

1.2. Dot Products, Orthogonality

- ▶ $v=(v_1, v_2, \dots, v_n)=v_1e_1 + v_2e_2 + \dots + v_ne_n$.
This is a unique expression.
- ▶ Distances when given position vectors:
 $D(P_1, P_2) = \|P_1 - P_2\| = ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}$ in 2-space.
- ▶ In 3-space
 $d(P_1, P_2) = \|P_2 - P_1\| = ((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2)^{1/2}$
- ▶ In n -space $u=(u_1, u_2, \dots, u_n)$, $v=(v_1, v_2, \dots, v_n)$,
Then $d(u, v) = ((u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2)^{1/2}$.

lengths

- ▶ Length, norm, magnitude of a vector
 $v=(v_1, \dots, v_n)$ is $\|v\| = (v_1^2 + v_2^2 + \dots + v_n^2)^{1/2}$.
- ▶ Examples $v=(1, 1, \dots, 1)$ $\|v\|=n/2$.
- ▶ Unit vectors $u=v/\|v\|$ corresponds to directions.
- ▶ Standard unit vectors
 $i=(1, 0)$, $j=(0, 1)$ in R^2
 $i=(1, 0, 0)$, $j=(0, 1, 0)$, $k=(0, 0, 1)$ in R^3
 $e_1=(1, 0, \dots, 0)$, $e_2=(0, 1, \dots, 0)$, ..., $e_n=(0, 0, \dots, 1)$ in R^n .
- ▶ Theorem. Two position vectors u, v in R^n .
 $d(u, v) \geq 0$, $d(u, v) = d(v, u)$, $d(u, v) = 0$ if and only if $u = v$.
- ▶ Proof: Use the formula.
- ▶ We now introduce dot product. Given two vectors, the dot product gives you a real number.
- ▶ The dot product generalizes length and angle and is useful to compute many quantities. In fact, it is more fundamental than angles. Given
 $u=(u_1, u_2, \dots, u_n)$, $v=(v_1, v_2, \dots, v_n)$,
 $u \cdot v = u_1v_1 + u_2v_2 + \dots + u_nv_n$.

Properties of dot products

▶ $\|v\| = (v \cdot v)^{1/2}$.

▶ Theorem 1.2.6

$u \cdot v = v \cdot u$, Symmetry

$u \cdot (v+w) = u \cdot v + u \cdot w$. distributivity

$k(u \cdot v) = (ku) \cdot v$ homogenous

$v \cdot v \geq 0$, and $v \cdot v = 0$ if and only if $v = 0$. positivity

▶ Theorem 1.2.7.

$0 \cdot v = v \cdot 0 = 0$

$(u+v) \cdot w = u \cdot w + v \cdot w$

$u \cdot (v-w) = u \cdot v - u \cdot w$, $(u-v) \cdot w = u \cdot w - v \cdot w$

$k(u \cdot v) = u \cdot (kv)$

10년 2월 3일

▶ We consider θ to be in $[0, \pi]$ interval.

▶ Orthogonality.

$u \cdot v = 0$ iff $\cos \theta = 0$ iff $\theta = \pi/2$.

Two nonzero vectors in 2- or 3-spaces are perpendicular if and only if their dot product is zero.

▶ See Example 5, 6. (See board)

▶ Definition. We extend the above formula to hold for n-space as well.

▶ Thus two vectors in n-spaces are *orthogonal* if their dot product is zero. A nonempty set of vectors is said to be an orthogonal set if each pair of distinct vectors are orthogonal.

▶ Use perpendicular for nonzero-vectors. 10년 2월 3일

▶ Theorems 1.2.6, 1.2.7 gives us a means to compute as one does with real numbers. (See board.)

▶ Theorem 1.2.8: u, v nonzero vectors in $\mathbb{R}^2, \mathbb{R}^3$.

If θ is an angle between u and v , then

$\cos \theta = u \cdot v / (\|u\| \|v\|)$ or

$\theta = \cos^{-1}(u \cdot v / (\|u\| \|v\|))$.

▶ Proof: Use cosine law

$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$.

Now $\|v-u\|^2 = (v-u) \cdot (v-u) = (v-u) \cdot v - (v-u) \cdot u = v \cdot v - u \cdot v -$

$v \cdot u + u \cdot u = \|v\|^2 - 2u \cdot v + \|u\|^2$.

$\|v\|^2 - 2u \cdot v + \|u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$.

We simplify to get above.

10년 2월 3일

▶ Zero vector 0 is orthogonal to every vector in \mathbb{R}^n . Actually, it is the only such vector in \mathbb{R}^n .

▶ $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

▶ Orthonormal set. Two vectors are orthonormal if they are orthogonal and have length 1. A set of vectors is *orthonormal* if every vector in the set has length 1 and each pair of vectors is orthogonal.

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Pythagoras theorem: If u and v are orthogonal vectors, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

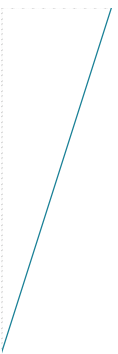
Proof: $\|u+v\|^2 = (u+v) \cdot (u+v) = \|u\|^2 + \|v\|^2 + 2u \cdot v = \|u\|^2 + \|v\|^2.$

Cauchy-Swartz inequality

$$(u \cdot v)^2 \leq \|u\|^2 \|v\|^2 \text{ or } |(u \cdot v)| \leq \|u\| \|v\|$$

Proof: If $u=0$ or $v=0$, then true.

(See board.)



10년 2월 3일

Theorem 1.2.14. Parallelogram equation for vectors.

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

- ▶ Proof: see board
- ▶ Triangle inequality: u, v, w vectors
 $d(u, v) \leq d(u, w) + d(w, v).$
- ▶ Proof: see board.



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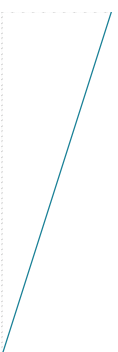
- ▶ Triangle inequality: u, v, w vectors.

$$\|u+v\| \leq \|u\| + \|v\|.$$

- ▶ Proof: $\|u+v\|^2 = (u+v) \cdot (u+v) =$

$$\|u\|^2 + 2(u \cdot v) + \|v\|^2 \leq \|u\|^2 + 2|u \cdot v| + \|v\|^2 \leq$$

$$\|u\|^2 + 2\|u\| \|v\| + \|v\|^2$$



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1.3. Vector equations of lines and planes



Lines

- ▶ General equation for lines in 2-space:
 $Ax+By=C$. (A, B not both zero)
- ▶ $Ax+By=0$ (passes origin)

- ▶ Another method (parametric equation): Let a line pass through x_0 .

If x is a point on the line, then $x-x_0$ is always parallel to a fixed vector say v .

- ▶ Thus $x-x_0=tv$ for some real number t .

- ▶ $x = x_0+tv$. (t is called a parameter)

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- ▶ Actually, one can turn the general equation to parametric equation in \mathbb{R}^2 and conversely.

- ▶ General to parametric: Find two points in it and use the two-point vector equation.

$$7x+5y=35. (5,0) \text{ and } (7,0).$$

$$X=(1-t)(0,7)+t(5,0). \quad x=5t, y=7-7t.$$

- ▶ Parametric to general: Eliminate t from the equation:

$$x=5t, y=7-7t. \text{ Then } 7x+5y = 35. \text{ This is the general equation.}$$

- ▶ Final comment: to give general equations for lines in 3-space, we need two equations.

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- ▶ $(x,y)=(x_0,y_0)+t(a,b)$.
- ▶ $x=x_0+ta, y=y_0+tb$.

- ▶ In 3-space, $(x,y,z)=(x_0,y_0,z_0)+t(a,b,c)$.
Thus, $x=x_0+ta, y=y_0+tb, z=z_0+tc$.

- ▶ Given two points x_1, x_0 in \mathbb{R}^2 or \mathbb{R}^3 , we try to find a line through them.

The line is parallel to $x_1 - x_0$.

$$\text{Thus } x = x_0 + t(x_1 - x_0) \text{ or } x = (1-t)x_0 + tx_1.$$

This is a *two-point vector equation*.

If t is in $[0, 1]$, then the point is in the segment with endpoints x_0, x_1 .

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A plane in \mathbb{R}^3

- ▶ From a plane S in \mathbb{R}^3 , we can obtain a point x_0 and a perpendicular vector n .

- ▶ From x_0 , and n , we can obtain a *point-normal equation* of S :
 $n \cdot (x-x_0)=0$.

- ▶ Conversely, any x satisfying the equation lies in S .

- ▶ $(A,B,C) \cdot (x-x_0, y-y_0, z-z_0)=0$.

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$$

$$Ax+By+Cz=D. \text{ (general equation of } S.)$$

Rmk: The coefficients give us the normal vector.

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- ▶ Actually S passes 0 if and only if $D=0$.

- ▶ There is also a parametric equation of a plane:

Given a plane W , let x_0 be a point and let v_1 and v_2 be two vectors parallel to W .

Then $t_1v_1+t_2v_2$ is also parallel to W for any real numbers t_1 and t_2 by parallelogram laws.

Thus $x_0+t_1v_1+t_2v_2$ lies in W .

Conversely, given any point x in W , $x-x_0$ is parallel to W and hence equals $t_1v_1+t_2v_2$ for some real numbers t_1 and t_2 .

Thus $x=x_0+t_1v_1+t_2v_2$ is the equation of points of W .

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- ▶ In general \mathbb{R}^n :

- ▶ A line through x_0 parallel to v :

$$X=x_0+tv.$$

- ▶ A plane through x_0 parallel to v_1, v_2 .

$$X=x_0+t_1v_1+t_2v_2$$

- ▶ Actually, we can do s -dimensional subspace with s parallel vectors. But we stop here.

- ▶ See Example 8 (page 34)

10년 2월 3일

- ▶ Examples: Given a point, and two vectors, find parametric equations.

- ▶ Given three points x_0, x_1, x_2 on W , find a parametric equation

$$x = x_0+t_1(x_1-x_0)+t_2(x_2-x_0).$$

- ▶ From general equation to a parametric equation. (Example 7)

Solution: is to find three distinct point and use the above.

- ▶ From parametric equation to a general equation. (not yet studied.)

10년 2월 3일

Comments on homework

- ▶ Ex set 1.2. Mostly computations.

- ▶ 1.2:13-16 use the definition

- ▶ 1.2: 32-35 Sigma notations (expect to know)

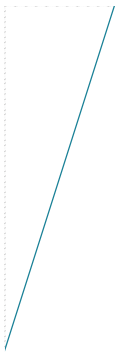
- ▶ 1.3: Two planes are parallel if their normal vectors are parallel. (perpendicular: the same)

- ▶ Finding normal vectors to the plane: Take the coefficients. (1.3:26-35)

- ▶ 1.3:37-38. Finding intersection line: Find two points in the intersections.

- ▶ 1.3:39-40. Use substitutions.

10년 2월 3일



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