Linear Algebra: Final Exam (2006 Spring)

Justify your answers fully.

1. (50 pts.) Find the minimal polynomial and the rational form of each of the following four matrices:

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{pmatrix}, \begin{pmatrix} 3 & -4 & -4 \\ -1 & 3 & 2 \\ 2 & -4 & -3 \end{pmatrix}, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Here $\theta \neq n\pi/2$ for $n \in \mathbf{Z}$.

2. Let V be an n-dimensional vector space over $\mathbf{C}$ and T a diagonalizable linear operator on V, $n \geq 2$.

(a) (10 pts.) Show that if T has n distinct characteristic values, and if $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of characteristic vectors for T show that $\alpha_1 + \cdots + \alpha_n$ is a cyclic vector for T.

(b) (10 pts.) Suppose T has only $n-1$ distinct characteristic values for $n \geq 2$. Discuss the cyclic decomposition of V for T and the cyclic vectors.

(c) (10 pts.) Show that if T has a cyclic vector, then T has n distinct characteristic values.

3. (30 pts.) Find a basis of the space of solutions of the ordinary differential equation

$$\frac{d^4 f}{dt^4} + 2\frac{d^3 f}{dt^3} - 2\frac{df}{dt} - 1 = 0.$$

In particular, what is the dimension of the solution space?

4. Let V be the vector space of $n \times n$-matrices over a complex field with a bilinear form given by $(A, B) = tr(AB^*)$ for $A, B \in V$. Let P be an element of V and define a linear operator $T_P : V \to V$ by $A \mapsto AP$ for $A \in V$.

(a) (5 pts.) Show that $(\cdot, \cdot)$ is an inner product.

(b) (15 pts.) Describe the adjoint linear operator of T_P with respect to the inner product.

5. (20 pts.) Write the following matrix B as NU where N is in $T^+(3)$ and $U \in U(3)$:

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$