2.1. Introduction to Systems of linear equations

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Linear Systems

- Recall lines and planes
- $x+y=4$, $x-5y=1$. Solve this system.
- $x+y-z=0$, $x+y+z=1$, $x-y=1$. Solve this system.

- System of linear equations:
- $a_{11} x_1+a_{12} x_2+\dots+a_{1n} x_n=b_1$
- $a_{21} x_1+a_{22} x_2+\dots+a_{2n} x_n=b_2$
- ....
- $a_{m1} x_1+a_{m2} x_2+\dots+a_{mn} x_n=b_m$
- The system is homogeneous if all b_i s are zero.

Theoretical: Do solutions exists and if so, how many and in what form?

- To see this, we play with lines first.
  - Two lines are parallel and are disjoint. -> no solution
  - Two lines are parallel and coincide. -> infinitely many solutions
  - Two lines are not parallel and intersect at unique point.-> unique solution
- In the 3-space, see Figure 2.1.2.

- Theorem 2.1.1. Every system of linear equations have zero, one, or infinitely many solutions.
- Proof: We will see this through Gaussian eliminations.
- Examples:
- $x-y=0$, $x+y=1$ -> $x-y=0$,$2y=1$-> $y=1/2$, $x=-1/2$
- $x-y=2$, $2x-2y=0$ -> $x-y=0$, $0=-4$ -> no solution
- $x-y=-1$, $3x-3y=-3$ -> $x-y=-1$, $0=0$ -> $y=t$, $x=t-1$.

Augmented matrices and elementary row operations

$$\begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & \vdots & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

Elementary operations

- These operations can be used to solve a system of equations
  1. Multiply an equation through by a nonzero constant
  2. Interchange to equations
  3. Add a multiple of one equation to another.
- For a matrix (an augmented one)

  Multiply a row through a nonzero constant

  Interchange two rows

  Add a multiple of one row to another.

- Example
- See Example 6
- Example 7. Write a vector w=(4,1,3) as a linear combination of vectors a=(1,1,1), b=(2,-1,2), c=(0,0,1).

  Solve W=c_1a+c_2b+c_3c.

  Solve (4,1,3)=c_1(1,1,1)+c_2(2,-1,2)+c_3(0,0,1)

  (4,1,3)=(c_1+2c_2,c_1-c_2,c_1+2c_2+c_3)

$$\begin{array}{ccccccc} c_1 & + & 2c_2 & & & = & 4 \\ c_1 & - & c_2 & & & = & 1 \\ c_1 & + & 2c_2 & + & c_3 & = & 3 \end{array}$$

Comments Ex.2.1

- 1-4 Linearity of equations
- 5-6 solution checking
- 7-8 Graphing problems
- 9,10 solving
- 11,12 Finding equations (Eliminate t)
- 15,16 number of solutions
- 17-24 System <-> augmented matrix
- 25,26 Elementary row operations

2.2. Solving equations by row reduction

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Row reduced echelon form

Reduced row echelon form

- Recall Ex. 6.

  A) If a row is not zero, the first nonzero element is 1. (Leading 1)

  B) A zero row lie below all nonzero rows.

  C) The leading 1 of the lower row starts later than the leading 1 of the upper row (up to here row echelon form)

  D) Each column containing a leading 1 is zero elsewhere.

Examples

- See Examples 1, 2.
- Once one obtains a reduced row echelon matrix, then what do we do?
- Example 4 (a) no solutions (contradiction always arises)
- Exampe 4 (b) OK. x+ 3z = -1, y-4z= 2

  z is not associated with any leading one.

  We call such z a free variable.

  Let assign t to z. Then x=-1-3t, y=2+4t, z=t

  Is the solution space. (infinite case)

  In the unique case there is no free variable.

- It is often desirable to write solution spaces as linear combinations of column vectors.
- For Example 4 (b)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1-3\mathrm{t} \\ 2+4\mathrm{t} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

Gauss Jordan Eliminations

- Using elementary row operations, one can make any matrix into a reduced row echelon form.
- Thus any system of equations can be solved.
- The basic step.

    1. Find a first column with a nonzero entry. (The column is usually the first one.)
    2. Interchange so that the nonzero entry is at the top row
    3. Multiply to make it to be 1.
    4. Use 1, to make the entries below it to become 0.

Now take the first row out of considerations and work in (m-1)xn matrix A' below.

We work the same way:

- Find the first column A' with nonzero entry.
- Interchange to move it to the first row of A'.
- Scalar-multiply to make it 1.
- Make the entries below 1 to be zero.

We obtain row-echelon form.

Now use each leading 1 to make the other entries in the same column become zero.

This is called the Gauss-Jordan elimination.

Gaussian elimination: not do the reduction.

Pivot positions: positions of leading 1s.

Pivot columns: columns that containing leading 1s.

Solving equations

- A system of linear equations
- Take an augmented matrix
- Gauss-Jordan elimination
- Make it back into a system of linear equations.
- Take free variables: variables not associate with leading 1s.
- Solve
- Finally make it into a linear combinations of column vectors.
- See Board for an example.

Back substitution

- One can use Gaussian elimination only.
- Make it back into a system of linear equations.
- In this case, we solve from the lowest equation.
- See the black board.
- Concluding Theorem: A system of linear equations have either no solution or a unique solution or infinitely many solutions.

Homogeneous linear system

- If the system of linear equations consists of only homogeneous linear equations, we call the system homogeneous.
- This is equivalent to all constant terms being 0.
- A homogeneous linear system always have at least one solution (0,0,…,0) a trivial solution.
- Theorem 2.2.1. A homogeneous system has either a unique trivial solution or infinitely many solutions.

Consequences

- Theorem 2.2.2. homogeneous linear system with n unknowns. Its reduced echelon matrix has r nonzero rows. Then the system has n-r free variables.
- This is actually the dimension of the solution space….

Computer implementations

- We always use largest nonzero element to put in the top rows.
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Comments on Ex. 2.2.

- 1-8 recognizing forms (reduced row echelon forms, row echelon forms…)
- 9-14. Solving given reduced echelon forms.
- 17-20 Back substitution
- 23-42 Solving. Gauss-Jordan
- 43-46 A bit tricky….
- 52…