Linear Algebra: Final Exam (2007 Spring)

Justify your answers fully.

1. Let the field be the field of real numbers. Which of the following functions on the set of 3×3-matrices over the reals are 3-linear and which are not?

   (a) (10pts.) $D(A) = A_{11} + A_{22} + A_{33}$.

   (b) (10pts.) $D(A) = (A_{11})^2 + A_{11}A_{22}$.

   (c) (10pts.) $D(A) = A_{11}A_{12}A_{33}$.

   (d) (10pts.) $D(A) = A_{13}A_{22}A_{31} + 5A_{12}A_{23}A_{31}$.

2. Let T be a linear operator on V. Suppose

$$V = W_1 \oplus \cdots \oplus W_k,$$

where each W_i is invariant under T. Let $T_i : W_i \to W_i$ be the induced map by restricting T.

   (a) (15pts.) Prove that the characteristic polynomial f of T is the product of the characteristic polynomials f_i of T_i.

   (b) (15pts.) Prove that the minimal polynomial p of T is the monic generator of the ideal $I = \bigcap_{i=1}^{k} I_i$ in $F[x]$ where I_i is the ideal generated by the minimal polynomials p_i of T_i.

3. Let V be the vector space of real valued polynomials of degree ≤ 3. Define the inner product

$$(f|g) := \int_0^1 f(t)g(t)dt$$

for $f, g \in V$.

   (a) (15pts.) Find g_1 in V so that $(f|g_1) = f(1) + f(-1)$ for all $f \in V$.

   (b) (15pts.) Let L be any linear functional on V. Show why one can find $g_0 \in V$ such that $(f|g_0) = L(f)$ for all $f \in V$.

4. Let T be a linear map $\mathbf{R}^3 \to \mathbf{R}^3$ corresponding to the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{bmatrix}.$$

   (a) (20pts.) Find vectors α_1, α_2 for the cyclic decomposition $\mathbf{R}^3 = Z(\alpha_1, T) \oplus Z(\alpha_2, T)$ of T where $Z(\alpha_i, T)$ are cyclic subspaces of $\mathbf{R}^3$ generated by α_i, $i = 1, 2$.

   (b) (15pts.) Find the minimal polynomial p and the T-annhilators q_i for α_i for $i = 1, 2$ and verify $f = q_1q_2$ where f is the characteristic polynomial.

   (c) (15pts.) Find the rational form of the matrix A.