Representation by matrices

Representation.

Basis change.

- T:V^n->W^m. $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}, \mathcal{B}' = \{\beta_1, \ldots, \beta_m\}$

$$T\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i, A(i,j) = A_{ij} \text{ mxn-matrix of} T.$$

$$\begin{array}{rcl} \alpha & = & x_1\alpha_1 + \cdots + x_n\alpha_n \\ T\alpha & = & T(\sum_{i=1}^{n} x_j\alpha_j \\ & = & \sum_{j=1}^{n} x_j T\alpha_j \\ & = & \sum_{j=1}^{n} x_j (\sum_{i=1}^{n} A_{ij}\beta_i) \\ & = & \sum_{j=1}^{n} (\sum_{i=1}^{m} x_j A_{ij})\beta_i \\ & = & \sum_{i=1}^{m} (\sum_{j=1}^{n} A_{ij} x_j)\beta_i \end{array}$$

$$(x_1, \ldots, x_n) \mapsto (\sum_{j=1}^{n} A_{1j}x_j, \ldots, \sum_{j=1}^{n} A_{mj}x_j)$$

$$[T\alpha]_{\mathcal{B}'} = A \cdot [\alpha]_{\mathcal{B}}$$

- T <-> matrix of T w.r.t B and B'
  - {T:V->W } $<\!\!-\!\!>_{B,B'}$ {A_{mxn} } 1-1 onto
  - L(V,W) $<\!\!-\!\!>_{B,B'}$ M(m,n) 1-1 onto and is a linear isomorphism
  - Example: $L(F^2,F^2) = M_{2x2}(F)$
  - $L(F^m,F^n) = M_{mxn}(F)$
- When W=V, we often use B'=B.

$$T\alpha_j = \sum_{i=1}^{n} A_{ij}\alpha_i, j = 1,\ldots,n \qquad [T]_{\mathcal{B}} = [A_{ij}]$$

$$[T\alpha]_{\mathcal{B}} = [T]_{\mathcal{B}}[\alpha]_{\mathcal{B}}$$

- Example: $V = F^{n\times 1}$, $W=F^{m\times 1}$,
  - $T:V\to W$ defined by $T(X)=AX$.
  - B, B' standard basis
  - Then $[T]_{B,B'} = A$.

$$T : C^2 \to C^2, \quad \begin{array}{rcl} y_1 & = & 2x_1 - x_2 \\ y_2 & = & x_1 + x_2 \end{array}$$

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

- Theorem: V,W,Z, T:V->W, U:W->Z basis B,B',B'', $A=[T]_{B,B'}$, $B=[U]_{B',B''}$. Then $C = AB = [U \bullet T]_{B,B''}$.
  - Matrix multiplications correspond to compositions.
- Corollary: $[UT]_B=[U]_B[T]_B$ when V=W=Z,B=B'=B''.
- Corollary. $[T^{-1}]_B = ([T]_B)^{-1}$
  - Proof: UT=I=TU. $U=T^{-1}$. $[U]_B[T]_B=[I]_B=[T]_B[U]_B$.

- Basis Change

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}, \mathcal{B}' = \{\alpha'_1, \ldots, \alpha'_n\}$$

$$\begin{aligned} \alpha'_j &= \sum_{i=1}^n P_{ij}\alpha_i \\ x_i &= \sum_{i=1}^n P_{ij}x'_j \text{ (P.51-52)} \\ [\alpha]_{\mathcal{B}} &= P[\alpha]_{\mathcal{B}'} \end{aligned}$$

$$\begin{aligned} [T\alpha]_{\mathcal{B}} &= [T]_{\mathcal{B}}[\alpha]_{\mathcal{B}} \\ [T\alpha]_{\mathcal{B}} &= P[T\alpha]_{\mathcal{B}'} \\ [T]_{\mathcal{B}}P[\alpha]_{\mathcal{B}'} &= P[T\alpha]_{\mathcal{B}'} \end{aligned}$$

$$\begin{aligned} P^{-1}[T]_{\mathcal{B}}P[\alpha]_{\mathcal{B}'} &= [T\alpha]_{\mathcal{B}'} \\ P^{-1}[T]_{\mathcal{B}}P &= [T]_{\mathcal{B}'} \end{aligned}$$

- Theorem 14: $[T]_{B'} = P^{-1}[T]_B P$. Let U be s.t. $Ua_j = a_j'$ j=1,.., n. $P=[P_1,\ldots,P_n]$. $P_j = [a'_j]_B$.
  - Then $[U]_B = P$ and $[T]_{B'} = [U]_B^{-1}[T]_B[U]_B$.
- Examples: $[T]_{\mathcal{B}} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$$\mathcal{B} = \{[1,0],[0,1]\}, \mathcal{B}' = \{[1,1],[1,-1]\}$$

$$[1,1] = 1[1,0] + 1[0,1], [1,-1] = 1[1,0] + (-1)[0,1]$$

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, P^{-1} = 1/2\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$[T]_{\mathcal{B}'} = (1/2)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (1/2)\begin{pmatrix} 3 & 3 \\ -1 & 3 \end{pmatrix}$$

- Example:

$$V = \{f : R \to R | f \text{ is a polynomial of degree} \leq 3\}$$

$$\mathcal{B} = \{f_1, f_2, f_3, f_4\}, f_i(x) = x^{i-1}, i = 1, 2, 3, 4.$$

  - B is a basis
  - Define $g_i(x) := (x+t)^{i-1}, i = 1, 2, 3, 4, t \in R$

$$\begin{aligned} g_1 &= f_1 = 1 \\ g_2 &= (x+t)^1 = t f_1 + f_2 \\ g_3 &= (x+t)^2 = t^2 f_1 + 2t f_2 + f_3 \\ g_3 &= t^3 f_1 + 3t^2 f_2 + 3t f_3 + f_4 \end{aligned}$$

$$P = \begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 1 & 3t \\ 0 & 0 & 0 & 1 \end{pmatrix}, P^{-1} = \begin{pmatrix} 1 & -t & t^2 & -t^3 \\ 0 & 1 & -2t & 3t^2 \\ 0 & 0 & 1 & -3t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- D:V->V is a differentiation.

$$[D]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[D]_{\mathcal{B}'} = P^{-1}[D]_{\mathcal{B}}P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  - not changed as you can compute from

$$Dg_1 = 0, Dg_2 = g_1, Dg_3 = 2g_2, Dg_4 = 3g_3$$

Linear functionals

- Linear functionals are another devices to. They are almost like vectors but are not vectors. Engineers do not distinguish them. Often one does not need to….
- They were used to be called covariant vectors. (usual vectors were called contravariant vectors) by Einstein and so on.
- These distinctions help.
- Dirac functionals are linear functionals.
- Many singular functions are really functionals.
- They are not mysterious things.

- In mathematics, we give a definition and the mystery disappears (in theory).
- f:V->F. V over F is a linear functional if
  - f(ca+b)=cf(a)+f(b), c in F, a,b in V.
- If V is finite dimensional, it is easy to classify:
- Define f:F^n->F by $f(x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n$
  - Then f is a linear functional and is represented by a row matrix $[a_1, \ldots, a_n]$
  - V is f.d case: A subspace N_f defined by one nonzero linear functional f. dim N_f = dim V – 1 (so-called hyperspace)

- Every linear functional is of this form:

$$f(x_1, \ldots, x_n) = f(\sum_{j=1}^{n} x_j \epsilon_j) = \sum_{j=1}^{n} x_j f(\epsilon_j) = \sum_{j=1}^{n} a_j x_j$$

- Example:
  C[a,b]={f:[a,b]->R|f is continuous}
  is a vector space over R.
  - L:C[a,b] -> R, L(g)=g(x) is a linear functional. x is some point of R.
  - L:C[a,b]->R, L(g) = $\int_a^b g(t)dt$ is a linear functional.

- V^* := L(V, F) is a vector space called the dual space.
- dim V^*= dim V by Theorem 5. Ch 3.
- Find a basis of V^*:
  - B={a_1,…,a_n} is a given basis of V.
  - By Theorem 1 (p.69), there exists unique $f_{i:}$V->F such that $f_i(a_j)=\delta_{ij}$ for I=1,…,n, j=1,…,n.
  - {f_1,…,f_n} is a basis of V^*: We only need to show they are linearly independent.

- Proof: Let $f = \sum_{i=1}^{n} c_i f_i$.

$$f(\alpha_j) = \sum_{i=1}^{n} c_i f_i(\alpha_j) = \sum_{i=1}^{n} c_i \delta_{ij} = c_j \quad --(*)$$

If $\sum c_i f_i = 0$, then $f(\alpha_j) = \sum_{i=1}^{n} c_i f_i(\alpha_j) = c_j = 0.\ \forall j = 1, ..., n, c_j = 0.$

- $B^*=\{f_1,\ldots,f_n\}$ is the dual basis of V^*.
- Theorem 15:
  - (1) B^* is a basis. (2) $f = \sum_{i=1}^{n} f(\alpha_i) f_i$
  - (3) $\alpha = \sum_{i=1}^{n} f_i(\alpha_i) \alpha_i$
  - Proof: (1) done. (2) from (*)

– Proof continued: (3)

$$\begin{aligned} \alpha &= \sum_{i=1}^n x_i \alpha_i \\ f_j(\alpha) &= \sum_{i=1}^n x_i f_j(\alpha_i) \\ &= \sum_{i=1}^n x_i \delta_{ij} = x_j \\ \alpha &= \sum_{i=1}^n f_i(\alpha) \alpha_i \end{aligned}$$

- Example: l(x,y) := 2x+y defined on F^2.
  - Basis [1,0], [0,1]
  - Dual basis $f_1(x,y):=x$, $f_2(x,y):=y$
  - $L=2f_1+f_2$. (f([1,0])=2, f([0,1])=1)
  - $[x,y]=x[1,0]+y[0,1]=f_1([x,y])[1,0]+f_2([x,y])[0,1]$

Annihilators

- Definition: S a subset of V.
  S^0 = annihilator(S) :={f:V->F|f(a)=0, for all a in V}.
- S^0 is a vector subspace of V^*.
- $\{0\}^0 = V^*$. $V^0=\{0\}$.
- Theorem: W subspace of V f.d.v.s over F. dim W+ dim W^0= dim V.

- Proof: $\{a_1,\ldots,a_k\}$ basis of W.
  - Extend to V. $\{a_1,\ldots,a_k,a_{k+1},\ldots,a_n\}$ basis of V.
  - $\{f_1,\ldots,f_k,f_{k+1},\ldots,f_n\}$ dual basis V*.
  - $\{f_{k+1},\ldots,f_n\}$ is a basis of W^0:
    - $f_{k+1},\ldots,f_n$ are zero on $a_1,\ldots,a_k$ and hence zero on W and hence in W^0.
    - $f_{k+1},\ldots,f_n$ are independent in W^0.
    - They span W^0 :
      - Let f be in W^0. Then

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i = \sum_{i=k+1}^{n} f(\alpha_i) f_i$$

- Corollary: W k-dim subspace of V. Then W is the intersection of n-k hyperspaces.
  - Proof: In the proof of the above theorem, W is precisely the set of vectors zero under $f_{k+1},\ldots,f_n$. Each f_i gives a hyperspace.
- Corollary: W_1,W_2 subspaces of f.d.v.s. V. Then $W_1=W_2$ iff $W_1^0=W_2^0$.
  - Proof: (->) obvious
    - (<-) If $W_1\neq W_2$, then there exists v in W_2 not in W_1 (w.l.o.g).
    - By above corollary, there exists f in V* s.t. f(v) $\neq$0 and f| W_1=0. Then f in W_1^0 but not in W_2^0.

- Solving a linear system of equations:
  $A_{11}x_1+\ldots+A_{1n}x_n=0,$
  $\ldots \qquad \ldots \qquad \ldots$
  $A_{m1}x_1+\ldots+A_{mn}x_n=0.$
- Let $f_i(x_1,\ldots,x_n)= A_{i1}x_1+\ldots+A_{in}x_n.$
- The solution space is the subspace of F^n of all v s.t. $f_i(v)=0$, $i=1,\ldots,m$.
- Dual point of view of this. To find the annihilators given a number of vectors:
  - Given vectors $a_i =(A_{i1},\ldots,A_{in})$ in F^n.
  - Let $f(x_1,\ldots,x_n)= c_1x_1+\ldots+c_nx_n$.

- The condition that f is in the annhilator of the subspace S span by a_i is : $\sum_{j=1}^{n} A_{ij} c_j = 0$
- The solution of the system AX=0 is S^0.
- Thus we can apply row-reduction techniques to solve for S^0.
- Example 24: a1=(2,-2,3,4,-1),
  - a2=(-1,1,2,5,2), a3=(0,0,-1,-2,3),
  - a4=(1,-1,2,3,0).

$$A = \begin{pmatrix} 2 & -2 & 3 & 4 & -1 \\ -1 & 1 & 2 & 5 & 2 \\ 0 & 0 & -1 & -2 & 3 \\ 1 & -1 & 2 & 3 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f(x_1, \ldots, x_5) = \sum_{j=1}^{5} c_j x_j, \quad \sum_{j=1}^{5} A_{ij} c_j = 0, \quad \sum_{j=1}^{5} R_{ij} c_j = 0$$

$$c_1 - c_2 - c_4 = 0, c_3 + 2c_4 = 0, c_5 = 0$$

$$c_2 = a, c_4 = b, c_1 = a + b, c_3 = -2b, c_5 = 0$$

$$f(x_1, x_2, x_3, x_4) = (a + b)x_1 + ax_2 - 2bx_3 + bx_4$$