Geometry of linear operators

Orthogonal opertors

Norm preserving operators

- Orthogonal <-> dot product preserving -> angle preserving, orthogonality preserving

Theorem 6.2.1 *If $T : R^n \to R^n$ is a linear operator on R^n, then the following statements are equivalent.*

(a) $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n. **[*T* orthogonal (i.e., length preserving)]**

(b) $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n. **[*T* is dot product preserving.]**

- Proof: (a)->(b). $||x+y||^2=(x+y).(x+y)$. $||x-y||^2=(x-y).(x-y)$.
  - By adding, we obtain $¼(||x+y||^2-||x-y||^2)=(x.y)$.
  - $T(x).T(y) = ¼(||Tx+Ty||^2-||Tx-Ty||^2)=1/4(||T(x+y)||^2 -||T(x-y)||^2) = 1/4(||x+y||^2-||x-y||^2)=(x.y)$.
  - (b)->(a) omit

Orthogonal operators preserve angles and orthogonality

- Θ=Arccos(x.y/(||x||||y||).
- If T is an orthogonal transformation R^n->R^n, then angle(Tx,Ty)=Arccos(Tx.Ty/||Tx||||Ty||) = Arccos(x.y/||x||||y||)=angle(x,y).
- Thus the orthogonal maps preserve angles and in particular orthogonal pair of vectors.
- Rotations and reflections are othogonal maps.
- An orthogonal projection is not an orthogonal map.
- The angle preserving means k times an orthogonal map.

Orthogonal matrices

Definition 6.2.2 A square matrix A is said to be ***orthogonal*** if $A^{-1} = A^T$.

- Or $AA^T=I$ or $A^TA=I$.
- Orthogonal matrix is always nonsingular.
- Example: Rotation and reflection matrices.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = I$$

- T_A is orthogonal <-> A is orthogonal: to be proved later

Theorem 6.2.3

(*a*) *The transpose of an orthogonal matrix is orthogonal.*

(*b*) *The inverse of an orthogonal matrix is orthogonal.*

(*c*) *A product of orthogonal matrices is orthogonal.*

(*d*) *If A is orthogonal, then* $\det(A) = 1$ *or* $\det(A) = -1$.

- Proof (a) $A^TA=I$. $A^T(A^T)^T=I$. A^T is orthogonal.
- (b) $(A^{-1})^T=(A^T)^T=A=(A^{-1})^{-1}$. A^{-1} is orthogonal.
- (c), (d) omit.

Theorem 6.2.4 *If A is an $m \times n$ matrix, then the following statements are equivalent.*

(a) $A^TA = I$.

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n.

(d) *The column vectors of A are orthonormal.*

- Proof: (a)->(b): $||Ax||^2$=Ax.Ax=x.A^TAx=x.Ix=$||x||^2$.
- (b)->(c): Theorem 6.2.1. with T(x)=Ax.
- (c)->(d): e_1,e_2,…,e_n are orthonormal. Since Ae_i.Ae_j =e_i.e_j for all i and j, Ae_1,Ae_2,…,Ae_n are orthonormal (see p.22-23). These are column vectors of A.
- (d)->(a): ij-th term of A^TA = a_i^T a_j= a_i.a_j where a_i is the ith column of A. This is 1 if i=j. 0 otherwise.

Theorem 6.2.5 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*a*) *A is orthogonal.*

(*b*) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n.

(*c*) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n.

(*d*) *The column vectors of A are orthonormal.*

(*e*) *The row vectors of A are orthonormal.*

- Proof: This is 6.2.4.
  (e) Since the transpose of A is also orthogonal.

- An operator T is orthogonal if and only if ||T(x)||=||x|| for all x.
- Thus, ||Ax||=||x|| for all x for the matrix A of T.
- Hence, we have by Theorem 6.2.5.

Theorem 6.2.6 *A linear operator $T: R^n \to R^n$ is orthogonal if and only if its standard matrix is orthogonal.*

Theorem 6.2.7 *If $T: R^2 \to R^2$ is an orthogonal linear operator, then the standard matrix for T is expressible in the form*

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad or \quad H_{\theta/2} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{10}$$

That is, T is either a rotation about the origin or a reflection about a line through the origin.

Contraction and dilations of R^2

- $T(x,y)=(kx,ky)$.
- T is a contraction of $0 \leq k < 1$.
- T is a dilation of $k > 1$.
- Horizontal compression with factor
- k: $T(x,y) = (kx,y)$ if $0 \leq k < 1$.
  - Horizontal expansion if $k > 1$.
- Vertical compression: $T(x,y) =(x, ky)$ if $0 \leq k < 1$.
  - Vertical expansion if $k > 1$.

- Shearing in the x-direction with factor k: $T(x,y)=(x+ky,y)$. This sends (x,y) to $(x+ky,y)$.
  - Thus, it preserves the y-coordinate and changes the x-coordinate by an amount proportional to y.
  - This sends a vertical line to a line of slope 1/k.
- Shearing in the y-direction with factor k: $T(x,y)=(x,y+kx)$. This send (x,y) to $(x,y+kx)$.
  - Thus it preserves the x-coorinates and changes the y-coordinates by an amount proportional to x.
  - This sends a horizontal line to a line of slope k.
- Example 6.

Linear operators on R^3.

- A orthogonal transformations in R^3 is classified:
  - A rotation about a line through the origin.
  - A reflection about a plane through the origin.
  - A rotation about a line L through the origin composed with a reflection about the plane P through the origin perpendicular to L.
- The first has det =1 and the other have determinant -1.
- Examples: Table 6.2.5.
- For rotations, the axis of rotation is the line fixed by the rotation. We obtain direction by u=wxT(w) for w in the perpendicular plane.
- Table 6.2.6.

General rotations

Theorem 6.2.8 *If* $\mathbf{u} = (a, b, c)$ *is a unit vector, then the standard matrix* $R_{\mathbf{u},\theta}$ *for the rotation through the angle* θ *about an axis through the origin with orientation* $\mathbf{u}$ *is*

$$R_{\mathbf{u},\theta} = \begin{bmatrix} a^2(1-\cos\theta) + \cos\theta & ab(1-\cos\theta) - c\sin\theta & ac(1-\cos\theta) + b\sin\theta \\ ab(1-\cos\theta) + c\sin\theta & b^2(1-\cos\theta) + \cos\theta & bc(1-\cos\theta) - a\sin\theta \\ ac(1-\cos\theta) - b\sin\theta & bc(1-\cos\theta) + a\sin\theta & c^2(1-\cos\theta) + \cos\theta \end{bmatrix} \tag{13}$$

- Suppose A is a rotation matrix. To find out the axis of rotation, we need to solve $(I-A)x=O$.
  - Once we know the line L of fixed points, we find the perpendicular plane P and a vector w in it.
  - Form $w \times Aw$. That is the direction of L.
  - The angle of rotation is
  - $\text{Angle}(w,Aw) = \text{ArcCos}(w.Aw/||w||||Aw||)$
- This is always less than or equal to π.
- Example 7.
- Actually, this is computable by $\cos\theta=(tr(A)-1)/2$ by using formula (13). Details omitted.
- We can also use $v=Ax+A^tx+[1-tr(A)]x$. x any vector, v is the axis direction.