1.2.Dot Products, Orthogonality

lengths

- Length, norm, magnitude of a vector $v=(v_1,\ldots,v_n)$ is $||v||= (v_1^2+v_2^2+\ldots+v_n^2)^{1/2}$.
- Examples $v=(1,1,\ldots,1)$ $||v||=n/2$.
- Unit vectors $u=v/||v||$ corresponds to directions.
- Standard unit vectors

  $i=(1,0)$, $j=(0,1)$ in R^2

  $i=(1,0,0)$, $j=(0,1,0)$, $k=(0,0,1)$ in R^3

  e_1=(1,0,..,0), e_2=(0,1,…,0), …,e_n=(0,0,…,1) in R^n.

- v=(v_1,v_2, …,v_n)= v_1e_1 + v_2e_2 + … +v_n e_n.
  This is a unique expression.
- Distances when given position vectors:
- D(P_1,P_2) = ||P_1P_2||=
  $((x\_2-x\_1)^2+(y\_2-y\_1)^2)^{1/2}$ in 2-space.
- In 3-space
  d(P_1,P_2) = ||P_2-P_1||=$((x\_2-x\_1)^2+(y\_2-y\_1)^2+(z\_2-z\_1)^2)^{1/2}$
- In n-space u=(u_1,u_2,…,u_n), v=(v_1,v_2,…,v_n),

Then d(u,v) = $((u\_1-v\_1)^2+(u\_2-v\_2)^2+\ldots+(u\_n-v\_n)^2)^{1/2}$.

- Theorem. Two position vectors u,v in R^n.
  d(u,v)≥0, d(u,v)=d(v,u), d(u,v)=0 if and only if u=v.
- Proof: Use the formula.

- We now introduce dot product. Given two vectors, the dot product gives you a real number.
- The dot product generalizes length and angle and is useful to compute many quantities. In fact, it is more fundamental than angles. Given u=(u_1,u_2,…,u_n), v=(v_1,v_2,…,v_n),

u•v = u_1v_1+u_2v_2+…+u_nv_n.

Properties of dot products

- $||v||=(v\bullet v)1/2$.
- Theorem 1.2.6
  - $u\bullet v=v\bullet u$, Symmetry
  - $u\bullet(v+w)= u\bullet v+u\bullet w$. distributivity
  - $k(u\bullet v)=(ku)\bullet v$ homogenous
  - $v\bullet v\geq 0$, and $v\bullet v=0$ if and only if $v=0$. positivity
- Theorem 1.2.7.
  - $0\bullet v=v\bullet 0=0$
  - $(u+v)\bullet w=u\bullet w+v\bullet w$
  - $u\bullet(v-w)=u\bullet v-u\bullet w$, $(u-v)\bullet w=u\bullet w-v\bullet w$
  - $k(u\bullet v)=u\bullet(kv)$

- Theorems 1.2.6,1.2.7 gives us a means to compute as one does with real numbers. (See board.)
- Theorem 1.2.8:u, v nonzero vectors in R^2, R^3. If θ is an angle between u and v, then
  - $\cos\theta = u\bullet v/(||u||||v||)$ or
  - $\theta=\cos^{-1}(u\bullet v/||u||||v||)$.
- Proof: Use cosine law
  - $||v-u||^2=||u||^2+||v||^2 – 2||u||||v||\cos\theta$.
  - Now$||v-u||2= (v-u)\bullet(v-u) = (v-u)\bullet v – (v-u)\bullet u = v\bullet v –u\bullet v-v\bullet u+u\bullet u= ||v||^2-2u\bullet v+||u||^2$.
  - $||v||^2-2u\bullet v+||u||^2 = ||u||^2+||v||^2 – 2||u||||v||\cos\theta$.
  - We simplify to get above.

- We consider θ to be in [0,π] interval.
- Orthogonality.
  - $u\bullet v=0$ iff $\cos\theta=0$ iff $\theta=\pi/2$.
  - Two nonzero vectors in 2- or 3-spaces are perpendicular if and only if their dot product is zero.
- See Example 5,6.(See board)
- Definition. We extend he above formula to hold for n-space as well.
- Thus two vectors in n-spaces are *orthogonal* if their dot product is zero. A nonempty set of vectors is said to be an orthogonal set if each pair of distinct vectors are orthogonal.
- Use perpendicular for nonzero-vectors.

- Zero vector 0 is orthorgonal to every vector in R^n. Actually, it is the only such vector in R^n.
- {(1,0,0),(0,1,0),(0,0,1)}
- Orthonormal set. Two vectors are orthonormal if they are orthogonal and have length 1. A set of vectors is *orthonormal* if every vector in the set has length 1 and each pair of vectors is orthogonal.

Phythagoras theorem: If u and v are orthogonal vectors, then

$||u+v||^2=||u||^2+||v||^2$.

Proof: $||u+v||^2 =(u+v).(u+v)=||u||^2+||v||^2+2u.v$

$= ||u||^2+||v||^2$.

Cauchy-Swartz inequality

$(u.v)^2 \le ||u||^2||v||^2$ or $|(u.v)|\le ||u||^2||v||^2$

Proof: If u=0 or v=0, then true.

(See board.)

- Triangle inequality: u,v,w vectors.

  $||u+v|| \le ||u||+||v||$.

- Proof: $||u+v||^2=(u+v).(u+v)=$

$||u||^2+2(u.v)+||v||^2 \le ||u||^2+2|u.v|+||v||^2\le$

$||u||^2+2||u||||v||+||v||^2$

Theorem 1.2.14. Parallelogram equation for vectors.

$||u+v||^2+||u-v||^2 = 2(||u||^2+||v||^2)$.

- Proof: see board
- Triangle inequality: u,v,w vectors $d(u,v)\le d(u,w)+d(w,v)$.
- Proof: see board.

1.3. Vector equations of lines and planes

Lines

- General equation for lines in 2-space:
  Ax+By= C. (A,B not both zero)
- Ax+By=0 (passes origin)

- Another method (parametric equation): Let a line pass through x_0.
  If x is a point on the line, then x-x_0 is always parallel to a fixed vector say v.
- Thus x-x_0=tv for some real number t.
- x = x_0+tv. (t is called a parameter)

- (x,y)=(x_0,y_0)+t(a,b).
- x=x_0+ta, y=y_0+tb.

- In 3-space, (x,y,z)=(x_0,y_0,z_0)+t(a,b,c).
  Thus, x=x_0+ta, y=y_0+tb, z=z_0+tc.

- Given two points x_1, x_0 in R^2 or R^3, we try to find a line through them.
  The line is parallel to x_1 – x_0.
  Thus x= x_0+t(x_1-x_0) or x= (1-t)x_0+tx_1.
  This is a *two-point vector equation*.
  If t is in [0,1], then the point is in the segment with endpoints x_0, x_1.

- Actually, one can turn the general equation to parametric equation in R^2 and conversely.
- General to parametric: Find two points in it and use the two-point vector equation.
  7x+5y=35. (5,0) and (7,0).
  X=(1-t)(0,7)+t(5,0).   x=5t, y=7-7t.
- Parametric to general: Eliminate t from the equation:
  x=5t, y= 7-7t. Then 7x+5y = 35. This is the general equation.
- Final comment: to give general equations for lines in 3-space, we need two equations.

A plane in R^3

- From a plane S in R^3, we can obtain a point x_0 and a perpendicular vector n.
- From x_0, and ***n***, we can obtain a *point-normal equation* of S:
  n.(x-x_0)=0.
- Conversely, any x satisfying the equation lies in S.
- (A,B,C).(x-x_0,y-y_0,z-z_0)=0.
  A(x-x_0)+B(y-y_0)+C(z-z_0)=0.
  Ax+By+Cz=D. (general equation of S.)
  Rmk:The coefficients give us the normal vector.

- Actually S passes 0 if and only if D=0.
- There is also a parametric equation of a plane:

  Given a plane W, let x_0 be a point and let v_1 and v_2 be two vectors parallel to W.

  Then $t_1v_1+t_2v_2$ is also parallel to W for any real numbers t_1 and t_2 by parallelogram laws.

Thus $x_0+t_1v_1+t_2v_2$ lies in W.

  Conversely, given any point x in W, $x- x_0$ is parallel to W and hence equals $t_1v_1+t_2v_2$ for some real numbers t_1 and t_2.

  Thus $x=x_0+t_1v_1+t_2v_2$ is the equation of points of W.

- Examples: Given a point, and two vectors, find parametric equations.
- Given three points x_0,x_1,x_2 on W, find a parametric equation

  $x = x_0+t_1(x_1-x_0)+t_2(x_2-x_0)$.
- From general equation to a parametric equation. (Example 7)

  Solution: is to find three distinct point and use the above.
- From parametric equation to a general equation.

(not yet studied.)

- In general R^n:
- A line through x_0 parallel to v:

  $X=x_0+tv$.
- A plane through x_0 parallel to v_1, v_2.

  $X=x_0+t_1v_1+t_2v_2$

- Actually, we can do s-dimensional subspace with s parallel vectors. But we stop here.
- See Example 8 (page 34)

Comments on homework

- Ex set 1.2. Mostly computations.
- 1.2:13-16 use the definition
- 1.2: 32-35 Sigma notations (expect to know)
- 1.3: Two planes are parallel if their normal vectors are parallel. (perpendicular: the same)
- Finding normal vectors to the plane: Take the coefficients. (1.3:26-35)
- 1.3:37-38. Finding intersection line: Find two points in the intersections.
- 1.3:39-40. Use substitutions.

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