6.3. Annihilating polynomials

Cayley-Hamilton theorem

Polynomials and transformations
• \((p+q)(T) = p(T)+q(T)\)
• \((pq)(T)=p(T)\cdot q(T)\)
• \(\text{Ann}(T) = \{p \in F[x] \mid p(T)=0\}\) is an ideal.

**Proof:**

– \(p, q \in \text{Ann}(T)\), \(k \in F\) \(\rightarrow\) \(p+kq(T)=0\) \(\rightarrow\) \(p+kq \in \text{Ann}(T)\).

– \(p \in \text{Ann}(T)\), \(q \in F[x]\) \(\rightarrow\) \(pq(T)=p(T)q(T)=0\) \(\rightarrow\) \(pq \in \text{Ann}(T)\).

• \(\text{Ann}(T)\) is strictly bigger than \(\{0\}\).
• **Proof:** We show that there exists a nonzero polynomial \( f \) in \( \text{Ann}(T) \) for any \( T:V^n \rightarrow V^n \).

• \( IT, T^2, \ldots, T^{n^2} \) : 1 + \( n^2 \) operators in \( \text{L}(V, V) \).

• \( \dim \text{L}(V, V) = n^2 \). Therefore, there exists a relation

\[
c_0 I + c_1 T + c_2 T^2 + \ldots + c_{n^2} T^{n^2} = 0
\]

• Every polynomial ideal is of form \( fF[x] \).
• Definition: $T: V \rightarrow V$. $V$ over $F$. (finite dim.)

The minimal polynomial of $T$ is the unique monic generator of $\text{Ann}(T)$.

• How to obtain the m.poly?

• Proposition: the m.poly $p$ is characterized by
  1. $P$ is monic.
  2. $P(T) = 0$
  3. No polynomial $f$ s.t. $f(T) = 0$ and has smaller degree than $p$. 
A similar operators have the same minimal polynomials: This follows from:

- \( f(GTG^{-1})=0 \iff f(T)=0 \).

Proof:
- \( (GTG^{-1})^i = GTG^{-1}GTG^{-1} \ldots GTG^{-1}=GT^iG^{-1} \)
- \( 0=c_0 I + c_1 GTG^{-1} + c_2 (GTG^{-1})^2 + \ldots + c_n (GTG^{-1})^n \iff \)
- \( 0=G(c_0 I + c_1 T + c_2 T^2 + \ldots + c_n T^n)G^{-1} \)
• **Theorem 3.** T a linear operator on $V^n$. (A nxn matrix). The characteristic and minimal polynomial of $T$(of A) have the same roots, except for multiplicities.

• **Proof:** $p$ a minimal polynomial of T
  – We show that
  – $p(c )=0$  $\iff$  $c$ is a characteristic value of T.
    • $(\Rightarrow)$ $p(c )=0$. $p=(x-c)q$.
    • $\deg q < \deg p$. $q(T)\neq 0$ since $p$ has minimal degree
    • Choose $b$ s.t. $q(T)b \neq 0$. Let $a=q(T)b$.
    • $0=p(T)b = (T-cI)q(T)b = (T-cI)a$
    • $c$ is a characteristic value.
• \((-\) Ta=ca. \(a \neq 0\).
• \(p(T)a=p(c)a\) by a lemma.
• \(a \neq 0, p(T)=0 -> p(c)=0\).

• **T diagonalizable. (We can compute the m.poly)**
  – \(c_1, \ldots, c_k\) distinct char. values.
  – Then \(p=(x-c_1)\cdots(x-c_k)\).

• **Proof:** If \(a\) is a char. Vector, then one of \(T-c_1I, \ldots, T-c_kI\) sends \(a\) to 0.
  – \((T-c_1I)\cdots(T-c_kI)a=0\) for all char. v. \(a\).
  – Characteristic vectors form a basis \(a_i\).
  – \((T-c_1I)\cdots(T-c_kI) = 0\).
  – \(p=(x-c_1)\cdots(x-c_k)\) is in \(Ann(T)\).
  – \(p\) has to be minimal since \(p\) has to have all these factors by Theorem 3.
• Caley-Hamilton theorem: T a linear operator V. If f is a char. poly. for T, then f(T)=0. i.e., f in Ann(T). The min. poly. p divides f.

• Proof: highly abstract:
  – K a commutative ring of poly in T.
  – \{a_1, \ldots, a_n\} basis for V.

\[
Ta_i = \sum_{j=1}^{n} A_{ji}a_j \quad \Rightarrow \quad \sum_{j=1}^{n} (\delta_{ij}T - A_{ji}I)a_j = 0
\]

for i=1,\ldots,n
Let B be a matrix in $K^{n \times n}$ with entries:
\[ B_{ij} = \delta_{ij} T - A_{ji} I \]

- e.g. $n=2$.
\[ B = \begin{bmatrix} T - A_{11} I & -A_{21} I \\ -A_{12} I & T - A_{22} I \end{bmatrix} \]

- $\det B = (T-A_{11}I)(T-A_{22}I)-A_{12}A_{21}I = f_T(T)$.

For all $n$, $\det B = f_T(T)$. $f_T$ char. poly. (omit proof)

We show $f_T(T)=0$ or equiv. $f(T)a_k=0$ for each $k$.

- $(6-6) \sum_{j=1}^{n} B_{ij} a_j = 0$ by (*).

- Let $B' = \text{adj } B$. 

\[
\sum_{j=1}^{n} B'_{ki} B_{ij} a_j = 0.
\]
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} B'_{ki} B_{ij} a_j = 0.
\]
\[
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} B'_{ki} B_{ij} \right) a_j = 0.
\]
\[
\sum_{j=1}^{n} \left( \delta_{kj} \det B \right) a_j = 0.
\]

- \( \det B a_k = 0. \) for each \( k. \)
- \( \det B = 0. \) \( f_T(T) = 0. \)

- **Characteristic polynomial gives informations on the factors of the minimal polynomials.**