformally flat,
    ★ projectively flat, or
    ★ affinely flat geometric structures.
• Determine the topology and geometry of the deformation space

\[ D(M) = \{ \text{geometric (projective) structures on } M \} / \text{isotopies} \]

• Rigid type geometric structures is

★ studied using Margulis type rigidity:
★ What about flexible types ones: such as
  ★ conformally flat,
  ★ projectively flat, or
  ★ affinely flat geometric structures.

• \( D(M) \) is closely related to the space of \( G \)-representations of \( \pi_1(M) \). The latter space is fairly hard to understand.
• Determine the topology and geometry of the deformation space

\[ D(M) = \{\text{geometric (projective) structures on } M\} / \text{isotopies} \]

• Rigid type geometric structures is
  ⋆ studied using Margulis type rigidity:
  ⋆ What about flexible types ones: such as
    * conformally flat,
    * projectively flat, or
    * affinely flat geometric structures.

• \( D(M) \) is closely related to the space of \( G \)-representations of \( \pi_1(M) \). The latter space is fairly hard to understand.

A rigid type geometry usually has \( G \)-invariant metrics on \( X \), and the universal cover of \((G, X)\)-manifold \( M \) is isomorphic to the universal cover of \( X \), i.e., \( \tilde{M} \) is complete and \( M = \tilde{X} / \Gamma \).
• Determine the topology and geometry of the deformation space

\[ D(M) = \{ \text{geometric (projective) structures on } M \} / \text{isotopies} \]

• Rigid type geometric structures is
  • studied using Margulis type rigidity:
  • What about flexible types ones: such as
    • conformally flat,
    • projectively flat, or
    • affinely flat geometric structures.

• \( D(M) \) is closely related to the space of \( G \)-representations of \( \pi_1(M) \). The latter space is fairly hard to understand.

A rigid type geometry usually has \( G \)-invariant metrics on \( X \), and the universal cover of \( (G, X) \)-manifold \( M \) is isomorphic to the universal cover of \( X \), i.e., \( M \) is complete and \( M = \tilde{X} / \Gamma \).
A flexible type geometry has no $G$-invariant metrics on $X$, and the universal cover of $(G, X)$-manifold $M$ can be complicated and immerses over $X$. $G$-representations of $\pi_1(M)$ are poorly understood.
Origins in Geometry

- **Cartan** defined projectively flat structures on manifolds as:
  - “geodesically Euclidean but with no metrics”, i.e.,
    - torsion-free
    - projectively flat (i.e., same geodesics structures as flat metrics)
  - affine connection on manifolds.
Origins in Geometry

- **Cartan** defined projectively flat structures on manifolds as:
  - “geodesically Euclidean but with no metrics”, i.e.,
    - torsion-free
    - projectively flat (i.e., same geodesics structures as flat metrics)
  - affine connection on manifolds.

- **Ehresmann** identifies this structure as having a maximal atlas of charts
  - to $\mathbb{RP}^n$
  - with transition maps in $\text{PGL}(n + 1, \mathbb{R})$. 
Origins in Geometry

- **Cartan** defined projectively flat structures on manifolds as:
  - “geodesically Euclidean but with no metrics”, i.e.,
    - torsion-free
    - projectively flat (i.e., same geodesics structures as flat metrics)
  - affine connection on manifolds.

- **Ehresmann** identifies this structure as having a maximal atlas of charts
  - to $\mathbb{RP}^n$
  - with transition maps in $\text{PGL}(n + 1, \mathbb{R})$.

- Some reasons for interest in projective structures:
  - 2-dimensional classical geometries: euclidean, spherical, hyperbolic geometries are projective geometries (and conformal geometries).
**Origins in Geometry**

- **Cartan** defined projectively flat structures on manifolds as:
  - “geodesically Euclidean but with no metrics”, i.e.,
    - torsion-free
    - projectively flat (i.e., same geodesics structures as flat metrics)
  - affine connection on manifolds.

- **Ehresmann** identifies this structure as having a maximal atlas of charts
  - to $\mathbb{R}P^n$
  - with transition maps in $\operatorname{PGL}(n+1, \mathbb{R})$.

- Some reasons for interest in projective structures:
  - 2-dimensional classical geometries: euclidean, spherical, hyperbolic geometries are projective geometries (and conformal geometries).
The study of the space of representations of the fundamental group of surfaces are related to

- Gauge theory: Hitchin-Teichmüller components,
- affine differential geometry (Calabi-Yau manifolds): Loftin, Labourie’s work, and
- some theoretical physics:
  - w3-algebra: Fock and Goncharov.
  - Quantization of 3-manifolds: Ovsienko and Duval
The study of the space of representations of the fundamental group of surfaces are related to

- Gauge theory: Hitchin-Teichmuller components,
- affine differential geometry (Calabi-Yau manifolds): Loftin, Labourie’s work, and
- some theoretical physics:
  - w3-algebra: Fock and Goncharov.
  - Quantization of 3-manifolds: Ovsienko and Duval

- 3-dimensional Thurston geometries are almost projective. ($\mathbb{R}P^3 \# \mathbb{R}P^3$ does not admit a projective structure (Cooper-Long).)
- There are many parallels to conformally flat structures: geometric analysis and so on.
The study of the space of representations of the fundamental group of surfaces are related to

• Gauge theory: Hitchin-Teichmuller components,
• affine differential geometry (Calabi-Yau manifolds): Loftin, Labourie’s work, and
• some theoretical physics:
  • w3-algebra: Fock and Goncharov.
  • Quantization of 3-manifolds: Ovsienko and Duval

• 3-dimensional Thurston geometries are almost projective. ($\mathbb{RP}^3 \# \mathbb{RP}^3$ does not admit a projective structure (Cooper-Long).)
• There are many parallels to conformally flat structures: geometric analysis and so on.
Most examples of projective manifolds are by taking quotients of a domain in $\mathbb{R}P^n$ by a discrete subgroup of PGL($n + 1$, $\mathbb{R}$).

The domains are usually **convex** and we call the quotient **convex projective manifold**. There are of course projective manifolds that are not from domains.

Let $H^n$ be the interior of an ellipsoid. Then $H^n$ is the hyperbolic space and $\text{Aut}(H^n)$ is the isometry group. $H^n/\Gamma$ has a canonical projective structure.
Most examples of projective manifolds are by taking quotients of a domain in $\mathbb{R}P^n$ by a discrete subgroup of $\text{PGL}(n+1, \mathbb{R})$.

- The domains are usually **convex** and we call the quotient **convex projective manifold**. There are of course projective manifolds that are not from domains.
- Let $H^n$ be the interior of an ellipsoid. Then $H^n$ is the hyperbolic space and $\text{Aut}(H^n)$ is the isometry group. $H^n/\Gamma$ has a canonical projective structure.

- J.L. Koszul showed that convexity is preserved if one slightly changed the projective structures. (Importance of convex projective manifolds)

- Kac-Vinberg were first to find examples of convex projective surfaces that are not hyperbolic. The examples are based on 2-dimensional Coxeter groups and easy matrix computations. These are related to Kac-Moody algebras.
Kobayashi and convex projective manifolds

- Kobayashi studied metrics on projective manifolds: He considers maps

\[ l \subset \mathbb{RP}^1 \rightarrow M \]

and take maximal ones. (l proper intervals or a complete real line.

This defines a pseudo-metric.
Kobayashi and convex projective manifolds

• Kobayashi studied metrics on projective manifolds: He considers maps

\[ l \subset \mathbb{RP}^1 \rightarrow M \]

and take maximal ones. (\(l\) proper intervals or a complete real line.

This defines a pseudo-metric.

• Kobayashi metric is a metric if and only if

★ there are no lines of length \(\pi\) if and only if
★ \(M = \Omega/\Gamma\) where \(\Omega\) is a convex domain in \(\mathbb{RP}^n\).
Kobayashi studied metrics on projective manifolds: He considers maps

\[ l \subset \mathbb{RP}^1 \to M \]

and take maximal ones. (\(l\) proper intervals or a complete real line.

This defines a pseudo-metric.

Kobayashi metric is a metric if and only if

★ there are no lines of length \(\pi\) if and only if
★ \(M = \Omega / \Gamma\) where \(\Omega\) is a convex domain in \(\mathbb{RP}^n\).
In this case, Kobayashi metric is Finsler and Hilbert

\[ d(p, q) = \log(o, s, q, p). \]

If \( \Omega = H^n \), the metric is the standard hyperbolic metric.
Brief history

• Benzecri (and Milnor) showed affine 2-manifold has Euler characteristic $= 0$ (the Chern conjecture for dimension 2).

• Benzecri studied convex domains that arise for convex projective manifolds. He showed that the boundary of $\Omega$ is $C^1$ and if $C^2$, it is an ellipsoid for 2-dimensional closed convex projective surfaces.

• Nagano and Yagi classified affine structures on tori.

• Goldman classified projective structures on tori. (His senior thesis)
Brief history

• Benzecri (and Milnor) showed affine 2-manifold has Euler characteristic $= 0$ (the Chern conjecture for dimension 2).

• Benzecri studied convex domains that arise for convex projective manifolds. He showed that the boundary of $\Omega$ is $C^1$ and if $C^2$, it is an ellipsoid for 2-dimensional closed convex projective surfaces.

• Nagano and Yagi classified affine structures on tori.

• Goldman classified projective structures on tori. (His senior thesis)
• **Grafting:** One can insert this type of annuli into a convex projective surface to obtain non-convex projective surfaces.

• **Convex decomposition theorem:** actually, one can show that this is all that can happen. (—)
• **Grafting:** One can insert this type of annuli into a convex projective surface to obtain non-convex projective surfaces.

• **Convex decomposition theorem:** actually, one can show that this is all that can happen.
Goldman’s classification of convex projective structures on surfaces:
Goldman’s classification of convex projective structures on surfaces: Determining the deformation space $D(\Sigma)$:

- The needed key is that

$$D(P) \rightarrow D(\partial P)$$

for a pair of pants $P$ is a principle fibration for a pair of pants $P$.
- First cut up the surface into pairs of pants.
- Each pair of pants is a union of two open triangles.
- We realize the triangles as geodesic ones.
- We investigate the projective invariants of union of four triangles in $\mathbb{RP}^2$. 
We generalized this to 2-orbifolds of negative Euler characteristics in a recent paper [CG], showing $D(\Sigma)$ is again homeomorphic to a cell.
Group theory and representations

- One can also consider $D(M)$ the quotient space of all pairs $(\text{dev}, h)$ where $\text{dev} : \tilde{M} \to \mathbb{R}P^n$ is an immersion equivariant with respect to a homomorphism

$$h : \pi_1(M) \to \text{PGL}(n + 1, \mathbb{R}).$$
Group theory and representations

• One can also consider $D(M)$ the quotient space of all pairs $(\text{dev}, h)$ where $\text{dev} : \tilde{M} \to \mathbb{R}P^n$ is an immersion equivariant with respect to a homomorphism

$$h : \pi_1(M) \to \text{PGL}(n + 1, \mathbb{R})$$.

• Sending $(\text{dev}, h)$ to $h$ gives us a map

$$\text{hol} : D(M) \to \text{Hom}(\pi_1(M), \text{PGL}(n + 1, \mathbb{R})/\text{PGL}(n + 1, \mathbb{R})$$,

which is a local homeomorphism if $M$ is a closed manifold (orbifold). Thus, our study is closed related to the study of $\text{PGL}(n + 1, \mathbb{R})$-representations.
Group theory and representations

- One can also consider $D(M)$ the quotient space of all pairs $(\text{dev}, h)$ where $\text{dev} : \tilde{M} \to \mathbb{R}P^n$ is an immersion equivariant with respect to a homomorphism

$$h : \pi_1(M) \to \text{PGL}(n + 1, \mathbb{R}).$$

- Sending $(\text{dev}, h)$ to $h$ gives us a map

$$\text{hol} : D(M) \to \text{Hom}(\pi_1(M), \text{PGL}(n + 1, \mathbb{R})/\text{PGL}(n + 1, \mathbb{R}),$$

which is a local homeomorphism if $M$ is a closed manifold (orbifold). Thus, our study is closed related to the study of $\text{PGL}(n + 1, \mathbb{R})$-representations.
• As stated earlier, Kac-Vinberg, Koszul started to study the deformations of representations $\Gamma \to \text{PGL}(n, \mathbb{R})$.

• There is a well-known deformation (Apanasov) called bending for projective and conformally flat structures.

• Johnson and Millson found that a certain hyperbolic manifold has a deformation space of projective structures that is singular. (They also worked out this for conformally flat structures also.)
• As stated earlier, Kac-Vinberg, Koszul started to study the deformations of representations \( \Gamma \to \text{PGL}(n, \mathbb{R}) \).

• There is a well-known deformation (Apanasov) called bending for projective and conformally flat structures.

• Johnson and Millson found that a certain hyperbolic manifold has a deformation space of projective structures that is singular. (They also worked out this for conformally flat structures also.)

• Recent work of Benoist (papers “Convex divisibles I-IV”):

  ★ **Theorem.** \( \Gamma \) an irreducible torsion-free subgroup of \( \text{GL}(m, \mathbb{R}) \). Then \( \Gamma \) acts on a proper convex cone \( C \) if and only if \( \Gamma \) is positive proximal.
  
  ★ If \( C \) is not a Lorentzian cone, then \( \Gamma \) is Zariski dense in \( \text{GL}(m, \mathbb{R}) \).

  ★ **Theorem.** Let \( \Gamma \) be a discrete torsion-free subgroup of \( \text{SL}(m, \mathbb{R}) \) acting on an open convex domain in \( \mathbb{R}P^{m-1} \). Let \( C \) be the corresponding cone on \( \mathbb{R}^m \). Then one of the following holds.
  
  ★ \( C \) is a product
\* $C$ is homogeneous
\* or $\Gamma$ is Zariski dense in $SL(m, \mathbb{R})$. 
\* $C$ is homogeneous
\* or $\Gamma$ is Zariski dense in $SL(m, \mathbb{R})$.
\* If the virtual center of $\Gamma_0$ is trivial, then

\[ E_{\Gamma_0} = \{ \rho \in H_{\Gamma_0} \mid \text{The image of } \rho \text{ divides a convex open domain in } \mathbb{R}P^{n-1} \} \]

is closed in

\[ H_{\Gamma_0} := Hom(\Gamma_0, PGL(m, \mathbb{R})) \]

The openness was obtained by Koszul.
\* $C$ is homogeneous
\* or $\Gamma$ is Zariski dense in $SL(m, \mathbb{R})$.
\* If the virtual center of $\Gamma_0$ is trivial, then

$$E_{\Gamma_0} = \{ \rho \in H_{\Gamma_0} \mid \text{The image of } \rho \text{ divides a convex open domain in } \mathbb{R}P^{n-1} \}$$

is closed in

$$H_{\Gamma_0} := Hom(\Gamma_0, PGL(m, \mathbb{R}))$$

The openness was obtained by Koszul.

\* Let $\Gamma$ be as above. Then the following conditions are equivalent:
\* $\Omega$ is strictly convex.
\* $\partial \Omega$ is $C^1$.
\* $\Gamma$ is Gromov-hyperbolic.
\* Geodesic flow on $\Omega/\Gamma$ is Anosov.
Some questions related to 3-manifold theory

- The abstract of a talk on October 26, 2004 in UC Santa Barbara:
Some questions related to 3-manifold theory

- The abstract of a talk on October 26, 2004 in UC Santa Barbara:
  Morwen Thistlethwaite, UTK (jointly with Cooper and Long)
  “Deforming closed hyperbolic 3-manifolds”

  The geometric structure on a hyperbolic 3-manifold determines a discrete faithful representation of its fundamental group into $PSL(2, \mathbb{C})$, or equivalently into $SO^+(3,1)$. 
Some questions related to 3-manifold theory

- The abstract of a talk on October 26, 2004 in UC Santa Barbara:
  Morwen Thistlethwaite, UTK (jointly with Cooper and Long)
  “Deforming closed hyperbolic 3-manifolds”

★ The geometric structure on a hyperbolic 3-manifold determines a discrete faithful representation of its fundamental group into $PSL(2, \mathbb{C})$, or equivalently into $SO^+(3, 1)$.
★ This representation is unique up to conjugation, but if we consider $G := SO^+(3, 1)$ as a subgroup of a larger group $\Gamma$, we can search for deformations of the $G$-representation into the group $\Gamma$. 
Some questions related to 3-manifold theory

- The abstract of a talk on October 26, 2004 in UC Santa Barbara:
  Morwen Thistlethwaite, UTK (jointly with Cooper and Long)
  “Deforming closed hyperbolic 3-manifolds”

- The geometric structure on a hyperbolic 3-manifold determines a discrete faithful representation of its fundamental group into $PSL(2, \mathbb{C})$, or equivalently into $SO^+(3, 1)$.
- This representation is unique up to conjugation, but if we consider $G := SO^+(3, 1)$ as a subgroup of a larger group $\Gamma$, we can search for deformations of the $G$-representation into the group $\Gamma$.
- Out of the first 1000 closed hyperbolic 3-manifolds in the Hodgson-Weeks census, a handful admit non-trivial deformations of their $SO^+(3, 1)$-representations into $SL(4, \mathbb{R})$; each resulting representation variety then gives rise to a family of real projective structures on the manifold.
We (J.R.Kim) studied the projective deformation spaces of knot complements numerically and found the dimension to be three. To determine the dimension of the projective deformation spaces, we need three more equations: The dimension is zero usually.

If there is an involution reversing a simple closed geodesic in a hyperbolic 3-manifold, we conjecture that the dimension is one.
Projective structures on 3-dimensional Coxeter orbifolds

We present some methodology to study Coxeter groups which are fundamental groups of 3-orbifolds with representations in $\text{SL}_\pm(4, \mathbb{R})$ and deformation spaces. These examples give us nontrivial deformation spaces of projective structures. (There are related examples by Benoist.)
Projective structures on 3-dimensional Coxeter orbifolds

- We present some methodology to study Coxeter groups which are fundamental groups of 3-orbifolds with representations in $\mathrm{SL}_\pm(4, \mathbb{R})$ and deformation spaces. These examples give us nontrivial deformation spaces of projective structures. (There are related examples by Benoist.)

- An \textit{n-dimensional orbifold} is a topological space which is locally modeled on orbit spaces of finite groups acting on open subsets of $\mathbb{R}^n$. An orbifold is good if its universal cover is a manifold.
Projective structures on 3-dimensional Coxeter orbifolds

- We present some methodology to study Coxeter groups which are fundamental groups of 3-orbifolds with representations in $\text{SL}_\pm(4, \mathbb{R})$ and deformation spaces. These examples give us nontrivial deformation spaces of projective structures. (There are related examples by Benoist.)

- An $n$-dimensional orbifold is a topological space which is locally modeled on orbit spaces of finite groups acting on open subsets of $\mathbb{R}^n$. An orbifold is good if its universal cover is a manifold.

- The fundamental group of the orbifold will be a Coxeter group with a presentation

$$R_i, i = 1, 2, \ldots, f : R_i^2 = 1, (R_i R_j)^{n_{ij}} = 1$$

where $R_i$ is associated with silvered sides and $R_i R_j$ has order $n_{ij}$ associated with an edge.
- Let $P$ be a fixed convex 3-dimensional polyhedron. Let us assign orders at each edge.
  - $e$ the number of edges
  - $e_2$ the numbers of edges of order-two among the edges.
  - $f$ be the number of sides.
Let $P$ be a fixed convex 3-dimensional polyhedron. Let us assign orders at each edge.

- $e$ the number of edges
- $e_2$ the numbers of edges of order-two among the edges.
- $f$ be the number of sides.

**Definition.** A reflection in $S^n$ is given by a great hypersphere of fixed points and a pair of antipodal points $p, -p$ mapping to each other, which we call antipodally-fixed points.
Given two reflections \( R_1 \) and \( R_2 \) in \( S^n \), \( n \geq 1 \) with respectively distinct hyperspheres and pairs of antipodally-fixed points we can define a \textit{dihedral angle} between the respective transverse hyperspheres of fixed points \( P_1 \) and \( P_2 \):

\[ \text{If } R_1 \circ R_2 \text{ is not order-two and antipodally-fixed points of } R_1 \text{ is not in the hypersphere of } R_2 \text{ and vice versa, } R_1 \circ R_2 \text{ will fix } P_1 \cap P_2 \text{ and} \]

\[ \sim \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & I^{n-1,n-1} \end{bmatrix} \]

if \( n - 3 < \text{tr}(R_1 \circ R_2) < n + 1 \). In this case, we define the dihedral angle to be \( \theta/2 \).
If $R_1 \circ R_2$ is of order-two and $P_1$ and $P_2$ meet, then we see that the respective antipodally-fixed points $p_1$ and $p_2$ satisfy $\pm p_1 \in P_2$ and $\pm p_2 \in P_1$. In this case, the dihedral angle is defined to be $\pi/2$. 
If \( R_1 \circ R_2 \) is of order-two and \( P_1 \) and \( P_2 \) meet, then we see that the respective antipodally-fixed points \( p_1 \) and \( p_2 \) satisfy \( \pm p_1 \in P_2 \) and \( \pm p_2 \in P_1 \). In this case, the dihedral angle is defined to be \( \pi/2 \).

We remove any vertex of \( P \) which has more than three edges ending or with orders of the edges ending there is not of form

\[
(2, 2, n), n \geq 2, (2, 3, 3), (2, 3, 4), (2, 3, 5),
\]

i.e., orders of spherical triangular groups. This make \( P \) into a possibly open 3-dimensional orbifold.
If $R_1 \circ R_2$ is of order-two and $P_1$ and $P_2$ meet, then we see that the respective antipodally-fixed points $p_1$ and $p_2$ satisfy $\pm p_1 \in P_2$ and $\pm p_2 \in P_1$. In this case, the dihedral angle is defined to be $\pi/2$.

- We remove any vertex of $P$ which has more than three edges ending or with orders of the edges ending there is not of form

$$(2, 2, n), n \geq 2, (2, 3, 3), (2, 3, 4), (2, 3, 5),$$

i.e., orders of spherical triangular groups. This make $P$ into a possibly open 3-dimensional orbifold.

- Let $\hat{P}$ denote the differentiable orbifold with sides silvered and the edge orders realized as assigned from $P$ with vertices removed. We say that $\hat{P}$ has a Coxeter orbifold structure.
If $R_1 \circ R_2$ is of order-two and $P_1$ and $P_2$ meet, then we see that the respective antipodally-fixed points $p_1$ and $p_2$ satisfy $\pm p_1 \in P_2$ and $\pm p_2 \in P_1$. In this case, the dihedral angle is defined to be $\pi/2$.

- We remove any vertex of $P$ which has more than three edges ending or with orders of the edges ending there is not of form

\[(2, 2, n), n \geq 2, (2, 3, 3), (2, 3, 4), (2, 3, 5),\]

i.e., orders of spherical triangular groups. This make $P$ into a possibly open 3-dimensional orbifold.

- Let $\hat{P}$ denote the differentiable orbifold with sides silvered and the edge orders realized as assigned from $P$ with vertices removed. We say that $\hat{P}$ has a **Coxeter orbifold structure**.

- Cone-type, product-type, finite fundamental group type
Coxeter orbifold won’t be studied. The other type Coxeter orbifolds are said to be of *normal type*.
Coxeter orbifold won’t be studied. The other type Coxeter orbifolds are said to be of normal type.

- $\hat{P}$ is orderable if the sides of $P$ can be ordered so that each side has no more than three edges of order-two or edges of sides of higher index.
Coxeter orbifold won’t be studied. The other type Coxeter orbifolds are said to be of normal type.

- $\hat{P}$ is orderable if the sides of $P$ can be ordered so that each side has no more than three edges of order-two or edges of sides of higher index.

- **Main Theorem.** Let $P$ be a fixed convex polyhedron and $\hat{P}$ be given a normal-type Coxeter orbifold structure. Let $k(P)$ be the dimension of the group of projective automorphisms acting on $P$. Suppose that $\hat{P}$ is orderable. Then the restricted deformation space of projective structures on the orbifold $\hat{P}$ is a smooth manifold of dimension $3f - e - e_2 - k(P)$ if it is not empty.
Coxeter orbifold won’t be studied. The other type Coxeter orbifolds are said to be of normal type.

- $\hat{P}$ is orderable if the sides of $P$ can be ordered so that each side has no more than three edges of order-two or edges of sides of higher index.

- Main Theorem. Let $P$ be a fixed convex polyhedron and $\hat{P}$ be given a normal-type Coxeter orbifold structure. Let $k(P)$ be the dimension of the group of projective automorphisms acting on $P$. Suppose that $\hat{P}$ is orderable. Then the restricted deformation space of projective structures on the orbifold $\hat{P}$ is a smooth manifold of dimension $3f - e - e_2 - k(P)$ if it is not empty.

- To prove the main theorem, we will be working with oriented real projective geometry: i.e., the geometry based on $(S^n, SL_{\pm}(n + 1, \mathbb{R}))$ where $S^n$ is the double cover of $\mathbb{R}P^n$ and $SL_{\pm}(n + 1, \mathbb{R})$ is the group of projective automorphisms on $S^n$. 
Coxeter orbifold won’t be studied. The other type Coxeter orbifolds are said to be of *normal type*.

- \( \hat{P} \) is *orderable* if the sides of \( P \) can be ordered so that each side has no more than three edges of order-two or edges of sides of higher index.

- **Main Theorem.** Let \( P \) be a fixed convex polyhedron and \( \hat{P} \) be given a normal-type Coxeter orbifold structure. Let \( k(P) \) be the dimension of the group of projective automorphisms acting on \( P \). Suppose that \( \hat{P} \) is orderable. Then the restricted deformation space of projective structures on the orbifold \( \hat{P} \) is a smooth manifold of dimension \( 3f - e - e_2 - k(P) \) if it is not empty.

- To prove the main theorem, we will be working with oriented real projective geometry: i.e., the geometry based on \((S^n, SL_\pm(n + 1, \mathbb{R}))\) where \( S^n \) is the double cover of \( \mathbb{R}P^n \) and \( SL_\pm(n + 1, \mathbb{R}) \) is the group of projective automorphisms on \( S^n \).
A projective structure on an orbifold corresponds to a unique \((S^n, \text{SL}_\pm (n + 1, \mathbb{R}))\)-structure and vice versa (see page 143 of Thurston’s book “Three-dimensional geometry and topology”).
A projective structure on an orbifold corresponds to a unique $(S^n, \text{SL}_\pm(n + 1, \mathbb{R}))$-structure and vice versa (see page 143 of Thurston’s book “Three-dimensional geometry and topology”).

Therefore, the deformation space of projective structures on an orbifold equals that of $(S^n, \text{SL}_\pm(n + 1, \mathbb{R}))$-structures on the orbifold. In this paper, we identify them.
A projective structure on an orbifold corresponds to a unique \((S^n, \text{SL}_\pm(n+1, \mathbb{R}))\)-structure and vice versa (see page 143 of Thurston’s book “Three-dimensional geometry and topology”).

Therefore, the deformation space of projective structures on an orbifold equals that of \((S^n, \text{SL}_\pm(n+1, \mathbb{R}))\)-structures on the orbifold. In this paper, we identify them.

Since \(S^n\), a subspace of dimension \(i\) of \(\mathbb{R}^{n+1}\) corresponds to a great sphere of dimension \(i-1\). They are totally geodesic. A pair of antipodal points are considered a great 0-sphere.
A projective structure on an orbifold corresponds to a unique \((S^n, SL_{\pm}(n + 1, \mathbb{R}))\)-structure and vice versa (see page 143 of Thurston’s book “Three-dimensional geometry and topology”).

Therefore, the deformation space of projective structures on an orbifold equals that of \((S^n, SL_{\pm}(n + 1, \mathbb{R}))\)-structures on the orbifold. In this paper, we identify them.

Since \(S^n\), a subspace of dimension \(i\) of \(\mathbb{R}^{n+1}\) corresponds to a great sphere of dimension \(i - 1\). They are totally geodesic. A pair of antipodal points are considered a great 0-sphere.

**Proposition.** Let \(p\) be a point of the deformation space \(D(\hat{P})\) of projective structures on the Coxeter orbifold \(\hat{P}\). Then the map

\[
hol : D(\hat{P}) \to \text{Hom}(\pi_1(\hat{P}), \hat{G})/\hat{G}
\]

induces a local homeomorphism near \(p\) from the restricted deformation space \(D_P(\hat{P})\) to \(\text{Hom}_P(\pi_1(\hat{P}), \hat{G})/\hat{G}\).
A projective structure on an orbifold corresponds to a unique \((S^n, SL_{\pm}(n+1, \mathbb{R}))\)-structure and vice versa (see page 143 of Thurston’s book “Three-dimensional geometry and topology”).

Therefore, the deformation space of projective structures on an orbifold equals that of \((S^n, SL_{\pm}(n+1, \mathbb{R}))\)-structures on the orbifold. In this paper, we identify them.

Since \(S^n\), a subspace of dimension \(i\) of \(\mathbb{R}^{n+1}\) corresponds to a great sphere of dimension \(i - 1\). They are totally geodesic. A pair of antipodal points are considered a great 0-sphere.

**Proposition.** Let \(p\) be a point of the deformation space \(D(\hat{P})\) of projective structures on the Coxeter orbifold \(\hat{P}\). Then the map

\[
hol : D(\hat{P}) \to \text{Hom}(\pi_1(\hat{P}), \hat{G})/\hat{G}
\]

induces a local homeomorphism near \(p\) from the restricted deformation space \(D_P(\hat{P})\) to \(\text{Hom}_P(\pi_1(\hat{P}), \hat{G})/\hat{G}\).
Thus, we reduce the proof to the study of representations with “fixed $P$".
Thus, we reduce the proof to the study of representations with “fixed $P$".
Thus, we reduce the proof to the study of representations with “fixed $P$”.

For a fixed convex polyhedron, we can change the antipodally-fixed points only.
Thus, we reduce the proof to the study of representations with “fixed $P$”.

For a fixed convex polyhedron, we can change the antipodally-fixed points only.

First, we embed the convex polyhedron $P$ in a $f$-dimensional simplex $\Delta$ with each side imbedded in the side of the simplex in a one-to-one manner.
Thus, we reduce the proof to the study of representations with “fixed $P$”.

- For a fixed convex polyhedron, we can change the antipodally-fixed points only.
- First, we embed the convex polyhedron $P$ in a $f$-dimensional simplex $\Delta$ with each side imbedded in the side of the simplex in a one-to-one manner.
- For a $f$-dimensional simplex, it is easy to solve the Coxeter edge relation conditions using the methods of Kac-Vinberg. This amounts to solving a multiplicative equations.
Thus, we reduce the proof to the study of representations with “fixed $P$”.

For a fixed convex polyhedron, we can change the antipodally-fixed points only.

First, we embed the convex polyhedron $P$ in a $f$-dimensional simplex $\Delta$ with each side imbedded in the side of the simplex in a one-to-one manner.

For a $f$-dimensional simplex, it is easy to solve the Coxeter edge relation conditions using the methods of Kac-Vinberg. This amounts to solving a multiplicative equations.

We will consider the high-dimensional space of all configurations of antipodally-fixed points. These points must lie on the subspace determined by the polyhedron $P$. 
Thus, we reduce the proof to the study of representations with “fixed $P$”.

- For a fixed convex polyhedron, we can change the antipodally-fixed points only.

- First, we embed the convex polyhedron $P$ in a $f$-dimensional simplex $\Delta$ with each side imbedded in the side of the simplex in a one-to-one manner.

- For a $f$-dimensional simplex, it is easy to solve the Coxeter edge relation conditions using the methods of Kac-Vinberg. This amounts to solving a multiplicative equations.

- We will consider the high-dimensional space of all configurations of antipodally-fixed points. These points must lie on the subspace determined by the polyhedron $P$.

- We define a function for each edge defined by computing
angles according to the given antipodally-fixed points and the functions describing the subspace conditions.
angles according to the given antipodally-fixed points and the functions describing the subspace conditions.

* We find gradient vectors for these functions and show how to find the dimension of their spans.
angles according to the given antipodally-fixed points and the functions describing the subspace conditions.

* We find gradient vectors for these functions and show how to find the dimension of their spans.

* We show by a matrix technique that the dimensions are independent of the antipodally-fixed points by natural cancellations following from orderability.
angles according to the given antipodally-fixed points and the functions describing the subspace conditions.

* We find gradient vectors for these functions and show how to find the dimension of their spans.

* We show by a matrix technique that the dimensions are independent of the antipodally-fixed points by natural cancellations following from orderability.
Pictures (due to Yves Benoist)
• One of the main questions currently is how to determine if the deformation space is empty or not.

• We also wish to understand about the hyperbolic polyhedrons which we are not necessarily studying in this paper.

• We could approach the Andreev theorem for hyperbolic polyhedra from projective sides if we accomplish all these.

References


