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Real projective orbifolds with ends and their deformation spaces

The deformation theory for the nice ones

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Preface

Let $G$ be a Lie group acting transitively on a manifold $X$. An $(X,G)$-geometry is given by this pair. Furthermore, an $(X,G)$-structure on an orbifold or a manifold is an atlas of charts to $X$ with transition maps in $G$. Here, we are concerned with $G = \text{PGL}(n+1,\mathbb{R})$ and $X = \mathbb{RP}^n$.

Cartan, Ehresmann, and others started the field of $(X,G)$-structures. Subjects of $(X,G)$-structures were popularized by Thurston and Goldman among many other people. These structures provide a way to understand representations and their deformations giving us viewpoints other than algebraic ones. Our deformation spaces often parameterize significant parts of the space of representations.

Since the examples are easier to construct, even now, we will be studying orbifolds, a natural generalization of manifolds. Also, computations can be done fairly well for simple examples. We began our study with Coxeter orbifolds where the computations are probably the simplest possible.

Thurston did use the theory of orbifolds in a deep way. The hyperbolization of Haken 3-manifold requires the uses of the deformation theory of orbifolds where we build from hyperbolic structures from handlebodies with “scalloped” orbifold structures. (See Morgan [144].) We do not really know how to escape this step, which was a very subtle point that some experts misunderstood. Also, orbifolds are natural objects obtained when we take quotients of manifolds by fibrations and so on. These are some of the reasons we study orbifolds instead of just manifolds.

Classically, conformally flat structures were studied much by differential geometers. Projectively flat structures were also studied from Cartan’s time. However, our techniques are much different from their approaches.

Convex real projective orbifolds are quotient spaces of convex domains on a projective space $\mathbb{RP}^n$ by a discrete group of projective automorphisms. Hyperbolic manifolds and many symmetric manifolds are natural examples. These can be deformed to one not coming from simple constructions. The study was initiated by Kuiper [125], Koszul [123], Benzetécri [25], Vey [161], and Vinberg [163], accumulating some class of results. Closed manifolds or orbifolds admit many such structures as shown first by Kac-Vinberg [113], Goldman [94], and Cooper-Long-
Thistlethwaite [70], [71]. Some parts of the theory for closed orbifolds were completed by Benoist [20] in the 1990s.

The topics of convex real projective structures on manifolds and orbifolds are currently developing. We present some parts. This book is mainly written for researchers in this field.

There are surprisingly many such structures coming from hyperbolic ones and deforming as shown by Vinberg for Coxeter orbifolds, Goldman for surfaces, and later by Cooper-Long-Thistlethwaite for 3-manifolds.

We compare these theories to the Mostow or Margulis type rigidity for symmetric spaces. The rigidity can be replaced by what is called the Ehresmann-Thurston-Weil principle that

- a subspace of the $G$-character space (variety) of the fundamental group of a manifold or orbifold $M$ classifies the $(X, G)$-structures on $M$ under the map

$$\text{hol} : \text{Def}_c(M) \to \text{Hom}_c(\pi_1(M), G)/G$$

where

- we define the deformation space

$$\text{Def}_c(M) := \{ (X, G) \text{-structures on } M \text{ satisfying some conditions denoted by } c \}/\sim$$

where $\sim$ is the isotopy equivalence relation, and

- $\text{Hom}_c(\pi_1(M), G)/G$ is the subspace of the character space $\text{Hom}(\pi_1(M), G)/G$ satisfying the corresponding conditions to $c$.

For closed real projective orbifolds, it is widely thought that Benoist’s work is quite an encompassing one. Hence, we won’t say much about this topic. (See Choi-Lee-Marquis for a survey [67].)

We focus on convex real projective orbifolds with ends, which we have now accumulated some number of examples. Basically, we will prove an Ehresmann-Thurston-Weil principle: We will show that the deformation space of properly convex real projective structures on an orbifold with some end conditions identifies under a map with the union of components of the subset of character spaces of the orbifold satisfying the corresponding conditions on end holonomy groups. Our conditions on the ends are probably very generic ones, and we have many examples of such deformations.

In fact, we are focusing on generic cases of lens-type or horospherical ends. To complete the picture, we need to consider all types of ends. Even with end vertex conditions, we are still to complete the picture leaving out the NPNC-ends. We hope to allow these types for our deformation spaces in the near future.

The book is divided into three parts:

(1) We will give some introduction and survey our main results and give examples where our theory is applicable.
(II) We will classify the types of ends we will work with. We use the uniform middle eigenvalue condition. The condition is used to prove the preservation of the convexity of the deformations.

(III) We will try to prove the Ehresmann-Thurston-Weil principle for the deformation spaces for our type of orbifolds. We show the local homeomorphism property and the closedness of the images for the maps from the deformations spaces to the character spaces restricted by the end conditions.

We give an outline at the beginning of each part.

As a motivation for our study, we say about some long-term goal: Deforming a real projective structure on an orbifold to an unbounded situation results in the actions of the fundamental group on affine buildings. This hopefully will lead us to some understanding of orbifolds and manifolds in particular of dimension three as indicated by Ballas, Cooper, Danciger, G.S. Lee, Leitner, Long, Thistlethwaite, and Tillmann.

There is a concurrent work by the group consisting of Cooper, Long, and Tillmann with Ballas and Leitner on the same subjects but with different conditions on ends. They impose the condition that the end fundamental groups to be amenable. However, we do not require the same conditions in this paper but instead we will use some type of norms of eigenvalue conditions to guarantee the convexity during the deformations. We note that their deformation spaces are somewhat differently defined. Of course, we benefited much from their work and insights in this book and are very grateful for their generous help and guidance. We also appreciate much help from Crampon and Marquis working also independently of the above group and us.

We will use the results of the whole of the monograph in Chapters 1 and 3. We need the results of Chapter 8 in Part 2 in the latter part of Chapter 4 in Part 2. Except for these, the logical dependence of the monograph is as ordered by the monograph. Appendix A depends only on Chapter 2, and the results are used in the monograph except for Chapter 2.

We mention that the definitions in Chapter 1 will be restated at most once more in various corresponding chapters to clarify with more materials. Chapter 2 will have many preliminary definitions that we will not repeat.

We need to lift the objects to \( S^n \) using Section 2.1.7. We give proofs in the book by considering objects to be in \( S^n \) and using the projective automorphism group \( SL_{\pm}(n+1, \mathbb{R}) \). We will use proof symbols:

- \([S^nS]\): at the end of the proof indicates that it is sufficient to prove for \( S^n \) since the conclusion does not involve \( \mathbb{R}P^n \) nor \( S^n \).
- \([S^nT]\): indicates that the version of the theorem, proposition, or lemma for \( S^n \) implies one for \( \mathbb{R}P^n \) often with the help of Proposition 2.13.
- \([S^nP]\): indicates that the proof of the theorem for \( S^n \) implies one for \( \mathbb{R}P^n \) often with the help of Proposition 2.13.

If we do not need to go to \( S^n \) to prove the result, we leave no mark except for the end of the proof.
This book generalizes and simplifies the earlier preprints of the author. We were able to drop many conditions in the earlier versions of the theorems overcoming many limitations. Some of the results were announced in some survey articles [57] and [58].

Finally, to better communicate the ideas, the author made some effort to make the material more clear and precise, entailing the trade-off of the writing being long, somewhat technical, and sometimes redundant.

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The study was begun with a conversation with Stephan Tillmann at the workshop titled “Manifolds at Melbourne 2006”. We began to work on this topic seriously from my sabbatical at the University of Melbourne from 2008. We thank Craig Hodgson and Gye-Seon Lee for working with me with many examples and their insights. These resulted in computing many examples and valuable ideas. We made much progress during the stay at the University of Melbourne during the spring of 2009. The idea of radial ends comes from the cooperation with them.

We thank David Fried for helping me to understand the issues with the distanced nature of the tubular actions and duality in Chapter 6. There is a subcase that David Fried proved, but it is not included here. We thank Yves Carrière with the general approach to study the non-discrete cases for nonproperly convex ends in Chapter 8. The Lie group theoretical approach of Riemannian foliations was a key idea for the non-discrete quasi-joined ends in Chapter 8.

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We thank Samuel Ballas, Daryl Cooper, Arielle Leitner, Darren Long, and Stephan Tillmann for explaining their work and help, and we thank Mickaël Crampon and Ludovic Marquis also. These discussions clarified some technical issues with ends. Their works were influential here.

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Part I
Introduction to orbifolds and real projective structures.
Part I aims to survey the main results of the monograph and give some preliminary definitions.

In Chapter 1, we give the introduction and state the main results of the monograph. Basic definitions associated with convex real projective structures on orbifolds and some brief history will be given here. We state our mini-main result Corollary 1.1 illustrating the main focus of the Ehresmann-Thurston-Weil principle. We will define ends and radial, horospherical, and totally geodesic ends and discuss the deformation spaces of real projective structures and the relevant parts of the character spaces. Next, we define stable properly convex real projective structures (SPC-structures) on orbifolds and strictly SPC-structures as well. We will prove the principle by dividing the task into two parts. Initially, we will explain that the map hol is a local homeomorphism. Then we explain that the image is closed. Finally, we end by stating the main results of the monograph, which will be proved in Part 3.

In Chapter 2, we go over basic preliminary materials. We begin with defining geometric structures and real projective structures, in particular convex ones. We discuss the ends of orbifolds. Affine orbifolds and affine suspensions of real projective orbifolds are defined. We discuss the linear algebra and estimations using it, orthopotent actions of Lie groups, higher-convergence group actions, attracting and repelling sets, convexity, the Benoist theory on convex divisible actions, and so on. We discuss the dual orbifolds of a given convex real projective orbifolds as given by Vinberg. Finally, we extend duality to all convex compact sets and discuss the geometric limits of the dual convex sets. Here we will use a slightly more generalized version of convexity.

In Chapter 3, we give examples to which our theory applies. Coxeter orbifolds and the orderability theory for Coxeter orbifolds using the Vinberg theory will be explained. We discuss the work jointly done with Gye-Seon Lee, Hodgson, and Greene. We state the work of Heusner-Porti on projective deformations of hyperbolic link complements. Also, we state some results on finite-volume convex real projective structures by Cooper-Long-Tillmann and Crampon-Marquis that these admit thick and thin decompositions. We also discuss our nicest cases Corollary 3.2 and 3.3 where the Ehresmann-Thurston-Weil principle holds in the simplest way: the deformation space of the orbifold identifies with a union of components of character space of the orbifold fundamental group. These include the Coxeter orbifolds admitting complete hyperbolic structures.
Chapter 1
Introduction

In Section 1.1, we define the real projective structures on orbifolds first describing as immersions from the universal covers to $\mathbb{RP}^n$ equivariant with respect to holonomy homomorphisms. We briefly discuss some history of the subject and the one for the orbifolds with ends. In Section 1.2, we will state the main result for a simple case. Here, every deformation space has some uniqueness property at each of the ends. In Section 1.3, we explain the ends, end neighborhoods, pseudo-ends, and pseudo-end fundamental groups. Then we explain the totally geodesic ends and lens associated with these. We explain the radial end including lens-shaped ones and horospherical ones, which generalize the notions of cusps. We will also explain the compatibility condition with the end compactification. These will be the main types of ends that we will consider. Then we will explain the uniform middle eigenvalue conditions relating these ends with eigenvalue conditions. In Section 1.4, we will introduce the deformation spaces and character spaces (varieties). The character spaces will be refined by applying many end conditions corresponding to conditions for lens-shaped ends and horospherical ends. These will be shown to be semi-algebraic subsets. In Section 1.5, we present the main results of the monograph: Ehresmann-Thurston-Weil principle. We define the stable properly-convex real projective structures on an orbifold and many associated deformation spaces. We will divide into two parts the openness and the closedness to prove later in Chapter 11. We will also use the fixed-point sections.

1.1 Introduction

Geometric $n$-dimensional orbifolds are mostly objects finitely covered by $n$-dimensional manifolds. These are very good orbifolds. Hence, each is of form $M/G$ for a manifold $M$ and a group $G$ acting on $M$ properly discontinuously but not necessarily freely as shown by Thurston. We may even assume that $G$ is finite in this monograph. (See Theorem 2.3.) We are interested in real projective structures on orbifolds, which are principally non-manifold ones. (See Section 2.1.3 for precise def-
The $n$-dimensional “bad” orbifolds form the majority outside the theory of geometric structures for $n \geq 4$, and they are not “bad”).

The real projective structures can be considered as torsion-free projectively flat affine connections on orbifolds. Another way to view is to consider one as an immersion from the universal cover $\tilde{\Sigma}$ of an orbifold $\Sigma$ to $\mathbb{R}^n$ equivariant with respect to a homomorphism $h: \pi_1(\Sigma) \to \text{PGL}(n+1, \mathbb{R})$ for the orbifold fundamental group $\pi_1(\Sigma)$ of $\Sigma$. (In this book, we assume that $n \geq 2$ for convenience.) These orbifolds have ends. We will study the cases when the ends are of specific type since otherwise it is almost impossible to study. The types that we consider are radial ends (R-ends) where concurrent projective geodesics foliate end neighborhoods. A hypersurface is a codimension-one submanifold or suborbifold of a manifold or an orbifold. Another type ones are totally geodesic ends (T-ends) when the closure of the end neighborhood can be compactified by an ideal totally geodesic hypersurface in some ambient real projective orbifold.

Kuiper, Benzécri, Koszul, Vey, and Vinberg might be the first people to consider these objects seriously as they are related to proper action of affine groups on affine cones in $\mathbb{R}^n$. We note here, of course, the older study of affine structures on manifolds with many major open questions.

Fig. 1.1 The developing images of convex $\mathbb{RP}^2$-structures on 2-orbifolds deformed from hyperbolic ones: $S^2(3,3,5)$.

1.1.1 Real projective structures on manifolds and orbifolds.

In general, the theory of geometric structures on manifolds with ends is not studied very well. We should try to obtain more results here and find what the appropriate conditions are. This question also seems to be related to making sense of the topological structures of ends of manifolds or orbifolds with many other geometric structures such as ones locally modeled on symmetric spaces. (See for example [115], [116], [100], and [101].) We devote Chapter 3 to examples.
1.1 Introduction

Given a vector space $V$, we let $\mathbb{P}(V)$ denote the space obtained by taking the quotient space of $V \setminus \{O\}$ under the equivalence relation

$$v \sim w \text{ for } v, w \in V \setminus \{O\} \text{ iff } v = sw, \text{ for } s \in \mathbb{R} \setminus \{0\}.$$ 

We let $[v]$ denote the equivalence class of $v \in V \setminus \{O\}$. For a subspace $W$ of $V$, we denote by $\mathbb{P}(W)$ the image of $W \setminus \{O\}$ under the quotient map, also said to be a subspace.

Recall that the projective linear group $\text{PGL}(n+1, \mathbb{R})$ acts on $\mathbb{RP}^n$, i.e., $\mathbb{P}(\mathbb{R}^{n+1})$, in a standard manner. Let $\mathcal{O}$ be a noncompact strongly tame $n$-orbifold where the orbifold boundary is not necessarily empty.

- A real projective orbifold is an orbifold with a geometric structure modeled on $(\mathbb{RP}^n, \text{PGL}(n+1, \mathbb{R}))$. (See Sections 2.1.4 and 2.1.5, [55], and Chapter 6 of [56].)
- A real projective orbifold also has the notion of projective geodesics as given by local charts and has a universal cover $\tilde{\mathcal{O}}$ where a deck transformation group $\pi_1(\mathcal{O})$ acting on.
- The underlying space of $\mathcal{O}$ is homeomorphic to the quotient space $\tilde{\mathcal{O}}/\pi_1(\mathcal{O})$.
- A real projective structure on $\mathcal{O}$ gives us a so-called development pair $(\text{dev}, h)$ where
  - $\text{dev} : \tilde{\mathcal{O}} \to \mathbb{RP}^n$ is an immersion, called the developing map, and
  - $h : \pi_1(\mathcal{O}) \to \text{PGL}(n+1, \mathbb{R})$ is a homomorphism, called a holonomy homomorphism, satisfying
    $$\text{dev} \circ \gamma = h(\gamma) \circ \text{dev} \text{ for } \gamma \in \pi_1(\mathcal{O}).$$
- The pair $(\text{dev}, h)$ is determined only up to the action
  $$g(\text{dev}, h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1}) \text{ for } g \in \text{PGL}(n+1, \mathbb{R})$$
  and any chart in the atlas extends to a developing map. (See Section 3.4 of [159].)
Let $\mathbb{R}^{n+1^*}$ denote the dual of $\mathbb{R}^{n+1}$. Let $\mathbb{RP}^n$ denote the dual projective space $\mathbb{P}(\mathbb{R}^{n+1^*})$. $\text{PGL}(n+1,\mathbb{R})$ acts on $\mathbb{RP}^n$ by taking the inverse of the dual transformation. Then $h : \pi_1(\mathcal{O}) \to \text{PGL}(n+1,\mathbb{R})$ has a dual representation $h^* : \pi_1(\mathcal{O}) \to \text{PGL}(n+1,\mathbb{R})$ sending elements of $\pi_1(\mathcal{O})$ to the inverse of the dual transformation of $\mathbb{R}^{n+1^*}$.

A projective map $f : \mathcal{O}_1 \to \mathcal{O}_2$ from a real projective orbifold $\mathcal{O}_1$ to $\mathcal{O}_2$ is a map so that for each $p \in \mathcal{O}_1$, there are charts $\phi_1 : U_1 \to \mathbb{RP}^n$, $p \in U_1$, and $\phi_2 : U_2 \to \mathbb{RP}^n$ where $f(p) \in U_2$ where $f(U_1) \subset U_2$ and $\phi_2 \circ \phi_1^{-1}$ is a projective map.

For an element $g \in \text{PGL}(n+1,\mathbb{R})$, we denote

$$g \cdot [w] := [\hat{g}(w)]$$

for $[w] \in \mathbb{RP}^n$ or

$$g \cdot [w] := [(\hat{g}^T)^{-1}(w)]$$

for $[w] \in \mathbb{RP}^n$.

where $\hat{g}$ is any element of $\text{SL}_+(n+1,\mathbb{R})$ mapping to $g$ and $\hat{g}^T$ the transpose of $\hat{g}$.

The complement of a codimension-one subspace of $\mathbb{RP}^n$ can be identified with an affine space $\mathbb{R}^n$. Affine transformations of $\mathbb{R}^n$ are the restrictions to $\mathbb{R}^n$ of projective transformations of $\mathbb{RP}^n$ fixing the subspace. We call the complement an affine subspace. It has a geodesic structure of a standard affine space. A convex domain in $\mathbb{RP}^n$ is a convex subset of an affine subspace. A properly convex domain in $\mathbb{RP}^n$ is a convex domain contained in a precompact subset of an affine subspace. A strictly convex domain in $\mathbb{RP}^n$ is a properly convex domain that contains no segment in the boundary. (We will use this definition of convexity except for Section 11.3. A slightly general definition will be used for $\mathbb{S}^n$ as defined in Section 2.1.6.)

A convex real projective orbifold is a real projective orbifold projectively diffeomorphic to the quotient $\Omega / \Gamma$ where $\Omega$ is a convex domain in an affine subspace of $\mathbb{RP}^n$ and $\Gamma$ is a discrete group of projective automorphisms of $\Omega$ acting properly. If an open orbifold has a convex real projective structure, it is covered by a convex domain $\Omega$ in $\mathbb{RP}^n$. For the developing map $\text{dev}$ with associated holonomy homomorphism $h$, the image of the developing map $\text{dev}(\hat{\mathcal{O}})$ for the universal cover $\hat{\mathcal{O}}$ of $\mathcal{O}$ is a convex domain. Here we may assume $\text{dev}(\hat{\mathcal{O}}) = \Omega$, and $\mathcal{O}$ is projectively diffeomorphic to $\text{dev}(\hat{\mathcal{O}})/h(\pi_1(\mathcal{O}))$. In our discussions, since $\text{dev}$ is an embedding and so is $h$, $\mathcal{O}$ will be regarded as an open domain in $\mathbb{RP}^n$ and $\pi_1(\mathcal{O})$ as a subgroup of $\text{PGL}(n+1,\mathbb{R})$ in such cases. A convex real projective orbifold is properly convex (resp. complete affine) if $\Omega$ is a properly convex domain (resp. an affine subspace in $\mathbb{RP}^n$). (We will often drop “real projective” from convex real projective orbifold or manifold or properly convex real projective orbifold or manifold) (See Section 2.1.5 for more details here.)

- Let $p : X \to Y$ be a covering map. We say that a map $f : C \to Y$ for a connected set $C$ lifts to $\tilde{f} : C \to X$ if we have $p \circ \tilde{f} = f$.
- A connected subset $C$ in $Y$ lifts to a set $C'$ in $X$ if $p|C'$ is a homeomorphism to $C$.
- An element $g$ of $\text{PGL}(n+1,\mathbb{R})$ lifts to an element $g'$ of $\text{GL}(n+1,\mathbb{R})$ if $g'$ goes to $g$ under the projection.
- Given a subgroup $\Gamma$ in $\text{PGL}(n+1,\mathbb{R})$, a lift in $\text{GL}(n+1,\mathbb{R})$ is any subgroup that maps to $\Gamma$ bijectively.
Remark 1.1 (Matrix convention) Given a vector space $V$, we denote by $S(V)$ the quotient space of

$$(V - \{O\})/\sim \text{ where } v \sim w \text{ iff } v = sw \text{ for } s > 0.$$ 

We will represent each element of $PGL(n+1, \mathbb{R})$ by a matrix of determinant $\pm 1$; i.e., $PGL(n+1, \mathbb{R}) = SL_{\pm}(n+1, \mathbb{R})/\langle \pm I \rangle$. Recall the covering map $S^n = S(\mathbb{R}^n+1) \to \mathbb{R}P^n$. For each $g \in PGL(n+1, \mathbb{R})$, there is a unique lift in $SL_{\pm}(n+1, \mathbb{R})$ preserving each component of the inverse image of $dev(\tilde{\sigma})$ under $p_{S^n}: S^n \to \mathbb{R}P^n$. We will use this representative. (See Section 2.1.6 and Section 2.1.7 also.)

One simple class of examples are obtained as follows: Let $B$ be the interior of the standard conic in an affine subspace $A^n \subset \mathbb{R}P^n$. The group of projective automorphisms of $B$ equals $PO(1,n) \subset PGL(n+1, \mathbb{R})$, and is the group of isometries of $B$ with the Hilbert metric; that is, the Klein model of the hyperbolic space.

Given any complete hyperbolic manifold (resp. orbifold) $M$, $M$ is of form $B/\Gamma$ for a discrete group $\Gamma \subset PO(1,n)$. Thus, $M$ admits a convex projective structure. $M$ with the real projective structures obtained here is called a hyperbolic manifold (resp. orbifold).

1.1.2 Real projective structures on closed orbifolds

We will briefly say about the results on convex real projective structures on closed orbifolds here and their deformation spaces. (For a more detailed survey, see Choi-Lee-Marquis [67].)

After Kuiper [125], Benzécri [25] started to consider higher-dimensional convex real projective manifolds. Let $\Omega/\Gamma$ be a closed real projective orbifold. He [25] showed that $\Omega$ has $C^{1,\e}$-boundary, and if $\Omega$ has $C^2$-boundary, then $\Omega$ is the interior of a conic.

However, Benzécri could not find examples not coming from classical geometry as Benoist informed us on one occasion. Kac-Vinberg [113] first found examples of deformations. Higher-dimensional examples were found by Vinberg [163]. Goldman [94] classified convex real projective structures on surfaces and showed that the deformation space is homeomorphic to cells.

After this, Benoist [20], [21], [22], [23], and [24] more or less completed the study of these orbifolds including their deformation spaces. Inkang Kim [119] simultaneously proved some of these results for closed 3-manifolds admitting hyperbolic structures.

D. Cooper, D. Long, and M. Thistlethwaite [70], [71] discovered that plenty of closed hyperbolic 3-manifolds deform to convex real projective 3-manifolds. These are closed manifolds.
1.1.3 Real projective structures on orbifolds with ends

Earlier near 2008, S. Tillmann found an example of a 3-orbifold obtained from pasting sides of a single ideal hyperbolic tetrahedron admitting a complete hyperbolic structure with cusps and a one-parameter family of real projective structure deformed from the hyperbolic one. (Porti-Tillmann [150] improved this to a two-dimensional family.)

Also, Marquis [136], [137] completed the end theory of 2-orbifolds. Earlier, Craig Hodgson, Gye-Seon Lee, and I found a few examples of deformations: 3-dimensional ideal hyperbolic Coxeter orbifolds without edges of order 3 has at least 6-dimensional deformation spaces in [63]. (See Example 3.1.)

Crampon and Marquis [74] and Cooper, Long, and Tillmann [73] have done similar studies with the finite Hilbert volume condition. In this case, only possible ends are horospherical ones. The work here studies more general type ends while we have benefited from their work. We will see that there are examples where horospherical R-ends deform to lens-shaped R-ends and vice versa (see also Example 3.1.) More recently, Cooper, Long, and Leitner [7] also have made more progress on the classification of the ends where they require that the ends have neighborhoods with nilpotent fundamental groups. (See [73], [129], [128], [130], and [7].) In part 2, we will also classify the ends but with different conditions, some of which overlap with theirs. Their group is also producing the theory of the deformations with the above restrictions. See Cooper-Long-Tillmann [72], Ballas-Marquis [9], and Ballas-Cooper-Leitner [7]. However, we note that their definition of the deformation spaces are more complicated with added structures.

But as Davis observed, there are many other types such as ones preserving subspaces of dimension greater than equal to 0. In fact, Cooper and Long found such an example from $S/\text{SL}(3, \mathbb{Z})$ for the space $S$ of unimodular positive definite bilinear forms. Since $S$ is a properly convex domain in $\mathbb{R}P^5$ and $\text{SL}(3, \mathbb{Z})$ acts projectively, $S/\text{SL}(3, \mathbb{Z})$ is a strongly tame properly convex real projective orbifold by the classical theory of lattices. Borel and Serre [32] classified these types of ends as arithmetic manifolds. The ends are not of the type studied here since these are not radial or totally geodesic ends.

It remains how to see for which of these types of real orbifolds, nontrivial deformations exist. For example, we can consider examples such as complete hyperbolic manifolds and compute the deformation spaces. From Theorem 1 in [63] with Coxeter orbifolds, we know that a complete hyperbolic Coxeter orbifold always deforms nontrivially. There is a six-dimensional space of deformations as convex real projective orbifolds with suitable ends. This will be explained in Chapter 3. Here a horospherical end changes into “lens-shaped R-ends” as first shown by Benoist [24]. (See also [55].) S. Ballas [5, 6, 4] also produced some results along this line for hyperbolic 3-manifolds. Choi, Green, and Lee also found a class of examples of a type of deformations of convex real projective structures on Coxeter orbifolds earlier (See [55], [63], and [65]) We state this as Theorem 3.1. (We will give more details on the examples in Chapter 3.)
1.2 A preview of the main results

We conjecture that maybe these types of real projective orbifolds with R-ends might be very flexible. We have many examples to be explained in Chapter 3. Also, Ballas, Danciger, and Lee [8] announced that they had found much evidence for this very recently in Cooperfest at Berkeley in May 2015.

Also, there are some related developments for the complex field $\mathbb{C}$ with the Ptolemy module in SnapPy as developed by S. Garoufalidis, D. Thurston, and C. Zickert [88]. (See https://www.math.uic.edu/t3m/SnapPy/ for the implementations.)

Remark 1.2 As a technical comment, we will look at each of our orbifolds in consideration as the interior of a compact orbifold with boundary fixed for that orbifold. This will simplify many things.

1.2 A preview of the main results

1.2.1 Our settings

Given an orbifold, we recall the notion of universal covering orbifold $\tilde{O}$ with the orbifold covering map $p_{\tilde{O}} : \tilde{O} \to O$ and the deck transformation group $\pi_1(O)$ so that $p_{\tilde{O}} \circ \gamma = p_{\tilde{O}}$ for $\gamma \in \pi_1(O)$. (See [158], [35], [54] and [56].) We hope to generalize these theories to noncompact orbifolds with some particular conditions on ends. In fact, we obtain a class containing complete hyperbolic orbifolds with finite volumes. These are $n$-orbifolds with compact suborbifolds whose complements are diffeomorphic to intervals times closed $(n-1)$-dimensional orbifolds. Such orbifolds are said to be strongly tame orbifolds. An end neighborhood is a component of the complement of a compact subset not contained in any compact subset of the orbifold. An end $E$ is an equivalence class of compatible exiting sequences of end neighborhoods. Because of this, we can associate an $(n-1)$-orbifold at each end $E$, and we define the end fundamental group $\pi_1(E)$ as a subgroup of the fundamental group $\pi_1(O)$ of the orbifold $O$, which is the image of the fundamental group of a product end-neighborhood of $E$. (See Section 1.3.1 for details.) We also put the condition on end neighborhoods to be foliated by radial lines or to have totally geodesic ideal boundary.

We concentrate on studying the ends that are well-behaved, i.e., ones that are foliated by lines or are totally geodesic. In this setting, we wish to study the deformation spaces of the convex real projective structures on orbifolds with some boundary conditions using the character spaces. Our main aim is

- to identify the deformation space of convex real projective structures on an orbifold $O$ with certain boundary conditions with an open subset of a semi-algebraic subset of $\text{Hom}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))/\text{PGL}(n+1, \mathbb{R})$ defined by conditions corresponding to the boundary conditions.

Deformation spaces are defined in Section 1.4. This is an example of the so-called Ehresmann-Thurston-Weil principle [165]. The precise statements are given in The-
The singularities of our orbifolds: The left one is the double of a tetrahedral reflection orbifold with orders 3, obtained by identifying two regular ideal tetrahedra by faces, and the right one is Tillmann’s orbifold in the 3-spheres, obtained from an ideal tetrahedra with four edges of angles $\pi/6$ and two edges of angles $\pi/3$ and gluing across the two edges by respective isometries. The white dots indicate the points removed. The edges are all of order 3.

Theorems 1.3 and 1.4. See Definition 1.4 for the conditions (IE) and (NA). We use the notion of strict convexity with respect to ends as defined in Definition 1.6. Our main result is the following corollary of Theorem 1.7:

**Corollary 1.1** Let $\mathcal{O}$ be a noncompact strongly tame SPC $n$-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical radial ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$, and that the nilpotent normal subgroups of every finite-index subgroup of $\pi_1(\mathcal{O})$ are trivial. Then $\text{hol}$ maps the deformation space $C\text{Def}_{\mathcal{O}, \text{u}, \text{lh}}$ of SPC-structures on $\mathcal{O}$ homeomorphically to a union of components of

$$\text{rep}_{\mathcal{O}, \text{u}, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).$$

The same can be said for $S\text{Def}_{\mathcal{O}, \text{u}, \text{lh}}$.

These terms will be defined more precisely later on in Sections 1.4.1 and 1.5.1.1. Roughly speaking, $C\text{Def}_{\mathcal{O}}$ (resp. $S\text{Def}_{\mathcal{O}}$) is the deformation space of properly convex (resp. strictly convex) real projective structures with conditions on ends with radial end-structure.

$$\text{rep}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$

is the space of characters, each of whose end holonomy group fixes a point. (We consider radial ends for now. See Section 1.3.3.)

$$C\text{Def}_{\mathcal{O}, \text{u}, \text{lh}}(\mathcal{O})$$

(resp. $S\text{Def}_{\mathcal{O}, \text{u}, \text{lh}}(\mathcal{O})$)

is the deformation space of properly convex (resp. strictly properly convex) real projective structures with conditions on ends that each end has a lens-cone neighborhood or a horospherical one, and each end holonomy group fixes a unique point.
1.3 End structures

\[ \text{rep}_{E_u,ulh}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \]

is the space of characters each of whose end holonomy group fixes a unique point and acts on a lens-cone or a horosphere.

As explained in Chapter 3 devoted to examples, our main examples satisfy this condition:

Suppose that a strongly tame properly convex 3-orbifold \( \mathcal{O} \) with radial ends admits a finite volume complete hyperbolic structure and has radial ends only, and finite-order elements generate every end fundamental group. Since finite-volume hyperbolic \( n \)-orbifolds satisfy (IE) and (NA) (see P.151 of [134] for example), the theory simplifies by Corollary 3.2, i.e., each radial end is always generalized lens-shaped or horospherical, so that

\[ \text{SDef}_{E_u,ulh}(\mathcal{O}) = \text{SDef}_{E}(\mathcal{O}). \]

The space under hol maps homeomorphically to a union of components of

\[ \text{rep}_{E}(\pi_1(\mathcal{O}), \text{PGL}(4, \mathbb{R})). \]

For a strongly tame Coxeter orbifold \( \mathcal{O} \) of dimension \( n \geq 3 \) admitting a complete finite-volume hyperbolic structure, hol is a homeomorphism from

\[ \text{SDef}_{E_u,ulh}(\mathcal{O}) = \text{SDef}_{E}(\mathcal{O}) \]

to a union of components of

\[ \text{rep}_{E}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \]

by Corollary 3.3. For this theory, we can consider a Coxeter orbifold based on a convex polytope admitting a complete hyperbolic structure with all edge orders equal to 3. More specifically, we can consider a hyperbolic ideal simplex or a hyperbolic ideal cube with such structures.

We mention that the remarkable work of Cooper-Long-Tillman [72] concentrates on openness. Their ends have nilpotent holonomy groups, and the character spaces are augmented with the deformation spaces of ends.

1.3 End structures

1.3.1 End fundamental groups

Let \( \mathcal{O} \) be a strongly tame real projective orbifold with the universal cover \( \tilde{\mathcal{O}} \) and the covering map \( p_{\mathcal{O}} \). A compact smooth orbifold \( \tilde{\mathcal{O}} \) whose interior is \( \mathcal{O} \) is called a compactification of \( \mathcal{O} \). There might be more than one compactifications. A strongly tame orbifold \( \mathcal{O} \) in our paper always will come with a compactification \( \tilde{\mathcal{O}} \) which is
a smooth orbifold with boundary. When we say $\partial$, we really mean $\partial$ with $\partial$. Each boundary component of $\partial$ is the ideal boundary component of $\partial$ and is an end of $\partial$.

An end neighborhood $U$ of $\partial$ is an open set $U$ where $\Sigma_E \cup U$ forms a neighborhood of an ideal boundary component $\Sigma_E$ corresponding to an end $E$.

Let $\hat{\partial}$ denote the universal cover or $\partial$ with the covering map $\hat{\rho}_\partial$. Let $\Gamma$ be the deck transformation group of $\hat{\partial} \to \partial$ which also restricts to the deck transformation group of $\hat{\partial} \to \partial$.

Each end neighborhood $U$, diffeomorphic to $S_E \times (0,1)$ for an $(n-1)$-orbifold $S_E$, of an end $E$ lifts to a connected open set $\hat{U}$ in $\hat{\partial}$. We choose $U$ and the diffeomorphism $f_U: U \to S_E \times (0,1)$ that $S_E \times (0,1]$ is also diffeomorphic to a tubular neighborhood of a boundary component of $\partial$ corresponding to $U$. A subgroup $\Gamma_U$ of $\Gamma$ acts on $\hat{U}$ where

$$p^{-1}_\partial(U) = \bigcup_{g \in \pi_1(\partial)} g(\hat{U}).$$

Each component $\hat{U}$ is said to be a proper pseudo-end neighborhood.

- An exiting sequence of sets $U_1, U_2, \ldots$ in $\hat{\partial}$ is a sequence so that for each compact subset $K$ of $\partial$ there exists an integer $N$ satisfying $p^{-1}_\partial(K) \cap U_i = \emptyset$ for $i > N$.
- A pseudo-end neighborhood sequence is an exiting sequence of proper pseudo-end neighborhoods

$$\{U_i | i = 1, 2, 3, \ldots\},$$

where $U_{i+1} \subset U_i$ for every $i$.

- Two pseudo-end sequences $\{U_i\}$ and $\{V_j\}$ are compatible if for each $i$, there exists $J$ such that $V_j \subset U_i$ for every $j$, $j > J$ and conversely for each $j$, there exists $I$ such that $U_i \subset V_j$ for every $i$, $i > I$.
- A compatibility class of a proper pseudo-end sequence is called a pseudo-end (p-end) of $\partial$. Each of these corresponds to an end of $\partial$ under the universal covering map $p_\partial$.
- For a pseudo-end $E$ of $\partial$, we denote by $\Gamma_E$ the subgroup $\Gamma_U$ where $U$ and $\hat{U}$ is as above. We call $\Gamma_E$ a pseudo-end fundamental group. We will also denote it by $\pi_1(E)$.
- A pseudo-end neighborhood $U$ of a pseudo-end $E$ is a $\Gamma_E$-invariant open set containing a proper pseudo-end neighborhood of $E$. A proper pseudo-end neighborhood is an example.

(From now on, we will replace “pseudo-end” with the abbreviation “p-end”.) (See Section 2.1.8 for details.)

As a summary, the set of boundary components of $\partial$ has a one-to-one correspondence with the set of p-ends of $\partial$.

**Proposition 1.1** Let $E$ be a p-end of a strongly tame orbifold $\partial$. The p-end fundamental group $\Gamma_E$ of $E$ is independent of the choice of $U$.

**Proof** Given $U$ and $U'$ that are end-neighborhoods for an end $E$, let $\hat{U}$ and $\hat{U}'$ be p-end neighborhoods for a p-end $E$ that are components of $p^{-1}(U)$ and $p^{-1}(U')$
1.3 End structures

respectively. Let $\tilde{U}''$ be the component of $p^{-1}(U'')$ that is a p-end neighborhood of $\tilde{E}$. Then $\Gamma_{\tilde{U}''}$ injects into $\Gamma_{\tilde{U}}$ since both are subgroups of $\Gamma$. Any $\mathcal{G}$-path in $U$ in the sense of Bridson-Haefliger [35] is homotopic to a $\mathcal{G}$-path in $U''$ by a translation in the $I$-factor. Thus, $\pi_1(U'') \to \pi_1(U)$ is surjective. Since $\tilde{U}$ is connected, any element $\gamma$ of $\Gamma_{\tilde{U}}$ is represented by a $\mathcal{G}$-path connecting $x_0$ to $\gamma(x_0)$. (See Example 3.7 in Chapter III. $\mathcal{G}$ of [35].) Thus, $\Gamma_{\tilde{U}''}$ is isomorphic to the image of $\pi_1(U'') \to \pi_1(U)$. Since $\tilde{U}$ is connected, any element $\gamma$ of $\Gamma_{\tilde{U}}$ is represented by a $\mathcal{G}$-path connecting $x_0$ to $\gamma(x_0)$. (See Example 3.7 in Chapter III. $\mathcal{G}$ of [35].) Thus, $\Gamma_{\tilde{U}''}$ is isomorphic to the image of $\pi_1(U'') \to \pi_1(U)$.

1.3.2 Totally geodesic ends

Suppose that an end $E$ of a real projective orbifold $\mathcal{O}$ of dimension $n \geq 2$ satisfies the following:

- The end has an end neighborhood homeomorphic to a closed connected $(n-1)$-dimensional orbifold $B$ times a half-open interval $(0,1)$.
- The end neighborhood completes to an orbifold $U'$ diffeomorphic to $B \times (0,1]$ in the compactification orbifold $\bar{\mathcal{O}}$.
- The subset of $U'$ corresponding to $B \times \{1\}$ is the ideal boundary component.
- Each point of the added boundary component has a neighborhood projectively diffeomorphic to the quotient orbifold of an open set $V$ in an affine half-space $P$ so that $V \cap \partial P \neq \emptyset$ by a projective action of a finite group. This implies that the developing map extends to the universal cover of the orbifold with $U'$ attached.

The completion is called a compactified end neighborhood of the end $E$. The boundary component $S_E$ is called the ideal boundary component of the end. Such ideal boundary components may not be uniquely determined as there are two projectively nonequivalent ways to add boundary components of elementary annuli (see Section 1.4 of [49]). Two compactified end neighborhoods of an end are equivalent if the end neighborhood contains a common end neighborhood whose compactification projectively embed into the compactified end neighborhoods. (See Definition 9.1 for more detail.) The equivalence class of compactified end-neighborhoods is called a totally geodesic end structure (T-end structure) for an end $E$.

We also define as follows:

- The equivalence class of the chosen compactified end neighborhood is called a totally geodesic end-structure of the totally geodesic end. The choice of the end structure is equivalent to the choice of the ideal boundary component.
- We will also call the ideal boundary $S_E$ the end orbifold (or end ideal boundary component) of the end.

$\mathbb{R}P^n$ has a Riemannian metric of constant curvature called the Fubini-Study metric. Recall that the universal cover $\tilde{\mathcal{O}}$ of $\mathcal{O}$ has a path-metric induced by
dev : \hat{\mathcal{O}} \to \mathbb{RP}^n. We can Cauchy complete \hat{\mathcal{O}} of this path-metric. The Cauchy completion is called the \textit{Kuiper completion} of \hat{\mathcal{O}}. (See [51].) Note we may sometimes use a lift dev : \hat{\mathcal{O}} \to \mathbb{S}^n lifting the developing map and use the same notation.

A \textit{T-end} is an end equipped with a T-end structure. A \textit{totally geodesic pseudo-end (T-p-end)} is a p-end \hat{E} corresponding to a T-end \textit{E}. There is a totally geodesic \((n-1)\)-dimensional domain \(\hat{S}_E\) in the Cauchy completion of \(\hat{\mathcal{O}}\) in the closure of a p-end neighborhood of \(\hat{E}\). Of course, \(\hat{S}_E\) covers \(S_E\). We call \(\hat{S}_E\) the p-end ideal boundary component. We will identify it with a domain in a hyperspace in \(\mathbb{RP}^n\) (resp. \(\mathbb{S}^n\)) when \(\text{dev}\) is a fixed map to \(\mathbb{RP}^n\) (resp. \(\mathbb{S}^n\)).

**Definition 1.1** A lens is a properly convex domain \(L\) in \(\mathbb{RP}^n\) so that \(\partial L\) is a union of two smooth strictly convex open disks. A properly convex domain \(L\) is a \textit{generalized lens} if \(\partial L\) is a union of two open disks, one of which is strictly convex and smooth and the other is allowed to be just a topological disk. A \textit{lens-orbifold} (or \textit{lens}) is a compact quotient orbifold of a lens by a properly discontinuous action of a projective group \(\Gamma\) preserving each boundary component. Also, the domains or an orbifold projectively diffeomorphic to a lens or lens-orbifolds are called lens.

(Lens condition for T-ends) The ideal boundary is identified by a projective map \(f\) as a totally geodesic suborbifold in the interior of a lens-orbifold in the ambient real projective orbifold containing \(\hat{\mathcal{O}}\) where \(f\) is a map from a neighborhood of the ideal boundary to a one-sided neighborhood in the lens-orbifold of the image.

If the lens condition is satisfied for a T-end, we will call it the \textit{lens-shaped T-end}. The intersection of a lens with \(\hat{\mathcal{O}}\) is called a \textit{lens end neighborhood} of the T-end. A corresponding T-p-end is said to be a \textit{lens-shaped T-p-end}.

In these cases, \(\hat{S}_E\) is a properly convex \((n-1)\)-dimensional domain, and \(S_E\) is a \((n-1)\)-dimensional properly convex real projective orbifold. We will call the cover \(L\) of a lens orbifold containing \(S_E\) the \textit{cocompactly acted lens (CA-lens)} of \(\hat{S}_E\) where we assume that \(\pi_1(\hat{E})\) acts properly and cocompactly on the lens. \(L \cap \hat{\mathcal{O}}\) is said to be \textit{lens p-end neighborhood} of \(\hat{E}\) or \(\hat{S}_E\).

We remark that for each component \(\partial_i L\) for \(i = 1, 2\) of \(L\), \(\partial_i L/\Gamma\) is compact and both are homotopy equivalent up to a virtual manifold cover \(L/\Gamma'\) of \(L/\Gamma\) for a finite index subgroup \(\Gamma'\). Also, the ideal boundary component of \(L/\Gamma'\) has the same homotopy type as \(L/\Gamma\) and is a compact manifold. (See Selberg’s Theorem 2.3.)

### 1.3.3 Radial ends

A segment is a convex arc in a 1-dimensional subspace of \(\mathbb{RP}^n\) or \(\mathbb{S}^n\). We will denote the closed segment by \([xy]\) if \(x\) and \(y\) are endpoints. It is uniquely determined by \(x\) and \(y\) if \(x\) and \(y\) are not antipodal. In the following, all the sets are required to be inside an affine subspace \(\mathbb{A}^n\) and its closure to be either in \(\mathbb{RP}^n\) or \(\mathbb{S}^n\).

Let \(\hat{\mathcal{O}}\) denote the universal cover of \(\mathcal{O}\) with the developing map \(\text{dev}\). Suppose that an end \(E\) of a real projective orbifold satisfies the following:
1.3 End structures

- The end has an end neighborhood $U$ foliated by properly embedded projective geodesics.
- Choose any map $f : \mathbb{R} \times [0, 1] \to \mathcal{O}$ so that $f|_{\mathbb{R} \times \{t\}}$ for each $t$ is a geodesic leaf of such a foliation of $U$. Then $f$ lifts to $\tilde{f} : \mathbb{R} \times [0, 1] \to \tilde{\mathcal{O}}$ where $\text{dev} \circ \tilde{f}|_{\mathbb{R} \times \{t\}}$ for each $t,t \in [0, 1]$, maps to a geodesic in $\mathbb{RP}^n$ ending at a point of concurrency common for every $t$.

The foliation is called a radial foliation and leaves radial lines of $E$. Two such radial foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of radial end neighborhoods of an end are equivalent if the restrictions of $\mathcal{F}_1$ and $\mathcal{F}_2$ in an end neighborhood agree. A radial end structure is an equivalence class of radial foliations.

We will fix a radial end structure for each end of $\mathcal{O}$ so that it is a restriction of a smooth foliation whose leaves ends transversely to the boundary components of $\bar{\mathcal{O}}$.

Recall that orbifold with boundary can be doubled by Propositions 4.4.3 and Section 4.6,1.2 of [56]. Since $\bar{\mathcal{O}}$ is given, the compatibility condition implies that there is actually a unique radial structure for each radial end by the orbifold version of Lemma 5.3 of Hirsch [106] by first doubling the orbifold along the boundary component. Here, the orbifold version of Lemma 5.3 has the analogous proof since the proof is a local one, and adding the finite group action is trivial here. A small minor change is needed since $\frac{\partial g}{\partial y}(x,0)$ may not be zero. However, to begin with, we may subtract $\frac{\partial g}{\partial y}(x,0) \cdot y$ off from $g$. Here an end neighborhood $U$ is compatible to $\bar{\mathcal{O}}$ if it a product form $\Sigma \times (0,1)$ where each $\Sigma \times \{t\}$ is transverse to the radial foliation for sufficiently small $t$ and the diffeomorphism $f : U \to \Sigma \times (0,1)$ extends to $U$ union the ideal boundary component corresponding to $E$ as a diffeomorphism.

A R-end is an end with a radial end structure. A radial pseudo-end (R-p-end) is a p-end with covering a radial end with induced foliation. Each lift of the radial foliation has a finite path-length induced from $\text{dev}$. A pseudo-end (p-end) vertex of a radial p-end neighborhood or a radial p-end is the common endpoint of concurrent lift of leaves of the radial foliation, which we obtain by Cauchy completion along the leaves. Note that $\text{dev}$ always extends to the pseudo-end vertex. The p-end vertex is defined independently of the choice of $\text{dev}$. We will identify with a point of $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) if $\text{dev}$ is an embedding to $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$).

(See Definition 9.1 for more detail.)

Remark 1.3 (End-compactification structures) If we have a compactification $\bar{\mathcal{O}}'$ of $\mathcal{O}$ not diffeomorphic to $\mathcal{O}$, and choose $\bar{\mathcal{O}}'$ instead of $\bar{\mathcal{O}}$, all these discussions have to take place with respect to $\bar{\mathcal{O}}'$. By the s-cobordism theorem of Mazur [139], Barden [12] and Stallings and the existence theorem 11.1 of Milnor [141], there are tame manifolds with more than one compactifications. (This is due to Benoit Kloeckner in the mathematics overflow. See also Section 9.1.3.)

Let $\mathbb{RP}^{n-1}_x$ denote the space of concurrent lines to a point $x$ where two lines are equivalent if they agree in a neighborhood of $x$. Now, $\mathbb{RP}^{n-1}$ is projectively diffeomorphic to $\mathbb{RP}^{n-1}$. The real projective transformations fixing $x$ induce real
projective transformations of $\mathbb{RP}^{n-1}$. Let $x \in \mathbb{S}^n$. The space $\mathbb{S}^n_1$ denotes the space of equivalence classes of concurrent lines ending at $x$ with orientation away from $x$ where two are considered equivalent if they agree on an open subset with a common boundary point $x$. An equivalence class here is called a direction from $x$. Note that $\mathbb{S}^n_1$ is well-defined on $\mathbb{RP}^n$ as well for $x \in \mathbb{RP}^n$.

For a subset $K$ in a convex domain $\Omega$ in $\mathbb{RP}^n$ or $\mathbb{S}^n$, let $x$ be a boundary point. We define $R_x(K)$ for a subset $K$ of $\Omega$ the space of directions of open rays from $x$ in $\Omega$. We defined $R_x(K) \subset \mathbb{S}^n_1$. Any projective group fixing $x$ induces an action on $\mathbb{S}^n_1$.

Following Lemma 1.1 gives us another characterization of $R$-end and the condition $R_x(\bar{U}) = R_x(\Omega)$.

**Lemma 1.1** Let $\Omega$ be a properly convex open domain in $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$, $n \geq 2$). Suppose that $\bigodot = \Omega / \Gamma$ is a noncompact strongly tame orbifold. Let $U$ be an end neighborhood and let $\bar{U}$ be a connected open set in $\Omega$ covering $U$. Let $\Gamma_\bar{U}$ denote the subgroup of $\Gamma$ acting on $\bar{U}$. Suppose that $\bar{U}$ is foliated by segments with a common endpoint $x$ in $\partial \Omega$. Suppose that $\Gamma_\bar{U}$ fixes $x$. Then the following hold:

- $\Gamma_\bar{U}$ acts properly on $R_x(\bar{U})$ if and only if every radial ray in $\bar{U}$ ending at $x$ maps to a properly embedded arc in $U$.
- If the above item holds, then $R_x(\Omega) = R_x(\bar{U})$ and $x$ is an $R$-p-end vertex of $\bigodot$.

**Proof** It is sufficient to prove for $\mathbb{S}^n$. The forward direction of the first item is clear since then the images of leaves under $\Gamma_\bar{U}$ does not accumulate.

For converse, suppose that $g_i(K) \cap K \neq \emptyset$ for infinitely many mutually distinct $g_i \in \Gamma_\bar{U}$ for a compact set $K \subset R_x(\bar{U})$. Then there exists a sequence $p_i \in K$ so that $g_i(p_i) \rightarrow p_\infty$, $p_\infty \in K$. We can choose a compact set $\tilde{K} \subset \Omega$ so that the ray $l_i$ ending at $x$ in direction $p_i$ has an endpoint $\tilde{p}_i \in \tilde{K}$ for each $i$.

The sequence $\{g_i(\tilde{p}_i)\}$ is bounded away from $x$: Suppose not. Let $\tilde{U}_{i,1}, \tilde{U}_{i,2}, \ldots, \tilde{U}_{i,j} \supset \tilde{U}_{i,j+1}$ for each $j = 1, 2, \ldots$, denote a sequence of pseudo-end neighborhoods containing $\bar{U}$. Then for each $j$ there exists infinitely many $g_i(K) \cap \tilde{U}_{i,j} \neq \emptyset$ and hence the image of $\tilde{K}$ and $U_{i,j}$ meets for infinitely many $j$. This contradicts the fact that $U_{i,j}$ is an exiting sequence in $\bigodot$.

We can choose $q_i$ on $l_i$ so that $g_i(q_i) \rightarrow q_\infty$ for $q_\infty$ in $\Omega$ since the direction of $g_i(l_i)$ is in $K$ and its endpoint is uniformly bounded away from $x$.

By Lemma 1.2, we can choose a point $\tilde{q}_i$ on $l_i$ so that $d_{\Omega}(q_i, \tilde{q}_i) < C$ for a uniform constant $C$. Thus, $d_{\Omega}(g_i(q_i), g_i(\tilde{q}_i)) < C$. We can choose a subsequence so that $g_i(\tilde{q}_i) \rightarrow q'_\infty$ for a point $q'_\infty$ in $\Omega$ with $d_{\Omega}(q_\infty, q'_\infty) \leq C$. This implies that $l_i$ is not properly embedded in $U$. This is a contradiction.

For the second item, we have $R_x(\bar{U}) \subset R_x(\Omega)$ clearly. Let $l$ be a line from $x$ in $U$, and let $m$ be any line from $x$ in $\Omega$. For a parametrization of $l$ by $[0, 1)$, we obtain $d_{\Omega}(l(t), m(t)) < C$, $t \in [0, 1)$, for a uniform constant $C > 0$ and a parameterization $m(t)$ of $m$ by Lemma 1.2. Since $l$ maps to a properly embedded arc in $U$, $d_{\Omega}(l(t), \partial \bar{U} \cap \Omega) \rightarrow \infty$ as $t \rightarrow 1$. This implies that $m(t) \in \bar{U}$ for sufficiently large $t$. Therefore $m$ has a direction in $R_x(\bar{U})$. Hence, we showed $R_x(\bar{U}) = R_x(\Omega)$.}

Two rays $l$ and $m$ with arc-length parameterization in $\Omega$ is asymptotic if

\[ \text{[S^mT]} \]
d_\Omega(l(t),m(t)) < C \text{ for a constant } C.

(See Section 3.11.3 of [79]).

Lemma 1.2 (Benoist [22]) Let \( l \) be a line in a properly convex open domain \( \Omega \) in \( \mathbb{R}^n \) (resp. \( S^n \), \( n \geq 2 \)), ending at \( x \in \partial \Omega \). Let \( m \) be a line ending at \( x \) also. Then for a parametrization \( l(t) \) of \( l \) there is a parametrization \( m(t) \) of \( m \) so that 
\[
d_\Omega(m(t),l(t)) < C \text{ for a constant } C \text{ independent of } t.
\]
That is, \( m \) and \( l \) are asymptotic rays.

Proof We will prove for \( S^n \). We choose a supporting hyperplane \( P \) at \( x \). Then \( P \cap \text{Cl}(\Omega) \) is a properly convex domain. We choose a codimension-one subspace \( Q \) of \( P \) disjoint from \( P \cap \text{Cl}(\Omega) \) and a parameter of hyperplanes \( P_t \) passing \( l(t) \) and containing \( Q \). We denote \( m(t) = m \cap P_t \). For convenience, we may suppose our interval is \([0,1]\) and that \( l(0) \) and \( m(0) \) are the beginning point of \( l \) and \( m \). Let \( J \) denote the 2-dimensional subspace containing \( l \) and \( m \). Now, \( m(t),l(t) \) are on a line \( P_t \cap J \). The function \( t \mapsto d_\Omega(l(t),m(t)) \) is eventually decreasing by the convexity of the 2-dimensional domain \( \Omega \cap J \) since we can draw four segments from \( x \) to \( l(t),m(t) \) and the endpoints of \( m(t)l(t) \) \( \cap \Omega \) and the segments to the endpoints always moves outward and the the segments to \( l(t),m(t) \) are constant. Hence, 
\[
d_\Omega(l(t),m(t)) \leq C d_\Omega(l(0),m(0)) \text{ for a constant } C' \geq 1.
\]
(See also Section 3.2.6 and 3.2.7 of [22] For eventual decreasing property, we don’t need the \( C^1 \)-boundary property of \( \Omega \).

Let \( \Omega \) be a properly convex domain in \( \mathbb{R}^p \) so that \( \mathcal{O} = \Omega / \Gamma \) for a discrete subgroup \( \Gamma \) of automorphisms of \( \Omega \). The space of radial lines in an R-end lifts to a space \( R_\ast(\Omega) \) of lines in \( \Omega \) ending at a point \( x \) of \( \partial \Omega \). By above \( \Gamma_x \) acts properly on \( R_\ast(\Omega) \). The quotient space \( R_\ast(\Omega)/\Gamma_x \) has an \( (n-1) \)-orbifold structure by Lemma 1.1 and the properness of the radial lines. The end orbifold \( \Sigma_E \) associated with an R-end is defined as the space of radial lines in \( \mathcal{O} \). It is clear that \( \Sigma_E \) can be identified with \( R_\ast(\Omega)/\Gamma_x \). Obviously, \( \Sigma_E \) is diffeomorphic to the component of \( \mathcal{O} - \mathcal{O} \) corresponding to \( E \). The space of radial lines in an R-end has the local structure of \( \mathbb{R}^{p+1} \) since we can lift a local neighborhood to \( \mathcal{O} \), and these radial lines lift to lines developing into concurrent lines. The end orbifold has an induced real projective structure of one dimension lower.

For the following, we may assume that all subsets here are bounded subsets of an affine subspace \( \mathbb{A}^n \).

- An \( n \)-dimensional submanifold \( L \) of \( \mathbb{A}^n \) is said to be a pre-horoball if it is strictly convex, and the boundary \( \partial L \) is diffeomorphic to \( \mathbb{R}^{n-1} \) and \( bdL - \partial L \) is a single point. The boundary \( \partial L \) is said to be a pre-horosphere.

- Recall that an \( n \)-dimensional subdomain \( L \) of \( \mathbb{A}^n \) is a lens if \( L \) is a convex domain and \( \partial L \) is a disjoint union of two smoothly strictly convex embedded open \( (n-1) \)-cells \( \partial_1 L \) and \( \partial_2 L \).

- A cone is a bounded domain \( D \) in an affine patch with a point in the boundary, called an end vertex \( v \) so that every other point \( x \in D \) has an open segment \( \overline{vx} \subset D \). A cone \( D \) is a join \( \{v\} \ast A \) for a subset \( A \) of \( D \) if \( D \) is a union of segments starting from \( v \) and ending at \( A \). (See Definition 2.7.)
• The cone \( \{ p \} \ast L \) over a lens-shaped domain \( L \) in \( \mathbb{A}^n \), \( p \not\in \text{Cl}(L) \) is a lens-cone if it is a convex domain and satisfies
  - \( \{ p \} \ast L = \{ p \} \ast \partial_+ L \) for one boundary component \( \partial_+ L \) of \( L \) and
  - every segment from \( p \) to \( \partial_+ L \) meets the other boundary component \( \partial_- L \) of \( L \) at a unique point.

• As a consequence, each line segment from \( p \) to \( \partial_+ L \) is transverse to \( \partial_+ L \). \( L \) is called the lens of the lens-cone. (Here different lenses may give the identical lens-cone.) Also, \( \{ p \} \ast L - \{ p \} \) is a manifold with boundary \( \partial_+ L \).

• Each of two boundary components of \( L \) is called a top or bottom hypersurface depending on whether it is further away from \( p \) or not. The top component is denoted by \( \partial_+ L \) and the bottom one by \( \partial_- L \).

• A cone \( \{ p \} \ast L \) is said to be a generalized lens-cone if
  - \( \{ p \} \ast L = \{ p \} \ast \partial_+ L \), \( p \not\in \text{Cl}(L) \) is a convex domain for a generalized lens \( L \), and
  - every segment from \( p \) to \( \partial_+ L \) meets \( \partial_- L \) at a unique point.

A lens-cone will of course be considered a generalized lens-cone.

• We again define the top hypersurface and the bottom one as above. They are denoted by \( \partial_+ L \) and \( \partial_- L \) respectively. \( \partial_+ L \) can be non-smooth; however, \( \partial_- L \) is required to be smooth.

• A totally-geodesic submanifold is a convex domain in a subspace. A cone-over a totally-geodesic submanifold \( D \) is a union of all segments with one endpoint a point \( x \) not in the subspace spanned by \( D \) and the other endpoint in \( D \). We denote it by \( \{ x \} \ast D \).

We apply these to ends:

Definition 1.2

Pre-horospherical R-end An R-p-end \( \tilde{E} \) of \( \tilde{\mathcal{O}} \) is pre-horospherical if it has a pre-horoball in \( \tilde{\mathcal{O}} \) as a p-end neighborhood, or equivalently an open p-end neighborhood \( U \) in \( \tilde{\mathcal{O}} \) so that \( \text{bd} U \cap \tilde{\mathcal{O}} = \text{bd} U - \{ v \} \) for a boundary fixed point \( v \). \( \tilde{E} \) is pre-horospherical if it has a pre-horoball in \( \tilde{\mathcal{O}} \) as a p-end neighborhood. We require that the radial foliation of \( \tilde{E} \) is the one where each leaf ends at \( v \).

Lens-shaped R-end An R-p-end \( \tilde{E} \) is lens-shaped (resp. generalized-lens-shaped), if it has a p-end neighborhood that is projectively diffeomorphic to the interior of \( L \ast \{ v \} \) under \( \text{dev} \) where

• \( L \) is a lens (resp. generalized lens) and
• \( h(\pi_1(\tilde{E})) \) acts properly and cocompactly on \( L \),
and every leaf of the radial foliation of the p-end neighborhood ends corresponds to a radial segment ending at v. In this case, the image L is said to be a CA-lens (resp. cocompactly acted generalized lens (generalized CA-lens)) of such a p-end. A p-end end neighborhood of $\tilde{E}$ is (generalized) lens-shaped if it is a (generalized) lens-cone p-end neighborhood of $\tilde{E}$.

An R-end of $\mathcal{O}$ is lens-shaped (resp. totally geodesic cone-shaped, generalized lens-shaped) if the corresponding R-p-end is lens-shaped (resp. totally geodesic cone-shaped, generalized lens-shaped). An end neighborhood of an end $E$ is (generalized) lens-shaped if it is a (generalized) lens-cone p-end neighborhood of $E$.

An end neighborhood is lens-shaped if it is a lens-shaped R-end neighborhood or T-end neighborhood. A p-end neighborhood is lens-shaped if it is a lens-shaped R-p-end neighborhood or T-p-end neighborhood. Of course it is redundant to say that R-end or T-end satisfies the lens condition dependent on its radial or totally geodesic end structure.

**Definition 1.3** A real projective orbifold with radial or totally geodesic ends is a strongly tame orbifold with a real projective structure where each end is an R-end or a T-end with an end structure given for each. An end of a real projective orbifold is (resp. generalized) lens-shaped or pre-horospherical if it is a (resp. generalized) lens-shaped or pre-horospherical R-end or if it is a lens-shaped T-end.

**1.3.4 Cusp ends**

A parabolic algebra $p$ is an algebra in a semi-simple Lie algebra $g$ whose complexification contains a maximal solvable subalgebra of $g$ (p. 279–288 of [160]). A parabolic group $P$ of a semi-simple Lie group $G$ is the full normalizer of a parabolic subalgebra.

An ellipsoid in $\mathbb{RP}^n = \mathbb{F}(\mathbb{R}^{n+1})$ (resp. in $\mathbb{S}^n = \mathbb{S}(\mathbb{R}^{n+1})$) is the projection $C - \{O\}$ of the null cone

$$C := \{x \in \mathbb{R}^{n+1}|B(x, x) = 0\}$$

for a nondegenerate symmetric bilinear form $B : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ of signature $(1,n)$. Ellipsoids are always equivalent by projective automorphisms of $\mathbb{RP}^n$. An ellipsoid ball is the closed contractible domain in an affine subspace $A^n$ of $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) bounded by an ellipsoid contained in $A^n$. A horoball is an ellipsoid ball with a point $p$ of the boundary removed. An ellipsoid with a point $p$ on it removed is called a horosphere. The vertex of the horosphere or the horoball is defined as $p$.

Let $U$ be a horoball with a vertex $p$ in the boundary of $B$. A real projective orbifold that is projectively diffeomorphic to an orbifold $U/\Gamma_p$ for a discrete subgroup $\Gamma_p \subset \text{PO}(1,n)$ fixing a point $p \in \text{bd}B$ is called a horoball orbifold. A cusp or horospherical end is an end with an end neighborhood that is such an orbifold. A cusp group is a subgroup of a parabolic subgroup of an isomorphic copy of $\text{PO}(1,n)$.
in \( \text{PGL}(n+1, \mathbb{R}) \) or in \( \text{SO}^+(1,n) \) in \( \text{SL}_{\pm}(n+1, \mathbb{R}) \). A cusp group is a \textit{unipotent cusp-group} if it is unipotent as well.

By Corollary 4.1, an end is pre-horospherical if and only if it is a cusp end. We will use the term interchangeably but not in Chapter 4 where we will prove this fact.

### 1.3.5 Examples

**Example 1.1** The interior of a finite-volume hyperbolic \( n \)-orbifold with rank \( n - 1 \) horospherical ends and totally geodesic boundary forms an example of a noncompact strongly tame properly convex real projective orbifold with radial or totally geodesic ends. For horospherical ends, the end orbifolds have Euclidean structures.

(Also, we could allow hyperideal ends by attaching radial ends. See Section 4.1.1.)

**Example 1.2** For examples, if the end orbifold of an R-end \( E \) is a 2-orbifold based on a sphere with three singularities of order 3, then a line of singularity is a leaf of a radial foliation. End orbifolds of Tillmann’s orbifold [150] and the the double of a tetrahedral reflection orbifold are examples. A double orbifold of a cube with edges having orders 3 only has eight such end orbifolds. (See Proposition 4.6 of [57] and their deformations are computed in [63]. Also, see Ryan Greene [98] for the theory.)

### 1.3.6 Uniform middle eigenvalue conditions

This corresponds to Part 2. In Chapters 4, 5, 6, and 8, we will classify ends that possibly arise in properly convex orbifolds. We don’t have a complete classification; however, we will use this. (Ballas, Cooper, Leitner, Long, and Tillmann assume the nilpotency of the end holonomy group and they have a classification [7], [72], and [130].)

We will show that the uniform middle eigenvalue condition given in Definitions 6.2 and 6.3 implies that the lens-condition holds for an R-end or a T-end. The work in Part 2 will show that the lens-conditions are stable ones; that is, a sufficiently small perturbation of the structures will keep the lens conditions for each ends. (See Chapter 6 for the definition of the uniform middle eigenvalue condition.)

Other types of ends are not stable. That is, we can transition between horospherical and lens-shaped ends within the character spaces that we define.

The middle eigenvalue conditions will be somewhat justified in Section A.2.
1.4 Deformation spaces and the spaces of holonomy homomorphisms

To discuss the deformation spaces, we introduce the following notions. The end will be either assigned an $R$-type or a $T$-type.

- An end of type $R$ ($R$-end) is required to be radial.
- A end of type $T$ ($T$-end) is required to a $T$-end of lens-type or be horospherical.

In this monograph, a strongly tame orbifold will always have such an assignment, and finite-covering maps will always respect the types. We will fix these types for our orbifolds in consideration.

An isotopy $i : \mathcal{O} \to \mathcal{O}$ is a self-diffeomorphism so that there exists a smooth orbifold map $J : \mathcal{O} \times [0, 1] \to \mathcal{O}$, so that

$$i_t : \mathcal{O} \to \mathcal{O} \text{ given by } i_t(x) = J(x, t)$$

are self-diffeomorphisms for $t \in [0, 1]$ and $i = i_1, i_0 = 1_{\mathcal{O}}$. We require $i_t$ to be restrictions of isotopies

$$\tilde{i}_t : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}} \text{ given by } \tilde{i}_t(x) = J(x, t)$$

are self-diffeomorphism for $t \in [0, 1]$ and $\tilde{J} : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$ is a smooth orbifold map.

Note that the radial structures for each R-end and the totally geodesic structure for each T-end is preserved since we required the radial foliations to extend to $\tilde{\mathcal{O}}$ smoothly and the ideal boundary component to be the boundary component of $\tilde{\mathcal{O}}$ by the compatibility condition above in Sections 1.3.2 and 1.3.3.

We define $\text{Def}_e(\mathcal{O})$ as the deformation space of real projective structures on $\mathcal{O}$ with end structures; more precisely, this is the quotient space of the real projective structures on $\mathcal{O}$ satisfying the above conditions for ends of type $R$ and $T$ under the isotopy equivalence relations. We define the topology more precisely in Section 9.1.4. (See [54], [36] and [93] for more details.)

1.4.1 The semi-algebraic properties of $\text{rep}^s(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$ and related spaces

Since $\mathcal{O}$ is strongly tame, the fundamental group $\pi_1(\mathcal{O})$ is finitely generated. Let $\{g_1, \ldots, g_m\}$ be a set of generators of $\pi_1(\mathcal{O})$. As usual $\text{Hom}(\pi_1(\mathcal{O}), G)$ for a Lie group $G$ has an algebraic topology as a subspace of $G^m$. This topology is given by the notion of algebraic convergence

$$\{h_j\} \to h \text{ if } \{h_j(g_j)\} \to h(g_j) \in G \text{ for each } j, j = 1, \ldots, m.$$ 

A conjugacy class of a representation is called a character in this monograph.

The $\text{PGL}(n+1, \mathbb{R})$-character space (variety) $\text{rep}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$ is the quotient space of the homomorphism space.
where $\text{PGL}(n+1, \mathbb{R})$ acts by conjugation

$$h(\cdot) \mapsto gh(\cdot)g^{-1} \text{ for } g \in \text{PGL}(n+1, \mathbb{R}).$$

Similarly, we define

$$\text{rep}(\pi_1(O), \text{SL}_\pm(n+1, \mathbb{R})) = \text{Hom}(\pi_1(O), \text{SL}_\pm(n+1, \mathbb{R}))/\text{SL}_\pm(n+1, \mathbb{R})$$

as the $\text{SL}_\pm(n+1, \mathbb{R})$-character space. This is not really a variety in the sense of algebraic geometry. We merely define this space as the quotient space for now, possibly non-Hausdorff one.

A representation or a character is stable if the orbit of it or its representative is closed and the stabilizer is finite under the conjugation action in $\text{Hom}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$ (resp. $\text{Hom}(\pi_1(O), \text{SL}_\pm(n+1, \mathbb{R}))$).

By Theorem 1.1 of [112], a representation $\rho$ is stable if and only if it is irreducible and no proper parabolic subgroup contains the image of $\rho$. The stability and the irreducibility are open conditions in the Zariski topology. Also, if the image of $\rho$ is Zariski dense, then $\rho$ is stable. $\text{PGL}(n+1, \mathbb{R})$ acts properly on the open set of stable representations in $\text{Hom}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$. Similarly, $\text{SL}_\pm(n+1, \mathbb{R})$ acts so on $\text{Hom}(\pi_1(O), \text{SL}_\pm(n+1, \mathbb{R}))$. (See [112] for more details.)

A representation of a group $G$ into $\text{PGL}(n+1, \mathbb{R})$ or $\text{SL}_\pm(n+1, \mathbb{R})$ is strongly irreducible if the image of every finite index subgroup of $G$ is irreducible. Actually, many of the orbifolds have strongly irreducible and stable holonomy homomorphisms by Theorem 1.2.

An eigen-1-form of a linear transformation $\gamma$ is a linear functional $\alpha$ in $\mathbb{R}^{n+1}$ so that $\alpha \circ \gamma = \lambda \alpha$ for some $\lambda \in \mathbb{R}$. We recall the lifting of Remark 1.1.

- \[ \text{Hom}_s(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \]

  to be the subspace of representations $h$ satisfying

  The vertex condition for \( \mathcal{R} \)-ends: \( h|\pi_1(\tilde{E}) \) has a nonzero common eigenvector of positive eigenvalues for a lift of $h(\pi_1(\tilde{E}))$ in $\text{SL}_\pm(n+1, \mathbb{R})$ for each $\mathcal{R}$-type p-end fundamental group $\pi_1(\tilde{E})$.

  The lens-condition for \( \mathcal{T} \)-ends: \( h|\pi_1(\tilde{E}) \) acts on a hyperspace $P$ for each $\mathcal{T}$-type p-end fundamental group $\pi_1(\tilde{E})$ and acts discontinuously and cocompactly on a lens $L$, a properly convex domain with $L^c \cap P = L \cap P \neq \emptyset$ or a horoball tangent to $P$.

- We denote by

  \[ \text{Hom}'(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \]

  the subspace of stable and irreducible representations, and define
1.4 Deformation spaces and the spaces of holonomy homomorphisms

\[ \text{Hom}_E^\varepsilon(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})) \]

to be

\[ \text{Hom}_E(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})) \cap \text{Hom}_E^\varepsilon(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})). \]

- We define
  \[ \text{Hom}_E(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})) \]
  to be the subspace of representations \( h \) where
  - \( h|\pi_1(\tilde{E}) \) has a unique common eigenspace of dimension 1 in \( \mathbb{R}^{n+1} \) with positive eigenvalues for its lift in \( \text{SL}_+(n+1, \mathbb{R}) \) for each p-end holonomy group \( h(\pi_1(\tilde{E})) \) of \( \mathbb{T} \)-type and
  - \( h|\pi_1(\tilde{E}) \) has a common null-space \( P \) of eigen-1-forms satisfying the following:
    - \( \pi_1(\tilde{E}) \) acts properly and cocompactly on a lens \( L \) with \( L \cap P = L' \cap P \) with nonempty interior in \( P \) or
    - \( H - \{ p \} \) for a horosphere \( H \) tangent to \( P \) at \( p \)
  and is unique such one for each p-end holonomy group \( h(\pi_1(\tilde{E})) \) of the p-end of \( \mathbb{T} \)-type.

For \( \mathbb{T} \)-ends, the lens condition is satisfied for a hyperplane \( P \) and \( P \) is unique one satisfying the condition in other words. We define

\[ \text{Hom}_{E,u}^\varepsilon(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})) \]

to be

\[ \text{Hom}^\varepsilon(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})) \cap \text{Hom}_{E,u}^\varepsilon(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})). \]

**Remark 1.4** The above condition for type \( \mathbb{T} \) generalizes the principal boundary condition for real projective surfaces of Goldman [94].

Since each \( \pi_1(\tilde{E}) \) is finitely generated and there is only finitely many conjugacy classes of \( \pi_1(\tilde{E}) \),

\[ \text{Hom}_E(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})) \]

is a closed semi-algebraic subset.

Define

\[ \text{Hom}_{E,f}(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})) \]

to be a subset of

\[ \text{Hom}_E(\pi_1(\partial), \text{PGL}(n+1, \mathbb{R})) \]

so that the p-end holonomy group of each R-p-end fixes finitely many points and the p-end holonomy group of each T-p-ends acts on finitely many hyperspaces. This is an open subset since we can use discriminants of characteristic polynomials of the holonomy matrices of the generators of the end fundamental groups.
Proposition 1.2

\[ \text{Hom}_{E,u}(\pi_1(\mathcal{O}), \text{PGL}(n+1,\mathbb{R})) \]

is an open subset of a semi-algebraic subset

\[ \text{Hom}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1,\mathbb{R})) \]

So is

\[ \text{Hom}_{E,u}^s(\pi_1(\mathcal{O}), \text{PGL}(n+1,\mathbb{R})). \]

Proof By Lemma 1.3, the condition that each p-end holonomy group has the unique fixed point is an open condition.

Let \( \tilde{E} \) be a T-p-end. Let \( h \in \text{Hom}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1,\mathbb{R})) \), and let \( G := h(\pi_1(\tilde{E})) \). Assume that \( G \) is not a cusp group. Let \( P \) be a hyperspace where \( G \) acts on.

Proposition 6.4 implies that the condition of the existence of the hyperspace \( P \) satisfying the lens-property is an open condition in \( \text{Hom}_{E,f}(\pi_1(\mathcal{O}), \text{PGL}(n+1,\mathbb{R})) \).

Suppose that there is another hyperspace \( P' \) with a lens \( L' \) satisfying the above properties. Then \( P \cap P' \) is also \( G \)-invariant. Hence, by Proposition 2.15, we obtain that \( \text{Cl}(P \cap L) \) is a join \( K \ast \{k\} \) for a properly convex domain \( K \) in \( P \cap P' \) and a point \( k \) in \( P - P' \). Similarly, exchanging the role of \( P \) and \( P' \), we obtain that there is a point \( k' \in P' - P \) fixed by \( G \). \( G \) acts on the one-dimensional subspace \( S_G \) containing \( k \) and \( k' \). There are no other fixed point on \( S_G \) since otherwise \( S_G \) is the set of fixed points and \( G \) acts on any hyperspace containing \( P \cap P' \) and a point on \( S_G \). This contradicts our assumption on \( G \). Hence, only \( k \) and \( k' \) are fixed points in \( S_G \) and \( P \cap P' \) and \( \{k,k'\} \) contain all the fixed points of \( G \).

Now, \( k' \) is the unique fixed point outside \( P \). The existence of lens for \( P \) tells us that \( k' \) must be a fixed point outside the closure of the lens. By Theorem 6.9, the existence of a lens for \( P \) tells us that every \( g \in G \), the maximum norm of eigenvalues of \( g \) associated with \( k \) and \( P \cap P' \) is greater than that of \( k' \).

Now, we switch the role of \( P \) and \( P' \). We can take a central element \( g' \) with the largest norm of eigenvalue at \( k' \) by the last item of Proposition 2.15 and the uniform middle eigenvalue condition from Theorem 6.9. This cannot happen by the above paragraph. Hence, \( P \) satisfying the lens-condition is unique.

Suppose that \( G \) is a cusp group. Then there exists a unique hyperspace \( P \) containing the fixed point of \( G \) tangent to horospheres where \( G \) acts on. Therefore,

\[ \text{Hom}_{E,u}(\pi_1(\mathcal{O}), \text{PGL}(n+1,\mathbb{R})) \]

is in an open subset of a union of semi-algebraic subsets of

\[ \text{Hom}_{E,f}(\pi_1(\mathcal{O}), \text{PGL}(n+1,\mathbb{R})). \]
Lemma 1.3. Let $V$ be a semi-algebraic subset of $\text{PGL}(n+1,\mathbb{R})^m$ (resp. $\text{SL}_{\pm}(n+1,\mathbb{R})^m$.) For each $(g_1,\ldots,g_m) \in V$, suppose that there is a function

$E : V \mapsto \mathbb{Z}$

where $E(g_1,\ldots,E_m)$ is the maximum of

$\{ \dim W | W \text{ is a subspace of fixed points of each } g_i, i = 1,\ldots,m \}$

where we define $\dim \emptyset = -1$. Then

$V \ni (g_1,\ldots,g_m) \mapsto \dim E(g_1,\ldots,g_m)$

is an upper semi-continuous function on $V$.

Proof. Suppose that we have a sequence $(g_i^{(j)})$ for each $i = 1,\ldots,m$ and suppose that $g_i^{(j)} \to g_i$ as $j \to \infty$ for each $i$. For any sequence of subspaces of fixed points of $g_1^{(j)},\ldots,g_m^{(j)}$, a limit subspace is contained in a subspace of fixed points of $g_1,\ldots,g_m$. 

We define

- $\text{rep}_{\rho}^E(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))$

  to be the set

  $\text{Hom}_{\rho}(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))/\text{PGL}(n+1,\mathbb{R})$.

- We denote by $\text{rep}_{\rho}^E(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))$ the subspace of

  $\text{rep}_{\rho}(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))$

  of stable and irreducible characters.

- We define

  $\text{rep}_{\rho,\mathcal{U}}(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))$

  to be

  $\text{Hom}_{\rho,\mathcal{U}}(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))/\text{PGL}(n+1,\mathbb{R})$.

- We define

  $\text{rep}_{\rho,\mathcal{U}}(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))$ := $\text{rep}_{\rho}^E(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R})) \cap \text{rep}_{\rho,\mathcal{U}}(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))$. \hfill (1.2)

Let $\rho \in \text{Hom}_{\rho}(\pi_1(E),\text{PGL}(n+1,\mathbb{R}))$ where $E$ is an end. Define
Hom_{E,par}(\pi_1(E), PGL(n+1, \mathbb{R}))

to be the subspace of representations where \(\pi_1(E)\) goes into a cusp group, i.e., a parabolic subgroup in a conjugated copy of \(PO(n, 1)\). By Lemma 1.4,

\[
\text{Hom}_{E,par}(\pi_1(E), PGL(n+1, \mathbb{R}))
\]
is a closed semi-algebraic set.

**Lemma 1.4** \(\text{Hom}_{E,par}(G, PGL(n+1, \mathbb{R}))\) is a closed algebraic set.

**Proof** Let \(P\) be a maximal parabolic subgroup of a conjugated copy of \(PO(n+1, \mathbb{R})\) that fixes a point \(x\). Then \(\text{Hom}(G, P)\) is a closed semi-algebraic set.

\[
\text{Hom}_{E,par}(G, PGL(n+1, \mathbb{R}))
\]
equals a union

\[
\bigcup_{g \in PGL(n+1, \mathbb{R})} \text{Hom}(G, gPg^{-1}),
\]

which is another closed semi-algebraic set. \(\square\)

Let \(E\) be an end orbifold of \(\mathcal{E}\). Given

\[
\rho \in \text{Hom}_{E}(\pi_1(E), PGL(n+1, \mathbb{R})),
\]

we define the following sets:

- Let \(E\) be an end of type \(\mathcal{R}\). Let

\[
\text{Hom}_{E,\mathcal{R}L}(\pi_1(E), PGL(n+1, \mathbb{R}))
\]
denote the space of representations \(h\) of \(\pi_1(E)\) where \(h(\pi_1(E))\) acts on a lens-cone \(\{p\} \star L\) for a lens \(L\) and \(p\) for \(p \not\in Cl(L)\) of a \(p\)-end \(\tilde{E}\) corresponding to \(E\) and acts properly and cocompactly on the lens \(L\) itself. Again, \(\{p\} \star L\) is assumed to be a bounded subset of an affine patch \(\mathbb{A}^n\). Thus, it is a union of open subsets of semi-algebraic sets in \(\text{Hom}_{E}(\pi_1(E), PGL(n+1, \mathbb{R}))\) by Proposition 6.4.

- Let \(E\) denote an end of type \(\mathcal{T}\). Let

\[
\text{Hom}_{E,\mathcal{T}L}(\pi_1(E), PGL(n+1, \mathbb{R}))
\]
denote the space of totally geodesic representations \(h\) of \(\pi_1(E)\) satisfying the following condition:

- \(h(\pi_1(E))\) acts on an lens \(L\) and a hyperspace \(P\) where
  - \(L \cap P = L'' \cap P \neq \emptyset\)
  - \(L/h(\pi_1(E))\) is a compact orbifold with two strictly convex boundary components.
1.4 Deformation spaces and the spaces of holonomy homomorphisms

is again a union of open subsets of the semi-algebraic sets

\[ \text{Hom}_E(\pi_1(E), \text{PGL}(n+1, \mathbb{R})) \]

by Proposition 6.4.

Let

\[ R_E : \text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \ni h \mapsto h|_{\pi_1(E)} \in \text{Hom}(\pi_1(E), \text{PGL}(n+1, \mathbb{R})) \]

be the restriction map to the p-end holonomy group \( h(\pi_1(E)) \) corresponding to the end \( E \) of \( \mathcal{O} \).

A representative set of p-ends of \( \mathcal{O} \) is the subset of p-ends where each end of \( \mathcal{O} \) has a corresponding p-end and a unique chosen corresponding p-end. Let \( \mathcal{R}_\mathcal{O} \) denote the representative set of p-ends of \( \mathcal{O} \) of type \( \mathcal{R} \), and let \( \mathcal{T}_\mathcal{O} \) denote the representative set of p-ends of \( \mathcal{O} \) of type \( \mathcal{T} \). We define a more symmetric space:

\[ \text{Hom}^{s}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \]

to be

\[ \left( \bigcap_{E \in \mathcal{R}_\mathcal{O}} R^{-1}_E \left( \text{Hom}_{s, \text{par}}(\pi_1(E), \text{PGL}(n+1, \mathbb{R})) \cup \text{Hom}_{s, \text{RL}}(\pi_1(E), \text{PGL}(n+1, \mathbb{R})) \right) \right) \cap \left( \bigcap_{E \in \mathcal{T}_\mathcal{O}} R^{-1}_E \left( \text{Hom}_{s, \text{par}}(\pi_1(E), \text{PGL}(n+1, \mathbb{R})) \cup \text{Hom}_{s, \text{TL}}(\pi_1(E), \text{PGL}(n+1, \mathbb{R})) \right) \right). \]

The quotient space of the space under the conjugation under \( \text{PGL}(n+1, \mathbb{R}) \) is denoted by

\[ \text{rep}_{s, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})). \]

We define

\[ \text{Hom}^{s}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \]

to be

\[ \text{Hom}^{s}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \cap \text{Hom}^{s}_{\text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})). \]

Hence, this is a union of open subsets of semialgebraic subsets in

\[ X := \text{Hom}^{s}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})). \]

We don’t claim that the union is open in \( X \). These definitions allow for changes between horospherical ens to lens-shaped radial ones and totally geodesic ones.

The quotient space of this space under the conjugation under \( \text{PGL}(n+1, \mathbb{R}) \) is denoted by

\[ \text{rep}^{s}_{\text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})). \]

Since
\[
\text{rep}_{E,u}^{\mathcal{F}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]
is the Hausdorff quotient of the above set with the conjugation \(\text{PGL}(n+1, \mathbb{R})\)-action, this is an open subset of a semi-algebraic subset by Proposition 1.2 and Proposition 1.1 of [112].

We define
\[
\text{Hom}_{E,u,lh}^{\mathcal{F}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]
to be the subset
\[
\text{Hom}_{E,u}^{\mathcal{F}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \cap \text{Hom}_{E,u,lh}^{\mathcal{F}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).
\]

The above shows

**Proposition 1.3**
\[
\text{rep}_{E,u,lh}^{\mathcal{F}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]
is an open subset of a semi-algebraic set in
\[
\text{rep}_{E,u}^{\mathcal{F}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).
\]

### 1.5 The Ehresmann-Thurston-Weil principle

Note that elements of \(\text{Def}_{E}(\mathcal{O})\) have characters in
\[
\text{rep}_{E}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).
\]

Denote by \(\text{Def}_{E,u}(\mathcal{O})\) the subspace of \(\text{Def}_{E}(\mathcal{O})\) of equivalence classes of real projective structures with characters in
\[
\text{rep}_{E,u}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]
and

- each lift of radial rays of the \(R\)-\(p\)-end neighborhood \(\hat{E}\) of \(\mathcal{O}\)-type ends at the point fixed by \(h(\pi_1(\hat{E}))\) and
- the hyperspace determined by the ideal boundary component of \(T\)-\(p\)-end or the hyperspace tangent to the horosphere of the \(p\)-end of \(\mathcal{T}\)-type coincides with the one on which \(h(\pi_1(\hat{E}))\) acts.

Also, we denote by \(\text{Def}_{E}^{\mathcal{F}}(\mathcal{O}) \subset \text{Def}_{E}(\mathcal{O})\) and \(\text{Def}_{E,u}^{\mathcal{F}}(\mathcal{O}) \subset \text{Def}_{E,u}(\mathcal{O})\) the subspaces of equivalence classes of real projective structures with stable and irreducible characters.

For technical reasons, we will be assuming \(\partial \mathcal{O} = \emptyset\) in most cases. Here, we are not yet concerned with the convexity of orbifolds. The following map \(\text{hol}\), the so-
1.5 The Ehresmann-Thurston-Weil principle

called *Ehresmann-Thurston map*, is induced by sending $(\text{dev}, h)$ to the conjugacy class of $h$ as isotopies preserve $h$:

**Theorem 1.1 (Theorem 9.1)** Let $\mathcal{O}$ be a noncompact strongly tame real projective $n$-orbifold, $n \geq 2$, with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends with end structures and given types $\mathcal{R}$ or $\mathcal{I}$. Assume $\partial \mathcal{O} = \emptyset$. Then the following map is a local homeomorphism:

$$\text{hol} : \text{Def}^I_{\mathcal{E},u}(\mathcal{O}) \to \text{rep}^I_{\mathcal{E},u}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).$$

Also, we define

$$\text{rep}^I_{\mathcal{E}}(\pi_1(\mathcal{O}), \text{SL}_{\pm}(n+1, \mathbb{R})), \text{rep}^I_{\mathcal{E},u}(\pi_1(\mathcal{O}), \text{SL}_{\pm}(n+1, \mathbb{R}))$$

similarly to Section 1.4 where the uniqueness of the p-end vertex for each p-end is only up to the antipodal map.

By lifting $(\text{dev}, h)$ by the method of Section 2.1.6, we obtain that

$$\text{hol} : \text{Def}^I_{\mathcal{E},u}(\mathcal{O}) \to \text{rep}^I_{\mathcal{E},u}(\pi_1(\mathcal{O}), \text{SL}_{\pm}(n+1, \mathbb{R}))$$

is a local homeomorphism.

**Remark 1.5** The restrictions of end types are necessary for this theorem to hold. (See Goldman [93], Canary-Epstein-Green [36], Bergeron-Gelander [28] and Choi [54] for many versions of results for closed manifolds and orbifolds.)

### 1.5.1 SPC-structures and its properties

**Definition 1.4** For a strongly tame orbifold $\mathcal{O}$,

*(IE)* $\mathcal{O}$ or $\pi_1(\mathcal{O})$ satisfies **infinite-index end-fundamental-group condition (IE)** if $[\pi_1(\mathcal{O}) : \pi_1(E)] = \infty$ for the end fundamental group $\pi_1(E)$ of each end $E$.

*(NA)* $\mathcal{O}$ or $\pi_1(\mathcal{O})$ satisfies the **non-annular property (NA)** if

$$\pi_1(E_1) \cap \pi_1(E_2)$$

is finite for two distinct p-ends $E_1, E_2$ of $\mathcal{O}$.

(NA) implies that $\pi_1(E)$ contains every element $g \in \pi_1(\mathcal{O})$ normalizing $\langle h \rangle$ for an infinite order $h \in \pi_1(E)$ for an end fundamental group $\pi_1(E)$ of an end $E$. These conditions are satisfied by complete hyperbolic manifolds with cusps. These are group-theoretical properties with respect to the end groups.

**Definition 1.5 (Definition 7.2)** An **stable properly convex structure (SPC-structure)** on an $n$-orbifold is the structure of a properly convex real projective orbifold with a stable and irreducible holonomy group.
Definition 1.6 (Definition 7.3) Suppose that $O$ has an SPC-structure. Let $\tilde{U}$ be the inverse image in $\tilde{O} \subset \mathbb{R}P^n$ of the union $U$ of some choice of a collection of disjoint end neighborhoods of $\tilde{O}$. If every straight arc and every non-$C^1$-point in $\text{bd}\, \tilde{O}$ are contained in the closure of a component of $\tilde{U}$, then $\tilde{O}$ is said to be strictly convex with respect to the collection of the ends. And $\tilde{O}$ is also said to have a stable properly and strictly convex structure (strict SPC-structure) with respect to the collection of ends.

By a strongly tame orbifold with real projective structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends, we mean one with a real projective structure that has $\mathcal{R}$-type or $\mathcal{I}$-type assigned for each end and each $\mathcal{R}$-end is either generalized lens-shaped or horospherical and each $\mathcal{I}$-end is lens-shaped or horospherical.

Notice that the definition depends on the choice of $U$. However, we will show that if each end is required to be a lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-end, then we show that the definition is independent of $U$ in Corollary 7.2.

We will prove the following in Section 7.3.

Theorem 1.2 Let $O$ be a noncompact strongly tame properly convex real projective $n$-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends and satisfies $(IE)$ and $(NA)$. Then the holonomy group is strongly irreducible and is not contained in a proper parabolic subgroup of $\text{PGL}(n+1, \mathbb{R})$ (resp. $\text{SL}_\pm(n+1, \mathbb{R})$). That is, the holonomy is stable.

1.5.1.1 Main theorems

We now state our main results:

- We define $\text{Def}_{\mathcal{E}, \text{lh}}(O)$ to be the subspace of $\text{Def}_{\mathcal{E}}(O)$ consisting of real projective structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends and stable irreducible holonomy homomorphisms.
- We define $\text{CDef}_{\mathcal{E}, \text{lh}}(O)$ to be the subspace of $\text{Def}_{\mathcal{E}}(O)$ consisting of SPC-structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends.
- We define $\text{CDef}_{\mathcal{E}, \text{u}, \text{lh}}(O)$ to be the subspace of $\text{Def}_{\mathcal{E}, \text{u}}(O)$ consisting of SPC-structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends.
- We define $\text{SDef}_{\mathcal{E}, \text{lh}}(O)$ to be the subspace of $\text{Def}_{\mathcal{E}, \text{lh}}(O)$ consisting of strict SPC-structures with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends.
- We define $\text{SDef}_{\mathcal{E}, \text{u}, \text{lh}}(O)$ to be the subspace of $\text{Def}_{\mathcal{E}, \text{u}, \text{lh}}(O)$ consisting of strict SPC-structures with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends.

We remark that these spaces are dual to the same type of the spaces but we switch the $\mathcal{R}$-end with $\mathcal{I}$-ends and vice versa by Proposition 6.11. Also by Corollary 7.7, for strict SPC-orbifolds with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends have only lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends.

The following theorems are to be regarded as examples of the so-called Ehresmann-Thurston-Weil principle.
Theorem 1.3 Let $\mathcal{O}$ be a noncompact strongly tame $n$-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathbb{R}$- or $\mathcal{T}$-ends. Assume $\partial \mathcal{O} = \emptyset$. Suppose that $\mathcal{O}$ satisfies (IE) and (NA). Then the subspace

$$C\text{Def}_v, u, \text{lh}(\mathcal{O}) \subset \text{Def}_v, u, \text{lh}(\mathcal{O})$$

is open.

Suppose further that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup. Then $\text{hol}$ maps $C\text{Def}_v, u, \text{lh}(\mathcal{O})$ homeomorphically to a union of components of

$$\text{rep}_{v, u, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})).$$

Theorem 1.4 Let $\mathcal{O}$ be a strict SPC noncompact strongly tame $n$-dimensional orbifold, $n \geq 2$, with lens-shaped or horospherical $\mathbb{R}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. Then

- $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to its end fundamental groups.
- The subspace $S\text{Def}_v, u, \text{lh}(\mathcal{O}) \subset \text{Def}_v, u, \text{lh}(\mathcal{O})$ of strict SPC-structures with lens-shaped or horospherical $\mathbb{R}$- or $\mathcal{T}$-ends is open.

Suppose further that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup. Then $\text{hol}$ maps the deformation space $S\text{Def}_v, u, \text{lh}(\mathcal{O})$ of strict SPC-structures on $\mathcal{O}$ with lens-shaped or horospherical $\mathbb{R}$- or $\mathcal{T}$-ends homeomorphically to a union of components of

$$\text{rep}_{v, u, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})).$$

Theorems 1.3 and 1.4 are proved by dividing into the openness result in Section 1.5.2 and the closedness result in Section 1.5.3.

1.5.2 Openness

For openness of $S\text{Def}_v, \text{lh}(\mathcal{O})$, we will make use of:

Corollary 1.2 (Corollary 10.2) Assume that $\mathcal{O}$ is a noncompact strongly tame SPC $n$-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathbb{R}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Let $E_1, \ldots, E_k$ be the ends of $\mathcal{O}$. Assume $\partial \mathcal{O} = \emptyset$. Then $\pi_1(\mathcal{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \ldots, \pi_1(E_k)$ if and only if $\mathcal{O}$ is strictly SPC with respect to ends $E_1, \ldots, E_k$.

Theorem 1.5 Let $\mathcal{O}$ be a noncompact strongly tame real projective $n$-orbifold, $n \geq 2$, and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. In $\text{Def}_v, \text{lh}(\mathcal{O})$, the subspace
CDef\(\mathcal{E}_{\mathcal{D},\mathcal{L}}(\mathcal{O})\) of \textit{SPC-structures} with generalized lens-shaped or horospherical \(\mathcal{R}\)- or \(\mathcal{F}\)-ends is open, and so is \(\text{SDef}_{\mathcal{E}_{\mathcal{D}},\mathcal{L}}(\mathcal{O})\).

**Proof** \(\text{Hom}_{\mathcal{E}_{\mathcal{D}},\mathcal{L}}^+(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))\) is an open subset of \(\text{Hom}_{\mathcal{E}_{\mathcal{D}}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))\) by Proposition 1.2. On \(\text{Hom}_{\mathcal{E}_{\mathcal{D}}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))\) has a uniqueness section defined by Lemma 1.5. Now, Theorem 1.6 proves the result. \(\Box\)

We are given a properly real projective orbifold \(\mathcal{O}\) with ends \(E_1, \ldots, E_{e_1}\) of \(\mathcal{R}\)-type and \(E_{e_1+1}, \ldots, E_{e_1+e_2}\) of \(\mathcal{F}\)-type. Let us choose representative p-ends \(\tilde{E}_1, \ldots, \tilde{E}_{e_1}\) and \(\tilde{E}_{e_1+1}, \ldots, \tilde{E}_{e_1+e_2}\). Again, \(e_1\) is the number of \(\mathcal{R}\)-type ends, and \(e_2\) the number of \(\mathcal{F}\)-type ends of \(\mathcal{O}\).

We define a subspace of \(\text{Hom}_{\mathcal{E}_{\mathcal{D}},\mathcal{L}}^+(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))\) to be as in Section 1.4.1.

Let \(\mathcal{V}\) be an open subset of a semi-algebraic subset of 

\[
\text{Hom}_{\mathcal{E}_{\mathcal{D}},\mathcal{L}}^+(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]

invariant under the conjugation action of \(\text{PGL}(n+1, \mathbb{R})\) so that the following hold:

- one can choose a continuous section \(s^{(1)}_{\mathcal{V}} : \mathcal{V} \to (\mathbb{R}^{n^*})^{e_1}\) sending a holonomy homomorphism to a common fixed point of \(\Gamma_{\tilde{E}_i}\) for \(i = 1, \ldots, e_1\) and

\[
s^{(1)}_{\mathcal{V}}(gh(\cdot)g^{-1}) = g \cdot s^{(1)}_{\mathcal{V}}(h(\cdot)) \quad \text{for} \quad g \in \text{PGL}(n+1, \mathbb{R}).
\]

\(s^{(1)}_{\mathcal{V}}\) is said to be a \textit{fixed-point section}.

If \(\tilde{E}_i\) for every \(i = 1, \ldots, e_1\) has a p-end neighborhood with a radial foliation with leaves developing into rays ending at the point of the \(i\)-th factor of \(s^{(1)}_{\mathcal{V}}\), we say that radial end structures are \textit{determined} by \(s^{(1)}_{\mathcal{V}}\).

Again we assume that \(\mathcal{V}\) is an open subset of a semi-algebraic subset of 

\[
\text{Hom}_{\mathcal{E}_{\mathcal{D}},\mathcal{L}}^+(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]

invariant under the conjugation action by \(\text{PGL}(n+1, \mathbb{R})\), and the following hold:

- one can choose a continuous section \(s^{(2)}_{\mathcal{V}} : \mathcal{V} \to (\mathbb{R}P^{n^*})^{e_2}\) sending a holonomy homomorphism to a common dual fixed point of \(\Gamma_{\tilde{E}_i}\) for \(i = e_1 + 1, \ldots, e_1 + e_2\),

\[
s^{(2)}_{\mathcal{V}}(gh(\cdot)g^{-1}) = (g^*)^{-1} \circ s^{(2)}_{\mathcal{V}}(h(\cdot)) \quad \text{for} \quad g \in \text{PGL}(n+1, \mathbb{R}),
\]

- letting \(P_{\mathcal{V}}(\tilde{E}_i)\) denote the null space of the \(i\)-th value of \(s^{(2)}_{\mathcal{V}}\) for \(i = e_1 + 1, \ldots, e_1 + e_2\), \(\Gamma_{\tilde{E}_i}\) acts on the hyperspace \(P_{\mathcal{V}}(\tilde{E}_i)\) satisfying the lens-condition for \(\tilde{E}_i\).

\(s^{(2)}_{\mathcal{V}}\) is said to be a \textit{dual fixed-point section}.

If each \(\tilde{E}_i\) for every \(i = e_1 + 1, \ldots, e_1 + e_2\)

- has a p-end neighborhood with the ideal boundary component in the hyperspace determined by the \(i\)-th factor of \(s^{(2)}_{\mathcal{V}}\) provided \(\tilde{E}_i\) is a T-end, or
Lemma 1.5 We can define section determined by $s$, we say that end structures for the totally geodesic end are determined by $s^{(2)}_y$.

We define $s^y : \mathcal{V} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{pe})^{e_2}$ as $s^{(1)}_y \times s^{(2)}_y$ and call it a fixing section.

**Proof** $s_u$ is a continuous function since a sequence of fixed points or dual fixed points of end holonomy group is a fixed point or a dual fixed point of the limit end hyperspace as the images.

We define $s_u : \text{Hom}_e,\text{u},\text{lh}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{pe})^{e_2}$ by choosing for each holonomy and each p-end the unique fixed point and the unique hyperspace as the images.

We call $s_u$ the uniqueness section.

Let $\mathcal{V}$ and $s_y : \mathcal{V} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{pe})^{e_2}$ be as above.

- We define $\text{Def}^e_{g,y,\text{lh}}(\mathcal{O})$ to be the subspace of $\text{Def}^e_{g,y}(\mathcal{O})$ of real projective structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-end structures determined by $s_y$ and holonomy homomorphisms in $\mathcal{V}$.

- We define $\text{CDef}^e_{g,y,\text{lh}}(\mathcal{O})$ to be the subspace consisting of SPC-structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-end structures determined by $s_y$ and holonomy homomorphisms in $\mathcal{V}$ in $\text{Def}^e_{g,y,\text{lh}}(\mathcal{O})$.

- We define $\text{SDef}^e_{g,y,\text{lh}}(\mathcal{O})$ to be the subspace consisting of strict SPC-structures with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-end structures determined by $s_y$ and holonomy homomorphisms in $\mathcal{V}$ in $\text{Def}^e_{g,y,\text{lh}}(\mathcal{O})$.

**Theorem 1.6** Let $\mathcal{O}$ be a noncompact strongly tame real projective $n$-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. Choose an open $\text{PGL}(n+1, \mathbb{R})$-conjugation invariant subset of a union of semialgebraic subsets of

$$\forall \subset \text{Hom}^e_{\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))},$$

and a fixing section $s_y : \mathcal{V} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{pe})^{e_2}$.

Then $\text{CDef}^e_{g,y,\text{lh}}(\mathcal{O})$ is open in $\text{Def}^e_{g,y,\text{lh}}(\mathcal{O})$, and so is $\text{SDef}^e_{g,y,\text{lh}}(\mathcal{O})$.

This is proved in Theorem 11.4.

By Theorems 1.5 and 1.1, we obtain:

**Corollary 1.3** Let $\mathcal{O}$ be a noncompact strongly tame real projective $n$-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. Then

$$\text{hol} : \text{CDef}^e_{g,u,\text{lh}}(\mathcal{O}) \to \text{rep}^e_{g,u,\text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$
is a local homeomorphism.
Furthermore, if $\mathcal{O}$ has a strict SPC-structure with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends, then so is

$$\text{hol} : \text{SDef}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \rightarrow \text{rep}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).$$

### 1.5.3 The closedness of convex real projective structures

The results here will be proved in Chapter 11 in Part 3.
We recall

$$\text{rep}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$

the subspace of stable irreducible characters of

$$\text{rep}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$

which is shown to be the open subset of a semi-algebraic subset in Section 1.4.1, and denote by $\text{rep}_{\mathcal{O}, \mathcal{U}, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$, an a union of open subsets of semialgebraic sets.

**Theorem 1.7** Let $\mathcal{O}$ be a noncompact strongly tame SPC $n$-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$, and that the nilpotent normal subgroups of every finite-index subgroup of $\pi_1(\mathcal{O})$ are trivial. Then the following hold:

- The deformation space $\text{CDef}_{\mathcal{O}, \mathcal{U}, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$ of SPC-structures on $\mathcal{O}$ with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends maps under $\text{hol}$ homeomorphically to a union of components of $\text{rep}_{\mathcal{O}, \mathcal{U}, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$.
- The deformation space $\text{SDef}_{\mathcal{O}, \mathcal{U}, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$ of strict SPC-structures on $\mathcal{O}$ with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends maps under $\text{hol}$ homeomorphically to the union of components of $\text{rep}_{\mathcal{O}, \mathcal{U}, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$.

**Proof** $\text{Hom}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$ is an open subset of $\text{Hom}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$ by Proposition 1.2. Corollary 11.4 proves this by the existence of the uniqueness section of Lemma 1.5.

The following is probably the most general result.

**Theorem 1.8 (Theorem 11.3)** Let $\mathcal{O}$ be a noncompact strongly tame SPC $n$-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. Then

- Suppose that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup and $\partial \mathcal{O} = \emptyset$. Then $\text{hol}$ maps the deformation space $\text{CDef}_{\mathcal{O}, \mathcal{U}, \text{lh}}(\mathcal{O})$ of SPC-structures on $\mathcal{O}$ with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends homeomorphically to a union of components of
1.5 The Ehresmann-Thurston-Weil principle

\[ \text{rep}_{E, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})). \]

- Suppose that every finite-index subgroup of \( \pi_1(\mathcal{O}) \) contains no nontrivial infinite nilpotent normal subgroup and \( \partial \mathcal{O} = \emptyset \). Then hol maps the deformation space \( S\text{Def}_{E, \text{lh}}(\mathcal{O}) \) of strict SPC-structures on \( \mathcal{O} \) with lens-shaped or horospherical \( \mathcal{R} \)- or \( \mathcal{T} \)-ends homeomorphically to a union of components of \[ \text{rep}_{E, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})). \]

For example, these apply to the projective deformations of hyperbolic manifolds with torus boundary as in [8].

1.5.4 Remarks

We give some remarks on our results here:

The theory here is by no means exhaustive final words. We have a somewhat complicated theory of ends, which are given in Part 2 of this monograph. Our boundary condition is very restrictive in some sense. With this, it could be said that the above theory is not so surprising.

Ballas, Cooper, Leitner, Long, and Tillmann have different restrictions on ends, and they are working with manifolds. The associated end neighborhoods have nilpotent holonomy groups. (See [73], [72], [129], [128], [130], and [8]). They are currently developing the theory of ends and the deformation theory based on this assumption. Of course, we expect to benefit and thrive from many interactions between the theories as it happens in multitudes of fields.

Originally, we developed the theory for orbifolds as given in papers of Choi [55], Choi, Hodgson, and Lee [63], and [42]. However, the recent examples of Ballas [5], [6], and Ballas, Danciger, and Lee [8] can be covered using fixing sections. Also, differently from the above work, we can allow ends with hyperbolic holonomy groups.

As suggested by Davis, one can look at ends with end holonomy groups acting on properly convex domains in totally geodesic subspaces of codimension between 2 and \( n-1 \). While they are perfectly reasonable to occur, in particular for Coxeter type orbifolds, we shall avoid these types as they are not understood yet, and we will hopefully study these in other papers. We will only be thinking of ends with end holonomy groups acting on codimension \( n \) or codimension 1 subspaces. However, we think that the other types of the ends do not change the theory present here in an essential way. (These are related to generalized Dehn surgery. See our article [66].)

The lens condition is very natural and is a stable condition. We think that all other types of interests are limit of the ends with lens conditions. (See the work mentioned above of Ballas, Cooper, and Leitner [7].) Horospherical \( \mathcal{R} \)- or \( \mathcal{T} \)-ends occur naturally from hyperbolic orbifolds and it a limit of lens-shaped ends. In this monograph, we study these generic cases. However, we question that it is sufficient to add NPNC-ends of Chapter 8 to the discussions to finish the study of deformation.
spaces of convex real projective structure with R-ends and T-ends. We will attempt this later. This monograph does finish the study in some cases of orbifolds as discussed in Chapter 3.
Chapter 2
Preliminaries

We will go over the underlying theory. In Section 2.1, we discuss the Hausdorff convergences of sequences of compact sets, Hilbert metrics, some orbifold topology, geometric structures on orbifolds, real projective structures on orbifolds, spherical real projective structures and liftings, ends, pseudo-ends of the orbifolds, end fundamental groups, and pseudo-end-fundamental groups. We also classify compact convex subsets of $\mathbb{S}^n$ in Proposition 2.5. In Section 2.2, we discuss that affine structures and affine suspensions of real projective orbifolds. In Section 2.3, we discuss the linear algebra and estimation to find convergences, orthopotent groups, proximal and semi-proximal actions, semi-simplicity, and the higher convergence groups. Higher convergence groups are generalizations of convergence groups. We will use the notion in Chapter 5 to develop a theory of flow decompositions. In Section 2.4, we explain the comprehensive Benoist theory on convex orbifolds, where he completed theories of Kuiper, Koszul, Vey, Vinberg, and so on, on divisible actions on convex linear cones as he terms them. In particular, the strict-join decomposition of properly convex orbifolds will be explained. Lemma 2.19 shows that the properly convex real projective structures are uniquely determined by holonomy groups, which is a somewhat commonly overlooked fact. In Section 2.5, we explain the duality theory of Vinberg. We introduce the augmented boundary of properly convex domains as the set of boundary points and the sharply supporting hyperplanes associated with these points. The duality map is extended to the augmented boundary. Duality is extended to sweeping actions also. The duality is extended to every compact convex set in $\mathbb{S}^n$, and we discuss the relationship between the duality and the geometric convergences of the sequences of properly convex sets.

2.1 Preliminary definitions

The following notation is used in the monograph. For a subset $A$ of a space $X$, we denote by $\text{Cl}_X(A)$ the closure of $A$ in $X$ and $\text{bd}_X(A)$ the boundary of $A$ in $X$. If $A$ is a domain of a subspace of $\mathbb{R}^n$ or $\mathbb{S}^n$, we denote by $\text{bd}A$ the topological boundary in
the subspace. The closure \( Cl(A) \) of a subset \( A \) of \( \mathbb{R}P^n \) or \( S^n \) is the topological closure in \( \mathbb{R}P^n \) or in \( S^n \). We will also denote by \( K^o \) the manifold or orbifold interior for a manifold or orbifold \( K \). Also, we may use \( K^o \) as the interior relative to the topology of \( P \) when \( K \) is a domain \( K \) in a totally geodesic subspace \( P \) in \( S^n \) or \( \mathbb{R}P^n \). Define \( \partial A \) for a manifold or orbifold \( A \) to be the manifold or orbifold boundary. (See Section 2.1.3.)

Let \( p, q \in S^n \). We also denote by \( pq \) a minor arc connecting \( p \) and \( q \) in a great circle in \( S^n \). If \( q \neq p \), this is unique. Otherwise, we need to specify a point in \( S^n \) not antipodal to both. We denote by \( p, zq \) the unique minor arc connecting \( p \) and \( q \) passing \( z \).

If \( p, q \in \mathbb{R}P^n \), then \( pq \) denote one of the closures of a component of \( l - \{p, q\} \) for a one dimensional projective line containing \( p, q \). When \( p, q \) is in a fixed ambient convex domain \( \Omega \), then \( pq \) is uniquely determined.

### 2.1.1 The Hausdorff distances used

We will be using the standard elliptic metric \( d \) on \( \mathbb{R}P^n \) (resp. in \( S^n \)) where the set of geodesics coincides with the set of projective geodesics up to parameterizations. Sometimes, these are called Fubini-Study metrics.

**Definition 2.1** Given a set, we define

\[
N_\varepsilon(A) := \{ x \in S^n | d(x, A) < \varepsilon \} \quad \text{resp.} \quad N_\varepsilon(A) := \{ x \in \mathbb{R}P^n | d(x, A) < \varepsilon \}.
\]

Given two subsets \( K_1 \) and \( K_2 \) of \( S^n \) (resp. \( \mathbb{R}P^n \)), we define the *Hausdorff distance* \( d_H(K_1, K_2) \) between \( K_1 \) and \( K_2 \) to be

\[
\inf \{ \varepsilon > 0 | K_2 \subset N_\varepsilon(K_1), K_1 \subset N_\varepsilon(K_2) \}.
\]

The simple distance \( d(K_1, K_2) \) is defined as

\[
\inf \{ d(x, y) | x \in K_1, K_2 \}.
\]

We say that a sequence \( \{A_i\} \) of compact sets *converges* to a compact subset \( A \) if \( \{d_H(A_i, A)\} \to 0 \). Here the limit is unique. Recall that every sequence of compact sets \( \{A_i\} \) in \( S^n \) (resp. \( \mathbb{R}P^n \)) has a convergent subsequence. The limit \( A \) is characterized as follows if it exists:

\[
A := \{ a \in H | a \text{ is a limit point of some sequence } \{a_i | a_i \in A_i\} \}.
\]

See Proposition E.12 of [15] for a proof since the Chabauty topology for a compact space is the Hausdorff topology (See also Munkres [147].)

We will use the same notation even when \( A_i \) and \( A \) are closed subsets of a fixed open domain using \( d_H \) and \( d \).
Proposition 2.1 (Benedetti-Petronio) A sequence \( \{A_i\} \) of compact sets in \( \mathbb{R}P^n \) (resp. \( S^n \)) converges to \( A \) in the Hausdorff topology if and only if the both of the following hold:

- If \( x_i \in A_i \), and \( \{x_i\} \to x \), where \( i \to \infty \), then \( x \in A \).
- If \( x \in A \), then there exists \( x_i \in A_i \) for each \( i \) such that \( \{x_i\} \to x \).

Proof Since \( S^n \) and \( \mathbb{R}P^n \) are compact, the Chabauty topology is same as the Hausdorff topology. Hence, this follows from Proposition E.12 of Benedetti-Petronio [15].

Lemma 2.1 Let \( \{g_i\} \) be a sequence of elements of \( \text{PGL}(n + 1, \mathbb{R}) \) (resp. \( \text{SL}_\pm(n + 1, \mathbb{R}) \)) converging to \( g_\infty \) in \( \text{PGL}(n + 1, \mathbb{R}) \) (resp. \( \text{SL}_\pm(n + 1, \mathbb{R}) \)). Let \( \{K_i\} \) be a sequence of compact set and let \( K \) be another one. Then \( \{K_i\} \to K \) if and only if \( \{g_i(K_i)\} \to g_\infty(K) \).

Proof We use the above point description of the geometric limit. [S^n S]

An \( n \)-hemisphere \( H \) in \( S^n \) supports a domain \( D \) if \( H \) contains \( D \). \( H \) is called a supporting hemisphere. An oriented hyperspace \( S \) in \( S^n \) supports a domain \( D \) if the closed hemisphere bounded in an inner-direction by \( S \) contains \( D \). \( S \) is called a supporting hyperspace. If a supporting hyperspace contains a boundary point \( x \) of \( D \), then it is called a sharply-supporting hyperspace at \( x \). If the boundary of a supporting \( n \)-hemisphere is sharply supporting at \( x \), then the hemisphere is called a sharply-supporting hemisphere at \( x \).

Proposition 2.2 Let \( K_i \) be a sequence of compact convex sets (resp. cells) of \( S^n \). Then up to choosing a subsequence \( K_i \to K \) to a compact convex set (resp. cell) \( K \) of \( S^n \). Also, a geometric limit must be a compact convex cell when \( K_i \) are compact convex cells. If \( K_i \) is in a fixed \( n \)-hemisphere, then so is \( K \).

Proof By Proposition 2.1, we can show this when \( K_i \) is a hemisphere. For other cases, consider sequences of segments and Proposition 2.1. □

The following is probably well-known.

Lemma 2.2 Suppose that one of the following holds:

- \( K_i \) for each \( i, i = 1, 2, \ldots \), is a compact convex domain, and \( K \) is one also in \( S^n \).
- \( K_i \) is a convex open domain, and \( K \) is one also in \( S^n \).
- \( K_i \) is a properly convex domain, and \( K \) is one also in \( \mathbb{R}P^n \).

Suppose that a sequence \( \{K_i\} \) geometrically converging to \( K \) with nonempty interior. Then \( \{\text{bd}K_i\} \to \text{bd}K \).

Proof We prove for \( S^n \). Suppose that a point \( p \) is in \( \text{bd}K \). Let \( B_\varepsilon(p) \) be an open \( \varepsilon \)-ball of \( p \). Suppose \( B_\varepsilon(p) \cap K_i = \emptyset \) for infinitely many \( i \). Then \( p \) cannot be a limit point of \( K \) by Proposition 2.1. This is a contradiction. Thus, \( B_\varepsilon(p) \cap K_i \neq \emptyset \) for \( i = N \) for some \( N \). Suppose that \( B_\varepsilon(p) \subset K_i \) for infinitely many \( i \). Then each point in \( B_\varepsilon(p) - K \) is a limit point of some sequence \( p_i, p_i \in K_i \), and hence \( B_\varepsilon(p) \subset K, p \in \text{bd}K \).
coordinates of \(K\), a contradiction. Hence, given \(\varepsilon > 0\), \(B_\varepsilon(p) \cap \partial K_i \neq \emptyset\) for \(i > M\) for some \(M\). Then \(p\) is a limit of a sequence \(p_i, p_i \in \partial K_i\).

Conversely, suppose that a sequence \(\{p_i\}, p_i \in \partial K_i\) where \(i \to \infty\) as \(j \to \infty\), converges to \(p\). Then \(p \in K\) clearly. Suppose that \(p \in K^o\). Then there is \(\varepsilon, \varepsilon > 0\), with \(B_\varepsilon(p) \subset K\). Now, \(K_i\) has a sharply supporting closed hemisphere \(H_i\) at \(p_i\) with \(K_i \subset H_i\). Since \(\{p_i\} \to p\), we may choose a subsequence \(j_i\) so that \(\{H_{i_k}\} \to H_\infty\) and \(d_H(H_{i_k}, H_\infty) < \varepsilon / 4\) for a hemisphere \(H_\infty\). Let \(q \in B_\varepsilon/4(p) - H_\infty\) so that \(d_H(q, H_\infty) > \varepsilon / 4\). Hence, \(B_\varepsilon/4(q) \subset B_\varepsilon(p) - H_k\) for all \(j\). Since \(K_k \subset H_k\), \(q\) is a limit of a sequence \(\{q_k\}\), \(q_k \in K_k\) converges to \(q\). However, since \(\{K_k\} \to K\) and \(q \in K\), this is a contradiction to Proposition 2.1. Hence, \(p \in \partial K\). Now, Proposition 2.1 proves \(\{\partial K_i\} \to \partial K\).

When \(K_i\) is an open domain in \(S^n\), we just need to take its closure and use the first part.

For the \(R^{p^n}\)-version, we lift \(K_i\) to \(S^n\) to properly convex domains \(K_i'\). Now, we may also choose a subsequence so that \(\{K_i'\}\) geometrically converges to a choice of a lift \(K'\) of \(K\) by Proposition 2.1. Since \(K'\) is properly convex, \(K'\) is in a bounded subset of an affine subspace of \(S^n\). Then the result follows from the \(S^n\)-version.

We note that the last statement is false if \(\{K_i\}\) geometrically converges to a hemisphere when lifted to \(S^n\).

**Theorem 2.1** Suppose that \(K_i\) and \(K\) are (resp. properly) convex compact balls of the same dimension in \(S^n\) (resp. \(R^{p^n}\)). Suppose that \(\{K_i\} \to K\). It follows that \(\{\partial K_i\} \to \partial K\). (2.1)

This holds also provided \(K_i\) and \(K\) are properly compact convex in \(R^{p^n}\) with \(\{K_i\} \to K\).

**Proof** Since \(K_i\) and \(K\) are of the same dimension, we find \(g_i \in SL_\pm(n + 1, R)\) so that \(g_i(K_i) = K\) and \(\{g_i\} \to g_\infty\) for \(g_\infty \in SL_\pm(n + 1, R)\). Then \(\{g_i(K_i)\} \to g_\infty(K)\).

Then \(\{\partial g_i(K_i)\} \to \partial g_\infty(K)\) by Lemma 2.2. Hence, \(\{\partial K_i\} \to \partial K\) by Lemma 2.1. \([S^nP]\)

### 2.1.2 The Hilbert metric

Let \(\Omega\) be a properly convex open domain. A line or a subspace of dimension-one in \(R^{p^n}\) has a 2-dimensional homogeneous coordinate system. Let \([o, s, q, p]\) denote the cross ratio of four points on a line as defined by

\[
\frac{\bar{\partial} - \bar{q} \cdot \bar{s} - \bar{p}}{\bar{s} - \bar{q} \cdot \bar{\partial} - \bar{p}}
\]

where \(\bar{\partial}, \bar{p}, \bar{q}, \bar{s}\) denote respectively the first coordinates of the homogeneous coordinates of \(o, p, q, s\) provided that the second coordinates equal 1. Define a metric
for \( p, q \in \Omega \), \( d_\Omega(p, q) = \log |[o,s,q,p]| \) where \( o \) and \( s \) are endpoints of the maximal segment in \( \Omega \) containing \( p, q \) where \( o, q \) separates \( p, s \). The metric is one given by a Finsler metric. (See [120].)

Assume that \( \{K_i\} \to K \) geometrically for a sequence of properly convex compact domains \( K_i \) and a properly convex compact domain \( K \). Suppose that two sequences of points \( \{x_i|x_i \in K_i\} \) and \( \{y_i|y_i \in K_i\} \) converge to \( x, y \in K \) respectively. Since the endpoints of a maximal segment always are in \( \partial K_i \) and \( \{\partial K_i\} \to \partial K \), Theorem 2.1 shows that

\[
\{d_{K_i}(x_i,y_i)\} \to d_K(x,y). \tag{2.2}
\]

We omit the details of the elementary proof.

**Lemma 2.3 (Cooper-Long-Tillman [73])** Let \( U \) be a convex subset of a properly convex domain \( V \) in \( \mathbb{S}^n \) (resp. \( \mathbb{RP}^n \)). Let

\[
U_\varepsilon := \{x \in V | d_V(x,U) \leq \varepsilon \}
\]

for \( \varepsilon > 0 \). Then \( U_\varepsilon \) is properly convex.

**Proof** Given \( u, v \in U_\varepsilon \), we find

\[
w, t \in \Omega \text{ so that } d_V(u,w) < \varepsilon, d_V(v,t) < \varepsilon.
\]

Then each point of \( \overline{wt} \) is within \( \varepsilon \) of \( \overline{w} \subset U \) in the \( d_V \)-metric. By Lemma 1.8 of [73], this follows.

**Proposition 2.3** Let \( \Omega \) be a properly convex domain in \( \mathbb{S}^n \) (resp. \( \mathbb{RP}^n \)). Then the group \( \text{Aut}(\Omega) \) of projective automorphisms \( \Omega \) is closed in \( \text{SL}_\pm(n+1, \mathbb{R}) \) (resp. \( \text{PGL}(n+1, \mathbb{R}) \)) acts on \( \Omega \). Also, the set of elements of \( g \) of \( \text{Aut}(\Omega) \) so that \( g(x) \in K \) for a compact subset \( K \) of \( \Omega \) is compact.

**Proof** We prove for \( \mathbb{S}^n \). Clearly, the limit of a sequence of elements in \( \text{Aut}(\Omega) \) is an isometry of the Hilbert metric of \( \Omega \). Hence, it acts on \( \Omega \).

For the second part, we take an \( n \)-simplex \( \sigma \) with a point \( x \) in the interior as a base point.

The space \( \mathcal{J}^n \) of nondegenerate convex \( n \)-simplices with base points in \( \mathbb{S}^n \) with Hausdorff topology is homeomorphic to \( \text{SL}_\pm(n+1, \mathbb{R}) \) since the action of \( \text{SL}_\pm(n+1, \mathbb{R}) \) is simply transitive on \( \mathcal{J}^n \).

The subspace of simplices of form \( g(\sigma) \) for \( g \) with \( g(x) \in K, g \in \text{Aut}(\Omega) \) is compact by the existence of the Hilbert metric: We can show this by using the invariants. The distance from each vertex to the hyperspace containing the remaining vertices is an invariant of the action.

Since \( \mathcal{J}^n \) is diffeomorphic to \( \text{SL}_\pm(n+1, \mathbb{R}) \), the closedness of \( \text{Aut}(\Omega) \) proves the result.

**Proposition 2.4** Let \( \Omega \) be a properly convex domain in \( \mathbb{S}^n \) (resp. \( \mathbb{RP}^n \)). Suppose that a discrete subgroup \( \Gamma \) of \( \text{SL}_\pm(n+1, \mathbb{R}) \) (resp. \( \text{PGL}(n+1, \mathbb{R}) \)) acts on \( \Omega \). Then \( \Omega/\Gamma \) is an orbifold.
Proof. The second part of Proposition 2.3 implies that $\Gamma$ acts properly discontinuously. We obtain that $\Omega/\Gamma$ is again a closed orbifold. (We need a slight modification of Proposition 3.5.7 of Thurston [159].)

\[\square\]

2.1.3 Topology of orbifolds

An $n$-dimensional orbifold structure on a Hausdorff space $X$ is given by maximal collection of charts $(U, \phi, G)$ satisfying the following conditions:

- $U$ is an open subset of $\mathbb{R}^n$ and $\phi : U \to X$ is a map and $G$ is a finite group acting on $U$,
- the chart $\phi : U \to X$ induces a homeomorphism $U/G$ to an open subset of $X$,
- the sets of form $\phi(U)$ covers $X$.
- for any pair of models $(U, \phi, G)$ and $(V, \psi, H)$ with an inclusion map $\iota : \phi(U) \to \phi(V)$ lifts to an embedding $U \to V$ equivariant with respect to an injective homomorphism $G \to H$. (compatibility condition)

An orbifold $O$ is a topological space with an orbifold structure. The boundary $\partial O$ of an orbifold is defined as the set of points with only half-open sets as models. (These are often distinct from topological boundary.)

Orbifolds are stratified by manifolds. Let $\mathcal{O}$ denote an $n$-dimensional orbifold with finitely many ends. We will require that $\mathcal{O}$ is strongly tame; that is, $\mathcal{O}$ has a compact suborbifold $K$ so that $\mathcal{O} - K$ is a disjoint union of end neighborhoods homeomorphic to closed $(n - 1)$-dimensional orbifolds multiplied by open intervals. Hence $\partial \mathcal{O}$ is a compact suborbifold. (See [158], [1], [114] and [56] for details.)

An orbifold covering map $p : \mathcal{O}_1 \to \mathcal{O}$ is a map so that for any point on $\mathcal{O}$, there is a connected open set $U \subset X$ with model $(\tilde{U}, \phi, G)$ as above whose inverse image $p^{-1}(U)$ is a union of connected open set $U_i$ of $\mathcal{O}_1$ with models $(\tilde{U}_i, \phi_i, G_i)$ for a subgroup $G_i \subset G$ and the induced chart $\phi_i : \tilde{U}_i \to \tilde{U}_i$.

We say that an orbifold is a manifold if it has a subatlas of charts with trivial local groups. We will consider good orbifolds only, i.e., covered by simply connected manifolds. In this case, the universal covering orbifold $\tilde{\mathcal{O}}$ is a manifold with an orbifold covering map $p : \tilde{\mathcal{O}} \to \mathcal{O}$. The group of deck transformations will be denote by $\pi_1(\mathcal{O})$ or $\Gamma$, and is said to be the fundamental group of $\mathcal{O}$. They act properly discontinuously on $\tilde{\mathcal{O}}$ but not necessarily freely.

2.1.4 Geometric structures on orbifolds

An $(X, G)$-structure on an orbifold $\mathcal{O}$ is an atlas of charts from open subsets of $X$ with finite subgroups of $G$ acting on them, and the inclusions always lift to restrictions of elements of $G$ in open subsets of $X$. This is equivalent to saying that the orbifold $\mathcal{O}$ has a simply connected manifold cover $\tilde{\mathcal{O}}$ with an immersion
2.1 Preliminary definitions

\( D : \tilde{\mathcal{O}} \to X \) and the fundamental group \( \pi_1(\mathcal{O}) \) acts on \( \tilde{\mathcal{O}} \) properly discontinuously so that \( h : \pi_1(\mathcal{O}) \to G \) is a homomorphism satisfying \( D \circ \gamma = h(\gamma) \circ D \) for each \( \gamma \in \pi_1(\mathcal{O}) \). Here, \( \pi_1(\mathcal{O}) \) is allowed to have fixed points with finite stabilizers. (We shall use this second more convenient definition here.) \((D, h(\cdot))\) is called a development pair and for a given \((X, G)\)-structure, it is determined only up to an action

\[
(D, h(\cdot)) \mapsto (k \circ D, kh(\cdot)k^{-1}) \quad \text{for } k \in G.
\]

Conversely, a development pair completely determines the \((X, G)\)-structure. (See Thurston [159] for the general theory of geometric structures.)

Thurston showed that an orbifold with an \((X, G)\)-structure is always good, i.e., covered by a manifold with an \((X, G)\)-structure. (See Proposition 13.2.1 of Chapter 13 of Thurston [158].) Hence, every geometric orbifold is of form \( \tilde{M}/\Gamma \) for a discrete group \( \Gamma \) acting on a simply connected manifold \( \tilde{M} \). Here, we have to understand \( \tilde{M}/\Gamma \) as having an orbifold structure coming from an atlas where each model set is based on a precompact open cell of \( \tilde{M} \) on which a finite subgroup of \( \Gamma \) acts. (See Theorem 4.23 of [56] for details.)

2.1.5 Real projective structures on orbifolds.

A cone \( C \) in \( \mathbb{R}^{n+1} - \{O\} \) is a subspace so that given a vector \( x \in C \), \( sx \in C \) for every \( s \in \mathbb{R}_+ \). A convex cone is a cone that is a convex subset of \( \mathbb{R}^{n+1} \) in the usual sense. A properly convex cone is a convex cone not containing a complete affine line.

Recall the real projective space \( \mathbb{RP}^n \) is defined as \( \mathbb{R}^{n+1} - \{O\} \) under the quotient relation \( v \sim w \) iff \( v = sw \) for \( s \in \mathbb{R} - \{O\} \).

- Given a vector \( v \in \mathbb{R}^{n+1} - \{O\} \), we denote by \([v] \in \mathbb{RP}^n\) the equivalence class.
- Let \( \Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{RP}^n \) denote the projection.
- Given a connected subset \( A \) of an affine subspace of \( \mathbb{RP}^n \), a cone \( C(A) \subset \mathbb{R}^{n+1} \) of \( A \) is given as a connected cone in \( \mathbb{R}^{n+1} \) mapping onto \( A \) under the projection \( \Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{RP}^n \).
- \( C(A) \) is unique up to the antipodal map \( x' : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) given by \( v \mapsto -v \).

The general linear group \( \text{GL}(n+1, \mathbb{R}) \) acts on \( \mathbb{R}^{n+1} \) and \( \text{PGL}(n+1, \mathbb{R}) \) acts faithfully on \( \mathbb{RP}^n \). Denote by \( \mathbb{R}_+ = \{ r \in \mathbb{R} | r > 0 \} \). The real projective sphere \( \mathbb{S}^n \) is defined as the quotient of \( \mathbb{R}^{n+1} - \{O\} \) under the quotient relation \( v \sim w \) iff \( v = sw \) for \( s \in \mathbb{R}_+ \). We will also use \( \mathbb{S}^n \) as the double cover of \( \mathbb{RP}^n \). Then \( \text{Aut}(\mathbb{S}^n) \), isomorphic to the subgroup \( \text{SL}_+ (n+1, \mathbb{R}) \) of \( \text{GL}(n+1, \mathbb{R}) \) of determinant \( \pm 1 \), double-covers \( \text{PGL}(n+1, \mathbb{R}) \). \( \text{Aut}(\mathbb{S}^n) \) acts as a group of projective automorphisms of \( \mathbb{S}^n \). A projective map of a real projective orbifold to another is a map that is projective by charts to \( \mathbb{RP}^n \). Let \( \Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{RP}^n \) be a projection and let \( \Pi' : \mathbb{R}^{n+1} - \{O\} \to \mathbb{S}^n \) denote one for \( \mathbb{S}^n \). An infinite subgroup \( \Gamma \) of \( \text{PGL}(n+1, \mathbb{R}) \) (resp. \( \text{SL}_+ (n+1, \mathbb{R}) \)) is strongly irreducible if every finite-index subgroup is irreducible. A subspace \( S \) of \( \mathbb{RP}^n \) (resp. \( \mathbb{S}^n \)) is the image of a subspace with the origin removed under the projection \( \Pi \) (resp. \( \Pi' \)).
A line in $\mathbb{RP}^n$ or $\mathbb{S}^n$ is an embedded arc in a 1-dimensional subspace. A **projective geodesic** is an arc in a projective orbifold developing into a line in $\mathbb{RP}^n$ or to a one-dimensional subspace of $\mathbb{S}^n$. A **great segment** is an embedded geodesic connecting a pair of antipodal points in $\mathbb{S}^n$ or the complement of a point in a 1-dimensional subspace in $\mathbb{RP}^n$. Sometimes open great segment is called a complete affine line. An affine subspace $A^n$ can be identified with the complement of a codimension-one subspace $\mathbb{RP}^{n-1}$ so that the geodesic structures are same up to parameterizations. A convex subset of $\mathbb{RP}^n$ is a convex subset of an affine subspace in this paper. A properly convex subset of $\mathbb{RP}^n$ is a precompact convex subset of an affine subspace. $\mathbb{R}^n$ identifies with an open half-space in $\mathbb{S}^n$ defined by a linear function on $\mathbb{R}^{n+1}$. (In this paper an affine subspace is either embedded in $\mathbb{RP}^n$ or $\mathbb{S}^n$.)

An $i$-dimensional complete affine subspace is a subspace of a projective orbifold projectively diffeomorphic to an $i$-dimensional affine subspace in some affine subspace $A^n$ of $\mathbb{RP}^n$ or $\mathbb{S}^n$.

Again an affine subspace in $\mathbb{S}^n$ is a lift of an affine subspace in $\mathbb{RP}^n$, which is the interior of an $n$-hemisphere. Convexity and proper convexity in $\mathbb{S}^n$ are defined in the same way as in $\mathbb{RP}^n$.

The complement of a codimension-one subspace $W$ in $\mathbb{RP}^n$ can be considered an affine space $A^n$ by correspondence

$$[1, x_1, \ldots, x_n] \to (x_1, \ldots, x_n)$$

for a coordinate system where $W$ is given by $x_0 = 0$. The group $\text{Aff}(A^n)$ of projective automorphisms acting on $A^n$ is identical with the group of affine transformations of form

$$x \mapsto Ax + b$$

for a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ and $b \in \mathbb{R}^n$. The projective geodesics and the affine geodesics agree up to parametrizations.

A subset $A$ of $\mathbb{RP}^n$ or $\mathbb{S}^n$ spans a subspace $S$ if $S$ is the smallest subspace containing $A$. We write $S = \langle A \rangle$. Of course, we use the same term for affine and vector spaces as well.

We will consider an orbifold $\mathcal{O}$ with a real projective structure: This can be expressed as

- having a pair $(\text{dev}, h)$ where $\text{dev} : \mathcal{O} \to \mathbb{RP}^n$ is an immersion equivariant with respect to
- the homomorphism $h : \pi_1(\mathcal{O}) \to \text{PGL}(n+1, \mathbb{R})$ where $\mathcal{O}$ is the universal cover and $\pi_1(\mathcal{O})$ is the group of deck transformations acting on $\mathcal{O}$.

$(\text{dev}, h)$ is only determined up to an action of $\text{PGL}(n+1, \mathbb{R})$ given by

$$g \circ (\text{dev}, h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1}) \text{ for } g \in \text{PGL}(n+1, \mathbb{R}).$$

$\text{dev}$ is said to be a developing map and $h$ is said to be a holonomy homomorphism and $(\text{dev}, h)$ is called a development pair. We will usually use only one pair where $\text{dev}$ is an embedding for this paper and hence identify $\mathcal{O}$ with its image. A holonomy is
an image of an element under \( h \). The holonomy group is the image group \( h(\pi_1(O)) \).

We denote by \( \text{Aut}(K) \) the group of projective automorphisms of a set \( K \) in some space with a projective structure. The Klein model of the hyperbolic geometry is given as follows: Let \( x_0, x_1, \ldots, x_n \) denote the standard coordinates of \( \mathbb{R}^{n+1} \). Let \( B \) be the interior in \( \mathbb{R}P^n \) or \( S^n \) of a standard ball that is the image of the positive cone of \( x_0^2 > x_1^2 + \cdots + x_n^2 \) in \( \mathbb{R}^{n+1} \). Then \( B \) can be identified with a hyperbolic \( n \)-space. The group of isometries of the hyperbolic space equals the group \( \text{Aut}(B) \) of projective automorphisms acting on \( B \). Thus, a complete hyperbolic manifold carries a unique real projective structure and is denoted by \( B/\Gamma \) for \( \Gamma \subset \text{Aut}(B) \). Actually, \( g(B) = g\text{Aut}(B)g^{-1} \) is the isometry group. (See [56] for details.)

A totally geodesic hypersurface \( A \) in \( \tilde{O} \) is a suborbifold of codimension-one where each point \( p \) in \( A \) has a neighborhood \( U \) in \( \tilde{O} \) so that \( D|_{\tilde{A}} \) has the image in a hyperspace. A suborbifold \( A \) is a totally geodesic hypersurface if it is covered by a one in \( \tilde{O} \).

### 2.1.6 Spherical real projective structures

We use a slightly different definition of convexity for \( S^n \).

**Definition 2.2** A convex segment is an arc contained in a great segment. A convex subset of \( S^n \) is a subset \( A \) where every pair of points of \( A \) connected by a convex segment.

It is easy to see that either a convex subset of \( S^n \) is contained in an affine subspace, it is in a closed hemisphere, or it is a great sphere of dimension \( \geq 1 \). In the first case, the set embeds to a convex set in \( \mathbb{R}P^n \) under the covering map.

**Proposition 2.5** A closed convex subset \( K \) of \( S^n \) is either a great sphere \( S^{i_0} \) of dimension \( i_0 \geq 1 \), or is contained in a closed hemisphere \( H^{i_0} \) in \( S^{i_0} \) and is one of the following:

- There exist a great sphere \( S^{j_0} \) of dimension \( j_0 \geq 0 \) in the boundary \( \partial K \) and a compact properly convex domain \( K_K \) in an independent subspace of \( S^{j_0} \) and \( K = S^{j_0} \ast K_K \), a strict join. Moreover, \( S^{j_0} \) is a unique maximal great sphere in \( K \).
- \( K \) is a properly convex domain in the interior of \( i_0 + 1 \)-hemisphere for some \( i \).

**Proof** Let \( S^{i_0} \) be the span of \( K \). Then \( K^{i_0} \) is not an empty domain in \( S^{i_0} \). The map \( x \mapsto d(x, K) \) is continuous on \( S^{i_0} \). Choose a maximum point \( x_0 \). If the maximum is \( < \pi/2 \), then the elliptic geometry tells us that there at least two point \( y, z \) of \( K \) closest to \( x_0 \) of same distance from \( x_0 \) since otherwise we can increase the value of \( d(\cdot, K) \) by moving \( x_0 \) slightly. Then there is a closer point on \( \tilde{V}^{i_0} \) in \( K \) to \( x_0 \). This is a contradiction. Hence, \( K = S^{i_0} \). Otherwise, \( K \) is a subset of an \( i_0 \)-hemisphere in \( S^{i_0} \). (See [48] also.)
The second part follows from Section 1.4 of [40]. (See also [76].) Hence, we obtain a unique maximal great sphere \( S^h \) in \( K \) which is contained in \( \partial K \), and \( K \) is a union of \( j_0 + 1 \)-hemispheres with common boundary \( S^j_0 \).

By choosing an independent subspace \( S^{n-j_0-1} \) to \( S^{j_0} \), each \( j_0 + 1 \)-hemisphere in \( K \) is transverse to \( S^{n-j_0-1} \) and hence meets it in a unique point. We let \( K_k \) denote the set of intersection points. Therefore, \( K = S^j_0 * K_k \).

There is a map \( K \to K_k \) given by sending a \( j_0 + 1 \)-hemisphere to the intersection point. Obviously, this is a restriction of projective diffeomorphism from the space of \( j_0 + 1 \)-hemispheres with boundary \( S^j_0 \) to \( S^{n-j_0-1} \). Since \( K \) cannot contain a higher-dimensional great sphere, it follows that \( K_k \) is properly convex also.

Recall that \( SL_+ (n+1, \mathbb{R}) \) is isomorphic to \( GL(n+1, \mathbb{R}) / \mathbb{R}_+ \). Then this group acts on \( S^n \) to be seen as a quotient space of \( \mathbb{R}^{n+1} - \{O\} \) by the equivalence relation

\[
\mathbf{v} \sim \mathbf{w}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{n+1} - \{O\} \text{ if } \mathbf{v} = s \mathbf{w} \text{ for } s \in \mathbb{R}_+.
\]

We let \( \langle \mathbf{v} \rangle \) denote the equivalence class of \( \mathbf{v} \in \mathbb{R}^{n+1} - \{O\} \). Given a vector subspace \( V \in \mathbb{R}^{n+1} \), we denote by \( S(V) \) the image of \( V - \{O\} \) under the quotient map. The image is called a subspace. A set of antipodal points is a subspace of dimension 0. There is a double covering map \( p_{\mathcal{S}^n} : \mathcal{S}^n \to \mathbb{R}P^n \) with the deck transformation group generated by \( \mathcal{A} \). This gives a projective structure on \( \mathcal{S}^n \). The group of projective automorphisms is identified with \( SL_\pm (n+1, \mathbb{R}) \). The notion of geodesics are defined as in the projective geometry: they correspond to arcs in great circles in \( \mathcal{S}^n \).

A collection of subspaces \( S(V_1), \ldots, S(V_m) \) (resp. \( P(V_1), \ldots, P(V_m) \)) are independent if the subspaces \( V_1, \ldots, V_m \) are independent.

The group \( SL_\pm (n+1, \mathbb{R}) \) of linear transformations of determinant \( \pm 1 \) maps to the projective group \( PGL(n+1, \mathbb{R}) \) by a double covering homomorphism \( \hat{g} \), and \( SL_\pm (n+1, \mathbb{R}) \) acts on \( \mathcal{S}^n \) lifting the projective transformations. The elements are also projective transformations.

We now discuss the standard lifting: A real projective structure on \( \mathcal{O} \) provides us with a development pair \( (dev, h) \) where \( dev : \mathcal{O}% => \mathbb{R}P^n \) is an immersion and \( h : \pi_1 (\mathcal{O}) \to PGL(n+1, \mathbb{R}) \) is a homomorphism. Since \( p_{\mathcal{S}^n} \) is a covering map and \( \mathcal{O} \) is a simply connected manifold, \( \mathcal{O} \) being a good orbifold, there exists a lift \( dev' : \mathcal{O} \to \mathcal{S}^n \) unique up to the action of \( \{1, \mathcal{A}\} \). This induces a spherical real projective structure on \( \mathcal{O} \) and \( dev' \) is a developing map for this real projective structure. Given a deck transformation \( \gamma : \mathcal{O} \to \mathcal{O} \), the composition \( dev' \circ \gamma \) is again a developing map for the real projective structure and hence equals \( h'(\gamma) \circ dev' \) for \( h'(\gamma) \in SL_\pm (n+1, \mathbb{R}) \). We verify that \( h' : \pi_1 (\mathcal{O}) \to SL_\pm (n+1, \mathbb{R}) \) is a homomorphism. Hence, \( (dev', h') \) gives us a spherical real projective structure, which induces the original real projective structure.

Given a projective structure where \( dev : \mathcal{O} \to \mathbb{R}P^n \) is an embedding to a properly convex open subset \( D \), the developing map \( dev \) lifts to an embedding \( dev' : \mathcal{O} \to \mathcal{S}^n \) to an open domain \( D \) without any pair of antipodal points. \( D \) is determined up to \( \mathcal{A} \).

We will identify \( \mathcal{O} \) with \( D \) or \( \mathcal{O} (D) \) and \( \pi_1 (\mathcal{O}) \) with \( \Gamma \). Then \( \Gamma \) lifts to a subgroup \( \Gamma' \) of \( SL_\pm (n+1, \mathbb{R}) \) acting faithfully and discretely on \( \mathcal{O} \). There is a unique way to lift so that \( D / \Gamma \) is projectively diffeomorphic to \( \mathcal{O} / \Gamma' \)
Theorem 2.2 There is a one-to-one correspondence between the space of real projective structures on an orbifold $\mathcal{O}$ with the space of spherical real projective structures on $\mathcal{O}$. Moreover, a real projective diffeomorphism of real projective orbifolds is an $\left(\mathbb{S}^n, \text{SL}_\pm(n+1, \mathbb{R})\right)$-diffeomorphism of spherical real projective orbifolds and vice versa.

Proof Straightforward. See p. 143 of Thurston [159] (see Section 2.1.7).

Again, we can define the radial end structures, horospherical, and totally geodesic ideal boundary for spherical real projective structures in obviously. Also, each end has $\mathfrak{H}$-type or $\mathcal{T}$-type assigned accordingly compatible with these definitions. They correspond directly in the following results also.

Proposition 2.6 (Selberg-Malcev) The holonomy group of a convex real projective orbifold is residually finite.

Proof In this case, $\text{dev} : \tilde{\mathcal{O}} \to \mathbb{R}^n$ always lifts an embedding to a domain in $\mathbb{S}^n$. $\Gamma$ also lifts to a group of projective automorphisms of the domain in $\text{SL}_\pm(n+1, \mathbb{R})$. The lifted group is residually finite by by Malcev [132]. Hence, $\Gamma$ is thus always residually finite.

Theorem 2.3 (Selberg) A real projective orbifold $S$ is covered finitely by a real projective manifold $M$ and $S$ is real projectively diffeomorphic to $M/G_1$ for a finite group $G_1$ of real projective automorphisms of $M$. An affine orbifold $S$ is covered finitely by an affine manifold $N$, and $S$ is affinely diffeomorphic to $N/G_2$ for a finite group $G_2$ of affine automorphisms of $N$. Finally, given a two convex real projective or affine orbifold $S_1$ and $S_2$ with isomorphic fundamental groups, one is a closed orbifold if and only if so is the other.

Proof Since $\text{Aff}(\mathbb{A}^n)$ is a subgroup of a general linear group, Selberg’s Lemma [153] shows that there exists a torsion-free subgroup of the deck transformation group. We can choose the group to be a normal subgroup and the second item follows.

A real projective structure induces an $\left(\mathbb{S}^n, \text{SL}_\pm(n+1, \mathbb{R})\right)$-structure and vice versa by Theorem 2.2. Also a real projective diffeomorphism of orbifolds is an $\left(\mathbb{S}^n, \text{SL}_\pm(n+1, \mathbb{R})\right)$-diffeomorphism and vice versa. We regard the real projective structures on $S$ and $M$ as $\left(\mathbb{S}^n, \text{SL}_\pm(n+1, \mathbb{R})\right)$-structures. We are done by Selberg’s lemma [153] that a finitely generated subgroup of a general linear group has a torsion-free normal subgroup of finite-index.

For the final item, we can take a torsion-free subgroup and the finite covers of $S_1$ and $S_2$ are manifold which are $K(\pi, 1)$ for identical $\pi$. Hence, the conclusion follows.
2.1.7 A comment on lifting real projective structures and conventions

We sharpen Theorem 2.2. Let SL\(_{\pm}(n+1, \mathbb{R})\) denote the component of SL\(_{\pm}(n+1, \mathbb{R})\) not containing I. A projective automorphism \(g\) of \(\mathbb{S}^n\) is orientation preserving if and only if \(g\) has a matrix in SL\((n+1, \mathbb{R})\). For even \(n\), the quotient map SL\((n+1, \mathbb{R})\) \(\to\) PGL\((n+1, \mathbb{R})\) is an isomorphism and so is the map SL\(_{-}(n+1, \mathbb{R})\) \(\to\) PGL\((n+1, \mathbb{R})\) for the component of SL\(_{\pm}(n+1, \mathbb{R})\) with determinants equal to \(-1\). For odd \(n\), the quotient map SL\((n+1, \mathbb{R})\) \(\to\) PGL\((n+1, \mathbb{R})\) is a 2 to 1 covering map onto its image component with deck transformations given by \(A \to \pm A\).

**Theorem 2.4** Let \(M\) be a strongly tame \(n\)-orbifold. Suppose that \(h : \pi_1(M) \to \text{PGL}(n+1, \mathbb{R})\) is a holonomy homomorphism of a real projective structure on \(M\) with radial or lens-shaped totally geodesic ends. Then the following hold:

- **Suppose that \(M\) is orientable.** We can lift to a homomorphism \(h' : \pi_1(M) \to \text{SL}(n+1, \mathbb{R})\), which is a holonomy homomorphism of the \((\mathbb{S}^n, \text{SL}_{\pm}(n+1, \mathbb{R}))\)-structure lifting the real projective structure.

- **Suppose that \(M\) is not orientable.** Then we can lift \(h\) to a homomorphism \(h' : \pi_1(M) \to \text{SL}_{\pm}(n+1, \mathbb{R})\) that is the holonomy homomorphism of the \((\mathbb{S}^n, \text{SL}_{\pm}(n+1, \mathbb{R}))\)-structure lifting the real projective structure so that the condition (\(\star\)) is satisfied.

\((\star)\) A deck transformation goes to a negative determinant matrix if and only if it reverses orientations.

In general a lift \(h'\) is unique if we require it to be the holonomy homomorphism of the lifted structure. For even \(n\), the lifting is unique if we require the condition (\(\star\)).

**Proof** For the first part, recall SL\((n+1, \mathbb{R})\) is the group of orientation-preserving linear automorphisms of \(\mathbb{R}^{n+1}\) and hence is precisely the group of orientation-preserving projective automorphisms of \(\mathbb{S}^n\). Since the deck transformations of the universal cover \(\tilde{M}\) of the lifted \((\mathbb{S}^n, \text{SL}_{\pm}(n+1, \mathbb{R}))\)-orbifold are orientation-preserving, the holonomy of the lift are in SL\((n+1, \mathbb{R})\). We use as \(h'\) the holonomy homomorphism of the lifted structure.

For the second part, we can double cover \(M\) by an orientable orbifold \(M'\) with an orientation-reversing \(\mathbb{Z}_2\)-action of the projective automorphism group generated by \(\phi : M' \to M', \phi\) lifts to \(\tilde{\phi} : \tilde{M}' \to \tilde{M}'\) for the universal covering manifold \(\tilde{M}' = \tilde{M}\) and hence \(h(\tilde{\phi}) \circ \text{dev} = \text{dev} \circ \tilde{\phi}\) for the developing map \(\text{dev}\) and the holonomy

\[h(\tilde{\phi}) \in \text{SL}_{\pm}(n+1, \mathbb{R}).\]

Then it follows from the first item since \(\text{dev}\) preserves orientations for a given orientation of \(\tilde{M}\). (See p. 143 of Thurston [159].)

The proof of the uniqueness is straightforward.

**Remark 2.1** (Convention on using spherical real projective structures) Suppose we are given a convex real projective orbifold of form \(\Omega/\Gamma\) for \(\Omega\) a convex domain in \(\mathbb{RP}^n\) and \(\Gamma\) a subgroup of PGL\((n+1, \mathbb{R})\). We can also think of \(\Omega\) as a domain in
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$\mathbb{S}^n$ and $\Gamma \subset \text{SL}_\pm(n+1, \mathbb{R})$. We can think of them in both ways and we will use a convenient one for the purpose.

2.1.7.1 Convex hulls

**Definition 2.3** Given a subset $K$ of a convex domain $\Omega$ of an affine subspace $\mathbb{A}^n$ in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$), the **convex hull** $\mathcal{CH}(K)$ of $K$ is defined as the smallest convex set containing $K$ in $\text{Cl}(\Omega) \subset \mathbb{A}^n$ where we required $\text{Cl}(\Omega)$ is a bounded subset of $\mathbb{A}^n$.

The convex hull is well-defined as long as $\Omega$ is properly convex. Otherwise, it may not be. This does not change the convex hull. (Usually it will be clear what $\Omega$ is by context but we will mention these.) For $\mathbb{RP}^n$, the convex hull depends on $\Omega$ but one can check that the convex hull is well-defined on $\mathbb{S}^n$ as long as $\Omega$ is properly convex.

Also, it is commonly well-known that each point of the convex hull of a set $K$ has a direction vector equal to a linear sum of at most $n+1$ vectors in the direction of $K$. Hence, the convex hull is a union of $n$-simplices with vertices in $K$. Also, if $K$ is compact, then the convex hull is also compact (See Berger [26].)

**Lemma 2.4** Let $\Omega$ be a convex open set. Let $\{K_i\}$ be a sequence for a compact set $K_i$ in a properly convex domain for each $i$. Suppose that $\{K_i\}$ geometrically converges to a compact set $K \subset \Omega$, Then $\{\mathcal{CH}(K_i)\} \to \mathcal{CH}(K)$.

**Proof** It is sufficient to prove for $\mathbb{S}^n$. We write each element of $\mathcal{CH}(K_i)$ as a finite sum $\left(\sum_{j=1}^{n+1} \lambda_{i,j} v_{i,j}\right)$ for $v_{i,j}$ in the direction of $K_i$ and $\lambda_{i,j} \geq 0$. Lemma 2.1 implies the result. 

2.1.8 p-ends, p-end neighborhoods, and p-end groups

By strong tameness, $\mathcal{O}$ has only finitely many ends $E_1, \ldots, E_m$, and each end has an end neighborhood diffeomorphic to $\Sigma_{E_i} \times (0, 1)$ for an orbifold $\Sigma_{E_i}$. (Here, $\Sigma_{E_i}$ may not be uniquely determined up to diffeomorphisms except for some clear situations as here.)

By an **exiting sequence** of sets $U_i$ of $\mathcal{O}$, we mean a sequence of neighborhoods $\{U_i\}$ so that $U_i \cap p^{-1}_\mathcal{O}(K) \neq \emptyset$ for only finitely many indices for each compact subset $K$ of $\mathcal{O}$.

Let us recall some facts from Section 1.3.1. Each end neighborhood $U$ diffeomorphic to $\Sigma_{E} \times (0, 1)$ of an end $E$ is covered by a connected open set $\hat{U}$ in $\mathcal{O}$, where a subgroup of deck transformations $\Gamma_\mathcal{O}$ acts on. $\hat{U}$ is a component of $p_\mathcal{O}^{-1}(U) = \bigcup_{g \in \pi_1(E)} g(\hat{U})$. Each component of form $g(\hat{U})$ is said to be a proper pseudo-end neighborhood.

- A **pseudo-end sequence** is an exiting sequence of proper pseudo-end neighborhoods $U_1 \supset U_2 \supset \cdots$. 

Two pseudo-end sequences are compatible if an element of one sequence is contained eventually in the element of the other sequence.

A compatibility class of a proper pseudo-end sequence is called a pseudo-end of $\hat{\mathcal{O}}$. Each of these corresponds to an end of $\mathcal{O}$ under the universal covering map $p_{\mathcal{O}}$.

For a pseudo-end $\hat{E}$ of $\hat{\mathcal{O}}$, we denote by $\Gamma_{\hat{E}}$ the subgroup $\Gamma_{\hat{O}}$ where $U$ and $\hat{U}$ is as above. We call $\Gamma_{\hat{E}}$ a pseudo-end fundamental group.

A pseudo-end neighborhood $U$ of a pseudo-end $\hat{E}$ is a $\Gamma_{\hat{E}}$-invariant open set containing a proper pseudo-end neighborhood of $\hat{E}$. (See Proposition 1.1 also.)

### 2.1.8.1 p-end vertices

Let $\mathcal{O}$ be a real projective orbifold with the universal cover $\hat{\mathcal{O}}$. We fix a developing map $\text{dev}$ in this subsection and identify with its image. Given a radial end of $\mathcal{O}$ and an end neighborhood $U$ of a product form $E \times [0, 1)$ with a radial foliation, we take a component $U_1$ of $p^{-1}(U)$ and the lift of the radial foliation. The developing images of leaves of the foliation end at a common point $x$ in $\mathbb{RP}^n$.

Recall that a p-end vertex of $\hat{\mathcal{O}}$ is the ideal point of leaves of $U_1$. (See Section 1.3.3.) When $\text{dev}$ is fixed, we can identify it with its image under $\text{dev}$. It will be denoted by $v_E$ if its p-end neighborhoods correspond to a p-end $\hat{E}$.

Let $\mathbb{S}^{n-1}_{v_E}$ denote the space of equivalence classes of rays from $v_E$ diffeomorphic to an $(n-1)$-sphere where $\pi_1(\hat{E})$ acts as a group of projective automorphisms. Here, $\pi_1(\hat{E})$ acts on $v_E$ and sends leaves to leaves in $U_1$.

Given a p-end $\hat{E}$ corresponding to $v_E$, we define $\Sigma_E := R_{v_E}(\hat{\mathcal{O}})$ the space of directions of developed leaves under $\text{dev}$ oriented away from $v_E$ into a p-end neighborhood of $\hat{\mathcal{O}}$ corresponding to $\hat{E}$. The space develops to $\mathbb{S}^{n-1}_{v_E}$ by $\text{dev}$ as an embedding to a convex open domain.

Recall that $\Sigma_E/\Gamma_{\hat{E}}$ is projectively diffeomorphic to the end orbifold to be denoted by $\Sigma_E$ or by $\Sigma_E$. (See Lemma 1.1.)

We may use the lifting of $\text{dev}$ to $\mathbb{P}$. The endpoint $x'$ of the lift of radial lines will be identified with the p-end vertex also when the lift of $\text{dev}$ is fixed. Here, we can canonically identify $\mathbb{S}^{n-1}_{x'}$ and $\mathbb{S}^{n-1}_x$ and the group actions of $\Gamma_{\hat{E}}$ on them.

### 2.1.8.2 p-end ideal boundary components

We recall Section 1.3.2. Let $E$ be an end of a strongly tame real projective orbifold $\mathcal{O}$. Given a totally geodesic end of $\mathcal{O}$ and an end neighborhood $U$ diffeomorphic to $S_E \times [0, 1)$ with an end-completion by a totally geodesic orbifold $S_E$, we take a component $U_1$ of $p^{-1}(U)$ and a convex domain $\hat{S}_E$, the ideal boundary component, developing to a totally geodesic hypersurface under $\text{dev}$. Here $\hat{E}$ is the p-end corresponding to $E$ and $U_1$. There exists a subgroup $\Gamma_{\hat{E}}$ acting on $\hat{S}_E$. Again $S_E := \hat{S}_E/\Gamma_{\hat{E}}$ is projectively diffeomorphic to the end orbifold to be denote by $S_E$ or $\hat{S}_E$. 
2.1 Preliminary definitions

- We call $S_E$ a p-end ideal boundary component of $\mathcal{E}$.
- We call $S_E$ an ideal boundary component of $\mathcal{E}$.

We may regard $S_E$ as a domain in a hyperspace in $\mathbb{R}P^n$ or $S^n$.

2.1.8.3 Lie group invariant p-end neighborhoods

We need the following lemma later.

A **p-end holonomy group** is the image of a p-end fundamental group under the holonomy homomorphism. If its universal cover $\tilde{\mathcal{E}}$ embeds to $S^n$ or $\mathbb{R}P^n$, then $h$ is injective and hence p-end holonomy group is isomorphic to the p-end fundamental group. A **end holonomy group** is the image of an end fundamental group.

**Lemma 2.5** Let $\mathcal{E}$ be a convex strongly tame real projective n-orbifold, and let $\tilde{\mathcal{E}}$ be its universal cover in $\mathbb{R}P^n$ (resp. in $S^n$). Let $U$ be a p-R-end neighborhood of a p-end $\tilde{E}$ in $\tilde{\mathcal{E}}$ where a p-end holonomy group $\Gamma_{\tilde{E}}$ acts on. Let $Q$ be a discrete subgroup of $\Gamma_{\tilde{E}}$. Suppose that $G$ is a connected Lie group virtually containing $Q$ so that $G/Q$ is compact. Assume that $G$ acts on the p-R-end vertex $v_{\tilde{E}}$ and $\Sigma_{\tilde{E}}$. Then $\bigcap_{g \in G} g(U)$ contains a non-empty $G$-invariant p-end neighborhood of $\tilde{E}$.

**Proof** We first assume that $\mathcal{E} \subset S^n$ and $Q \subset G$. Let $F$ be the compact fundamental domain of $G$ under $G \cap Q$. It is sufficiently to prove for the case when $U$ is a proper p-end neighborhood since for any open set $V$ containing $U$, $\bigcap_{g \in G} g(V)$ contains a p-end neighborhood $\bigcap_{g \in G} g(U)$. Hence, we assume that $\text{bd}U / \Gamma_{\tilde{E}}$ is a smooth compact surface. Let $F_U$ denote the fundamental domain of $\text{bd}U$.

Let $F$ be a compact fundamental domain of $G$ with respect to $Q$. Let $L$ be a compact subset of $\Sigma_{\tilde{E}}$ and let $\hat{L}$ denote the union of all maximal open segments with endpoints $v_{\tilde{E}}$ in the direction of $L$.

We claim that $\bigcap_{g \in F} g(U) = \bigcap_{g \in G} g(U)$ contains an open set in $\hat{L}$. We show this by proving that $\bigcap_{g \in F} g(U) \cap l$ for any maximal $l$ in $\hat{L}$ has a lower bound on its $d$-length. The lower bound is uniform for $L$.

Suppose not. Then there exists sequence $g_i \in F$ and maximal segment $l_i$ in $\hat{L}$ so that the sequence of $d$-length of $g_i(U) \cap l_i$ from $v_{\tilde{E}}$ goes to 0 as $i \to \infty$. The endpoint of $g_i(U) \cap l_i$ equals $g_i(y_i)$ for $y_i \in \text{bd}U$. This implies that $\{g_i(y_i)\} \to v_{\tilde{E}}$.

Now, $y_i$ corresponds to a direction $u_i \in \Sigma_{\tilde{E}}$. Since $F$ is a compact set, $u_i$ corresponds to a point of a compact set $F^{-1}(L)$, which corresponds to a compact set $\hat{F}_U$ of $\text{bd}U$ with directions in $F^{-1}(L)$. Hence, $y_i \in \hat{F}_U$, a compact set. Since $v_{\tilde{E}}$ is a fixed point of $G$, and $y_i \subset \hat{F}_U$ for a compact subset $\hat{F}_U$ of $S^n$ not containing $v_{\tilde{E}}$, this shows that $g_i$ form an unbounded sequence in $\text{SL}_{\pm}(n+1, \mathbb{R})$. This is a contradiction to $g_i \in F$.

We have a nonempty set

$$\hat{U} := \bigcap_{g \in G} g(U) = \bigcap_{g \in F} g(U)$$
containing an open set in $U$. $G$ acts on $\hat{U}$ clearly. We take the interior of $\hat{U}$. If $G$ only virtually contains $\Gamma_{\hat{E}}$, we just need to add finitely many elements to the above arguments.

2.2 Affine orbifolds

An affine orbifold is an orbifold with a geometric structure modeled on $(\mathbb{A}^n, \text{Aff}(\mathbb{A}^n))$. An affine orbifold has a notion of affine geodesics as given by local charts. Recall that a geodesic is complete in a direction if the affine geodesic parameter is infinite in the direction.

- An affine orbifold has a parallel end if the corresponding end has an end neighborhood foliated by properly embedded affine geodesics parallel to one another in charts and each leaf is complete in one direction. We assume that the affine geodesics are leaves assigned as above.
  - We obtain a smooth complete vector field $X_E$ in a neighborhood of $E$ for each end following the affine geodesics, which is affinely parallel in the flow; i.e., leaves have parallel tangent vectors. We call this an end vector field.
  - We denote by $X_\partial$ the vector field partially defined on $\partial$ by taking the union of vector fields defined on some mutually disjoint neighborhoods of the ends using the partition of unity.
  - The oriented direction of the parallel end is uniquely determined in the developing image of each p-end neighborhood of the universal cover of $\partial$.
  - Finally, we put a fixed complete Riemannian metric on $\partial$ so that for each end there is an open neighborhood where the metric is invariant under the flow generated by $X_\partial$. Note that such a Riemannian metric always exists.

- An affine orbifold has a totally geodesic end $E$ if each end can be completed by a totally geodesic affine hypersurface. That is, there exists a neighborhood of the end $E$ diffeomorphic to $\Sigma_E \times [0, 1)$ for an $(n-1)$-orbifold $\Sigma_E$ that compactifies to an orbifold diffeomorphic to $\Sigma_E \times [0, 1]$, and each point of $\Sigma_E \times \{1\}$ has a neighborhood affinely diffeomorphic to a neighborhood of a point $p$ in $\partial H$ for a half-space $H$ of an affine space. This implies the fact that the corresponding p-end holonomy group $h(\pi_1(\hat{E}))$ for a p-end $\hat{E}$ going to $E$ acts on a hyperspace $P$ corresponding to $E \times \{1\}$.

Recall that an orbifold is a topological space stratified by open manifolds (See Chapter 4 of [56]). An affine or projective orbifold is triangulated if there is a smoothly embedded $n$-cycle consisting of geodesic $n$-simplices on the compactified orbifold relative to ends by adding an ideal point to a radial end and an ideal boundary to each totally geodesic ends. where the interiors of $i$-simplices in the cycle are mutually disjoint and are embedded in strata of the same or higher dimension.
2.2 Affine orbifolds

2.2.1 Affine suspension constructions

The affine subspace \( \mathbb{R}^{n+1} \) is a dense open subset of \( \mathbb{RP}^{n+1} \) which is the complement of \((n+1)\)-dimensional projective space \( \mathbb{RP}^{n+1} \). Thus, an affine transformation is a restriction of a unique projective automorphism acting on \( \mathbb{R}^{n+1} \). The group of affine transformations \( \text{Aff}(\mathbb{A}^{n+1}) \) is isomorphic to the group of projective automorphisms acting on \( \mathbb{R}^{n+1} \) by the restriction homomorphism.

A dilatation \( \gamma \) in an affine subspace \( \mathbb{R}^{n+1} \) is a linear transformation with respect to an affine coordinate system so that all its eigenvalues have norm \( >1 \) or \( <1 \). Here, \( \gamma \) is an expanding map in the dynamical sense. A scalar dilatation is a dilatation with a single eigenvalue.

An affine orbifold \( \partial \) is radiant if \( h(\pi_1(\partial)) \) fixes a point in \( \mathbb{R}^{n+1} \) for the holonomy homomorphism \( h : \pi_1(\partial) \to \text{Aff}(\mathbb{A}^{n+1}) \). A real projective orbifold \( \partial \) of dimension \( n \) has a developing map \( \text{dev}' : \partial \to S^n \) and the holonomy homomorphism \( h' : \pi_1(\partial) \to \text{SL}_+(n+1, \mathbb{R}) \). We regard \( S^n \) is embedded as a unit sphere in \( \mathbb{R}^{n+1} \) temporarily. We obtain a radiant affine \((n+1)\)-orbifold by taking \( \partial \) and \( \text{dev}' \) and \( h' \): Define \( D'' : \hat{\partial} \times \mathbb{R}_+ \to \mathbb{R}^{n+1} \) by sending \((x,t)\) to \( t\text{dev}'(x)\). For each element of \( \gamma \in \pi_1(\partial) \), we define the transformation \( \gamma' \) on \( \hat{\partial} \times \mathbb{R}_+ \) from

\[
\gamma'(x,t) = (\gamma(x), \theta(\gamma)||h'(\gamma)(t\text{dev}'(x))||)
\]

for a homomorphism \( \theta : \pi_1(\partial) \to \mathbb{R}_+ \). (2.3)

Also, there is a transformation \( S_t : \hat{\partial} \times \mathbb{R}_+ \to \hat{\partial} \times \mathbb{R}_+ \) sending \((x,t)\) to \((x,st)\) for \( s \in \mathbb{R}_+ \). Thus,

\[
\hat{\partial} \times \mathbb{R}_+ / \langle S_\rho, \pi_1(\partial) \rangle, \rho \in \mathbb{R}_+, \rho > 1
\]

is an affine orbifold with the fundamental group isomorphic to \( \pi_1(\partial) \times \mathbb{Z} \) where the developing map is given by \( D'' \) the holonomy homomorphism is given by \( h' \) and sending the generator of \( \mathbb{Z} \) to \( S_\theta \). We call the result the affine suspension of \( \partial \), which of course is radiant. The representation of \( \pi_1(\partial) \times \mathbb{Z} \) with the center \( \mathbb{Z} \) mapped to a scalar dilatation is called an affine suspension of \( h \). A special affine suspension is an affine suspension with \( \theta \equiv 1 \) identically.

There is a variation called generalized affine suspension. Here we use any \( \gamma \) that is a dilatation and normalizes \( h'(\pi_1(\partial)) \) and we deduce that

\[
\hat{\partial} \times \mathbb{R}_+ / \langle \gamma, \pi_1(\partial) \rangle
\]

is an affine orbifold with the fundamental group isomorphic to \( \langle \pi_1(\partial), \mathbb{Z} \rangle \). (See Sullivan-Thurston [157], Barbot [10] and Choi [52] also.)

**Definition 2.4** We denote by \( C(\partial) \) the manifold \( \hat{\partial} \times \mathbb{R} \) with the structure given by \( D'' \), and say that \( C(\partial) \) is the affine suspension of \( \partial \).

Let \( \Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \), given by \( v \to tv, t \in \mathbb{R}_+ \), be a one-parameter family of dilations fixing a common point. A family of self-diffeomorphisms \( \Psi_t \) on an affine orbifold \( M \) lifting to \( \hat{\Psi}_t : \hat{M} \to \hat{M} \) so that \( D \circ \hat{\Psi}_t = S_t \circ D \) for \( t \in \mathbb{R} \) is called a group of radiant flow diffeomorphisms.
Lemma 2.6 Let $\mathcal{O}$ be a strongly tame real projective $n$-orbifold.

- An affine suspension $\mathcal{O}'$ of $\mathcal{O}$ always admits a group of radiant flow diffeomorphisms. Here, $\{\Phi_t\}$ is a circle and all flow lines are closed.
- Conversely, if there exists a group of radiant flow diffeomorphisms where all orbits are closed and have the homology class $[\ast \times S^1]$ on $\mathcal{O} \times S^1$ with an affine structure, then $\mathcal{O} \times S^1$ is affinely diffeomorphic to one obtained by an affine suspension construction from a real projective structure on $\mathcal{O}$.

**Proof** The first item is clear by the above construction.

The generator of the $\pi_1(S^1)$-factor goes to a scalar dilatation since it induces the identity map on the space of directions of radial segments from the global fixed point. Thus, each closed curve along $\ast \times S^1$ gives us a nontrivial homology. The homology direction of the flow equals $[\ast \times S^1] \in H_1(\mathcal{O} \times S^1; \mathbb{R})$. By Theorem D of [83], there exists a connected cross-section homologous to

$$[\mathcal{O} \times \ast] \in H_n(\mathcal{O} \times S^1, V \times S^1; \mathbb{R}) \cong H_1(\mathcal{O} \times S^1; \mathbb{R})$$

where $V$ is the union of the disjoint end neighborhoods of product forms in $\mathcal{O}$. By Theorem C of [83], any cross-section is isotopic to $\mathcal{O} \times \ast$. The radial flow is transverse to the cross-section isotopic to $\mathcal{O} \times \ast$ and hence $\mathcal{O}$ admits a real projective structure. It follows easily now that $\mathcal{O} \times S^1$ is an affine suspension. (See [10] for examples.)

An affine suspension of a horospherical orbifold is called a **suspended horoball orbifold**. An end of an affine orbifold with an end neighborhood affinely diffeomorphic to this is said to be of **suspended horoball type**. This has also a parallel end since the fixed point in the boundary of $\mathbb{R}^n$ gives a unique direction.

**Proposition 2.7** Under the affine suspension construction, a strongly tame real projective $n$-orbifold has radial, totally geodesic, or horospherical ends if and only if the affine $(n + 1)$-orbifold affinely suspended from it has parallel, totally geodesic, or suspended horospherical ends.

Again affine $(n + 1)$-orbifold suspended have type $\mathcal{R}$- or $\mathcal{T}$-ends if the corresponding real projective $n$-orbifold has $\mathcal{R}$- or $\mathcal{T}$-ends in correspondingly.

### 2.3 The needed linear algebra

Here, we will collect the linear algebra we will need in this monograph. A source is a comprehensive book by Hoffman and Kunz [107].

**Definition 2.5** Given an eigenvalue $\lambda$ of an element $g \in SL_{\pm}(n + 1, \mathbb{R})$, a **C-eigenvector** $v$ is a nonzero vector in

$$RE_\lambda(g) := \mathbb{R}^{n+1} \cap (\ker(g - \lambda I) + \ker(g - \bar{\lambda} I)), \lambda \neq 0, \Im \lambda \geq 0$$
A \( \mathbb{C} \)-fixed point is the direction of a \( \mathbb{C} \)-eigenvector in \( \mathbb{R}P^n \) (resp. \( S^n \) or \( \mathbb{C}P^n \)).

Any element of \( g \) has a primary decomposition. (See Section 6.8 of [107].) Write the minimal polynomial of \( g \) as \( \prod_{i=1}^{m} (x - \lambda_i)^{r_i} \) for \( r_i \geq 1 \) and mutually distinct complex numbers \( \lambda_1, \ldots, \lambda_m \). Define

\[
C_{\lambda_i}(g) := \ker(g - \lambda_i I)^{r_i} \subset \mathbb{C}^{n+1}
\]

where \( r_i = r_j \) if \( \lambda_i = \lambda_j \). Then the primary decomposition theorem states

\[
\mathbb{C}^{n+1} = \bigoplus_{i=1}^{m} C_{\lambda_i}(g).
\]

A real primary subspace is the sum \( \mathbb{R}^{n+1} \cap (C_{\lambda_i}(g) \oplus \overline{C}_{\lambda_i}(g)) \) for \( \lambda \) an eigenvalue of \( g \).

A point \( [v] \), \( v \in \mathbb{R}^{n+1} \), is affiliated with a norm \( \mu \) of an eigenvalue if

\[
v \in \mathcal{A}_\mu(g) := \bigoplus_{i : |\lambda_i| = \mu} C_{\lambda_i}(g) \cap \mathbb{R}^{n+1}.
\]

Let \( \mu_1, \ldots, \mu_t \) denote the set of distinct norms of eigenvalues of \( g \). We also have \( \mathbb{R}^{n+1} = \bigoplus_{i=1}^{t} \mathcal{A}_{\mu_i}(g) \). Here, \( \mathcal{A}_\mu(g) \neq \{0\} \) if \( \mu \) equals \( |\lambda_i| \) for at least one \( i \).

**Proposition 2.8** Let \( g \) be an element of \( \text{PGL}(n + 1, \mathbb{R}) \) (resp. \( \text{SL}_+ (n + 1, \mathbb{R}) \)) acting on \( \mathbb{R}P^n \) (resp. \( S^n \)). Let \( V \) and \( W \) be independent complementary subspaces where \( g \) acts on. Suppose that every norm of the eigenvalue of any eigenvector in the direction of \( V \) is strictly larger than any norms of the eigenvalues of the vectors in the direction of \( W \). Let \( V^3 \) be the subspace that is the join of the \( \mathbb{C} \)-eigenspaces of \( V \). Then

- for \( x \in \mathbb{R}P^n - \Pi(V) - \Pi(W) \) (resp. \( S^n - \Pi(V') - \Pi(W') \)), \( \{g^n(x)\} \) accumulates to only points in \( \Pi(V^3) \) (resp. \( \Pi(V^3) \)) as \( n \to \infty \).
- Let \( U \) be a neighborhood of \( x \) in \( \mathbb{R}P^n - \Pi(V) - \Pi(W) \) (resp. \( S^n - \Pi(V') - \Pi(W') \)). Each point of an open subset of \( \Pi(V^3) \) (resp. \( \Pi(V^3) \)) is realized as a limit point of \( \{g^n(y)\} \) as \( n \to \infty \) for some \( y \) in \( U \).

**Proof** It is sufficient to prove for \( \mathbb{C}^{n+1} \) and \( \mathbb{C}P^n \). We write the minimal polynomial of \( g \) as \( \prod_{i=1}^{m} (x - \lambda_i)^{r_i} \) for \( r_i \geq 1 \) and mutually distinct complex numbers \( \lambda_1, \ldots, \lambda_m \).

Let \( W^C \) be the complexification of the subspace corresponding to \( W \) and \( V^C \) the one for \( V \). Then \( C_{\lambda_i}(g) \) is a subspace of \( W^C \) or \( V^C \) by elementary linear algebra. Now, we write the matrix of \( g \) determined only up to \( \pm I \) in terms of above primary decomposition spaces. Then we write the matrix in the Jordan form in an upper triangular form. The diagonal terms of the matrix of \( g^n \) dominates nondiagonal terms. The lemma easily follows.

The last part follows by writing \( x \) and \( y \) in terms of vectors in directions of \( V \) and \( W \) and other \( g \)-invariant subspaces. \( \square \)
2.3.1 Nilpotent and orthopotent groups

Let $U$ denote a maximal nilpotent subgroup of $\text{SL}_+(n+1,\mathbb{R})$ given by upper triangular matrices with diagonal entries equal to 1. We let $U_{\mathbb{C}}$ denote the group of by upper triangular matrices with diagonal entries equal to 1 in $\text{SL}_+(n+1,\mathbb{C})$.

Let $O(n+1)$ denote the orthogonal group of $\mathbb{R}^{n+1}$ with the standard hermitian inner product.

**Lemma 2.7** The matrix of $g \in \text{Aut}(\mathbb{S}^n)$ can be written under an orthogonal coordinate system as $k(g)a(g)n(g)$ where $k(g)$ is an element of $O(n+1)$, $a(g)$ is a positive diagonal element, and $n(g)$ is real unipotent. Also, diagonal elements of $a(g)$ are the norms of eigenvalues of $g$ as elements of $\text{Aut}(\mathbb{S}^n)$.

**Proof** Let $v_1,\ldots,v_{n+1}$ denote the basis vectors of $\mathbb{C}^{n+1}$ that are chosen from the Jordan decomposition of $g$ with the same norms of eigenvalues. Let $\lambda_j$ denote the norms of eigenvalues associated with $v_j$ in the nonincreasing order for $j = 1,\ldots,l+1$. Assume $\lambda_1,\ldots,\lambda_l$ are real and $\lambda_{l+2j-1} = \lambda_{l+2j}$ for each $j = 1,\ldots,(n+1-l)/2$ and are nonreal. Assume that $v_1,\ldots,v_l$ are real vectors.

Now we fix a standard hermitian inner product on $\mathbb{C}^{n+1}$. We obtain vectors $v'_1,\ldots,v'_{n+1}$ by the Gram-Schmidt orthonormalization process starting from $v_1$ and continuing to $v_2,\ldots$.

Then $g$ is written as an upper triangular matrix with block blocked by $l$ $1 \times 1$-matrices and $(n+1-l)/2$ number of $2 \times 2$-matrices with conjugate eigenvalues. For each pair of coordinates for these $2 \times 2$-matrices, we introduce a real coordinate system so that $k'(g)$ is a real rotation on each block with respect to the standard metric. Under the new coordinate system, $a'(g)$ is unchanged and $n'(g)$ is still unipotent. Finally, we can check that the new coordinate system is real and orthogonal. $n'(g)$ is real since $k'(g)$ and $a'(g)$ are real and $g$ has a real matrix under the real coordinate system. (See also Proposition 2.1 of Kostant [121].) \hfill \Box

Recall that all maximal unipotent subgroups are conjugate to each other in $\text{SL}_+(n+1,\mathbb{R})$. (See Section 21.3 of Humphreys [108].) We define

$$U' := \bigcup_{k \in O(n+1)} kUk^{-1} = \bigcup_{k \in \text{SL}_+(n+1,\mathbb{R})} kUk^{-1}.$$  

The second equality is explained: Each maximal unipotent subgroup is characterized by a maximal flag. Each maximal unipotent subgroup is conjugate to a standard lower triangular unipotent group by an orthogonal element in $O(n+1)$ since $O(n+1)$ acts transitively on the maximal flag space.
Corollary 2.1 Suppose that we have for a positive constant \( C_1 \), and an element \( g \in SL_{\pm}(n+1, \mathbb{R}) \),
\[
\frac{1}{C_1} \leq \lambda_{n+1}(g) \leq \lambda_1(g) \leq C_1
\]
for the minimal norm \( \lambda_{n+1}(g) \) of the eigenvalue of \( g \) and the maximal norm \( \lambda_1(g) \) of the eigenvalues of \( g \). Then \( g \) is in a bounded distance from \( U' \) with the bound depending only on \( C_1 \).

Proof Let us fix an Iwasawa decomposition \( SL_{\pm}(n+1, \mathbb{R}) = O(n+1)D_{n+1}U \) for a positive diagonal group \( D_{n+1} \). By Lemma 2.7, we can find an element \( k \in O(n+1) \) so that
\[
g = kk(g)k^{-1}ka(g)k^{-1}kn(g)k^{-1}
\]
where \( k(g) \in O(n+1), a(g) \in D^+_n, n(g) \in U' \). Then \( kk(g)k^{-1} \in O(n+1) \) and \( ka(g)k^{-1} \) is uniformly bounded from \( I \) by a constant depending only on \( C_1 \) by assumption.

A subset of a Lie group is of polynomial growth if the volume of the ball \( B_R(I) \) radius \( R \) is less than or equal to a polynomial of \( R \). As usual, the metric is given by the standard positive definite left-invariant bilinear form that is invariant under the conjugations by the compact group \( O(n+1) \).

Lemma 2.8 \( U' \) is of polynomial growth in terms of the distance from \( I \).

Proof Let \( Aut(S^n) \) have a left-invariant Riemannian metric. Clearly \( U \) is of polynomial growth by Gromov [99] since \( U \) is nilpotent. Given fixed \( g \in O(n+1) \), the distance between \( gug^{-1} \) and \( u \) for \( u \in U' \) is proportional to a constant \( c_x, c_x > 1 \), multiplied by \( d(u, I) \). Choose \( u \in U' \) which is unipotent. We can write \( u(s) = \exp(s\mathbf{u}) \) where \( \mathbf{u} \) is a nilpotent matrix of unit norm. \( g(t) := \exp(tx) \) for \( x \) in the Lie algebra of \( O(n+1) \) of unit norm. For a family of \( g(t) \in O(n+1) \), we define
\[
u(t, s) = g(t)u(t)s^{-1} = \exp(sAd_{g(t)}u).
\]

We compute
\[
u(t, s)^{-1}\frac{du(t, s)}{dt} = u(t, s)^{-1}(xu(t, s) - u(t, s)x) = (Ad_{u(t, s)^{-1}} - I)(x).
\]

Since \( \mathbf{u} \) is nilpotent, \( Ad_{u(t, s)^{-1}} - I \) is a polynomial of variables \( t, s \). The norm of \( du(t, s)/dt \) is bounded above by a polynomial in \( s \) and \( t \). The conjugation orbits of \( O(n+1) \) in \( Aut(S^n) \) are compact. Also, the conjugation by \( O(n+1) \) preserves the distances of elements from \( I \) since the left-invariant metric \( \mu \) is preserved by conjugation at \( I \) and geodesics from \( I \) go to geodesics from \( I \) of same \( \mu \)-lengths under the conjugations by (2.5). Hence, we obtain a parametrization of \( U' \) by \( U \) and \( O(n+1) \) where the volume of each orbit of \( O(n+1) \) grows polynomially. Since \( U \) is of polynomial growth, \( U' \) is of polynomial growth in terms of the distance from \( I \).
Theorem 2.5 (Zassenhaus [168]) For every discrete group $G$ of $\text{GL}(n+1, \mathbb{R})$, all of which have the shape in a complex basis in $\mathbb{C}^n$

\[
\begin{pmatrix}
e^{i\theta_1} & \cdots & * \\
0 & e^{i\theta_2} & \cdots & * \\
0 & 0 & \ddots & \cdots \\
0 & 0 & \cdots & e^{i\theta_{n+1}}
\end{pmatrix},
\]

there exists a positive number $\varepsilon$, so that all the matrices $A$ from $G$ which satisfy the inequalities $|e^{i\theta_j} - 1| < \varepsilon$ for every $j = 1, \ldots, n+1$ are contained in the radical of the group, i.e., the subgroup $G_u$ of elements of $G$ with only unit eigenvalues.

An element $g$ of $\text{GL}(n+1, \mathbb{R})$ (resp. $\text{PGL}(n+1, \mathbb{R})$) is said to be unit-norm-eigenvalued if it (resp. its representative) has only eigenvalues of norm 1. A group is unit-norm-eigenvalued if all of its elements are unit-norm-eigenvalued.

A subgroup $G$ of $\text{SL}_\pm(n+1, \mathbb{R})$ is orthopotent if there is a flag of subspaces $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_m = \mathbb{R}^{n+1}$ preserved by $G$ so that $G$ acts as an orthogonal group on $Y_{j+1}/Y_j$ for each $j = 0, \ldots, m-1$ for some choices of inner-products. (See D. Fried [84].)

Theorem 2.6 Let $G$ be a unit-norm-eigenvalued subgroup of $\text{SL}_\pm(n+1, \mathbb{R})$. Then $G$ is orthopotent, and the following hold:

- If $G$ is discrete, then $G$ is virtually unipotent.
- If $G$ is a connected Lie group, then $G$ is an extension of a solvable group by a compact group; i.e., $G/S$ is a compact group for a normal solvable group $S$ in $G$.
- If $G$ is contractible, then $G$ is a simply connected solvable Lie group.

Proof By Corollary 2.1, $G$ is in $U'$.

Suppose that $G$ is discrete. Then $G$ is of polynomial growth by Lemma 2.8. By Gromov [99], $G$ is virtually nilpotent.

Choose a finite-index normal nilpotent subgroup $G'$ of $G$. Since $G'$ is solvable, Theorem 3.7.3 of [160] shows that $G'$ can be put into an upper triangular form for a complex basis. Let $G'_n$ denote the subset of $G'$ with only elements with all eigenvalues equal to 1. $G'_n$ is a normal subgroup since it is in an upper triangular form. The map $G' \rightarrow G'/G'_n$ factors into a map $G' \rightarrow (\mathbb{S}^1)^n$ by taking the complex eigenvalues. By Theorem 2.5, the image is a discrete subgroup of $(\mathbb{S}^1)^n$. Hence, $G'/G'_n$ is finite where $G'_n$ is unipotent. (Another proof is given in the Remark of page 124 of Jenkins [111].)

Suppose that $G$ is a connected Lie group. Elements of $G$ have only unit-norm eigenvalues. Since $a(g) = 1$ for $a(g)$ in the proof of Lemma 2.7 for all $g \in G$, the proof of Corollary 2.1 shows that $G \subset KU K$ for a compact Lie group $K$ since $G$ is unit-norm-eigenvalued. Since $U$ is a distal group, $G$ is a distal group, and hence $G$ is orthopotent by [68] or [143].

Since $G \subset KU K$, it is of polynomial growth, Corollary 2.1 of Jenkins [111] implies that $G$ is an extension of a solvable Lie group by a compact Lie group.
If \( G \) is contractible, \( G \) then can only be an extension by a finite group. Since \( G \) is determined by its Lie algebra, \( G \) must be solvable by the second item. \( \square \)

**2.3.2 Elements of dividing groups**

Suppose that \( \Omega , \Omega \subset \mathbb{S}^n \) (resp. \( \subset \mathbb{R}^p \)), is an open domain that is properly convex but not necessarily strictly convex. Let \( \Gamma , \Gamma \subset \text{SL}_{\pm}(n+1,\mathbb{R}) \) (resp. \( \subset \text{PGL}(n+1,\mathbb{R}) \)), be a discrete group acting on \( \Omega \) so that \( \Omega /\Gamma \) is compact.

An element of \( \Gamma \) is said to be **elliptic** if it is conjugate to an element of a compact subgroup of \( \text{PGL}(n+1,\mathbb{R}) \) or \( \text{SL}_{\pm}(n+1,\mathbb{R}) \).

**Lemma 2.9** Suppose that \( \Omega \) is a properly convex domain in \( \mathbb{R}^p \) (resp. in \( \mathbb{S}^n \)), and \( \Gamma \) is a group of projective automorphisms of \( \Omega \). Suppose that \( \Omega /\Gamma \) is an orbifold. Then an element \( g \) of \( \Gamma \) is elliptic if and only if \( g \) fixes a point of \( \Omega \) if and only if \( g \) is of finite order.

**Proof** Let us assume \( \Omega \subset \mathbb{S}^n \). Let \( g \) be an elliptic element of \( \Gamma \). Take a point \( x \in \Omega \). Let \( x \) denote a vector in a cone \( C(\Omega) \subset \mathbb{R}^{n+1} \) corresponding to \( x \). Then the orbits \( \{g^n(x) | n \in \mathbb{Z} \} \) has a compact closure. There is a fixed vector in \( C(\Omega) \), which corresponds to a fixed point of \( \Omega \).

If \( x \) is a point of \( \Omega \) fixed by \( g \), then it is in the stabilizer group. Since \( \Omega /\Gamma \) is an orbifold, \( g \) is of finite order.

If \( g \) is of finite order, \( g \) is certainly elliptic. \( [\mathbb{S}^n \Gamma] \)

We recall the definitions of Benoist [23]: For an element \( g \) of \( \text{SL}_{\pm}(n+1,\mathbb{R}) \), we denote by \( \lambda_1(g), \ldots , \lambda_{n+1}(g) \) the sequence of the norms of eigenvalues of \( g \) with repetitions by their respective multiplicities. The first one \( \lambda_1(g) \) is called the **spectral radius** of \( g \).

Assume \( \lambda_1(g) \neq \lambda_{n+1}(g) \) for the following definitions.

- An element \( g \) of \( \text{SL}_{\pm}(n+1,\mathbb{R}) \) is **proximal** if \( \lambda_1(g) \) has multiplicity one.
- \( g \) is **positive proximal** if \( g \) is proximal and \( \lambda_1(g) \) is an eigenvalue of \( g \).
- An element \( g \) of \( \text{SL}_{\pm}(n+1,\mathbb{R}) \) is **semi-proximal** if \( \lambda_1(g) \) or \( -\lambda_1(g) \) is an eigenvalue of \( g \).
- An element \( g \) of \( \text{SL}_{\pm}(n+1,\mathbb{R}) \) is **positive semi-proximal** if \( \lambda_1(g) \) is an eigenvalue of \( g \). (Definition 3.1 of [23].)

- \( g \) is called **positive bi-proximal** if \( g \) and \( g^{-1} \) is both positive proximal.
- \( g \) is called **positive bi-semi-proximal** if \( g \) and \( g^{-1} \) is both positive semi-proximal.

Of course, the proximality is a stronger condition than semi-proximality.

Let \( \Omega \) be a properly convex open domain in \( \mathbb{S}^n \). For each positive bi-semi-proximal element \( g \in \Gamma \) acting on \( \Omega \), we have two disjoint compact convex subspaces

\[
A_g := A \cap \text{Cl}(\Omega) \quad \text{and} \quad R_g := R \cap \text{Cl}(\Omega)
\]
for the eigenspace $A$ associated with the largest of eigenvalues of $g$ and the eigenspace $R$ associated with the smallest of the eigenvalues of $g$. Note $g|A_g$ and $g|R_g$ are both identity maps. Here, $A_g$ is associated with $\lambda_1(g)$ and $R_g$ is with $\lambda_n(g)$, which is an eigenvalue as well. $A_g$ is called an attracting fixed subset and $R_g$ a repelling fixed subset.

Let $g$ be a positive bi-semi-proximal element. For $g, g \in \mathrm{SL}_+(n + 1, \mathbb{R})$,

- we denote $V^A_g := \ker(g - \lambda_1(g))^{m_1} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ where $m_1$ is the multiplicity of the eigenvalue $\lambda_1(g)$ in the characteristic polynomial of $g$, and
- by $V^R_g = \ker(g - \lambda_n(g))^{m_n} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ where $m_n$ is the multiplicity of the eigenvalue $\lambda_n(g)$.

We denote $\hat{A}_g = \langle V^A_g \rangle \cap \text{Cl}(\Omega)$ and $\hat{R}_g = \langle V^R_g \rangle \cap \text{Cl}(\Omega)$. Clearly,

\[ A_g \subset \hat{A}_g \text{ and } R_g \subset \hat{R}_g. \]

**Lemma 2.10 (Lemma 3.2 of [23])** Suppose a nonidentity projective automorphism $g$ acts on a properly convex domain. Then $g$ is positive bi-semi-proximal.

We generalize Proposition 5.1 of Benoist [22]. By Theorem 2.3 following from Selberg’s Lemma [153], there is a finite index subgroup $\Gamma' \subset \Gamma$ elements of $\Gamma'$ are not elliptic. (In fact a finite manifold cover is enough.)

**Theorem 2.7 (Benoist [23])** Suppose that $\Omega$ is properly convex but not necessarily strictly convex in $\mathbb{P}^n$. Let $\Gamma$ be a discrete group acting on $\Omega$ so that $\Omega / \Gamma$ is compact and Hausdorff. Let $\Gamma'$ be the finite index subgroup of $\Gamma$ without torsion. Then each nonidentity element $g, g \in \Gamma'$ is positive bi-semi-proximal with following properties:

- $\lambda_1(g) > 1$,
- $A_g, A_g \subset \text{bd}\Omega, R_g, \hat{R}_g \subset \text{bd}\Omega$ are proper convex subsets in the boundary.
- $\dim A_g = \dim \ker(g - \lambda_1 I) - 1$ and $\dim R_g = \dim \ker(g - \lambda_n I) - 1$.
- Let $K$ be a compact set in $\Omega$. Then $\{g^i(K) | n \geq 0\}$ has the limit set in $A_g$, and $\{g^i(K) | n < 0\}$ has the limit set in $R_g$.

Furthermore, if $\Omega$ is strictly convex, then $A_g = \hat{A}_g$ is a point in $\text{bd}\Omega$ and $R_g = \hat{R}_g$ is a point in $\text{bd}\Omega$ and $g$ is positive bi-proximal.

**Proof** By Proposition 2.9, every nonidentity element $g$ of $\Gamma'$ has a norm of eigenvalue $> 1$. By Lemma 2.10, $g$ is positive bi-semi-proximal.

By Proposition 2.8, $A_g$ is a limit point of $\{g^i(x) | i > 0\}$. Hence, $A_g$ is not empty and $A_g \subset \text{bd}\Omega$. Similarly $R_g$ is not empty as well.

Then $A_g$ equals the intersection $\langle V_1 \rangle \cap \text{Cl}(\Omega)$ for the eigenspace $V_1$ of $g$ corresponding to $\lambda_1(g)$. Since $g$ fixes each point of $\langle V_1 \rangle$, it follows that $A_g$ is a compact convex subset of $\text{bd}\Omega$. Similarly, $R_g$ is a compact convex subset of $\text{bd}\Omega$.

Suppose that $A_g \cap \Omega \neq \emptyset$. Then $g$ acts on the open properly convex domain $\hat{A}_g \cap \Omega$ as a unit-norm-eigenvalued element. By Lemma 2.11 applied to $\hat{A}_g \cap \Omega$, we obtain a contradiction again by obtaining a sequence of closed curves of $d_\Omega$-lengths in $\Omega / \Gamma$ converging to 0 which is impossible for a closed orbifold. Thus, $\hat{A}_g \subset \text{bd}\Omega$. 
As above, it is a compact convex subset. Similarly, $\hat{R}_g$ is a compact convex subset of $bd\Omega$.

The second item follows from the second item of Proposition 2.8 applied to an open subset of $\Omega$.

Suppose that $\Omega$ is strictly convex. Then

$$\dim A_g = 0, \dim \hat{A}_g = 0, \dim \hat{R}_g = 0, \dim \hat{R}_g = 0$$

by the strict convexity. Proposition 5.1 of [22] proves that $g$ is proximal. $g^{-1}$ is also proximal by the same proposition. These are positive proximal since $g$ acts on a proper cone. Hence, $g$ is positive bi-proximal. □

Note here that $\hat{A}_g$ may contain $A_g$ properly and $\hat{R}_g$ may contain $R_g$ properly also.

**Proposition 2.9** Let $\Omega$ be a properly convex domain in $S^n$. Suppose that $\Gamma$ is a discrete group acting on $\Omega$ so that $\Omega/\Gamma$ is compact and Hausdorff. Let $g$ be a non-torsion non-identity element as in Theorem 2.7. Then $g$ has some norm of eigenvalues $>1$. Furthermore, $g$ does not act with a single norm of eigenvalues on any subspace $Q$ with $Q \cap \Omega \neq \emptyset$. In particular $g$ is not orthopotent, and furthermore, $g$ cannot be unipotent.

**Proof** Notice it is sufficient to prove for the case of $Q$ since we can let $Q = S^n$. Suppose that $g$ acts with a single norm of eigenvalues on a subspace $Q$ with $Q \cap \Omega \neq \emptyset$. Applying Lemma 2.11 where $n$ is replaced by the dimension of $Q$, we obtain $0$ as the infimum of the Hilbert lengths of closed curves in a compact orbifold $\Omega/\Gamma$. Since $\Omega/\Gamma$ is a compact orbifold, there should be a positive lower bound. This is a contradiction. □

The above and the following propositions are related to Section 2, 3 of [73], using somewhat different approaches. We denote by $\|\cdot\|$ a standard Euclidean norm of a vector space over $\mathbb{R}$.

**Lemma 2.11** Suppose that $\Omega$ is a properly convex domain in $S^n$. Suppose that an infinite-order element $g$ acts on $\Omega$ with only single norm of eigenvalue. Then

$$\inf_{y \in \Omega} \{d_{\Omega}(y, g(y)) | y \in \Omega \cap Q\} = 0.$$ 

**Proof** $g$ fixes a point $x$ in $Cl(\Omega)$ by the Brouwer-fixed-point theorem. If $g$ fix a point in $\Omega$, we are done. Assume $x \in bd\Omega$ is a fixed point of $g$.

We prove by induction. When $\dim \Omega = 0, 1$, this is clearly obvious. Suppose that we proved the conclusion when $\dim \Omega = n - 1, n \geq 2$. We now assume that $\dim \Omega = n$.

We may assume that $\Omega \subset \mathbb{A}^n$ for an affine space $\mathbb{A}^n$ since $\Omega$ is properly convex. We choose a coordinate system where $x$ is the origin of $\mathbb{A}^n$. Then $g$ has a form of a rational map. We denote by $Dg_x$ the linear map that is the differential of $g$ at $x$.

Let $S_r$ denote the similarity transformation of $\mathbb{A}^n$ fixing $x$. Then we obtain

$$S_r \circ g \circ S_{1/r} : S_r(\Omega) \to S_r(g(\Omega)).$$
Recall the definition of the linear map $Dg_x : \mathbb{R}^n \to \mathbb{R}^n$ is one satisfying
\[
\lim_{y, u \to 0} \frac{\|g(y) - g(u) - Dg_x(y - u)\|}{\|y - u\|} \to 0.
\]
Hence,
\[
\lim_{r \to \infty} r \left\| g(S_{1/r}(y)) - g(S_{1/r}(u)) - S_{1/r}Dg_x(y - u) \right\| \to 0.
\]
Setting $u = 0$, we obtain
\[
\lim_{r \to \infty} \left\| S_r \circ g \circ S_{1/r}(y) - Dg_x(y) \right\| \to 0.
\]
We obtain that as $r \to \infty$, $\{S_r \circ g \circ S_{1/r}\}$ converges to $Dg_x$ on a sufficiently small open ball around $x$.

Also, it is easy to show that as $r \to \infty$, $\{S_r(\Omega)\}$ geometrically converges to a cone $\Omega_{x,\infty}$ with the vertex at $x$ on which $Dg_x$ acts on.

Let $x_n$ be an affine coordinate function for a sharply-supporting hyperspace of $\Omega$ taking $0$ value at $x$. It will be specified a bit later. For now any such one will do.

Let $x(t)$ be a projective geodesic with $x(0) = x$ at $t = 0$ and $x(t) \in \Omega, x_n(x(t)) = t$ for $t > 0$ and let $u = d\dot{x}(t)/dt \neq 0$ at $t = 0$. We assumed in the premise that $g$ is unit-norm-eigenvaluated. Then
\[
\lim_{r \to 0} d_{\Omega}(g(x(t)), x(t)) = d_{\Omega_{x,\infty}}(Dg_x(u), u)
\] (2.6)
considering $u$ as an element of the cone $\Omega_{x,\infty}$: This follows from
\[
d_{\Omega}(g \circ S_{1/r} \circ S_r(x(t)), x(t)) = d_{S_r(\Omega)}(S_r \circ g \circ S_{1/r}(S_r(x(t))) , S_r(x(t)))
\]
since $S_r : (\Omega, d_{\Omega}) \to (S_r(\Omega), d_{S_r(\Omega)})$ is an isometry. We set $x(t) = x(1/r)$ and obtain $S_r(x(1/r)) \to u$ as $r \to \infty$. Since $S_r(\Omega) \to \Omega_{x,\infty}$ as $r \to \infty$, (2.2) shows that
\[
\{d_{S_r(\Omega)}(S_r \circ g \circ S_{1/r}(S_r(x(1/r))) , S_r(x(1/r)))\} \to d_{\Omega_{x,\infty}}(Dg_x(u), u) \text{ as } r \to \infty.
\] (2.7)

Now, $\Omega_{x,\infty}^\circ$ is a convex cone of form $C(U)$ for a convex open domain $U$ in the infinity of $\mathbb{A}^n$. The space $U^\circ$ of sharply-supporting hyperspaces of $\Omega_{x,\infty}^\circ$ at $x$ is a convex compact set.

Since $g$ acts on a ball $U^\circ$, $g$ fixes a point by the Brouwer-fixed-point theorem, which corresponds to a hyperspace. Let $P$ be a hyperspace in $\mathbb{A}^n$ passing $x$ sharply-supporting $\Omega_{x,\infty}^\circ$, invariant under $Dg_x$.

We now choose the affine coordinate $x_n$ for $P$ so that $P$ is the zero set. First, suppose that $\Omega_{x,\infty}$ is a properly convex cone. Projecting $\Omega_{x,\infty}^\circ$ to the space $\mathbb{S}_{x}^{n-1}$ of rays starting from $x$, we obtain a properly convex open domain $\Omega_1 = R_{x}(\Omega)$. Here, $\dim \Omega_1 \leq n - 1$.

By the induction hypothesis on dimension $\dim \Omega_1$, since the $Dg_x$-action has only one-norm of the eigenvalues, we can find a sequence $\{z_i\}$ in $\Omega_1$ so that $\{d_{\Omega_1}(Dg_x(z_i), z_i)\} \to 0$. Since $\Omega_{x,\infty}$ is a proper convex cone in $\mathbb{A}^n$, we choose a
sequence \( u_i \in \Omega_{x,\infty} \) with \( x_n(u_i) = 1 \) and \( u_i \) has the direction of \( z_i \) from \( x \). Let \( u_i \) denote the vector in the direction of \( \frac{x_i}{\|x_i\|} \) on \( \mathbb{R}^n \) where \( x_n(u_i) = 1 \). Since \( g \) is unit-norm-eigenvalued, \( x_n(Dg(x_i)) = 1 \) also. Hence, the geodesic to measure the Hilbert metric from \( u_i \) to \( Dg(x_i) \) is on \( x_n = 1 \). Let \( P_1 \) denote the affine subspace given by \( x_n = 1 \). The projection \( \Omega_{x,\infty} \cap P_1 \to \mathcal{U} \) from \( x \) is a projective diffeomorphism and hence is an isometry. Therefore, \( d_{\Omega_{x,\infty}}(Dg(x_i), u_i) \to 0 \).

We can find arcs \( x_i(t) \) with

\[
x_i(x_i(t)) = t \quad \text{and} \quad dx_i(t)/dt = u_i \quad \text{at} \quad t = 0.
\]

Also, we find a sequence of points \( \{x_i(t_i)\}, t_i \to 0 \), so that

\[
d_\Omega(g(x_i(t_i)), x_i(t_i)) = d_{\Omega_{x,\infty}}(S_{1/t_i} \circ g \circ S_{1/t_i}(x_i(t_i))), S_{1/t_i}(x_i(t_i))).
\]

Since \( \{S_{1/t_i}(\operatorname{Cl}(\Omega))\} \to \operatorname{Cl}(\Omega_{x,\infty}) \), and \( \{S_{1/t_i}(x_i(t_j))\} \to u_i \) as \( j \to \infty \), we obtain

\[
\{d_\Omega(g(x_i(t_j)), x_i(t_j))\} \to d_{\Omega_{x,\infty}}(Dg(x_i), u_i)
\]

by \( (2.7) \).

By choosing \( j_i \) sufficiently large for each \( i \), we obtain

\[
\{d_\Omega(g(x_i(t_{j_i})), x_i(t_{j_i}))\} \to 0.
\]

Suppose that \( \Omega_{x,\infty} \) is not a properly convex cone. Now, \( \Omega_1 = R_+(\Omega) \) is a convex but not properly convex domain. By Proposition 2.5, such a set is foliated by complete affine spaces of dimension \( j, j > 0 \). Hence, \( \Omega_1 \) is foliated by complete affine half-spaces of dimension \( j + 1 \). The quotient space \( O_x := \Omega_1/\sim \) with equivalence relationship given by the foliation is a properly convex open domain of dimension \( < n \). Recall that there is a pseudo-metric \( d_{O_x} \) on \( \Omega_1 \). Note that the projection \( \pi : \Omega_1 \to O_x \) is projective and \( d_{O_x}(y, z) = d_{O_x}(\pi(y), \pi(z)) \) for all \( x, y \in \Omega_1 \), which is fairly easy to show. The differential \( Dg_x \) induces a projective map \( D'g_x : O_x \to O_x \).

Since \( \dim O_x < \dim \Omega_1 \), we have by the induction a sequence \( y_i \in O_x \) so that \( d_{O_x}(y_i, D'g_x(y_i)) \to 0 \) as \( i \to \infty \). We take an inverse image \( z_i \) in \( \Omega_1 \) of \( y_i \). Then \( d_{\Omega_1}(z_i, Dg_x(z_i)) = d_{O_x}(y_i, D'g_x(y_i)) \to 0 \) as \( i \to \infty \). Similarly to above, we obtain the desired result. \( \square \)

### 2.3.3 The higher-convergence-group

For this section, we only work with \( S^n \) since only this version is needed. We considering \( SL_\pm(n+1, \mathbb{R}) \) as an open subspace of \( M_{n+1}(\mathbb{R}) \). We can compactify \( SL_\pm(n+1, \mathbb{R}) \) as \( S(M_{n+1}(\mathbb{R})) \). Denote by \( \langle g \rangle \) the equivalence class of \( g \in SL_\pm(n+1, \mathbb{R}) \).

**Theorem 2.8 (The higher-convergence-group property)** Let \( g_i \) be any unbounded sequence of projective automorphisms of a properly convex domain \( \Omega \) in \( S^n \). We
consider \( g_i \in \text{SL}_\pm(n+1, \mathbb{R}) \) according to convention 1.1. Then we can choose a subsequence of \( \{\langle g_i \rangle\} \), \( g_i \in \text{SL}_\pm(n+1, \mathbb{R}) \), converging to \( \langle g_\infty \rangle \) in \( S(M_{n+1}(\mathbb{R})) \) for \( g_\infty \in M_{n+1}(\mathbb{R}) \) where the following hold:

- \( g_\infty \) is undefined on \( S(\ker g_\infty) \) and the range is \( S(\text{Im} g_\infty) \).
- \( \dim S(\ker g_\infty) + \dim S(\text{Im} g_\infty) = n-1 \).
- For every compact subset \( K \) of \( \mathbb{R}^n - S(\ker g_\infty) \), \( \{g_i(K)\} \to K_\infty \) for a subset \( K_\infty \) of \( S(\text{Im} g_\infty) \).
- Given a convergent subsequence \( \{g_i\} \) as above, \( \{\langle gg_\infty \rangle \} \) is also convergent to \( \langle gg_\infty \rangle \) and \( S(\ker gg_\infty) = S(\ker g_\infty) \) and \( S(\text{Im} gg_\infty) = gS(\text{Im} g_\infty) \).
- \( \{\langle g, g \rangle\} \) is also convergent to \( \langle g_\infty, g \rangle \) and

\[
S(\ker g_\infty g) = g^{-1}(S(\ker g_\infty)) \text{ and } S(\text{Im} g_\infty g) = S(\text{Im} g_\infty).
\]

**Proof** Since \( S(M_{n+1}(\mathbb{R})) \) is compact, we can find a subsequence of \( g_i \) converging to an element \( \langle g_\infty \rangle \). The second item is the consequence of the rank and nullity of \( g_\infty \). The third item follows by considering the compact open topology of maps and \( g_i \), divided by its maximal norm of the matrix entries.

The two final item are straightforward. 

**Lemma 2.12** \( \langle g_\infty \rangle \) can be obtained by taking the limit of \( g_i/m(g_i) \) in \( M_{n+1}(\mathbb{R}) \) first and then taking the direction where \( m(g_i) \) is the maximal norm of elements of \( g_i \) in the matrix form of \( g_i \).

**Proof** This follows since \( g_i/m(g_i) \) does not go to zero. 

**Definition 2.6** An unbounded sequence \( \{g_i\} \), \( g_i \in \text{SL}_\pm(n+1, \mathbb{R}) \), so that \( \{\langle g_i \rangle\} \) is convergent in \( S(M_{n+1}(\mathbb{R})) \) is called a convergence sequence. In the above \( g_\infty \in M_{n+1}(\mathbb{R}) \) is called a convergence limit, determined only up to a positive constant. The element \( \langle g_\infty \rangle \in S(M_{n+1}(\mathbb{R})) \) where \( \{\langle g_i \rangle\} \to \langle g_\infty \rangle \) is called a convergence limit.

We may also do this for \( \text{PGL}(n+1, \mathbb{R}) \). An unbounded sequence \( \{g_i\} \), \( g_i \in \text{PGL}(n+1, \mathbb{R}) \) so that \( \{[g_i]\} \) is convergent in \( \mathbb{P}(M_{n+1}(\mathbb{R})) \) is called a convergence sequence. Also, the element \( [g_\infty] \in \mathbb{P}(M_{n+1}(\mathbb{R})) \) is \( \{[g_i]\} \to [g_\infty] \) is called a convergence limit.

We have more requirements: We use the KAK-decomposition (or polar decomposition) of Cartan for \( \text{SL}_\pm(n+1, \mathbb{R}) \). We may write \( g_i = k_i d_i k_i^{-1} \) where \( k_i, k_i \in O(n+1, \mathbb{R}) \) and \( d_i \) is a positive diagonal matrix with a nonincreasing set of elements

\[
a_{1,j} \geq a_{2,j} \geq \cdots \geq a_{n+1,j}.
\]

Let \( S([1, m]) \) the subspace spanned by \( e_1, \ldots, e_m \) and \( S([m+1, n+1]) \) the subspace spanned by \( e_{m+1}, \ldots, e_{n+1} \). We assume that \( \{k_i\} \) converges to \( k_\infty \), \( \{k_i\} \) converges to \( \hat{k}_\infty \), and \( \{d_i, \ldots, d_{n+1}\} \) is convergent in \( \mathbb{R}^{m,n} \). We will further require this for convergence sequences.

For sequence in \( \text{PGL}(n+1, \mathbb{R}) \), we may also write \( g_i = k_i d_i k_i^{-1} \) where \( d_i \) is represented by positive diagonal matrices as above. Then we require as above.
This of course generalized the convergence sequence ideas, without the second set of requirements above, for $\text{PSL}(2, \mathbb{R})$ as given by Tukia (see [2]).

Given a convergence sequence $\{g_i\}, g_i \in \text{Aut}(\mathbb{S}^n)$, we define

\[ \hat{A}_*(\{g_i\}) := \mathbb{S}(\text{Im} g_\infty) \]  
\[ \hat{N}_*(\{g_i\}) := \mathbb{S}(\ker g_\infty) \]  
\[ A_*(\{g_i\}) := \mathbb{S}(\text{Im} g_\infty) \cap \text{Cl}(\Omega) \]  
\[ N_*(\{g_i\}) := \mathbb{S}(\ker g_\infty) \cap \text{Cl}(\Omega) \]

For a matrix $A$, we denote by $|A|$ the maximum of the norms of entries of $A$. Let $U$ be an orthogonal matrix in $\text{O}(n+1, \mathbb{R})$. Then we obtain

\[ \frac{1}{n+1} |A| \leq |AU| \leq n+1 |A| \] (2.12)

since the entries of $AU$ are dot products of rows of $A$ with elements of $U$ whose entries are bounded above by 1 and below by $-1$. Hence, we obtain for $g = kD\hat{k}^{-1}$ for $k, \hat{k} \in \text{O}(n+1, \mathbb{R})$ and $D$ diagonal as above.

\[ \frac{1}{(n+1)^2} |D| \leq |g| \leq (n+1)^2 |D| \] (2.13)

Recall Definition 2.5, we obtain

**Theorem 2.9** Let $\{g_i\}, g_i \in \text{Aut}(\mathbb{S}^n)$, be a convergence sequence. We consider $g_i \in \text{SL}_\pm(n+1, \mathbb{R})$ according to convention 1.1. Then we may assume that the following holds up to a choice of subsequence of $g_i$:

- there exists $m_a, 1 \leq m_a < n+1$, where $\{a_{j,i}/a_{1,i}\} \to 0$ for $j > m_a$ and $a_{j,i}/a_{1,i} > \varepsilon$ for $j \leq m_a$ for a uniform $\varepsilon > 0$.
- there exists $m_r, 1 \leq m_r < n+1$, where $a_{j,i}/a_{n+1,i} < C$ for $j \geq m_r$ for a uniform $C > 1$, and $\{a_{j,i}/a_{n+1,i}\} \to \infty$ for $j < m_r$.
- $\hat{N}_*(\{g_i\})$ is the geometric limit of $k_i([m_a+1, n+1])$.
- $A_*(\{g_i\})$ is the geometric limit of $A^0(g_i) = k_i([1, m_a])$.
- $\{g_i^{-1}\}$ is also a convergent sequence up to a choice of subsequences, and $A_*(\{g_i^{-1}\}) \subset \hat{N}_*(\{g_i\})$.

**Proof** We choose a subsequence so that $m_a$ and $m_r$ are defined respectively and $\{k_i\}, \{\hat{k}_i\}$ form convergent sequences. We denote $D_\infty$ as the limit of $\{D_i/D_i\}$ and $k_\infty$ and $\hat{k}_\infty$ as the limit of $\{k_i\}$ and $\{\hat{k}_i\}$. Then we obtain by (2.13),

\[ \frac{1}{(n+1)^3} |k_\infty \circ D_\infty \circ \hat{k}_\infty^{-1}(v)| \leq |g_\infty(v)| \leq (n+1)^3 |k_i \circ D_\infty \circ \hat{k}_i^{-1}(v)| \]

for every $v \in \mathbb{R}^{n+1}$. Thus, $k_\infty \circ D_\infty \circ \hat{k}_\infty^{-1}(v) = 0$ if and only if $g_\infty(v) = 0$ and, moreover, images and null spaces of $k_\infty \circ D_\infty \circ \hat{k}_\infty^{-1}$ and $g_\infty$ coincide. Hence, we obtain that
Hence, the first four items follow.

The last item follows by considering the third and fourth items and the fact that \( m_r \geq m_a \).

We define \( \hat{g} \) for each \( i \),

\[
F^p(g_i) := k_i \mathcal{S}([1, m_r - 1]), \quad \text{and} \quad R^p(g_i) := \hat{k}_i \mathcal{S}([m_r, n + 1]).
\]

We define \( \hat{R}_n \) as the geometric limit of \( \{ R^p(g_i) \} \), and \( \hat{F}_n \) as the geometric limit of \( \{ F^p(g_i) \} \). We also define

\[
\mathcal{R}_n := \hat{R}_n \cap \mathcal{C}(\Omega), \quad \mathcal{F}_n := \hat{F}_n \cap \mathcal{C}(\Omega).
\]

**Lemma 2.13** Suppose that \( \{ g_i \} \) and \( \{ g_i^{-1} \} \) are set-convergent sequences. Then \( \hat{R}_n \) is the geometric limit of \( \{ R_n \} \) and \( \hat{F}_n \) is the geometric limit of \( \{ F_n \} \).

**Proof** For \( g_i = k_i d_i \hat{K}_i^{-1} \), we have \( g_i^{-1} = \hat{K}_i^{-1} d_i^{-1} k_i \). Hence \( A^p(g) = \hat{k}_i \mathcal{S}([m_r, n + 1]) = R^p(g_i) \). We also have \( F^p(g_i) = \hat{k}_i \mathcal{S}([m_a, n + 1]) \). The result follows.

**Lemma 2.14** We also have \( \hat{A}_n \) and \( \hat{R}_n \) for positive bi-symmetric element \( g \) with \( \lambda_1(g) > 1 \).

**Proof** We consider \( g \) in \( \mathbb{C}^{n+1} \). Write \( g \) in the coordinate system where the complexification is of the Jordan form. Let \( V_\mathfrak{g} \) denote \( \mathfrak{R}_\lambda \) in \( \mathbb{R}^{n+1} \) which is a \( g \)-invariant subspace from (2.4). There is a complementary \( g \) which is a direct sum of \( \mathfrak{R}_\mu \) for \( \mu < \lambda_1 \). Then we use Proposition 2.8 applied to \( \Pi(V_\mathfrak{g}) \) and \( \Pi(N_\mathfrak{g}) \).

For the second part, we use \( g^{-1} \) and argue using obvious facts \( \hat{R}_n = \hat{A}_n^{-1} \) and \( \hat{A}_n \).

**Proposition 2.10** \( \hat{A}_n \) contains an open subset of \( \hat{A}_n \), and hence \( \hat{A}_n \) is a \( g \)-invariant subspace. Also, \( \hat{R}_n \) contains an open subset of \( \hat{R}_n \), and hence \( \hat{R}_n \) is also \( g \)-invariant.

**Proof** We write \( g_i = k_i D_i \hat{K}_i^{-1} \). By Theorem 2.9, \( \hat{A}_n \) is the geometric limit of \( k_i \mathcal{S}[1, m_a] \) for some \( m_a \) as above. \( g_i(U) = k_i D_i(V_i) \) for an open set \( U \subset \mathcal{V} \) and \( V_i = \hat{K}_i^{-1}(U) \). Since \( \hat{K}_i^{-1} \) is a \( \mathfrak{d} \)-isometry, \( V_i \) is an open set containing a closed ball \( B_i \) of fixed radius \( \varepsilon \). \( D_i \) converges to a diagonal matrix \( D_\infty \). We may assume without loss of generality that \( \{ B_i \} \to \mathcal{B}_\infty \) where \( \mathcal{B}_\infty \) is a ball of radius \( \varepsilon \). We may assume \( B_i \cap \mathcal{B}_\infty \supset B \) for a fixed ball of radius \( \varepsilon/2 \) for sufficiently large \( i \). Then \( \{ D_i(B) \} \to D_\infty \subset \mathcal{S}[1, m_a] \). Here, \( D_\infty \) is a subset of \( \mathcal{S}[1, m_a] \) containing an
open set. Since \( \{ D_t(B_i) \} \) geometrically converges to a subset containing \( D_\infty(B) \), up to a choice of subsequence. Thus, \( \{ k_i D_t(V_i) \} \) geometrically converges to a subset containing \( k_i D_\infty(B) \) by Lemma 2.1.

For the second part, we use the sequence \( g_i^{-1} = \hat{k}_i D_i^{-1}k_i^{-1} \) and Lemma 2.13.

**Lemma 2.15** Suppose that \( \Gamma \) acts properly discontinuously on a properly convex open domain \( \Omega \) and \( \{ g_i \} \) is a set-convergent sequence in \( \Gamma \). Suppose that \( \{ g_i \} \) is not bounded in \( \text{SL}_+(n+1, \mathbb{R}) \) and is a set-convergent sequence. Then the following hold:

1. \( \hat{\mathcal{R}}_s(\{ g_i \}) \cap \Omega = \emptyset \).
2. \( \hat{\mathcal{A}}_s(\{ g_i \}) \cap \Omega = \emptyset \).
3. \( \hat{\mathcal{F}}_s(\{ g_i \}) \cap \Omega = \emptyset \), and
4. \( \hat{\mathcal{N}}_s(\{ g_i \}) \cap \Omega = \emptyset \).

**Proof** (ii) Suppose not. Since \( \hat{\mathcal{A}}_s(\{ g_i \}) \cap \Omega \neq \emptyset \), \( \mathcal{A}_s(\{ g_i \}) \) meets \( \Omega \). Since \( \mathcal{A}_s(\{ g_i \}) \) is the set of points of limits \( g_i(x) \) for \( x \in \Omega \), the proper discontinuity of the action of \( \Gamma \) shows that \( \mathcal{A}_s(\{ g_i \}) \) does not meet \( \Omega \).

(iv) For each \( x \in \Omega \), a fixed ball \( B \) in \( \Omega \) centered at \( x \) does not meet \( \hat{k}_i(\mathbb{S}([m_i + 1, n + 1])) \) for infinitely many \( i \). Otherwise \( \{ g_i^{m_i}(B) \} \) converges to a nonproperly convex set in \( \text{Cl}(\Omega) \) as \( m \to \infty \), a contradiction. Hence, the second item follows.

The remainder follows by changing \( g_i \) to \( g_i^{-1} \) and Lemma 2.13.

**Theorem 2.10** Let \( \{ g_i \} \) be a set-convergence sequence in \( \Gamma \) acting properly discontinuously on a properly convex domain \( \Omega \). Then

\[
\mathcal{A}_s(\{ g_i \}) = \hat{\mathcal{A}}_s(\{ g_i \}) \cap \text{Cl}(\Omega) = \hat{\mathcal{A}}_s(\{ g_i \}) \cap \text{bd}\Omega, \tag{2.14}
\]

\[
\mathcal{N}_s(\{ g_i \}) = \hat{\mathcal{N}}_s(\{ g_i \}) \cap \text{Cl}(\Omega) = \hat{\mathcal{N}}_s(\{ g_i \}) \cap \text{bd}\Omega, \tag{2.15}
\]

\[
\mathcal{R}_s(\{ g_i \}) = \hat{\mathcal{R}}_s(\{ g_i \}) \cap \text{Cl}(\Omega) = \hat{\mathcal{R}}_s(\{ g_i \}) \cap \text{bd}\Omega, \tag{2.16}
\]

\[
\mathcal{F}_s(\{ g_i \}) = \hat{\mathcal{F}}_s(\{ g_i \}) \cap \text{Cl}(\Omega) = \hat{\mathcal{F}}_s(\{ g_i \}) \cap \text{bd}\Omega \tag{2.17}
\]

are subsets of \( \text{bd}\Omega \) and they are nonempty sets. Also, we have

\[
\hat{\mathcal{A}}_s(\{ g_i \}) \subset \mathcal{F}_s(\{ g_i \}), \quad \mathcal{A}_s(\{ g_i \}) \subset \hat{\mathcal{F}}_s(\{ g_i \}), \tag{2.18}
\]

\[
\hat{\mathcal{R}}_s(\{ g_i \}) \subset \mathcal{N}_s(\{ g_i \}), \quad \mathcal{R}_s(\{ g_i \}) \subset \hat{\mathcal{N}}_s(\{ g_i \}). \tag{2.19}
\]

**Proof** By Lemma 2.15, we only need to show the respective sets are not empty. By the third item of Theorem 2.8, a point \( x \) in \( \mathcal{A}_s(\{ g_i \}) \cap \text{Cl}(\Omega) \) is a limit of \( g_i(y) \) for some \( y \in \Omega \). Since \( \Gamma \) acts properly discontinuously, \( x \not\in \Omega \) and \( x \in \text{bd}\Omega \). By taking \( \{ g_i^{-1} \} \), we obtain \( \hat{\mathcal{R}}_s(\{ g_i \}) \cap \text{Cl}(\Omega) \neq \emptyset \). Since \( \hat{\mathcal{R}}_s(\{ g_i \}) \subset \hat{\mathcal{N}}_s(\{ g_i \}) \) and \( \mathcal{A}_s(\{ g_i \}) \subset \mathcal{F}_s(\{ g_i \}) \), the rest follows. The last collections are from definitions.

**Proposition 2.11** For an automorphism \( g \) of \( \Omega \), and a set-convergence sequence \( \{ g_i \} \), the following hold:
\[ \hat{A}_*(\{gg_i\}) = g(\hat{A}_*(\{g_i\})), \hat{A}_*(\{g_i, g_j\}) = \hat{A}_*(\{g_i\}), \]  
\[ \hat{N}_*(\{gg_i\}) = \hat{N}_*(\{g_i\}), \hat{N}_*(\{g_i, g_j\}) = g^{-1}(\hat{N}_*(\{g_i\})), \]  
\[ \hat{F}_*(\{gg_i\}) = g(\hat{F}_*(\{g_i\})), \hat{F}_*(\{g_i, g_j\}) = \hat{F}_*(\{g_i\}), \]  
\[ \hat{R}_*(\{gg_i\}) = \hat{R}_*(\{g_i\}), \hat{R}_*(\{g_i, g_j\}) = g^{-1}(\hat{R}_*(\{g_i\})). \]  

Proof: The fourth and fifth items of Theorem 2.8 imply the first and second lines here. The third line follows from the second line by Lemma 2.13. Also, the fourth line follows from the first line by Lemma 2.13. □

Of course, there are \( \mathbb{RP}^n \)-versions of the results here. However, we do not state these.

2.4 Convexity and convex domains

2.4.1 Convexity

In the following, a zero-dimensional sphere \( S_0^\infty \) denotes a pair of antipodal points.

Proposition 2.12

- A real projective \( n \)-orbifold is convex if and only if the developing map sends the universal cover to a convex domain in \( \mathbb{RP}^n \) (resp. \( S^n \)).
- A real projective \( n \)-orbifold is properly convex if and only if the developing map sends the universal cover to a precompact properly convex open domain in an affine patch of \( \mathbb{RP}^n \) (resp. \( S^n \)).
- If a convex real projective \( n \)-orbifold is not properly convex and not complete affine, then its holonomy is reducible in \( \text{PGL}(n+1, \mathbb{R}) \) (resp. \( \text{SL}_\pm(n+1, \mathbb{R}) \)). In this case, \( \hat{O} \) is foliated by affine subspaces \( l \) of dimension \( i \) with the common boundary \( \text{Cl}(l) - l \) equal to a fixed subspace \( \mathbb{RP}_i^{n-1} \) (resp. \( S_i^{n-1} \)) in \( \text{bd} \hat{O} \). Furthermore, this holds for any convex domain \( \mathbb{RP}^n \) (resp. \( S^n \)) and the projective group action on it.

Proof: We prove for \( S^n \) first. Since a convex domain is projectively diffeomorphic to a convex domain in an affine space as defined in Section 1.1.1, the developing map must be an embedding. The converse is also trivial. (See Proposition A.2 of [51].)

The second follows immediately.

For the final item, a convex subset of \( S^n \) is a convex subset of an affine subspace \( \mathbb{A}^n \), isomorphic to an affine space, which is the interior of a hemisphere \( H \). We may assume that \( D^n \neq \emptyset \) by restricting to a spanning subspace of \( D \) in \( S^n \). Let \( D \) be a convex subset of \( H^n \). If \( D \) is not properly convex, the closure \( \text{Cl}(D') \) must have a pair of antipodal points in \( H \). They must be in \( \text{bd} H \). A great open segment \( l \) must connect this antipodal pair in \( \text{bd} H \) and pass an interior point of \( D' \). If a subsegment of \( l \) is in \( \text{bd}D' \), then \( l \) is in a sharply supporting hyperspace and \( l \) does not pass an interior point of \( D' \). Thus, \( l \subset D' \). Hence, \( D \) contains a complete affine line.
Thus, $D$ contain a maximal complete affine subspace. Two such complete maximal affine subspaces do not intersect since otherwise a larger complete affine subspace of higher dimension is in $D$ by convexity. We showed in [40] that the maximal complete affine subspaces foliated the domain. (See also [76].) The foliation is preserved under the group action since the leaves are lower-dimensional complete affine subspaces in $D$. This implies that the common boundary of the closures of the affine subspaces is a lower-dimensional subspace. These subspaces are preserved under the group action. Hence, the holonomy group is reducible.

For the $\mathbb{R}P^n$-version, we use the double covering map $p_{2^n}$ mapping an open hemisphere to an affine subspace. \[ [S^n T]\]

**Proposition 2.13** Let $\Omega$ be a properly convex domain in $\mathbb{S}^n$. The image $\Omega'$ be the image of $\Omega$ under the double covering map $p_{2^n}$. Then the restriction $\text{Cl}(\Omega) \rightarrow \text{Cl}(\Omega')$ is one-to-one and onto.

**Proof** This follows since we can find an affine subspace $A^n$ containing $\text{Cl}(\Omega)$. Since the covering map restricts to a homeomorphism on $A^n$, this follows. \[ \square \]

**Definition 2.7** Given a convex set $D$ in $\mathbb{R}P^n$, we obtain a connected cone $C(D)$ in $\mathbb{R}^{n+1} - \{O\}$ mapping to $D$, determined up to the antipodal map. For a convex domain $D \subseteq \mathbb{S}^n$, we have a unique domain $C(D) \subset \mathbb{R}^{n+1} - \{O\}$.

A join of two properly convex subsets $A$ and $B$ in a convex domain $D$ of $\mathbb{R}P^n$ (resp. $\mathbb{S}^n$) is defined as

$$ A * B := \{ \{t \mathbf{x} + (1 - t) \mathbf{y} \} \mid \mathbf{x}, \mathbf{y} \in C(D), [\mathbf{x}] \in A, [\mathbf{y}] \in B, t \in [0, 1] \} $$

(resp. $A * B := \{ \langle t \mathbf{x} + (1 - t) \mathbf{y} \rangle \mid \mathbf{x}, \mathbf{y} \in C(D), \langle \mathbf{x} \rangle \in A, \langle \mathbf{y} \rangle \in B, t \in [0, 1] \} $)

where $C(D)$ is a cone corresponding to $D$ in $\mathbb{R}^{n+1}$. The definition is independent of the choice of $C(D)$ in $\mathbb{S}^n$. In $\mathbb{R}P^n$, the join may depend on the choice $C(D)$. Note we use $p * B = \{p\} * B$ interchangeably for a point $p$.

**Definition 2.8** Let $C_1, \ldots, C_m$ respectively be cones in a set of independent vector subspaces $V_1, \ldots, V_m$ of $\mathbb{R}^{n+1}$. In general, a sum of convex sets $C_1, \ldots, C_m$ in $\mathbb{R}^{n+1}$ in independent subspaces $V_i$ is defined by

$$ C_1 + \cdots + C_m := \{ v \mid v = c_1 + \cdots + c_m, c_i \in C_i \}. $$

A strict join of convex sets $\Omega_i$ in $\mathbb{S}^n$ (resp. in $\mathbb{R}P^n$) is given as

$$ \Omega_1 * \cdots * \Omega_m := \Pi'(C_1 + \cdots + C_m) (\text{resp. } \Pi(C_1 + \cdots + C_m)) $$

where each $C_i - \{O\}$ is a convex cone with image $\Omega_i$ for each $i$ for the projection $\Pi'$ (resp. $\Pi$).
2.4.2 The flexibility of boundary

The following lemma gives us some flexibility of boundary. A smooth hypersurface embedded in a real projective manifold is called strictly convex if under a chart to an affine subspace, it maps to a hypersurface which is defined by a real function with positive Hessians at points of the hypersurface.

Lemma 2.16 Let $M$ be a strongly tame properly convex real projective orbifold with strictly convex $\partial M$. We can modify $\partial M$ inward $M$ and the result bound a strongly tame or compact properly convex real projective orbifold $M'$ with strictly convex $\partial M'$.

Proof Let $\Omega$ be a properly convex domain covering $M$. We may assume that $\Omega \subset \mathbb{S}^n$. We may modify $M$ by pushing $\partial M$ inward. We take an arbitrary inward vector field defined on a tubular neighborhood of $\partial M$. (See Section 4.4 of [56] for the definition of the tubular neighborhoods.) We use the flow defined by them to modify $\partial M$. By the $C^2$-convexity condition, for sufficiently small change the image of $\partial M$ is still strictly convex and smooth. Let the resulting compact $n$-orbifold be denoted by $M'$. $M'$ is covered by a subdomain $\Omega'$ in $\Omega$.

Since $M'$ is a compact suborbifold of $M$, $\Omega'$ is a properly embedded domain in $\Omega$ and thus, $\partial \Omega' \cap \Omega = \partial \Omega'$. $\partial \Omega'$ is a strictly convex hypersurface since so is $\partial M'$. This means that $\Omega'$ is locally convex. A locally convex closed subset of a convex domain is convex by Lemma 2.17.

Hence, $\Omega'$ is convex and hence is properly convex being a subset of a properly convex domain. So is $M'$.

Remark 2.2 Thus, by choosing one in the interior, we may assume without loss of generality that a strictly convex boundary component can be pushed out to a strictly convex boundary component.

2.4.3 Needed convexity facts

We will use the following many times in the monograph.

Lemma 2.17 (Chapter 11 of [158]) Let $K$ be a closed subset of a convex domain $\Omega$ in $\mathbb{R}^n$ (resp. $\mathbb{S}^n$) so that each point of $\partial K$ has a convex neighborhood. Then $K$ is a convex domain.

Proof Assume $\Omega \subset \mathbb{S}^n$. We can connect each pair of points by a broken projective geodesics. Then local convexity shows that we can make the number of geodesic segments to go down by one using triangles. Finally, we may obtain a geodesic segment connecting the pair of points.
2.4 Convexity and convex domains

Lemma 2.18 Let $\Omega$ be a properly convex domain in $\mathbb{R}P^n$ (resp. in $\mathbb{S}^n$). Let $\sigma$ be a convex domain in $\text{Cl}(\Omega) \cap P$ for a subspace $P$. Then either $\sigma \subset \partial \Omega$ or $\sigma^o$ is in $\Omega$.

Proof Assume $\Omega \subset \mathbb{S}^n$. Suppose that $\sigma^o$ meets $\partial \Omega$ and is not contained in it entirely. Since $\Omega$ is convex and $\partial \Omega$ is closed, $\sigma \cap \Omega$ for a segment $s$ in $\sigma$ can have only one component which must be open. Since the complement of $\sigma^o \cap \partial \Omega$ is a relatively open set in $\sigma^o$, we can find a segment $s \subset \sigma^o$ with a point $z$ so that a component $s_1$ of $s - \{z\}$ is in $\partial \Omega$ and the other component $s_2$ is disjoint from it.

We may perturb $s$ in a 2-dimensional totally geodesic space containing $s$ and so that the new segment $s' \subset \text{Cl}(\Omega)$ meets $\partial \Omega$ only in its interior point. This contradicts the fact that $\Omega$ is convex by Theorem A.2 of [51].

2.4.4 The Benoist theory.

In late 1990s, Benoist more or less completed the theory of the divisible action as started by Benzécri, Vinberg, Koszul, Vey, and so on in the series of papers [22], [21], [23], [24], [18], [17]. The comprehensive theory will aid us much in this paper.

Proposition 2.14 (Corollary 2.13 [23]) Suppose that a discrete subgroup $\Gamma$ of $\text{SL}_{\pm}(n, \mathbb{R})$ (resp. $\text{PGL}(n, \mathbb{R})$), $n \geq 2$, acts on a properly convex $(n-1)$-dimensional open domain $\Omega$ in $\mathbb{S}^{n-1}$ (resp, $\mathbb{R}P^{n-1}$) so that $\Omega/\Gamma$ is a compact orbifold. Then the following statements are equivalent.

- Every subgroup of finite index of $\Gamma$ has a finite center.
- Every subgroup of finite index of $\Gamma$ has a trivial center.
- Every subgroup of finite index of $\Gamma$ is irreducible in $\text{SL}_{\pm}(n, \mathbb{R})$ (resp. in $\text{PGL}(n, \mathbb{R})$).
  That is, $\Gamma$ is strongly irreducible.
- The Zariski closure of $\Gamma$ is semisimple.
- $\Gamma$ does not contain an infinite nilpotent normal subgroup.
- $\Gamma$ does not contain an infinite abelian normal subgroup.

Proof Corollary 2.13 of [23] considers $\text{PGL}(n, \mathbb{R})$ and $\mathbb{R}P^{n-1}$. However, the version for $\mathbb{S}^{n-1}$ follows from this since we can always lift a properly convex domain in $\mathbb{R}P^{n-1}$ to one $\Omega$ in $\mathbb{S}^{n-1}$ and the group to one in $\text{SL}_{\pm}(n, \mathbb{R})$ acting on $\Omega$ by Theorem 2.4.\[\square\]

The center of a group $G$ is denoted by $Z(G)$. A virtual center of a group $G$ is a subgroup of $G$ each of whose elements is centralized by a finite index subgroup of $G$. The group with properties above is said to be the group with a trivial virtual center.

Theorem 2.11 (Theorem 1.1 of [23]) Suppose that a virtual-center-free discrete subgroup $\Gamma$ of $\text{SL}_{\pm}(n, \mathbb{R})$ (resp. $\text{PGL}(n, \mathbb{R})$), $n \geq 2$, acts on a properly convex $(n-1)$-dimensional open domain $\Omega \subset \mathbb{S}^{n-1}$ so that $\Omega/\Gamma$ is a compact orbifold. Then every representation of a component of $\text{Hom}(\Gamma, \text{SL}_{\pm}(n, \mathbb{R}))$ (resp. $\text{Hom}(\Gamma, \text{PGL}(n, \mathbb{R}))$)...
containing the inclusion representation also acts on a properly convex \((n-1)\)-dimensional open domain cocompactly.

When \(\Omega/\Gamma\) admits a hyperbolic structure and \(n = 3\), Inkang Kim [119] proved this simultaneously for a component.

**Proposition 2.15 (Theorem 1.1. of Benoist [21])** Suppose that a discrete subgroup \(\Gamma\) of \(\text{SL}_n(n,\mathbb{R})\) (resp. \(\text{PGL}(n,\mathbb{R})\)), \(n \geq 2\), acts on a properly convex \((n-1)\)-dimensional open domain \(\Omega\) in \(\mathbb{S}^{n-1}\) (resp. \(\mathbb{R}P^{n-1}\)) so that \(\Omega/\Gamma\) is a compact orbifold. Then

- \(\Omega\) is projectively diffeomorphic to the interior of a strict join \(K := K_1 \ast \cdots \ast K_{l_0}\)
  where \(K_i\) is a properly convex open domain of dimension \(n_i \geq 0\) in the subspace \(\mathbb{S}^{n_i}\) in \(\mathbb{S}^n\) (resp. \(\mathbb{R}P^{n_i}\) in \(\mathbb{R}P^n\)). \(K_i\) corresponds to a convex cone \(C_i \subset \mathbb{R}^{n_i+1}\) for each \(i\).
- \(\Omega\) is the image of \(C_1 + \cdots + C_{l_0}\).
- Let \(\Gamma_i^i\) be the image of \(\Gamma^i\) to \(K_i^i\) for the restriction map of the subgroup \(\Gamma^i\) of \(\pi_1(\Sigma)\) acting on each \(K_i^i\), \(j = 1, \ldots, l_0\). We denote by \(\Gamma_i^i\) an arbitrary extension of \(\Gamma^i\) by requiring it to act trivially on \(K_j^i\) for \(j \neq i\) and to have 1 as the eigenvalue associated with vectors in their directions.
- The subgroup corresponding to \(\mathbb{R}^{l_0-1}\) acts trivially on each \(K_j^i\) and form a positive diagonalizable matrix group.
- The fundamental group \(\pi_1(\Sigma)\) is virtually isomorphic to a subgroup of \(\mathbb{R}^{l_0-1} \times \Gamma_1 \times \cdots \times \Gamma_{l_0}\) for \((l_0 - 1) + \sum_{i=1}^{l_0} n_i = n\).
- \(\pi_1(\Sigma)\) acts on \(K_0\) cocompactly and discretely and in a semisimple manner (Theorem 3 of Vey [161]).
- The Zariski closure of \(\Gamma^i\) equals \(\mathbb{R}^{l_0-1} \times G_1 \times \cdots \times G_{l_0}\). Each \(\Gamma_i\) acts on \(K_{l_0}\) cocompactly, and \(G_j\) is a simple Lie group (Remark after Theorem 1.1 of [21]), and \(G_j\) acts trivially on \(K_m\) for \(m \neq j\).
- A virtual center of \(\pi_1(\Sigma)\) of maximal rank is isomorphic to \(\mathbb{Z}^{l_0-1}\) corresponding to the subgroup of \(\mathbb{R}^{l_0-1}\). (Proposition 4.4 of [21].)

The group indicated by \(\mathbb{Z}^{l_0-1}\) is in the virtual center of \(\pi_1(\Sigma)\). See Example 5.5.3 of Morris [146] for a group acting properly on a product of two hyperbolic spaces but restricts to a non-discrete group for each factor space.

**Corollary 2.2** Assume as in Proposition 2.15. Then every normal solvable subgroup of a finite-index subgroup \(\Gamma^i\) of \(\Gamma\) is virtually central in \(\Gamma\).

**Proof** If \(\Gamma\) is virtually abelian, this is obvious.

Suppose that \(\Omega\) is properly convex. Let \(G\) be a normal solvable subgroup of a finite-index subgroup \(\Gamma^i\) of \(\Gamma\). We may assume without loss of generality that \(\Gamma^i\) acts on each \(K_i^i\) by taking a further finite index subgroup and replacing \(G\) by a finite index subgroup of \(G\). Now, \(G\) is a normal solvable subgroup of the Zariski closure \(\mathcal{Z}(\Gamma^i)\). By Theorem 1.1 of [21], \(\mathcal{Z}(\Gamma^i)\) equals \(G_1 \times \cdots \times G_i \times \mathbb{R}^{l_i-1}\) and \(K = K_1 \ast \cdots \ast K_i\) where \(G_i\) is reductive and the following holds:

- if \(K_i\) is homogeneous, then \(G_i\) is simple and \(G_i\) is commensurable with \(\text{Aut}(K_i)\).
• Otherwise, $K_i^\circ$ is divisible and $G_i$ is a union of components of $\text{SL}_\pm(V_i)$

The image of $G$ into $G_i$ by the restriction homomorphism to $K_i$ is a normal solvable subgroup of $G_i$. Since $G_i$ is virtually simple, the image is a finite group. Hence, $G$ must be virtually a subgroup of the diagonalizable group $\mathbb{R}^{d-1}_+$ and hence is virtually central in $\Gamma'$. 

If $l_0 = 1$, $\Gamma$ is strongly irreducible as shown by Benoist. However, the images of these groups will be subgroups of $\text{PGL}(m, \mathbb{R})$ and $\text{SL}_\pm(m, \mathbb{R})$ for $m \leq n$. If $l_0 > 1$, we say that such an image in $\Gamma$ is virtually factorizable. Otherwise, such an image a non-virtually-factorizable group.

An action of a projective group $G$ on a properly convex domain $\Omega$ is sweeping if $\Omega/G$ is compact but not necessarily Hausdorff. A dividing action is sweeping.

Recall the commutant $H$ of a group acting on a properly convex domain is the maximal diagonalizable group commuting with the group. (See Vey [161].)

We have a useful theorem:

**Theorem 2.12 (directly from Proposition 3 of Vey [161])** Suppose that a projective group $\Gamma$ acts on a properly convex open domain $\Omega$ in $\mathbb{R}P^n$ (resp. in $\mathbb{S}^n$), with a sweeping action. Then $\Omega$ equals a convex hull of the orbit $\Gamma(x)$ for any $x \in \Omega$.

We define $\mathbb{S}^{n*} := S(\mathbb{R}^{n+1}, \mathbb{S})$, i.e., the sphericalization of the dual space $\mathbb{R}^{n+1}$. 

**Theorem 2.13** Suppose that $G$ acts on a convex domain $\Omega$ in $\mathbb{R}P^n$ (resp. in $\mathbb{S}^n$), so that $\Omega/G$ is a compact orbifold. Then if $G$ have only unit-norm eigenvalued elements, then $\Omega$ is complete affine.

**Proof** Theorem 2.6 shows that $G$ is orthopotent and has a unipotent group $U$ of finite index. A unipotent group has a global fixed point in $\mathbb{S}^n$, and so does $U^\circ$. Thus, there exists a hyperspace $P$ in $\mathbb{S}^n$ where $U$ acts on $P \cap \text{Cl}(\Omega) \neq \emptyset$ and $P \cap \Omega = \emptyset$ by Proposition 2.20. Thus, $\Omega$ is in an affine subspace $A$ bounded by $P$. Also, $G$ acts on $A$ as a group of affine transformations since every projective action on a complete affine space is affine. (See Berger [26].) Orthopotent groups are distal. Lemma 2 of Fried [84] implies the conclusion.

**Proposition 2.16** Assume as in Proposition 2.15. Then $K$ is the closure of the convex hull of $\bigcup_{g \in \mathbb{Z}^0 \cdot 1} A_g$ for the attracting limit set $A_g$ of $g$. Also, for any partial join $\hat{K} := K_{i_1} \ast \cdots \ast K_{i_j}$ for a subcollection $\{i_1, \ldots, i_j\}$, the closure of the convex hull of $\bigcup_{g \in \mathbb{Z}^0 \cdot 1} A_g \cap \hat{K}$ equals $\hat{K}$.

**Proof** We take a finite-index normal subgroup $\Gamma'$ of $\Gamma$ so that $\mathbb{Z}^0 \cdot 1$ is the center of $\Gamma$. Using Theorem 2.3, we may assume that $\Gamma'$ is torsion-free. Note that $kA_g$ for any $k \in \Gamma'$ equals $A_{kg^{-1}} = A_g$ since $kgk^{-1} = g$. Thus, $\Gamma'$ acts on $\bigcup_{g \in \mathbb{Z}^0 \cdot 1} A_g$ since it is a $\Gamma'$-invariant set. The interior $C$ of the convex hull of $\bigcup_{g \in \mathbb{Z}^0 \cdot 1} A_g$ is a subdomain in $K''$. Since $C/\Gamma' \to K''/\Gamma'$ is a homotopy equivalence of closed manifolds, we obtain $C = K''$ and $\text{Cl}(C) = K$ by Lemma 2.19.
For the second part, if the closure of the convex hull of $\bigcup_{g \in \mathbb{Z}^n} A_g \cap \hat{K}$ is a proper subset of $\hat{K}$, then the closure of the convex hull of $\bigcup_{g \in \mathbb{Z}^n} A_g$ is a proper subset of $K$. This is a contradiction.

**Lemma 2.19 (Domains for holonomy)** Suppose that $\Omega$ is an open domain in an open hemisphere in $\mathbb{S}^{n-1}$ (resp. in $\mathbb{R}^{n-1}$) where a projective group $\Gamma$ acts on so that $\Omega/\Gamma$ is a closed orbifold. Suppose that $\Gamma$ acts on a compact properly convex domain $K$. Then

- $K^o = R(\Omega)$ where $R$ is a diagonalizable projective automorphism commuting with a finite-index subgroup of $\Gamma$ with eigenvalues $\pm 1$ only and is a composition of reflections commuting with one-another.
- In fact $K = K_1 \ast \cdots \ast K_k$ where $K_j = K \cap P_j$, $J = 1, \ldots, k$, for a virtually invariant subspace $P_j$ of $\Gamma$ where $R$ equals I or $\mathcal{A}$.
- $\Omega$ has to be properly convex.
- If $K^o$ meets with $\Omega$, then $K^o = \Omega$.

**Proof** It is sufficient to prove for $\mathbb{S}^1$. We prove by an induction on dimension. For $\mathbb{S}^1$, this is clear.

By Theorem 2.3, there is a torsion-free finite-index subgroup $\Gamma'$ of $\Gamma$. Suppose that $\Omega \cap K^o \neq \emptyset$. Then $(\Omega \cap K^o)/\Gamma'$ is homotopy equivalent to $\Omega/\Gamma'$, a closed manifold. Hence, $\Omega \cap K^o = \Omega$ and $\Omega \subset K^o$. Similarly, $K^o \subset \Omega$. We obtain $K^o = \Omega$.

Also, if $\Omega \cap \mathcal{A}(K^o) \neq \emptyset$, then $\mathcal{A}(K^o) = \Omega$. The lemma is proved for this case.

Proposition 2.4 shows that $K^o/\Gamma$ is again a closed orbifold. Suppose that $\Gamma$ is not virtually factorizable with respect to $K$. Then $\Gamma$ is strongly irreducible by Benoist [21]. $K$ contains the attracting fixed points of bi-semi-proximal element $g$ of $\Gamma$. This implies that $\text{Cl}(\Omega) \cap \text{Cl}(K) \neq \emptyset$ or $\text{Cl}(\Omega) \cap \mathcal{A}(\text{Cl}(K)) \neq \emptyset$ since $\Omega$ contains a generic point of $\mathbb{S}^n$. By the above paragraph, we may suppose that the intersection is in $\text{bd}\Omega \cap \text{bd}K$ or $\text{bd}\Omega \cap \mathcal{A}(\text{bd}K)$. Then this is a compact convex set invariant under $\Gamma$. Hence, $\Gamma$ is reducible, a contradiction.

Now suppose that $\Gamma$ is virtually factorizable. Then there exists a diagonalizable free abelian group $D$ of rank $k-1$ for some $k \geq 2$ in the virtual center of $\Gamma$ by Proposition 2.15. $D$ acts trivially on a finite set of minimal subspace $P_1, \ldots, P_k$ by Proposition 2.15. Since $\Gamma$ permutes these subspaces, a torsion-free finite-index subgroup $\Gamma'$ of $\Gamma$ acts on $P_1, \ldots, P_k$. Let’s denote $P_j := P_1 \ast \cdots \ast P_{j-1} \ast P_{j+1} \ast \cdots \ast P_k$. Then $\Omega$ is disjoint from $P_j$ for each $j$ since otherwise $(\hat{P}_j \cap \Omega)/\Gamma' \rightarrow \Omega/\Gamma'$ is a homotopy equivalence of different dimensional manifolds.

However, $P_j \cap \text{Cl}(\Omega) \neq \emptyset$ since we can choose a sequence $\{g_i\}$ of elements $g_i \in D$ so that the associated eigenvalues for $P_j$ goes to zero and the other eigenvalues goes to infinity while their ratios are uniformly bounded as $i \rightarrow \infty$.

Again, define $K_j := K \cap P_j$. Since $K$ is properly convex, $K_1 \ast \cdots \ast K_k \subset K$. Since the action of $\Gamma$ on $K^o$ is cocompact and proper, Proposition 2.15 shows that $K = K_1 \ast \cdots \ast K_k$. We have a projection for $K^o \rightarrow K_j^o$ for each $j$ obtained from the join structure. Then the action of $\Gamma$ on $K_j$ is cocompact since otherwise $K^o/\Gamma$ cannot be compact. Also, the action of $\Gamma'$ on $P_j$ is irreducible by Benoist [21].
We can find a sequence in \( D \) converging to a projection \( \Pi_j \) to each \( P_j \) with the undefined space \( \hat{P}^j \). We define domains \( \Omega_j := \text{Cl}(\Pi_j(\Omega)) \) in \( P_j \). Since \( \Omega \) is in an open subset in a hemisphere, there exists a convex hull of \( \text{Cl}(\Omega) \), and hence so has \( \Omega_j \) for each \( j = 1, \ldots, m \). Then \( \Omega_j \) is properly convex by the third item of Proposition 2.12 and the irreducibility of the action in each factor \( K_j \) in Proposition 2.15. Hence \( \Omega \) is in a properly convex domain \( \Omega_1 \cdots \Omega_m \).

By Theorem 2.14 of Kobayashi, \( \Omega \) equals the interior of \( \Omega_1 \cdots \Omega_m \). Suppose that \( \Omega_j^0 \cap K_j^0 \neq \emptyset \) or \( \Omega_j \cap \mathcal{A}(K_j^0) \neq \emptyset \). Theorem 2.12 shows that \( K_j^0 = \Omega_j^0 \) or \( \mathcal{A}(K_j^0) = \Omega_j^0 \) since \( \Gamma \) acts on a convex domain \( \Omega_j^0 \). Suppose that \( \Omega_j^0 \cap K_j^0 \) or \( \Omega_j^0 \cap \mathcal{A}(K_j^0) \) are empty for all \( j \). Then \( \text{Cl}(\Omega_j) \cap K_j \neq \emptyset \) or \( \text{Cl}(\Omega_j) \cap \mathcal{A}(K_j) \neq \emptyset \) by Proposition 2.20. Since such intersection has a unique minimal subspace containing it, this contradicts the irreducibility of \( \Gamma' \)-action on \( P_j \).

Hence, it follows that \( K' := (K_1' \cdots K_m')^0 \) is a subset of \( \Omega \) for \( K_j' = K_j \) or \( K_j' = \mathcal{A}(K_j) \). Again \( K'/\Gamma' \to \Omega/\Gamma' \) is a homotopy equivalence and hence \( K' = \Omega \).

The action of projective automorphisms restricting to I or \( \mathcal{A} \) on each \( P_j \) gives us the final part. (See Theorem 4.1 of [67] also.)

We have the following useful result.

**Corollary 2.3.** Let \( \{h_i : \Gamma \to \text{SL}_+(n, \mathbb{R})\} \) be a sequence of faithful discrete representations so that \( \Theta_i := \Omega_i/h_i(\Gamma) \) is a closed real projective orbifold for a properly convex domain \( \Omega_i \) in \( \mathbb{S}^n \) for each \( i \). Suppose that \( \{h_i \} \to h_\infty \) algebraically, \( h_\infty \) is faithful with discrete image, and \( \{\text{Cl}(\Omega_i)\} \) geometrically converges to a properly convex domain \( \text{Cl}(\Omega_\infty) \) with nonempty interior \( \Omega_\infty \). Then \( h_\infty(\Gamma) \) acts on the interior \( \Omega_\infty \) so that the following hold:

- \( \Omega_\infty/h_\infty(\Gamma) \) is a closed real projective orbifold.
- \( \Omega_\infty/h_\infty(\Gamma) \) is diffeomorphic to \( \Theta_i \) for sufficiently large \( i \).
- If \( U \) is a properly convex domain where \( h_\infty(\Gamma) \) acts so that \( U/h_\infty(\Gamma) \) is an orbifold, then \( U = \Omega_\infty \) or \( J(\Omega_\infty) \) where \( J \) is a projective automorphism commuting with \( h_\infty(\Gamma) \). In particular, if \( \Gamma \) is non-virtually-factorizable, then \( J = 1 \) or \( \mathcal{A} \).

**Proof** By Proposition 2.4, the quotient \( \Omega_\infty/h_\infty(\Gamma) \) is an orbifold. For the second item, see the proof of Theorem 4.1 of [67]. The third item follows from Lemma 2.19. \( \square \)

**Theorem 2.14 (Kobayashi [120])** Suppose that a closed real projective orbifold has a developing map into a properly convex domain \( D \) in \( \mathbb{R}P^n \) (resp. in \( \mathbb{S}^n \)). Then the orbifold is projectively diffeomorphic to \( \Omega/\Gamma' \) for the holonomy group \( \Gamma' \) and the minimal \( \Gamma' \)-invariant open domain \( \Omega \) in \( \mathbb{D}^n \).

**Proof** This follows since all maximal segments in \( D \) are of \( d \)-length \( \leq \pi - \varepsilon_0 \) for a uniform \( \varepsilon_0 > 0 \). Hence, the Kobayashi metric is well-defined proving that the orbifold is properly convex. \( \square \)
2.4.5 Technical propositions.

By the following, the first assumption of Theorem 6.8 are needed only for the conclusion of the theorem to hold.

**Proposition 2.17** If a group $G$ of projective automorphisms acts on a strict join $A = A_1 * A_2$ for two compact convex sets $A_1$ and $A_2$ in $\mathbb{S}^n$ (resp. in $\mathbb{R}P^n$), then $G$ is virtually reducible.

**Proof** We prove for $\mathbb{S}^n$. Let $x_1, \ldots, x_{n+1}$ denote the homogeneous coordinates. There is at least one set of strict join sets $A_1, A_2$. We choose a maximal number collection of compact convex sets $A'_1, \ldots, A'_m$ so that $A$ is a strict join $A'_1 * \cdots * A'_m$. Here, we have $A'_i \subset S_i$ for a subspace $S_i$ corresponding to a subspace $V_i \subset \mathbb{R}^{n+1}$ that form an independent set of subspaces.

We claim that $g \in G$ permutes the collection $\{A'_1, \ldots, A'_m\}$: Suppose not. We give coordinates so that for each $i$, there exists some index set $I_i$ so that elements of $A'_i$ satisfy $x_j = 0$ for $j \in I_i$ and elements of $A$ satisfy $x_i \geq 0$. Then we form a new collection of nonempty sets

$$ J' := \{A'_i \cap g(A'_i) | 0 \leq i, j \leq n, g \in G\} $$

with more elements. Since

$$ A = g(A) = g(A'_1) * \cdots * g(A'_l), $$

we can show that each $A'_i$ is a strict join of nonempty sets in

$$ J'_i := \{A'_i \cap g(A'_j) | 0 \leq j \leq l, g \in G\} $$

using coordinates. $A$ is a strict join of the collection of the sets in $J'$, a contraction to the maximal property.

Hence, by taking a finite index subgroup $G'$ of $G$ acting trivially on the collection, $G'$ is reducible. $\square$

**Proposition 2.18** Suppose that a set $G$ of projective automorphisms in $\mathbb{S}^n$ (resp. in $\mathbb{R}P^n$) acts on subspaces $S_1, \ldots, S_l_0$ and a properly convex domain $\Omega \subset \mathbb{S}^n$ (resp. $\subset \mathbb{R}P^n$), corresponding to independent subspaces $V_1, \ldots, V_{l_0}$ so that $V_i \cap V_j = \{0\}$ for $i \neq j$ and $V_1 \oplus \cdots \oplus V_{l_0} = \mathbb{R}^{n+1}$. Let $\Omega_i := \text{Cl}(\Omega) \cap S_i$ for each $i, i = 1, \ldots, l_0$. Let $\lambda_i(g)$ denote the largest norm of the eigenvalues of $g$ restricted to $V_i$. We assume that

- for each $S_i$, $G_i := \{g|S_i | g \in G\}$ forms a bounded set of automorphisms, and
- for each $S_i$, there exists a sequence $\{g_{i,j} \in G\}$ which has the property

$$ \left\{ \frac{\lambda_k(g_{i,j})}{\lambda_k(g_{i,j})} \right\} \to \infty \text{ for each } k, k \neq i \text{ as } j \to \infty. $$
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Then $\Cl(\Omega) = \Omega_1 \ast \cdots \ast \Omega_{l_0}$ for $\Omega_j \neq \emptyset, j = 1, \ldots, l_0$.

Proof First, $\Omega \subset \Cl(\Omega)$ by definition. Each element of a strict join has a vector that is a linear combination of elements of the vectors in the directions of $\Omega_1, \ldots, \Omega_{l_0}$, Hence, we obtain

$\Omega_1 \ast \cdots \ast \Omega_{l_0} \subset \Cl(\Omega)$

since $\Cl(\Omega)$ is convex.

Let $z = [v_z]$ for a vector $v_z$ in $\mathbb{R}^{n+1}$. We write $v_z = v_1 + \cdots + v_{l_0}$, $v_j \in V_j$ for each $j, j = 1, \ldots, l_0$, which is a unique sum. Then $z$ determines $z_i = [v_i]$ uniquely.

Let $z$ be any point. We choose a subsequence of $\{g_{i,j}\}$ so that $\{g_{i,j}S_i\}$ converges to a projective automorphism $g_{i,\infty} : S_i \to S_i$ and $\lambda_{i,j} \to \infty$ as $j \to \infty$. Then $g_{i,\infty}$ also acts on $\Omega_i$. By Proposition 2.8, $\{g_{i,j}(z_i)\} \to g_{i,\infty}(z_i) = z_{i,\infty}$ for a point $z_{i,\infty} \in S_i$. We also have

$$z_i = g_{i,\infty}^{-1}(g_{i,\infty}(z_i)) = g_{i,\infty}^{-1}(\lim g_{i,j}(z_i)) = g_{i,\infty}^{-1}(z_{i,\infty}). \quad (2.24)$$

Now suppose $z \in \Cl(\Omega)$. We have $\{g_{i,j}(z)\} \to z_{i,\infty}$ by the eigenvalue condition. Thus, we obtain $z_{i,\infty} \in \Omega_i$ as $z_{i,\infty}$ is the limit of a sequence of orbit points of $z$. Hence we also obtain $z_i \in \Omega_i$ by (2.24). We obtain $\Omega_i \neq \emptyset$. This also shows that $\Cl(\Omega) = \Omega_1 \ast \cdots \ast \Omega_{l_0}$ since $z \in \{z_1\} \ast \cdots \ast \{z_{l_0}\}$.

For the $\mathbb{RP}^n$-version, we lift $\Omega$ to an open hemisphere in $\mathbb{S}^n$. Then the $\mathbb{S}^n$-version implies the $\mathbb{RP}^n$-version. 

[§nT]

2.5 The Vinberg duality of real projective orbifolds

The duality is a natural concept in real projective geometry and it will continue to play an essential role in this theory as well.

2.5.1 The duality

We start from linear duality. Let $\Gamma$ be a group of linear transformations $\text{GL}(n + 1, \mathbb{R})$. Let $\Gamma^*$ be the affine dual group defined by $\{g^* | g \in \Gamma\}$. Suppose that $\Gamma$ acts on a properly convex cone $C$ in $\mathbb{R}^{n+1}$ with the vertex $O$.

- An open convex cone $C^*$ in $\mathbb{R}^{n+1}$ is dual to an open convex cone $C$ in $\mathbb{R}^{n+1}$ if $C^* \subset \mathbb{R}^{n+1}$ is the set of linear functionals taking positive values on $\Cl(C) - \{O\}$. $C^*$ is a cone with the origin as the vertex again. Note $(C^*)^* = C$, and $C$ must be properly convex since otherwise $C^*$ cannot be open. We generalize the notion in Section 2.5.4.
- Now $\Gamma^*$ will acts on $C^*$. A central dilatational extension $\Gamma'$ of $\Gamma$ by $\mathbb{Z}$ is given by adding a scalar dilatation by a scalar $s > 1$ for the set $\mathbb{R}_+$ of positive real numbers.
• The dual $\Gamma^\ast$ of $\Gamma'$ is a central dilatation extension of $\Gamma^\ast$. Also, if $\Gamma'$ is cocompact on $C$ if and only if $\Gamma^\ast$ is on $C^\ast$. (See [91] for details.)
• Given a subgroup $\Gamma$ in $\text{PGL}(n + 1, \mathbb{R})$, the dual group $\Gamma^\ast$ is the image in $\text{PGL}(n + 1, \mathbb{R})$ of the dual of the inverse image of $\Gamma$ in $\text{SL}_+(n + 1, \mathbb{R})$.
• A properly convex open domain $\Omega$ in $P(\mathbb{R}^{n+1})$ is dual to a properly convex open domain $\Omega^\ast$ in $P(\mathbb{R}^{n+1}, \ast)$ if $\Omega$ corresponds to an open convex cone $C$ and $\Omega^\ast$ to its dual $C^\ast$. We say that $\Omega^\ast$ is dual to $\Omega$. We also have $\langle \Omega^\ast \rangle^\ast = \Omega$ and $\Omega$ is properly convex if and only if so is $\Omega^\ast$.
• We call $\Gamma$ a dividing group if a central dilatational extension acts cocompactly on $C$ with a Hausdorff quotient. $\Gamma$ is dividing if and only if so is $\Gamma^\ast$.
• Define $\mathbb{S}^n := \mathbb{S}(\mathbb{R}^{n+1})$. For an open properly convex subset $\Omega$ in $\mathbb{S}^n$, the dual domain is defined as the quotient in $\mathbb{S}^n$ of the dual cone of the cone $C_\Omega$ corresponding to $\Omega$. The dual set $\Omega^\ast$ is also open and properly convex but the dimension may not change. Again, we have $\langle \Omega^\ast \rangle^\ast = \Omega$.
• If $\Omega$ is a compact properly convex domain but not necessarily open, then we define $\overline{\Omega}$ to be the closure of the dual domain of $\Omega^\ast$. This definition agrees with the definition given in Section 2.5.4 for any compact convex domains since a sharply supporting hyperspace can be perturbed to a supporting hyperspace that is not sharply supporting. (See Berger [26].)
• Given a properly convex domain $\Omega$ in $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$), we define the augmented boundary of $\Omega$

$$bd^{Ag} \Omega := \{(x, H) | x \in bd\Omega, x \in H, H \text{ is an oriented sharply supporting hyperspace of } \Omega \} \subset S^n \times \mathbb{S}^n. \quad (2.25)$$

Define the projection

$$\Pi^{Ag}_\Omega : bd^{Ag} \Omega \rightarrow bd\Omega$$

by $(x, H) \mapsto x$. Each $x \in bd\Omega$ has at least one oriented sharply supporting hyperspace. An oriented hyperspace is an element of $\mathbb{S}^n$ since it is represented as a linear functional. Conversely, an element of $\mathbb{S}^n$ represents an oriented hyperspace in $\mathbb{S}^n$. (Clearly, we can do this for $\mathbb{R}P^n$ and the dual space $\mathbb{R}P^{n*}$ but we consider only nonoriented supporting hyperspaces.)

**Theorem 2.15** Let $A$ be a subset of $bd\Omega$. Let $A' := \Pi^{Ag,-1}_\Omega(A)$ be the subset of $bd^{Ag}(A)$. Then $\Pi^{Ag}_\Omega | A' : A' \rightarrow A$ is a quasi-fibration.

**Proof** We take a Euclidean metric on an affine subspace containing $\text{Cl}(\Omega)$. The sharply supporting hyperspaces at $x$ can be identified with unit normal vectors at $x$. Each fiber $\Pi^{Ag,-1}_\Omega(x)$ is a properly convex compact domain in a sphere of unit vectors through $x$. We find a continuous section defined on $bd\Omega$ by taking the center of mass of each fiber with respect to the Euclidean metric. This gives a local coordinate system on each fiber by giving the origin, and each fiber is a compact convex domain containing the origin. Then the quasi-fibration property is clear now. [S\textsuperscript{T}]
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Remark 2.3 We notice that for properly convex open or compact domains \( \Omega_1 \) and \( \Omega_2 \) in \( \mathbb{S}^n \) (resp. in \( \mathbb{R}P^n \)) we have

\[
\Omega_1 \subset \Omega_2 \text{ if and only if } \Omega_2^* \subset \Omega_1^*
\]  

(2.26)

Remark 2.4 We are given a strict join \( A \ast B \) for a properly convex compact \( k \)-dimensional domain \( A \) in \( \mathbb{R}P^k \subset \mathbb{R}P^n \) and a properly convex compact \( n-k-1 \)-dimensional domain \( B \) in the complementary \( \mathbb{R}P^{n-k-1} \subset \mathbb{R}P^n \). Let \( A_{\mathbb{R}P^k} \) denote the dual in \( \mathbb{R}P^k \) of \( A \) in \( \mathbb{R}P^k \) and \( B_{\mathbb{R}P^{n-k-1}} \) the dual domain in \( \mathbb{R}P^{n-k-1} \) of \( B \) in \( \mathbb{R}P^{n-k-1} \). \( \mathbb{R}P^k \) embeds into \( \mathbb{R}P^n \) as \( \mathbb{P}(V_1) \) for the subspace \( V_1 \) of linear functionals cancelling vectors in directions of \( \mathbb{R}P^{n-k-1} \) and \( \mathbb{R}P^{n-k-1} \) embeds into \( \mathbb{R}P^n \) as \( \mathbb{P}(V_2) \) for the subspace \( V_2 \) of linear functionals nullifying the vectors in directions of \( \mathbb{R}P^k \). These will be denoted by \( \mathbb{R}P^\dagger \) and \( \mathbb{R}P^{n-k-1\dagger} \) respectively.

Then we have

\[
(A \ast B)^* = A_{\mathbb{R}P^k}^\dagger \ast B_{\mathbb{R}P^{n-k-1}}^\dagger.
\]  

(2.27)

This follows from the definition and realizing every linear functional as a sum of linear functionals in the direct-sum subspaces.

Suppose that \( A \subset \mathbb{S}^k \) and \( B \subset \mathbb{S}^{n-k-1} \) respectively are \( k \)-dimensional and \( (n-k-1) \)-dimensional domains where \( \mathbb{S}^k \) and \( \mathbb{S}^{n-k-1} \) are complementary subspaces in \( \mathbb{S}^n \). Suppose that \( A \) and \( B \) have respective dual sets \( A_{\mathbb{S}^k} \subset \mathbb{S}^k \), \( B_{\mathbb{S}^{n-k-1}} \subset \mathbb{S}^{n-k-1} \). We embed \( \mathbb{S}^k \) and \( \mathbb{S}^{n-k-1} \) to \( \mathbb{S}^n \) as above. We denote the images by \( \mathbb{S}^{k\dagger} \) and \( \mathbb{S}^{n-k-1\dagger} \) respectively. Then the above equation also holds.

An element \( (x, H) \) is \( \text{bd}^A \Omega \) if and only if \( x \in \text{bd} \Omega \) and \( h \) is represented by a linear functional \( \alpha_H \) so that \( \alpha_H(y) > 0 \) for all \( y \) in the open cone \( C(\Omega) \) corresponding to \( \Omega \) and \( \alpha_H(v) = 0 \) for a vector \( v \), representing \( x \).

Let \( (x, H) \in \text{bd}^A \Omega \). The dual cone \( C(\Omega)^* \) consists of all nonzero 1-forms \( \alpha \) so that \( \alpha(y) > 0 \) for all \( y \in \text{Cl}(C(\Omega)) \) \( - \{O\} \). Thus \( \alpha(v) = 0 \) for all \( \alpha \in C^* \) and \( \alpha_H(v) = 0 \), and \( \alpha_H \notin C(\Omega)^* \) since \( v \in \text{Cl}(C(\Omega)) \) \( - \{O\} \). But \( H \in \text{bd} \Omega^* \) as we can perturb \( \alpha_H \) so that it is in \( C^* \). Thus, \( x \) is a sharply supporting hyperspace at \( H \in \text{bd} \Omega^* \). We define a duality map

\[
\mathcal{D}^A : \text{bd}^A \Omega \rightarrow \text{bd}^A \Omega^*
\]

given by sending \( (x, H) \) to \( (H, x) \) for each \( (x, H) \in \text{bd}^A \Omega \).

Proposition 2.19 Let \( \Omega \) and \( \Omega^* \) be dual open domains in \( \mathbb{S}^n \) and \( \mathbb{S}^n \) (resp. \( \mathbb{R}P^n \) and \( \mathbb{R}P^n \)).

(i) There is a proper map \( \Pi^A : \text{bd}^A \Omega \rightarrow \text{bd} \Omega \) given by sending \( (x, H) \) to \( x \).

(ii) A projective automorphism group \( \Gamma \) acts properly on a properly convex open domain \( \Omega \) if and only if so \( \Gamma^* \) acts on \( \Omega^* \) (Vinberg’s Theorem 2.16).

(iii) There exists a duality map

\[
\mathcal{D}^A : \text{bd}^A \Omega \rightarrow \text{bd}^A \Omega^*
\]
which is a homeomorphism.

(iv) Let \( A \subset \text{bd}^A \Omega \) be a subspace and \( A^* \subset \text{bd}^A \Omega^* \) be the corresponding dual subspace \( \mathcal{D}^A_{\Omega} (A) \). A group \( \Gamma \) acts on \( A \) so that \( A/\Gamma \) is compact if and only if \( \Gamma^* \) acts on \( A^* \) and \( A^*/\Gamma^* \) is compact.

**Proof** We will prove for \( \mathbb{S}^n \) first. (i) Each fiber is a closed set of hyperspaces at a point forming a compact set. The set of sharply supporting hyperspaces at a compact subset of \( \text{bd} \Omega \) is closed. The closed set of hyperspaces having a point in a compact subset of \( \mathbb{S}^n+1 \) is compact. Thus, \( \Pi_{\text{Ag}}^A \) is proper. Clearly, \( \Pi_{\text{Ag}}^A \) is continuous, and it is an open map since \( \text{bd} \Omega \) is given the subspace topology from \( \mathbb{S}^n \times \mathbb{S}^n^* \) with a product topology where \( \Pi_{\text{Ag}}^A \) extends to a projection.

(ii) See Chapter 4 of [91] or Vinberg [162].

(iii) \( \mathcal{D}_{\Omega}^A \) has the inverse map \( \mathcal{D}_{\Omega}^{A^*} \).

(iv) The item is clear from (iii).

**Definition 2.9** The two subgroups \( G_1 \) of \( \Gamma \) and \( G_2 \) of \( \Gamma^* \) are dual if sending \( g \mapsto g^{-1} \) gives us an isomorphism \( G_1 \to G_2 \). A set in \( A \subset \text{bd} \Omega \) is dual to a set \( B \subset \text{bd} \Omega^* \) if \( \mathcal{D}_{\Omega}^A : \Pi_{\text{Ag}}^{-1}(A) \to \Pi_{\text{Ag}}^{-1}(B) \) is a one-to-one and onto map.

**Remark 2.5** For an open subspace \( A \subset \text{bd} \Omega \) that is smooth and strictly convex, \( \mathcal{D}_{\Omega}^A \) induces a well-defined map

\[
A \subset \text{bd} \Omega \to A' \subset \text{bd} \Omega^*
\]

since each point has a unique sharply supporting hyperspace for an open subspace \( A' \). The image of the map \( A' \) is also smooth and strictly convex by Lemma 2.20. We will simply say that \( A' \) is the image of \( \mathcal{D} \).

We say that a two-sided open hypersurface is convex polyhedral if it is a union of locally finite collection of compact polytopes in hyperspaces meeting one another in strictly convex angles where the convexity is towards one-side.

**Lemma 2.20** Let \( \Omega^* \) be the dual of a properly convex open domain \( \Omega \) in \( \mathbb{R} \mathbb{P}^n \) (resp. in \( \mathbb{S}^n \)). Then

(i) \( \text{bd} \Omega \) is \( C^1 \) and strictly convex at a point \( p \in \text{bd} \Omega \) if and only if \( \text{bd} \Omega^* \) is \( C^1 \) and strictly convex at the unique corresponding point \( p^* \).

(ii) \( \Omega \) is an ellipsoid if and only if so is \( \Omega^* \).

(iii) \( \text{bd} \Omega^* \) contains a properly convex domain \( D = P \cap \text{bd} \Omega^* \) open in a totally geodesic hyperspace \( P \) if and only if \( \text{bd} \Omega \) contains a vertex \( p \) with \( R_p(\Omega) \) a properly convex domain. In this case, \( \mathcal{D}_{\Omega}^A \) sends the pair of \( p \) and the associated sharply supporting hyperspace of \( \Omega \) to the pairs of the totally geodesic hyperspace containing \( D \) and points of \( D \). Moreover, \( D \) and \( R_p(\Omega) \) are properly convex, and the projective dual of \( D \) is projectively diffeomorphic to \( R_p(\Omega) \).

(iv) Let \( S \) be a convex polyhedral open subspace of \( \text{bd} \Omega \). Then \( S \) is dual to a convex polyhedral open subspace of \( \text{bd} \Omega^* \).
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Proof We first prove for $\mathbb{S}^n$. (i) The one-to-one map $\phi: A^0 \Omega \to \Omega$ sends each pair $(x,H)$ of a point of $\partial \Omega$ and the sharply supporting hyperplane to a pair of $(H,x)$ where $H$ is a point of $\partial \Omega^*$ and $x$ is a sharply supporting hyperplane at $H$ of $\Omega^*$.

The fact that $\partial \Omega$ is $C^1$ and strictly convex implies that for $x \in \partial \Omega$, $H$ is unique, and for $H$, there is only one point of $\partial \Omega$ where $H$ meets $\partial \Omega$. Also, this is equivalent to the fact that for each $H \in \partial \Omega^*$, the supporting hyperspace $x$ is unique and for each $x$, there is one point of $\partial \Omega^*$ where $x$ meets $\partial \Omega^*$. This shows that $\partial \Omega^*$ is strictly convex and $C^1$.

(ii) Let $\mathbb{R}^{n+1}$ have the standard Lorentz inner product $B$. Let $C$ be the open positive cone. Then the space of linear functionals positive on $C$ is in one-to-one correspondence with vectors in $C$ using the isomorphism $C^* \to C$ given by $\phi \mapsto v_\phi$ so that $\phi = B(v_\phi, \cdot)$.

(iii) Suppose that $R_p(\Omega)$ is properly convex. We consider the set of hyperspaces sharply supporting $\Omega$ at $p$. This forms a properly convex domain: Let $v$ be the vector in $\mathbb{R}^{n+1}$ in the direction of $p$. Then we find the set of linear functionals positive on $C(\Omega)$ but being zero on $v$. Let $V$ be a complementary space of $v$ in $\mathbb{R}^{n+1}$. Let $A$ be given as the affine subspace $V + \{v\}$ of $\mathbb{R}^{n+1}$. We choose $V$ so that $C_v := C(C(\Omega)) \cap A$ is a bounded convex domain in $A$. We give $A$ a linear structure so that $v$ corresponds to the origin. We identify this space with $V$. The set of linear functionals positive on $C(\Omega)$ and $0$ at $v$ is identical with that of linear functionals on $\mathbb{R}^n$ positive on $C_v$: we define

$$C(D) := \{ f \in \mathbb{R}^{n+1} \mid f(C(\Omega)) - \{ t v \mid t \geq 0 \} > 0, f(v) = 0 \}$$

$$\cong \widetilde{C}_v := \{ g \in V^* \mid g(C_v - \{0\}) > 0 \} \subset \mathbb{R}^{n+1*}.$$

Here $\cong$ indicates a linear isomorphism, which follows by the decomposition $\mathbb{R}^{n+1} = \{ tv \mid t \in \mathbb{R} \} \oplus V$. Define $R_v'(C_v)$ as the equivalence classes of properly convex segments in $C_v$ ending at $v$ where two segments are equivalent if they agree in an open neighborhood of $v$. $R_v(\Omega)$ is identical with $R_v'(C_v)$ by the projectivization $S : \mathbb{R}^{n+1} - \{O\} \to \mathbb{S}^n$. Hence $R_v'(C_v)$ is a properly convex open domain in $S(V)$. Since $R_v'(C_v)$ is properly convex, the interior of the spherical projectivization $S(C_v) \subset S(V^*)$ is dual to the properly convex domain $R_v'(C_v) \subset \mathbb{S}^n(V)$.

Again we have a projection $S : \mathbb{R}^{n+1*} - \{O\} \to \mathbb{S}^{n*}$. Define $D := S(C(D)) \subset \mathbb{S}^{n*}$. Since $R_v'(C_v)$ corresponds to $R_v(\Omega)$, and $S(C_v)$ corresponds to $D$, the duality follows. Also, $D \subset \partial \Omega^*$ since points of $D$ are oriented sharply supporting hyperspaces to $\Omega$ by Proposition 2.19 (iii). (Here, we can also use Proposition 6.1.)

(iv) From (iii) each vertex of a convex polyhedral subspace of $S$ correspond to a compact convex polytope in the dual subspace. Also, we can check that each side of dimension $i$ correspond to a side of dimension $n-i-1$. 

$[\mathbb{S}^n]$
2.5.2 The duality of convex real projective orbifolds with strictly convex boundary

Since $\mathcal{O} = \Omega / \Gamma$ for an open properly convex domain $\Omega$ in $\mathbb{R}^n$ (resp. in $\mathbb{S}^n$) the dual orbifold $\mathcal{O}^* = \Omega^* / \Gamma^*$ is a properly convex real projective orbifold. The dual orbifold is well-defined up to projective diffeomorphisms.

**Theorem 2.16 (Vinberg)** Let $\mathcal{O}$ be a strongly tame properly convex real projective open or closed orbifold. The dual orbifold $\mathcal{O}^*$ is diffeomorphic to $\mathcal{O}$. For proof, see Chapter 6 of [91].

The map given by Vinberg [162] is called the Vinberg duality diffeomorphism. For an orbifold $\mathcal{O}$ with boundary, the map is a diffeomorphism in the interiors $\mathcal{O}^\circ \rightarrow \mathcal{O}^*^\circ$. Let $\tilde{\mathcal{O}}$ denote the properly convex projective domain covering $\mathcal{O}$. Also, $\mathcal{D}^{Ag}_{\tilde{\mathcal{O}}} \partial \mathcal{O} \rightarrow \partial \mathcal{O}^*$ (We conjecture that they form a diffeomorphism $\mathcal{O} \rightarrow \mathcal{O}^*$ up to isotopies. We also remark that $\mathcal{D}^{\mathcal{O}^*}_{\mathcal{O}} \circ \mathcal{D}^{\mathcal{O}}_{\mathcal{O}}$ may not be identity as shown by Vinberg.)

For each $p \in \Omega$, let $p_{V,\Omega}$ denote the vector in $\mathcal{C}(\Omega)$ with $f_V^{-1}(p_{V,\Omega}) = 1$ for the Koszul-Vinberg function $f_V$ for $\mathcal{C}(\Omega)$. (See (11.1).) Define $p^*_{V,\Omega}$ as the 1-form $\mathcal{D}^{f_{V,\Omega}}_{\mathcal{O}}(p_{V,\Omega})$, and also define $p^*_{V}$ as $(\mathcal{D}^{f_{V,\Omega}}_{\mathcal{O}}(p_{V,\Omega}))$. We obtain a compactification of $\mathcal{O}$ by defining $\mathcal{C}l^{Ag}_{\mathcal{O}}(\mathcal{O}) = \mathcal{O} \cup \partial^{Ag}\mathcal{O}$ by defining for any sequence $p_i \in \mathcal{O}$, we form a pair $(p_i, p^*_{V,\Omega})$ where $p^*_{V,\Omega}$ is the 1-form in $\mathbb{R}^n$ given by $Df_{V,\Omega}(p_{V,\Omega})$.

Clearly, a limit point of $\{p^*_{V,\Omega}\}$ is a supporting 1-form of $\mathcal{C}(\Omega)$ since it supports a properly convex domain $f_V^{-1}(1, \infty) \subset \mathcal{C}(\Omega)$. We say that $p_i$ converges to an element of $\partial^{Ag}\mathcal{O}$ if this augmented sequence converges to it.

**Theorem 2.17** Let $\Omega$ be a properly convex domain in $\mathbb{R}^n$ (resp. in $\mathbb{S}^n$), and let $\Omega^*$ be its dual in $\mathbb{R}^n$ (resp. in $\mathbb{S}^n$). Then the Vinberg duality diffeomorphism $\mathcal{D}_\mathcal{O} : \Omega \rightarrow \Omega^*$ extends to a homeomorphism $\mathcal{D}^{Ag}_{\mathcal{O}} : \mathcal{C}l^{Ag}(\mathcal{O}) \rightarrow \mathcal{C}l^{Ag}(\Omega)$. Moreover for any projective group $\Gamma$ acting on it, $\mathcal{D}^{Ag}_{\mathcal{O}}$ is equivariant with respect to the duality map $\Gamma \rightarrow \Gamma^*$ given by $g \mapsto g^{*^{-1}}$.

**Proof** We assume $\Omega \subset \mathbb{S}^n$. The continuity follows from the paragraph above the theorem since $\mathcal{D}_\mathcal{O}$ is induced by $(p, p^*_{V,\Omega}) \rightarrow (p^*_{V,\Omega}, p)$, and $\mathcal{D}^{Ag}_{\mathcal{O}}$ is a map switching the orders of the pairs also.

Proposition 2.19 shows the injectivity of $\mathcal{D}^{Ag}_{\mathcal{O}}$. The map is surjective since so is $\mathcal{D}_\mathcal{O}$ and $\mathcal{D}^{Ag}_{\mathcal{O}}$.

The equivariance follows since so are $\mathcal{D}_\mathcal{O}$ and $\mathcal{D}^{Ag}_{\mathcal{O}}$.  \[S^nT\]
2.5.3 Sweeping actions

The properly convex open set $D$ in $\mathbb{R}P^n$ (resp. $\mathbb{S}^n$) has a Hilbert metric. Also the group $\text{Aut}(K)$ of projective automorphisms of $K$ in $\text{SL}_+(n+1, \mathbb{R})$ is a locally compact closed group.

**Lemma 2.21** Let $D$ be a properly convex open domain in $\mathbb{R}P^n$ (resp. $\mathbb{S}^n$) with $\text{Aut}(D)$ of smooth projective automorphisms of $D$. Let a group $G$ act isometrically on an open domain $D$ faithfully with $G \to \text{Aut}(D)$ is an embedding. Suppose that $D/G$ is compact. Then the closure $\bar{G}$ of $G$ is a Lie subgroup acting on $D$ properly, and there exists a smooth Riemannian metric on $D$ that is $\bar{G}$-invariant.

**Proof** Assume $D \subset \mathbb{S}^n$. Since $\bar{G}$ is in $\text{SL}_+(n+1, \mathbb{R})$, the closure $\bar{G}$ is a Lie subgroup acting on $D$ properly. Suppose that $D \subset \mathbb{S}^n$.

One can construct a Riemannian metric $\mu$ with bounded entries. Let $\phi$ be a function supported on a compact set $F$ so that $G(F) \supset D$ where $\phi|F > 0$. Given a bounded subset of $\bar{G}$, the elements are in a bounded subset of the projective automorphism group $\text{SL}_+(n+1, \mathbb{R})$. A bounded subset of projective automorphisms have uniformly bounded set of derivatives on $\mathbb{S}^n$ up to the $m$-th order for any $m$. We can assume that the derivatives of the entries of $\phi \mu$ up to the $m$-th order are uniformly bounded above. Let $d\eta$ be the left-invariant measure on $\bar{G}$.

Then $\{g^* \phi \mu | g \in \bar{G}\}$ is an equicontinuous family on any compact subset of $D$ up to any order. For $J \subset D$, $\text{supp}(g^* \phi \mu) \cap J \neq \emptyset$ for $g$ in a compact set of $\bar{G}$. Thus the integral

$$\int_{g \in \bar{G}} g^* \phi \mu d\eta$$

of $g^* \phi \mu$ for $g \in \bar{G}$ is a $C^\infty$-Riemannian metric and that is positive definite. This bestows us a $C^\infty$-Riemannian metric $\mu_D$ on $D$ invariant under $\bar{G}$-action. 

By Lemma 2.21, there exists a Riemannian metric on a properly convex domain $\Omega$ invariant under $\text{Aut}(\Omega)$. Hence, we can define a frame bundle $\mathbb{F}\Omega$ where $\text{Aut}(\Omega)$ acts freely.

We generalize Proposition 2.15.

**Proposition 2.20** Suppose that a projective group $G$ acts on an $n$-dimensional properly convex open domain $\Omega$ in $\mathbb{S}^n \times \mathbb{R}P^n$ as a sweeping action. Then the following hold:

- Let $L$ be any subspace where $G$ acts on. Then $L \cap \text{Cl}(\Omega) \neq \emptyset$ but $L \cap \Omega = \emptyset$.
- If $G$ acts on a compact properly convex set, then it must meet $\text{Cl}(\Omega) \cup \mathcal{A}(\Omega)$.
- Suppose that $G$ is semi-simple. Then all the items up to the last one in the conclusion of Proposition 2.15 with $G$ replacing $\pi_1(\tilde{E})$ without discreteness hold. In particular, $\text{Cl}(\Omega) = K_1 \ast \cdots \ast K_0$ for properly convex sets $K_1, \ldots, K_0$.
- Suppose that $G$ is semi-simple. The closure of $G$ in $\text{Aut}(K)$ has a virtual center containing a group of diagonalizable projective automorphisms isomorphic to $\mathbb{Z}^{l_0-1}$ acting trivially on each $K_i$. 

\textbf{Proof} Assume \( \Omega \subset S^n \). Suppose that \( L \cap \text{Cl}(\Omega) = \emptyset \). Then there is a lower bound to the \( d \)-distance from \( \text{bd}\Omega \) to \( L \). Let \( x \in \Omega \). We denote the space of maximal open segments containing \( x \) and ending in \( L \) by \( L_{\Omega,x} \). This is a set homeomorphic to \( S^{\dim L} \).

Let \( l_+ \) denote the endpoint of \( L \cap l \) ahead of \( x \). Let \( l_{\Omega,0} \) denote the endpoint of \( L \cap \Omega \) ahead of \( x \), and \( l_{\Omega,1} \) denote the endpoint of \( L \cap \Omega \) after \( x \). We define a function \( f : \Omega \to [0, \infty) \) given by

\[
f(x) = \inf\{\log(l_+, l_{\Omega,0}, x, l_{\Omega,1}) | l \in L_{\Omega,x}\}
\]

where the logarithm measures the Hilbert distance between \( l_{\Omega,0} \) and \( x \) on the properly convex segment with endpoints \( l_+ \) and \( l_{\Omega,1} \). This is a continuous positive function. As \( x \to \text{bd}\Omega \), \( f(x) \to 0 \).

Since \( f(g(x)) = f(x) \) for all \( x \in \Omega \) and \( g \in G \), \( f \) induces a continuous map \( \bar{f} : \Omega / G \to (0, \infty) \). Here, \( \bar{f} \) can take as close value to 0 as one wishes. This contradicts the compactness of \( \Omega / G \).

Suppose that \( L \cap \Omega \neq \emptyset \). Then \( G \) acts on the convex domain \( L \cap \Omega \) open in \( L \). We define a function \( f : \Omega \to [0, \infty) \) given by measuring the Hilbert distance from \( L \cap \Omega \). Then \( f(x) \to \infty \) as \( x \to \text{bd}\Omega - L \). Again \( f(g(x)) = f(x) \) for all \( x \in \Omega, g \in G \). This induces \( \bar{f} : \Omega / G \to [0, \infty) \). Since \( \bar{f} \) can take as large value as one wishes for, this contradicts the compactness of \( \Omega / G \).

For the second item, suppose that such a set \( K \) exists. \( K \) and \( \mathcal{A}(K) \) are disjoint from \( \text{Cl}(\Omega) \). For \( x \in \Omega \), we define \( K_{\Omega,x} \) to be the space of oriented open segments containing \( x \) and ending in \( K \) and \( \mathcal{A}(K) \). We define \( l_+ \) to be the first point of \( K \cap l \) ahead of \( x \). Then a similar argument to the above proof applies and we obtain a contradiction.

Now, we go to the third item. Let \( G \) have a \( G \)-invariant decomposition \( \mathbb{R}^n = V_1 \oplus \cdots \oplus V_{l_0} \) where \( G \) acts irreducibly. This item follows by Lemma 2 of [161] since any decomposition of \( \mathbb{R}^n \) gives rise to a diagonalizable commutant of rank \( l_0 \).

For the fourth item, we prove for the case when \( G \) has a \( G \)-invariant decomposition \( \mathbb{R}^n = V_1 \oplus V_2 \). Then by the second item, \( G \) acts on \( K = K_1 \ast K_2 \) for properly convex domain \( K_i \subset S(V_i) \) for \( i = 1, 2 \). \( G \) acts cocompactly on \( K_0 \) and \( G \) is a subgroup of \( G_1 \times G_2 \times \mathbb{R}^+ \) where \( G_i \) is isomorphic to \( G / K_i \) extended to act trivially on \( K_{i+1} \) with \( G_i |_{V_{i+1}} = 1 \) where the indices in mod 2.

The closure \( \bar{G} \) of \( G \) in \( \text{Aut}(K) \) is a subgroup of \( \bar{G}_1 \times \bar{G}_2 \times \mathbb{R}^+ \) for the closure \( \bar{G}_i \) of \( G_i \) in \( \text{Aut}(K_i) \) for \( i = 1, 2 \).

\[
\bar{G} \subset \{(g_1, g_2, r) | g_i \in \bar{G}_i, i = 1, 2, r \in \mathbb{R}^+_+\}.
\]

For a fixed pair \((g_1, g_2)\), if there are more than one associated \( r \), then we obtain by taking differences that \((1, 1, r)\) is in the group \( \bar{G} \) for \( r \neq 1 \). This implies that \( \bar{G} \) contains a nontrivial subgroup of \( \mathbb{R}^+_+ \).

Otherwise, \( \bar{G} \) is in a graph of homomorphism \( \lambda : \bar{G}_1 \times \bar{G}_2 \to \mathbb{R}^+_+ \). An orbit of an action of this on the manifold \( \mathbb{F}K_i^0 \times \mathbb{F}K_i^0 \times \mathbb{R}^+ \) is in the orbit of the image of \( \lambda \). Hence, each orbit of a compact set meets \((y_1, y_2) \times (0, 1)\) for \( y_i \in \mathbb{F}K_i^0, i = 1, 2 \), at a compact set. Thus, we do not have a cocompact action.
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Furthermore, if we have a $G$-invariant decomposition $K_1 \ast \cdots \ast K_m$, we can use the decomposition $K_1 \ast (K_2 \ast \cdots \ast K_m)$. Now, we use the induction, to obtain the result. [S\textsuperscript{T}]

**Proposition 2.21 (Lemma 1 of Vey [161])** Suppose that a projective group $G$ acts on an $(n-1)$-dimensional properly convex open domain $\Omega$ as a sweeping action. Then the dual group $G^*$ acts on $\Omega^*$ as a sweeping action also.

**Proof** The Vinberg duality map in Theorem 2.16 is a diffeomorphism $\Omega \to \Omega^*$. This map is equivariant under the duality homomorphism $g \mapsto g^*$ for each $g \in G$. Here, $G$ does not need to be a dividing action. \hfill \square

### 2.5.4 Extended duality

We can generalize the duality for convex domains as was done at the beginning of Section 2.5; however, we don’t generalize for $\mathbb{R}P^n$. Given a closed convex cone $C_1$ in $\mathbb{R}^{n+1}_+$, consider the set of linear functionals in $\mathbb{R}^{n+1}_+$ taking nonnegative values in $C_1$. This forms a closed convex cone. We call this a dual cone of $C_1$ and denote it by $C_1^\ast$.

A closed cone $C_2$ in $\mathbb{R}^{n+1}_+$ is *dual* to a closed convex cone $C_1$ in $\mathbb{R}^{n+1}_+$ if $C_2$ is the closure of the set of linear functionals taking nonnegative values in $C_1$.

For a convex compact set $U$ in $S^n$, we form a corresponding convex cone $C(U)$. Then we form $C(U)^\ast$ and the image of its projection a convex compact set $U^*$ in $S^n$. Clearly, $(U^*)^\ast = U$ for a compact convex set $U$ by definition.

Also, the definition agrees with the previous definition defined for properly convex domains. This is straightforward: Functions in $C(U)^\ast$ can be approximated arbitrarily by functions strictly positive on $C(U)$.

Recall the classification of compact convex sets in Proposition 2.5.

Let $S^{i_0} \subset S^n$ be a pair of complementary subspaces of $S^n$. We denote by $S^{i_0} \subset S^n$ to be a great sphere of dimension $i_0$ corresponding to a subspace of linear functionals taking zero values in vectors in directions of $S^{n-i_0-1}$ in $S^n$. Hence, $S^{i_0} = S^{n-i_0-1} \subset S^n$ by definition. We can consider it the dual of $S^{i_0}$ in $S^n$ independent of $S^{n-i_0-1}$. Also, $S^{n-i_0-1} \subset S^n$ be defined similarly.

A *proper-subspace dual* $K^\dagger$ with respect to $S^{i_0}$ of a properly convex domain $K$ in $S^n$ is the dual domain as obtained from considering $S^{i_0}$ and corresponding vector subspace only.

**Proposition 2.22**

- Let $S^{i_0}$ be a great sphere of dimension $i_0$.
- Let $S^{j_0}$ be a one of dimension $j_0$ with $i_0 + j_0 + 1 \leq n$ independent of $S^{i_0}$.
- We also have the join $S^{i_0+j_0+1}$ of $S^{i_0}$ and $S^{j_0}$ and its complementary subspace $S^{n-i_0-j_0-2}$.
Let $\mathbb{S}^{n-i_0-1}$ be one of dimension $n - i_0 - 2$ complementary to $\mathbb{S}^0$ where $\mathbb{S}^{n-i_0-j_0-2} \subset \mathbb{S}^{n-i_0-1}$.

- We define $\mathbb{S}^0$ as $\mathbb{S}^{n-i_0-1}$ and $\mathbb{S}^{n-i_0-j_0-2}$ as $\mathbb{S}^{0}$. 

Let $U$ be a convex compact proper set in $\mathbb{S}^0$. Then the following hold:

(i) $U$ is a great $i_0$-sphere if and only if $U^*$ is a great $n - i_0 - 1$-sphere. $U^*$ is not convex if and only if $i_0 = n - 1$.

(ii) If $U$ is a join of a properly convex domain $K$ of dimension $i_0$ in a great sphere $\mathbb{S}^0$ and a complementary great sphere $\mathbb{S}^{0*}$ of dimension $i_0$ properly dual to $K$ in $\mathbb{S}^0$ if $i_0 + j_0 + 1 < n$.

(iii) If $U$ is a properly convex compact $n$-dimensional domain if and only if so is $U^*$.

(iv) If $U$ is not properly convex and has a nonempty interior, then $U^*$ has an empty interior.

(vi) In particular, if $U$ is a hemispherical domain in $\mathbb{S}^0$, then $U^*$ is a point and vice versa.

(vii) If $U_0 \neq \emptyset$ and $U^* \neq \emptyset$, then $U$ and $U^*$ are properly convex domains in $\mathbb{S}^n$.

(xi) $U$ is contained in a hemisphere if and only if $U^*$ is contained in a hemisphere.

**Proof**

(i) Suppose that $U$ is a great $i_0$-sphere. Then $C(U)$ is a subspace of dimension $i_0 + 1$. The set of linear functionals taking 0 values on $C(U)$ form a subspace of dimension $n - i_0$. Hence, $U^* = S(C(U))$ is a great sphere of dimension $n - i_0 - 1$. The converse is also true.

(ii) Suppose that $U$ is not a great sphere. Proposition 2.5 shows us that $U$ is contained in an $n$-hemispherical.

Let $\mathbb{S}^{m_0}$ be the span of $U$. Here, $m_0 = i_0 + j_0 + 1$. Then $U = \mathbb{S}^{m_0} \ast K^{m_0}$ for a great sphere $\mathbb{S}^{m_0}$ and a properly convex domain $K^{m_0}$ in a great sphere of dimension $i_0$ in $\mathbb{S}^{m_0}$.

$C(U)$ is a closed cone in the vector subspace $\mathbb{R}^{m_0+1}$. Then $C(U) = \mathbb{R}^{m_0+1} + C(K^{m_0})$ where $C(K^{m_0}) \subset \mathbb{R}^{m_0+1}$ for independent subspaces $\mathbb{R}^{j_0+1}$ and $\mathbb{R}^{j_0+1}$. Let $C(U)'$ denote the dual of $C(U)$ in $\mathbb{R}^{m_0+1}$. For $f \in C(U)'$, $f = 0$ on $\mathbb{R}^{i_0+1}$, and $f|_{\mathbb{R}^{i_0+1}}$ takes a value $\geq 0$ in $C(K^{m_0})$. Hence,

$$f : \mathbb{R}^{m_0+1} = \mathbb{R}^{j_0+1} \oplus \mathbb{R}^{i_0+1} \rightarrow \mathbb{R} \text{ is in } \{0\} \oplus C(K^{m_0}).$$

Denote the projection of $C(U)'$ in $\mathbb{S}^{m_0}$ by $U'$.

Suppose $m_0 = n$. Then we showed the second case of (ii).

Suppose $m_0 < n$. Then decompose $\mathbb{R}^{i_0+1} = \mathbb{R}^{n-m_0} \oplus \mathbb{R}^{m_0+1}$. We obtain that $f \in C(U)'$ is a sum $f_1 + f_2$ where $f_1$ is an element of $C(U)'$ extended by setting
2.5 The Vinberg duality of real projective orbifolds

$f_1|\mathbb{R}^{n-m_0} \oplus \{O\} = 0$ and $f_2$ is any linear functional satisfying $f_2|\{O\} \oplus \mathbb{R}^{m_0+1} = 0$ where we indicate by $\{O\}$ the trivial subspaces of the complements. Hence, $\langle f_2 \rangle \in \mathbb{S}^{m_0} = \mathbb{S}^{n-i_0-j_0-2\dagger}$. Hence, $U^*$ is a strict join of $U'$ and a great sphere $\mathbb{S}^{n-i_0-j_0-2\dagger}$.

(iii) This is obtained by taking the dual of the second case of (ii).

(iv) Since the definition agrees with classical one for properly convex domains, this follows. Also, one can derive this as contrapositive of (ii) and (iii) since domains and their duals not covered by (ii) and (iii) are the properly convex domains.

(v) $U$ is as in the second case of (ii).

(vi) If $U$ has the empty interior, then $U$ is covered by (ii) and (iii) or is a great sphere of dimension $< n$. (iii) corresponds to the case when the dual of $K$ has nonempty interior.

(vii) The forward part is given by (iii) where $K$ is a singleton in $\mathbb{S}^0$ and $i_0 = 0$. The converse part is the second case of (ii) where $K$ is of dimension zero and $j_0 = n - 1$.

(viii) The item (v) shows this using $(U^*)^* = U$.

(x) Proposition 2.5 shows that (ii), (iii), and (iv) cover all compact convex sets that are not great spheres.

□

We also note that for any properly convex domain $K$, $K \subset H^k \subset \mathbb{S}^k$ for a open hemisphere $H_k$, and a great sphere $\mathbb{S}^j$ in an independent space, the interior of $K \ast \mathbb{S}^j \subset H^k \ast \mathbb{S}^j$ is in an affine space $H^{k+j+1} = H^k \times H^{j+1} \subset \mathbb{S}^n$. Hence, a join is really a product of a certain form. We call this an affine form of a strict join.

2.5.5 Duality and geometric limits

Define the thickness of a properly convex domain $\Delta$ is given as

$$\min\{\max\{d(x, bd\Delta)|x \in \Delta\}, \max\{d(y, bd\Delta^*)|y \in \Delta^*\}\}$$

for the dual $\Delta^*$ of $\Delta$.

![Fig. 2.1 The diagram for Lemma 2.22.](image)
Lemma 2.22 Let $\Delta$ be a properly convex open (resp. compact) domain in $\mathbb{R}P^n$ (resp. $S^n$) and its dual $\Delta^*$ in $\mathbb{R}P^n*$ (resp. $S^n*$). Let $\varepsilon$ be a positive number less than the thickness of $\Delta$ and less than $\frac{1}{2}d(\Delta', \mathbb{A}(\Delta'))$ and $\frac{1}{2}d(\Delta^*, \mathbb{A}(\Delta^*))$ for a lift $\Delta'$ of $\Delta$ to $S^n$ (resp. $\Delta' = \Delta$). Then the following hold:

- $N_\varepsilon(\Delta) \subset (\Delta^* - \text{Cl}(N_\varepsilon(\text{bd}\Delta^*)))^*$.
- If two properly convex open domains $\Delta_1$ and $\Delta_2$ are of Hausdorff distance $< \varepsilon$ for $\varepsilon$ less than the thickness of each $\Delta_1$ and $\Delta_2$, then $\Delta_1^*$ and $\Delta_2^*$ are of Hausdorff distance $< \varepsilon$.
- Furthermore, if $\Delta_2 \subset N_\varepsilon(\Delta_1)$ and $\Delta_1 \subset N_\varepsilon(\Delta_2)$ for $0 < \varepsilon' < \varepsilon$, then we have $\Delta_2^* \supset \Delta_1^* - \text{Cl}(N_\varepsilon(\text{bd}\Delta_1^*))$ and $\Delta_1^* \supset \Delta_2^* - \text{Cl}(N_\varepsilon(\text{bd}\Delta_2^*))$.

Proof Using the double covering map $p_{2n} : S^n \rightarrow \mathbb{R}P^n$ of unit spheres in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1}$, we take components of $\Delta$ and $\Delta^*$. It is easy to show that the result for properly convex open domains in $S^n$ and $S^n*$ is sufficient.

For elements $\phi \in S^n$, and $x \in S^n$, we say $\phi(x) < 0$ if $f(\phi) < 0$ for $\phi = [f], x = [v]$ for $f \in \mathbb{R}^{n+1}, v \in \mathbb{R}^{n+1}$. Also, we say $\phi(x) > 0$ if $f(\phi) > 0$ for $\phi = [f], x = [v]$ for $f \in \mathbb{R}^{n+1}, v \in \mathbb{R}^{n+1}$.

For the first item, let $y \in N_\varepsilon(\Delta)$. Suppose that $\phi(y) < 0$ for $\phi \in \text{Cl}((\Delta^* - \text{Cl}(N_\varepsilon(\text{bd}\Delta^*))) \neq \emptyset$.

Since $\phi \in \Delta^*$, the set of positive valued points of $S^n$ under $\phi$ is an open hemisphere $H$ containing $\Delta$ but not containing $y$. The boundary $\text{bd}H$ of $H$ has a closest point $z \in \text{bd}\Delta$ of distance $< \varepsilon$. The closest point $z'$ to $z$ on $\text{bd}H$ is in $N_\varepsilon(\Delta)$ since $y$ is in $N_\varepsilon(\Delta) - H$ and $z'$ is closest to $\text{bd}\Delta$. The great circle $S^1$ containing $z$ and $z'$ are perpendicular to $\text{bd}H$ since $\overline{zz'}$ is minimizing lengths. Hence $S^1$ passes the center of the hemisphere. One can push the center of the hemisphere on $S^1$ until it becomes a sharply supporting hemisphere to $\Delta$. The corresponding $\phi'$ is in $\Delta^*$ and the distance between $\phi$ and $\phi'$ is less than $\varepsilon$. This is a contradiction. Thus, the first item holds (See Figure 2.1.)

For the final item, we have that $\Delta_2 \subset N_\varepsilon(\Delta_1), \Delta_1 \subset N_\varepsilon(\Delta_2)$ for $0 < \varepsilon' < \varepsilon$.

Hence, $\Delta_2 \subset (\Delta_1^* - \text{Cl}(N_\varepsilon(\text{bd}\Delta_1^*)))^*$, and $\Delta_2^* \supset \Delta_1^* - \text{Cl}(N_\varepsilon(\text{bd}\Delta_1^*))$ by (2.26), which proves the third item where we need to switch 1 and 2 also. We obtain $N_\varepsilon(\Delta_1^*) \supset \Delta_1^*$ and conversely. The second item follows.

The following may not hold for $\mathbb{R}P^n$:

Proposition 2.23 Suppose that $\{K_i\}$ is a sequence of properly convex domains in $S^n$ geometrically converging to a compact convex set $K$. Then $\{K_i^*\}$ geometrically converges to the compact convex set $K^*$ dual to $K$.

Proof Recall the compact metric space of all compact subsets of $S^n$ with the Hausdorff metric $d_H$. (See p.280-281 of Munkres [147].) $K_i$ is a Cauchy sequence under the Hausdorff metric $d_H$. By Lemma 2.22, $K_i^*$ is also a Cauchy sequence under the
hausdorff metric of $d_H$ of $\mathbb{S}^n$. The hausdorff metric of the space of all compact subsets of $\mathbb{S}^n$ is a compact metric space.

Since each $K_i$ is contained in an $n$-hemisphere corresponding to the linear functional $\phi_i$ with $\phi_i(C(K_i)) \geq 0$, we deduce that $K$ is contained in an $n$-hemisphere.

Let $K_\omega$ denote the limit of the Cauchy sequence $\{K_i\}$. We will show $K^\omega = K^\star$.

First, we show $K^\omega \subset K^\star$. Let $\phi_\omega$ be a limit of a sequence $\phi_i$ for $\phi_i \in C(K_i)$ for each $i$. By Proposition 2.1, it will be sufficient to show $\langle \phi_\omega \rangle \in K^\star$ for every such $\phi_\omega$.

We may assume that their euclidean norms are 1 always with the standard euclidean metric on $\mathbb{R}^n$. We will show that $\phi_\omega(C(K)) \geq 0$.

Let $S^n_1$ denote the unit sphere in $\mathbb{R}^{n+1}$ with a fubini-study path-metric $d_1$. The projection $\mathbb{S}^n \to \mathbb{S}^n_1$ is an isometry from $d$ to $d_1$. Then $C(K_i) \cap \mathbb{S}^n_1 \to C(K) \cap \mathbb{S}^n_1$ geometrically under the hausdorff metric $d_{H,1}$ associated with $d_1$. Let $N_\epsilon(U)$ denote the $\epsilon$-neighborhood of a subset $U$ of $\mathbb{S}^n_1$ under $d_1$. Since $K_i \to K$, we find a sequence $\epsilon_i$ so that $N_{\epsilon_i}(C(K_i)) \cap \mathbb{S}^n_1 \supset C(K) \cap \mathbb{S}^n_1$ and $\epsilon_i \to 0$ as $i \to \infty$.

For any $\phi$ in $\mathbb{R}^{n+1}$ of unit norm, for every pair of points $x, y \in S^n_1$ with $\phi(x) \geq 0$,

\[ d_1(x, y) \leq \delta \implies \phi(y) \geq -\delta : \quad (2.28) \]

This follows by integrating along the geodesic from $x$ to $y$ considering $\phi$ as a 1-form of norm 1.

Since $\min\{\phi_i|N_\epsilon(C(K_i)) \cap \mathbb{S}^n_1\} \geq -\epsilon_i$ by (2.28), we obtain $\phi_i|C(K) \cap \mathbb{S}^n_1 \geq -\epsilon_i$ for sufficiently large $i$. Since $\epsilon_i \to 0$, we obtain $\phi_\omega|C(K) \cap \mathbb{S}^n_1 \leq 0$, and $\phi_\omega \in C(K)^\star$.

Conversely, we show $K^\star \subset K^\omega$. Let $\phi \in C(K)^\star$. Then $\phi|C(K) \cap \mathbb{S}^n_1 \leq 0$. Define $\epsilon_i = \min\{\phi(C(K_i) \cap \mathbb{S}^n_1)\}$. If $\epsilon_i \geq 0$ for sufficiently large $i$, then $\phi \in C(K_i)^\star$ for sufficiently large $i$ and $\langle \phi \rangle \in K^\omega$ by Proposition 2.1, and we are finished in this case.

Suppose $\epsilon_i < 0$ for infinitely many $i$. By taking a subsequence if necessary, we assume that $\epsilon_i < 0$ for all $i$. Let $H_\phi, H_\phi \subset \mathbb{S}^n$, be the hemisphere determined by the nonnegative condition of $\phi$. Then $K_i - H_\phi \neq \emptyset$ for every $i$. Choose a point $y_i$ in $K_i$ of the maximal distance from $H_\phi$. Then $d_1(y_i, H_\phi) \leq \delta_i$ for $0 < \delta_i \leq \pi/2$. Since $\{K_i\} \to K$, we deduce $\{\delta_i\} \to 0$ as $i \to \infty$ obviously. Assume $\delta_i < \pi/4$ without loss of generality.

Define a distance function $f_1(\cdot) := d_1(\cdot, H_\phi) : \mathbb{S}^n \to \mathbb{R}_+$. Then $y_i$ is contained in a smooth sphere $S_\delta$ at the level $\delta_i$ with a fixed center $x_\phi$. Also, $K_i$ is contained in the complement of the convex open ball $B_\delta$ bounded by $S_\delta$.

Now, we will use convex affine geometry. Let $H_i$ denote the hemisphere whose boundary contains $y_i$ and is tangent to $S_\delta$ and disjoint from $B_\delta$. Then $y_i$ is a unique maximum point of $f_1|K_i$ since otherwise we will have a point with smaller $f_1$ by the convexity of $K_i$ and $B_\delta$. And $K_i$ is disjoint from $B_\delta$ as $y_i$ is the unique maximum point. Since $K_i$ and $\text{Cl}(B_\delta)$ are both convex and meets only at $y_i$, $\partial H_i$ supports $K_i$ and $\text{Cl}(B_\delta)$ by the hyperplane separation theorem applied to $C(K_i)$ and $C(\text{Cl}(B_\delta))$.

We obtained $K_i \subset H_i$. Let $\phi_i$ be the linear functional of unit norm corresponding to $H_i$. Then $\phi_i|C(K_i) \geq 0$. Let $s_i$ be the shortest segment from $y_i$ to $\partial H_i$ with the other endpoint $x_i \in \partial H_i$. The center $\mathcal{A}(x_\phi)$ of $H_\phi$ is on the great circle $\hat{s}_i$ containing $s_i$. The center of $H_i$ is on $\hat{s}_i$ and of distance $\delta_i$ from $\mathcal{A}(x_\phi)$ since $d_1(y_i, x_i) = \delta_i$. 


This implies that \( d(\phi, \phi_i) = \delta_i \). Since \( \delta_i \to 0 \), we obtain \( \{ \phi_i \} \to \phi \) and \( K^* \subset K^\infty \) by Proposition 2.1. \( \square \)
Chapter 3
Examples of properly convex real projective orbifolds with ends: cusp openings

We give examples where our theory applies to. We explain the theory of convex projective structures on Coxeter orbifolds and the orderability theory for Coxeter orbifolds. Our work jointly done with Gye-Seon Lee and Craig Hodgson generalizing the work of Benoist and Vinberg will be discussed. We also explain the vertex orderable Coxeter orbifolds. We state the work of Heusner-Porti on projective deformations of the hyperbolic link complement and the subsequent work by Ballas. Also, we state some nice results on finite volume convex real projective structures by Cooper-Long-Tillmann and Crampon-Marquis on horospherical ends and thick and thin decomposition. We also present the nicest cases where the Ehresmann-Thurston-Weil principle applies in the simplest way. For this, we need the principal results from Parts 2 and 3. Examples are orbifolds admitting complete hyperbolic structures with end fundamental groups finitely generated by torsion elements. Finally, we give computations of a specific case. As stated in the premise, this chapter will use the results of the whole monograph freely.

3.1 History of examples

Originally, Vinberg [163] investigated convex real projective Coxeter orbifolds as linear groups acting on convex cones. The groundbreaking work also produced many examples of real projective orbifolds and manifolds suitable to our study. For example, see Kac and Vinberg [113] for the deformation of triangle groups. However, the work was reduced to studying some Cartan forms with rank equal to $n + 1$ for $n = \dim \mathcal{O}$. The method turns out to be a bit hard in computing actual examples.

Later, Benoist [24] worked out some examples on prisms. Generalizing this, Choi [55] studied the orderability of Coxeter orbifolds after conversing with Kapovich about the deformability. This produced many examples of noncompact orbifolds with properly convex projective structures by the work of Vinberg. Later, Marquis [135] generalized the technique to study the convex real projective structures based
3 Examples of properly convex real projective orbifolds

on Coxeter orbifolds with truncation polytopes as base spaces. These are compact orbifolds, and so we will not mention these.

For compact hyperbolic 3-manifolds, Cooper-Long-Thistlethwaite [70] and [71] produced many examples with deformations using numerical methods. Some of these are exact computations.

We now discuss the noncompact strongly tame orbifolds with convex real projective structures.

Also, Choi, Hodgson, and Lee [63] computed the deformation spaces of convex real projective structures of some complete hyperbolic Coxeter orbifolds with or without ideal vertices, and Choi and Lee [65] showed that all compact hyperbolic weakly orderable Coxeter orbifolds have the local deformation spaces of dimension $e_+ - 3$ where $e_+$ is the number of ridges with order $\geq 2$. These Coxeter orbifolds form a large class of Coxeter orbifolds.

We can generalize these to complete hyperbolic Coxeter orbifolds that are weakly orderable with respect to ideal vertices. Lee, Marquis, and I will prove in later papers related ideal-vertex-orderable Coxeter 3-orbifolds have smooth deformation spaces of computable dimension.

For noncompact hyperbolic 3-manifolds, Porti and Tillmann [150], Cooper-Long-Tillmann [73], and Crampon-Marquis [74] made theories where the ends were restricted to be horospherical. Ballas [5] and [6] made initial studies of deformations of complete hyperbolic 3-manifolds to convex real projective ones. Cooper, Long, and Tillmann [72] have produced a deformation theory for convex real projective manifolds parallel to ours with different types of restrictions on ends, such as requiring the end holonomy group to be abelian. They also concentrate on the openness of the deformation spaces. We will provide our theory in Part 3.

3.2 Examples and computations

We will give some series of examples due to the author and many other people. Here, we won’t give compact examples since we already gave a survey in Choi-Lee-Marquis [67].

Given a polytope $P$, a face is a codimension-one side of $P$. A ridge is the codimension-two side of $P$. When $P$ is 3-dimensional, a ridge is called an edge.

We will concentrate on $n$-dimensional orbifolds whose base spaces are homeomorphic to convex Euclidean polyhedrons and whose faces are silvered and each ridge is given an order. For example, a hyperbolic polyhedron with edge angles of form $\pi/m$ for positive integers $m$ will have a natural orbifold structure like this.

**Definition 3.1** A Coxeter group $\Gamma$ is an abstract group defined by a group presentation of form

$$(R_i; (R_iR_j)^{n_{ij}}), i, j \in I$$

where $I$ is a countable index set, $n_{ij} \in \mathbb{N}$ is symmetric for $i, j$ and $n_{ii} = 1$, $n_{ij} \geq 2$ for $i \neq j$. 
The fundamental group of the orbifold will be a Coxeter group with a presentation
\[ R_i, i = 1, 2, \ldots, f, (R_i R_j)^{n_{ij}} = 1 \]
where \( R_i \) is associated with silvered sides and \( R_i R_j \) has order \( n_{ij} \) associated with the edge formed by the intersection of the \( i \)-th and \( j \)-th sides.

Let us consider only the 3-dimensional orbifolds for now. Let \( P \) be a fixed convex 3-polyhedron. Let us assign orders at each edge. We let \( e \) be the number of edges and \( e_2 \) be the numbers of edges of order-two. Let \( f \) be the number of sides.

For any vertex of \( P \), we will remove the vertex unless the link in \( P \) form a spherical Coxeter 2-orbifold of codimension 1. This make \( P \) into a 3-dimensional orbifold.

Let \( \hat{P} \) denote the differentiable orbifold with sides silvered and the edge orders realized as assigned from \( P \) with the above vertices removed. We say that \( \hat{P} \) has a Coxeter orbifold structure.

In this chapter, we will exclude a cone-type Coxeter orbifold whose polyhedron has a side \( F \) and a vertex \( v \) where all other sides are adjacent triangles to \( F \) and contains \( v \) and all ridge orders of \( F \) are 2. Another type we will not study is a product-type Coxeter orbifold whose polyhedron is topologically a polygon times an interval and ridge orders of top and the bottom sides are all 2. These are essentially lower-dimensional orbifolds. Finally, we will not study Coxeter orbifolds with finite fundamental groups. If \( \hat{P} \) is none of the above type, then \( \hat{P} \) is said to be a normal-type Coxeter orbifold.

A huge class of examples are obtainable from complete hyperbolic 3-polytopes with dihedral angles that are submultiples of \( \pi \). (See Andreev [3] and Roerder [151].)

**Definition 3.2** The deformation space \( \mathcal{D}(\hat{P}) \) of projective structures on a Coxeter orbifold \( \hat{P} \) is the space of all projective structures on \( \hat{P} \) quotient by isotopy group actions of \( \hat{P} \).

This definition was also used in a number of papers [55], [64], and [63]. The topology on \( \mathcal{D}(\hat{P}) \) is given by as follows: \( \mathcal{D}(\hat{P}) \) is a quotient space of the space of the development pairs \( (\text{dev}, h) \) with the compact open \( C^r \)-topology, \( r \geq 2 \), for the maps \( \text{dev} : \tilde{P} \to \mathbb{RP}^n \).

We will explain that the space is identical with \( \text{CDef}_E(\hat{P}) \) in Proposition 9.4. Also, \( \text{CDef}_E(\hat{P}) = \text{CDef}_{E,u,lh}(\hat{P}) \) by Corollary 3.3.

A point \( p \) of \( \mathcal{D}(\hat{P}) \) gives a fundamental polyhedron \( P \) in \( \mathbb{RP}^3 \), well-defined up to projective automorphisms. By Proposition 9.4, \( \mathcal{D}(\hat{P}) \) can be identified with \( \text{CDef}_E(\hat{P}) \). We concentrate on the space of \( p \in \mathcal{D}(\hat{P}) \) giving a fundamental polyhedron \( P \) fixed up to projective automorphisms. This space is called the restricted deformation space of \( \hat{P} \) and denoted by \( \mathcal{D}_P(\hat{P}) \). A point \( t \) in \( \mathcal{D}_P(\hat{P}) \) is said to be hyperbolic if a hyperbolic structure on \( \hat{P} \) induces the projective structure; that is, it is projectively diffeomorphic to \( \mathbb{B} / \Gamma \) for a standard unit ball \( \mathbb{B} \) and a discrete group \( \Gamma \subset \text{Aut}(\mathbb{B}) \). A point \( p \) of \( \mathcal{D}(\hat{P}) \) always determines a fundamental polyhedron \( P \) up to projective automorphisms because \( p \) determines reflections corresponding to sides up to conjugations also. We wish to understand the space where the funda-
mental polyhedron is always projectively equivalent to \( P \). We call this the *restricted deformation space* of \( \hat{P} \) and denote it by \( \mathcal{D}_P(\hat{P}) \).

The work of Vinberg [163] implies that each element of \( \mathcal{D}(\hat{P}) \) gives a convex projective structure (see Theorem 2 of [55]). That is, the image of the developing map of the orbifold universal cover of \( \hat{P} \) is projectively diffeomorphic to a convex domain in \( \mathbb{RP}^3 \), and the holonomy is a discrete faithful representation.

Now, we state the key property in this chapter:

**Definition 3.3** We say that \( P \) is *orderable* if we can order the sides of \( P \) so that each side meets sides of higher index in less than or equal to 3 edges.

A pyramid with a complete hyperbolic structure and dihedral angles that are submultiples of \( \pi \) is an obvious example. See Proposition 4 of [55] worked out with J. R. Kim.

An example is a drum-shaped convex polyhedron which has top and bottom sides of the same polygonal type and each vertex of the bottom side is connected to two vertices on the top side and vice versa. Another example will be a convex polyhedron where the union of triangles separates each pair of the interiors of nontriangular sides. In these examples, since nontriangular sides are all separated by the union of triangular sides, the sides are either level 0 or level 1, and hence they satisfy the trivalent condition. A dodecahedron would not satisfy the condition.

If \( P \) is compact, then Marquis [135] showed that \( P \) is a truncation polytope; that is, one starts from a tetrahedron and cut a neighborhood of a vertex so as to change the combinatorial type near that vertex only. Many of these can be realized as a compact hyperbolic polytope with dihedral angle submultiples of \( \pi \) by the Andreev theorem [151]. If \( P \) is not compact, we do not have the classification. Also, infinitely many of these can be realized as a complete hyperbolic polytope with dihedral angles that are submultiples of \( \pi \). (D. Choudhury was first to show this.)

**Definition 3.4** We denote by \( k(P) \) the dimension of the projective group acting on a convex polyhedron \( P \).

The dimension \( k(P) \) of the subgroup of \( G \) acting on \( P \) equals 3 if \( P \) is a tetrahedron and equals 1 if \( P \) is a cone with base a convex polyhedron which is not a triangle. Otherwise, \( k(P) = 0 \).

**Theorem 3.1** Let \( P \) be a convex polyhedron and \( \hat{P} \) be given a normal-type Coxeter orbifold structure. Let \( k(P) \) be the dimension of the group of projective automorphisms acting on \( P \). Suppose that \( \hat{P} \) is orderable. Then the restricted deformation space of projective structures on the orbifold \( \hat{P} \) is a smooth manifold of dimension \( 3f - e - e_2 - k(P) \) if it is not empty.

If we start from a complete hyperbolic polytope with dihedral angles that are submultiples of \( \pi \), we know that the restricted deformation space is not empty.

If we assume that \( P \) is compact, then we refer to Marquis [135] for the complete theory. However, the topic is not within the scope of this monograph.
3.2 Examples and computations

Definition 3.5 Let $P$ be a 3-dimensional hyperbolic Coxeter polyhedron, and let $\hat{P}$ denote its Coxeter orbifold structure. Suppose that $t$ is the corresponding hyperbolic point of $\mathcal{D}_P(\hat{P})$. We call a neighborhood of $t$ in $\mathcal{D}_P(\hat{P})$ the local restricted deformation space of $P$. We say that $\hat{P}$ is projectively deformable relative to the mirrors, or simply deforms rel mirrors, if the dimension of its local restricted deformation space is positive. Conversely, we say that $\hat{P}$ is projectively rigid relative to the mirrors, or rigid rel mirrors, if the dimension of its local restricted deformation space is 0.

The following theorem describes the local restricted deformation space for a class of Coxeter orbifolds arising from ideal hyperbolic polyhedra, i.e. polyhedra with all vertices on the sphere at infinity.

Theorem 3.2 (Choi-Hodgson-Lee [63]) Let $P$ be an ideal 3-dimensional hyperbolic polyhedron whose dihedral angles are all equal to $\pi/3$, and suppose that $\hat{P}$ is given its Coxeter orbifold structure. If $P$ is not a tetrahedron, then a neighborhood of the hyperbolic point in $\mathcal{D}_P(\hat{P})$ is a smooth 6-dimensional manifold.

The main ideas in the proof of Theorem 3.2 are as follows. We first show that $\mathcal{D}_P(\hat{P})$ is isomorphic to the solution set of a system of polynomial equations following ideas of Vinberg [163] and Choi [55]. Since the faces of $P$ are fixed, each projective reflection in a face of the polyhedron is determined by a reflection vector $b_i$. We then compute the Jacobian matrix of the equations for the $b_i$ at the hyperbolic point. This reveals that the matrix has exactly the same rank as the Jacobian matrix of the equations for the Lorentzian unit normals of a hyperbolic polyhedron with the given dihedral angles. By the infinitesimal rigidity of the hyperbolic structure on $\hat{P}$, this matrix is of full rank and has the kernel of dimension six; the result then follows from the implicit function theorem. In fact, we can interpret the infinitesimal projective deformations as applying infinitesimal hyperbolic isometries to the reflection vectors.

We can generalize the above theorem slightly as Hodgson pointed out.

Definition 3.6 Given a hyperbolic $n$-orbifold $X$ with totally geodesic boundary component diffeomorphic to an $(n-1)$-orbifold $\Sigma$. Let $\hat{X}$ denote the universal cover in the Klein model $\mathbb{B}$ in $\mathbb{S}^n$. Let $\Gamma$ be the group of deck transformations considered as projective automorphisms of $\mathbb{S}^n$. Then a complete hyperbolic hyperspace $\hat{\Sigma}$ covers $\Sigma$. Every component of the inverse image of $\Sigma$ is of form $g(\hat{\Sigma})$ for $g \in \pi_1(X)$. A point $v_\Sigma \in \mathbb{S}^n - \mathbb{B} - \partial f(\mathbb{B})$ is projectively dual to the hyperspace containing $\Sigma$ with respect to the bilinear form $B$. (See Section 1.3.4.) Then we form the join $C := \{v_\Sigma\} + \Sigma - \{v_\Sigma\}$. Then we form $\hat{C} := X \cup \bigcup_{g \in \Gamma} g(C)$. $\hat{C}/\Gamma$ is an $n$-orbifold with radial ends. We call the ends the hyperideal ends.

A point of $\mathcal{D}_P(\hat{P})$ corresponding to a hyperbolic $n$-orbifold with hyperideal ends added will be called a hyperbolic point again. An 3-dimensional hyperbolic polyhedron with possibly hyperideal vertices is a compact convex polyhedron with vertices outside $\mathbb{B}$ removed where no interior of a 1-dimensional edge is outside $\mathbb{B}$. We will generalize this further in Section 4.1.1.
Corollary 3.1 (Choi-Hodgson-Lee) Let $P$ be an ideal 3-dimensional hyperbolic polyhedron with possibly hyperideal vertices whose dihedral angles are of form $\pi/p$ for integers $p \geq 3$, and suppose that $\hat{P}$ is given its Coxeter orbifold structure. If $P$ is not a tetrahedron, then a neighborhood of the hyperbolic point in $D_{\mathcal{P}}(\hat{P})$ is a smooth 6-dimensional manifold.

We did not give proof for the case when some edges orders are greater than equal to 4 in the article [63]. We can allow any of our end orbifold to be a $(p,q,r)$-triangle reflection orbifold for $p,q,r \geq 3$. The same proof will apply as first observed by Hodgson: We modify the proof of Theorem 1 of the article in Section 3.3 of [63]. Let $\partial_{\infty} \hat{P}$ denote the union of end orbifolds of $\hat{P}$ which are orbifolds based on 2-sphere with singularities admitting either a Euclidean or hyperbolic structures. Let $h : \pi_1(\hat{P}) \to \text{PO}(3,1) \subset \text{PGL}(4,\mathbb{R})$ denote the holonomy homomorphism associated with the convex real projective structure induced from the hyperbolic structure. We just need to show

$$H^1(\hat{P}, so(3,1)_{Adh}) = 0, \quad H^1(\partial_{\infty} P, so(3,1)_{Adh}) = 0.$$ 

Recall that a $(p,q,r)$-triangle reflection orbifold for $1/p + 1/q + 1/r < 1$ has a rigid hyperbolic and conformal structure. By Corollary 2 of [156], the representation to $\text{PO}(3,1)$ is rigid. The first part of the equation follows. The second part also follows by Corollary 2 of [156]. These examples are convex by the work of Vinberg [163]. Corollary 3.2 implies the proper convexity.

We comment that we are using Theorem 7 (Sullivan rigidity) of [156] as the generalization of the Garland-Raghunathan-Weil rigidity [87] [164].

3.2.1 Vertex orderable Coxeter orbifolds

3.2.1.1 Vinberg theory

Let $\hat{P}$ be a Coxeter orbifold of dimension $n$. Let $P$ be the fundamental convex polytope of $\hat{P}$. The reflection is given by a point, called a reflection point, and a hyperplane. Let $R_i$ be a projective reflection on a hyperspace $S_i$ containing a side of $P$. Then we can write

$$R_i := 1 - \alpha_i \otimes v_i$$

where $\alpha_i$ is zero on $S_i$ and $v_i$ is the reflection vector and $\alpha_i(v_i) = 2$.

Given a reflection group $\Gamma$, we form a Cartan matrix $A(\Gamma)$ given by $a_{ij} := \alpha_i(v_j)$. Vinberg [163] proved that the following conditions are necessary and sufficient for $\Gamma$ to be a linear Coxeter group:

(C1) $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$.
(C2) $a_{ii} = 2$; and
(C3) for $i \neq j$, $a_{ij}a_{ji} \geq 4$ or $a_{ij}a_{ji} = 4\cos^2(\frac{\pi}{n_{ij}})$ an integer $n_{ij}$. 
The Cartan matrix is a $f \times f$-matrix when $P$ has $f$ sides. Also, $a_{ij} = a_{ji}$ for all $i, j$ if $\Gamma$ is conjugate to a reflection group in $O^+(1, n)$. This condition is the condition of $\hat{P}$ to be a hyperbolic Coxeter orbifold.

The Cartan matrix is determined only up to an action of the group $D_{f,f}$ of non-singular diagonal matrices:

$$A(\Gamma) \to DA(\Gamma)D^{-1} \text{ for } D \in D_{f,f}.$$  

This is due to the ambiguity of choices

$$\alpha_i \mapsto c_i \alpha_i, v_i \mapsto \frac{1}{c_i} v_i, c_i > 0.$$  

Vinberg showed that the set of all cyclic invariants of form $a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_k j_k}$ classifies the isomorphic linear Coxeter group generated by reflections up to the conjugation.

### 3.2.1.2 The classification of convex real projective structures on triangular reflection orbifolds

We will follow Kac-Vinberg [113]. Let $\hat{T}$ be a 2-dimensional Coxeter orbifold based on a triangle $T$. Let the edges of $T$ be silvered. Let the vertices be given orders $p, q, r$  where $1/p + 1/q + 1/r \leq 1$. If $1/p + 1/q + 1/r < 1$, then the universal cover $\hat{T}$ of $T$ is a properly convex domain or a complete affine plane by Vinberg [163]. We can find the topology of $\mathfrak{D}(\hat{T})$ as Goldman did in his senior thesis [90]. We may put $T$ as a standard triangle with vertices $e_1 := [1, 0, 0], e_2 := [0, 1, 0], e_3 := [0, 0, 1]$.

Let $R_i$ be the reflection on a line containing $[e_{i-1}], [e_{i+1}]$ and with a reflection vertex $[v_i]$. Let $\alpha_i$ denote the linear function on $\mathbb{R}^3$ taking zero values on $e_{i-1}$ and $e_{i+1}$. We choose $v_i$ to satisfy $\alpha_i(v_i) = 2$.

When $1/p + 1/q + 1/r = 1$, the triangular orbifold admits a compatible Euclidean structure. When $1/p + 1/q + 1/r < 1$, the triangular orbifold admits a hyperbolic structure not necessarily compatible with the real projective structure.

A linear Coxeter group $\Gamma$ is hyperbolic if and only if the Cartan matrix $A$ of $\Gamma$ is indecomposable, of negative type, and equivalent to a symmetric matrix of signature $(1, n)$.

Assume that no $p, q, r$ is 2 and $1/p + 1/q + 1/r < 1$. Let $a_{ij}$ denote the entries of the Cartan matrix. It satisfies

$$a_{12} a_{21} = 4 \cos^2 2 \pi / p, a_{23} a_{32} = 4 \cos^2 2 \pi / q, a_{13} a_{31} = 4 \cos^2 2 \pi / r.$$  

There are only two cyclic invariants $a_{12} a_{23} a_{31}$ and $a_{13} a_{32} a_{21}$ satisfying

$$a_{12} a_{23} a_{31} a_{13} a_{32} a_{21} = 64 \cos^2 2 \pi / p \cos^2 2 \pi / q \cos^2 2 \pi / r.$$  

Then the triple invariant $a_{12} a_{23} a_{31} \in \mathbb{R}^+$ classifies the conjugacy classes of $\Gamma$. A single point of $\mathbb{R}^+$ corresponds to a hyperbolic structure. For different points, they are properly convex by [61].
Since \( a_{ij} = a_{ji} \) for geometric cases, we obtain that

\[
a_{12}a_{23}a_{31} = 2^3 \cos(\pi/p) \cos(\pi/q) \cos(\pi/r)
\]
gives the unique hyperbolic points.

We define for this orbifold \( \mathcal{D}(\hat{T}) := \mathbb{R}_+ \) the space of the triple invariants. A unique point correspond to a Euclidean or hyperbolic structure.

**Example 3.1 (Lee’s example)** Consider the Coxeter orbifold \( \hat{P} \) with the underlying space on a polyhedron \( P \) with the combinatorics of a cube with all sides mirrored and all edges given order 3 but with vertices removed. By the Mostow-Prasad rigidity and the Andreev theorem, the orbifold has a unique complete hyperbolic structure. There exists a six-dimensional space of real projective structures on it by Theorem 3.2 where one has a projectively fixed fundamental domain in the universal cover.

There are eight ideal vertices of \( P \) corresponding to eight ends of \( \hat{P} \). Each end orbifold is a 2-orbifold based on a triangle with edges mirrored, and vertex orders are all 3. Each end orbifold has a real projective structure and hence is characterized by the triple invariant. Thus, each end has a neighborhood diffeomorphic to the 2-orbifold multiplied by \((0,1)\). The eight triple invariants are related when we are working on the restricted deformation space since the deformation space is only six-dimensional.

### 3.2.1.3 The end mappings

We will give some explicit conjectural class of examples where we can control the end structures. We worked this out with Greene, Gye-Seon Lee, and Marquis starting from the workshop at the ICERM in 2014.

**Definition 3.7** Let \( P \) be a simple 3-polytope given a natural number \( \geq 2 \) on each edge. Let \( V \) be a set of ideal vertices chosen ahead so that \( P - V \) has an orbifold structure \( \hat{P} \). Let the faces of \( \hat{P} \) be given an ordering. Each face has at most one vertex in \( V \). Each vertex in \( V \) has three edges ending there and the edges have order 3 only. For each face \( F_i \), let \( N_i \) denote the number of edges of order 2 or in the faces of higher ordering. Then \( \hat{P} \) is \( V \)-orderable if

- \( N_i \leq 1 \) for any face \( F_i \) containing a vertex in \( V \) and
- \( N_i \leq 3 \) for face \( F_i \) containing no vertex in \( V \).

Let \( \partial V \) denote the disjoint union of end orbifolds corresponding to the set of ideal vertices \( V \).

**Conjecture 3.1 (Choi-Greene-Lee-Marquis [62])** Suppose that \( P \) with a set of vertices \( V \) is \( V \)-orderable, and \( P \) admits a Coxeter orbifold structure with a convex real projective structure. Then the function \( \mathcal{D}(\partial V) \to \mathcal{D}(\partial V) \) is onto.

We also make a generalization to “weakly \( V \)-orderable orbifolds”, where we let \( N_i \) be redefined as the the number of edges of order 2 and in the faces of higher ordering.
3.3 Some relevant results

Conjecture 3.2.1 (Choi-Greene-Lee-Marquis [62]) Suppose that $P$ with a set of vertices $V$ is weakly $V$-orderable. Suppose $P$ admits a Coxeter orbifold structure with a complete hyperbolic structure. Then the function $\mathcal{D}(\mathcal{O}) \to \mathcal{D}(\partial V \mathcal{O})$ is locally surjective at the hyperbolic point.

3.3 Some relevant results

For closed hyperbolic manifolds, the deformation spaces of convex structures on manifolds were extensively studied by Cooper-Long-Thistlethwaite [70] and [71].

3.3.1 The work of Heusener-Porti

Definition 3.8 Let $N$ be a closed hyperbolic manifold of dimension equal to 3. We consider the holonomy representation of $N$

$$\rho : \pi_1(N) \to \text{PSO}(3,1) \hookrightarrow \text{PGL}(4,\mathbb{R}).$$

A closed hyperbolic three manifold $N$ is called *infinitesimally projectively rigid* if

$$H^1(\pi_1(N), \mathfrak{sl}(4,\mathbb{R})_{\text{Ad} \rho}) = 0.$$

Definition 3.9 Let $M$ denote a compact three-manifold with boundary a union of tori and whose interior is hyperbolic with finite volume. $M$ is called *infinitesimally projectively rigid relative to the cusps* if the inclusion $\partial M \to M$ induces an injective homomorphism

$$H^1(\pi_1(M), \mathfrak{sl}(4,\mathbb{R})_{\text{Ad} \rho}) \to H^1(\partial M, \mathfrak{sl}(4,\mathbb{R})_{\text{Ad} \rho}).$$

Theorem 3.3 (Heusener-Porti [105]) Let $M$ be an orientable 3-manifold whose interior has a complete hyperbolic metric with finite volume. If $M$ is infinitesimally projectively rigid relative to the cusps, then infinitely many Dehn fillings on $M$ are infinitesimally projectively rigid.

Theorem 3.4 (Heusener-Porti [105]) Let $M$ be an orientable 3-manifold whose interior has a complete hyperbolic metric of finite volume. If a hyperbolic Dehn filling $N$ on $M$ satisfies:

(i) $N$ is infinitesimally projectively rigid,

(ii) the Dehn filling slope of $N$ is contained in the (connected) hyperbolic Dehn filling space of $M$,

then infinitely many Dehn fillings on $M$ are infinitesimally projectively rigid.
The complete hyperbolic manifold $M$ that is the complement of a figure-eight knot in $S^3$ is infinitesimally projectively rigid. Then infinitely many Dehn fillings on $M$ are infinitesimally projectively rigid.

They showed the following:

- For a sufficiently large positive integer $k$, the homology sphere obtained by $\frac{1}{k}$-Dehn filling on the figure eight knot is infinitesimally not projectively rigid. Since the Fibonacci manifold $M_k$ is a branched cover of $S^3$ over the figure eight knot complements, for any $k \in \mathbb{N}$, the Fibonacci manifold $M_k$ is not projectively rigid.
- All but finitely many punctured torus bundles with tunnel number one are infinitesimally projectively rigid relative to the cusps. All but finitely many twist knots complements are infinitesimally projectively rigid relative to the cusps.

3.3.2 Ballas’s work on ends.

The following are from Ballas [5] and [6].

- Let $M$ be the complement in $S^3$ of $4_1$ (the figure-eight knot), $5_2$, $6_1$, or $5_2^*$ (the Whitehead link). Then $M$ does not admit strictly convex deformations of its complete hyperbolic structure.
- Let $M$ be the complement of a hyperbolic amphichiral knot, and suppose that $M$ is infinitesimally projectively rigid relative to the boundary and the longitude is a rigid slope. Then for sufficiently large $n$, there is a one-dimensional family of strictly convex deformations of the complete hyperbolic structure on $M(m/0)$ for $m \in \mathbb{Z}$.
- Let $M$ be the complement in $S^3$ of the figure-eight knot. There exists $\varepsilon$ such that for each $s \in (\varepsilon, \varepsilon)$, $\rho_s$ is the holonomy of a finite volume properly convex projective structure on $M$ for a parameter $\rho_s$ of representations $\pi_1(M) \to \text{PGL}(4, \mathbb{R})$. Furthermore, when $s \neq 0$, this structure is not strictly convex.

We also note the excellent work of Ballas, Danciger, and Lee [8] experimenting with more of these and finding a method to glue along tori for deformed hyperbolic 3-manifolds to produce convex real projective 3-manifolds that does not admit hyperbolic structures.

3.3.3 Finite volume strictly convex real projective orbifolds with ends

We summarize the main results of two independent groups. The Hilbert metric is a complete Finsler metric on a properly convex set $\Omega$. This is the hyperbolic metric in the Klein model when $\Omega$ is projectively diffeomorphic to a standard ball. A simplex with its Hilbert metric is isometric to a normed vector space, and appears in a natural geometry on the Lie algebra $\text{sl}(n, \mathbb{R})$. A singular version of this metric arises in the
study of certain limits of projective structures. The Hilbert metric has a Hausdorff
measure and hence a notion of finite volume. (See [73].)

**Theorem 3.5** (Choi [50], Cooper-Long-Tillmann [73], Crampon-Marquis [74])
For each dimension \( n \geq 2 \) there is a Margulis constant \( \mu_n > 0 \) with the following
property. If \( M \) is a properly convex projective \( n \)-manifold and \( x \) is a point in \( M \),
then the subgroup of \( \pi_1(M, x) \) generated by loops based at \( x \) of length less than \( \mu_n \)
is virtually nilpotent. In fact, there is a nilpotent subgroup of index bounded above
by \( m = m(n) \). Furthermore, if \( M \) is strictly convex and finite volume, this nilpotent
subgroup is abelian. If \( M \) is strictly convex and closed, this nilpotent subgroup is
trivial or infinite cyclic.

**Theorem 3.6** (Cooper-Long-Tillmann [73], Crampon-Marquis [74]) Each end
of a strictly convex projective manifold or orbifold of finite volume is horospherical.

**Theorem 3.7** ((Relatively hyperbolic). Cooper-Long-Tillmann [73], Crampon-Marquis [74]) Suppose that \( M = \Omega / \Gamma \) is a properly convex manifold of finite volume which is the interior of a compact manifold \( N \), and the holonomy of each component of \( \partial N \) is topologically parabolic. Then the following are equivalent:

1. \( \Omega \) is strictly convex,
2. \( \partial \Omega \) is \( C^1 \),
3. \( \pi_1(N) \) is hyperbolic relative to the subgroups of the boundary components.

Here, the definition of the term “topologically parabolic” is according to [73]. This
is not a Lie group definition but a topological definition. We have found a general-
ization Theorem 10.3 and its converse Theorem 10.5 in Chapter 10.

### 3.4 Nicest cases

We will now present the cases when the theory presented in this monograph works best.

**Definition 3.10** A countable group \( G \) satisfies the property the central normal-nilpotent-subgroup condition (NS) if every normal solvable subgroup \( N \) of a finite-index subgroup \( G' \) is virtually central in \( G' \); that is, \( N \cap G'' \) is central in \( G'' \) for a finite-index subgroup \( G'' \) of \( G' \).

By Corollary 2.2, the fundamental group of a closed orbifold admitting a properly
convex structure has the property (NS). Clearly a virtually abelian group satisfies
(NS). Obviously, the groups of Benoist are somewhat related to this condition. (see
Proposition 2.14.)

**Example 3.2** Let \( M \) be a complete hyperbolic 3-orbifold and each end orbifold has
a sphere or a disk as the base space. The end fundamental group is generated by a
finite order elements. By Lemma 3.1, a properly convex real projective structure on
\( M \) has lens-shaped or horospherical radial ends only.
We need the end classification results from Chapters 4, 6, and 8 to prove the following. Let \( g \in \pi_1(O) \). Using the choice of representing matrix of \( g \) as in Remark 1.1, we let \( \lambda_\nu(g) \) denote the eigenvalue of holonomy of \( g \) associated with the vector in direction of \( x \) if \( x \) is a fixed point of \( g \).

The holonomy group of \( \pi_1(O) \) can be lifted to \( SL_+(n+1, \mathbb{R}) \) so that \( \lambda_\nu(g) = 1 \) for the holonomy of every \( g \in \pi_1(O) \) where \( \nu_E \) is a \( p \)-end vertex of a \( p \)-end \( E \) corresponding to \( E \). Then we say that \( E \) or \( \tilde{E} \) satisfies the unit-middle-eigenvalue condition with respect to \( \nu_E \) or the \( p \)-end structure.

Suppose that \( E \) is a \( T \)-end. If the hyperspace containing the ideal boundary component \( \tilde{S}E \) of \( p \)-end \( \tilde{E} \) of \( E \) corresponds to 1 as the eigenvalue of the dual of the holonomy of every \( g \in \pi_1(\tilde{E}) \), then we say we say that \( E \) or \( \tilde{E} \) satisfies the unit middle eigenvalue condition with respect to \( \tilde{S}E \) or the \( T \)-p-end structure.

**Lemma 3.1** Suppose that \( O \) is a strongly tame convex real projective orbifold with radial ends. Assume that the end fundamental group \( \pi_1(E) \) of an end \( E \) satisfies (NS). Let \( E \) be an \( R \)-end, or is a \( T \)-end. Suppose that one of the following holds:

- \( \pi_1(E) \) is virtually generated by finite order elements or is simple, or
- the end holonomy group of \( E \) satisfies the unit middle eigenvalue condition.

Then the following hold:

- the end \( E \) is either a properly convex generalized lens-shaped \( R \)-end or a lens-shaped \( T \)-end, or is horospherical.
- If the end \( E \) furthermore has a virtually abelian end holonomy group, then \( E \) is a lens-shaped \( R \)-end, a lens-shaped \( T \)-end, or is a horospherical end.

**Proof** We suppose that \( O \) is a convex domain in \( \mathbb{S}^n \). First, let \( E \) be an \( R \)-end. The map

\[
 g \in \Gamma_\tilde{E} \mapsto \lambda_\nu(g) \in \mathbb{R}_+
\]

is a homomorphism. Thus, \( \lambda_\nu(g) = 1 \) for \( g \in \Gamma_\tilde{E} \) since the end holonomy group is simple or virtually generated by the finite order elements.

Each \( R \)-end is either complete, properly convex, or is convex but not properly convex and not complete by Section 4.1.

Suppose that \( \tilde{E} \) is complete. Then Theorem 4.3 shows that either \( \tilde{E} \) is horospherical or each element \( g, \tilde{g} \in \pi_1(E) \) has at most two norms of eigenvalues where two norms for an element are realized. Since the multiplication of all eigenvalues equals 1, we obtain \( \lambda^{r+1-r}(g)\lambda_\nu(g)^r = 1 \) for some integer \( r, 1 \leq r \leq n \) and the other norm \( \lambda_i(g) \) of the eigenvalues. The second case cannot happen.

Suppose that \( \tilde{E} \) is properly convex. Then the uniform middle eigenvalue condition holds by Remark 6.1 since \( \lambda_\nu(g) = 1 \) for all \( g \). (See Definition 6.2.) By Theorem 6.2, \( \tilde{E} \) is of generalized lens-type.

Finally, Corollary 8.8 rules out the case when \( \tilde{E} \) is convex but not properly convex and not complete.

Now, let \( E \) be a \( T \)-end. By dualizing the above, \( E \) satisfies the uniform middle eigenvalue condition (see Definition 6.3). Theorem 6.9 implies the result. [\( \mathbb{S}^n \)]
Theorem 3.8 Suppose that $\mathcal{O}$ is a strongly tame properly convex real projective orbifold with R-ends or T-ends. Suppose that each end fundamental group satisfies property (NS) and is virtually generated by finite order elements, or is simple or satisfies the unit middle eigenvalue condition. Then the holonomy is in
\[ \text{Hom}_{E,u,lh}^s(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})). \]

Proof Suppose that $E$ is an R-end. Let $\tilde{E}$ be a p-end corresponding to $E$ and $v_{\tilde{E}}$ be the p-end vertex. By Lemma 3.1, we obtain the R-end is lens-type or horospherical.

We prove the uniqueness of the fixed point under $h(\pi_1(\tilde{E}))$: Suppose that $x$ is another fixed point of $h(\pi_1(\tilde{E}))$. Since $\pi_1(\tilde{E})$ is as in the premise, the eigenvalue $\lambda_x(g)$ for every $g \in \pi_1(\tilde{E})$ associated with $x$ is always 1. In the horospherical case $x = v_{\tilde{E}}$ since the cocompact lattice action on a cusp group fixes a unique point in $\mathbb{R}P^n$.

Now consider the lens case. The uniform middle eigenvalue condition with respect to $v_{\tilde{E}}$ and $x$ holds by Remark 6.1 since $\lambda_x(g) = 1$ for all $g$. Lemma 3.1 shows that $\pi_1(\tilde{E})$ acts on a lens-cone with vertex at $x$. Proposition 6.8 implies the uniqueness of the p-end vertex.

Suppose that $E$ is a T-end. The proof of Proposition 6.2 shows that the hyperspace containing $\tilde{S}_{\tilde{E}}$ corresponds to $v_{\tilde{E}}$ for the R-p-end $\tilde{E}$ corresponding to the dual of the T-p-end $\tilde{E}$ and vice versa. Hence, the result follows from the R-end part of the proof.

This was proved by Marquis in Theorem A of [138] when the orbifold is a Coxeter one.

Theorems 3.8, 1.7, and 1.2 imply the following:

Corollary 3.2 Let $\mathcal{O}$ be a noncompact strongly tame SPC $n$-dimensional orbifold with R-ends and T-ends and satisfies (IE) and (NA). Suppose that each end fundamental group is generated by finite order elements or is simple. Suppose each end fundamental group satisfies (NS). Assume $\partial \mathcal{O} = \emptyset$, and that the nilpotent normal subgroups of every finite-index subgroup of $\pi_1(\mathcal{O})$ are trivial. Then
\[ \text{CDef}_E(\mathcal{O}) = \text{CDef}_{E,u,lh}(\mathcal{O}) \]
and $\text{hol}$ maps the deformation space $\text{CDef}_E(\mathcal{O})$ of SPC-structures on $\mathcal{O}$ homeomorphic to a union of components of
\[ \text{rep}_{E,u,lh}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \]
which is also a union of components of
\[ \text{rep}_{E,u,lh}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})), \text{rep}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})). \]

The same can be said for
\[ \text{SDef}_E(\mathcal{O}) = \text{SDef}_{E,u,lh}(\mathcal{O}). \]
These type of deformations from structures with cusps to ones with lens-shaped ends are realized in our main examples as stated in Section 3.2. We need the restrictions on the target space since the convexity of $\mathcal{O}$ is not preserved under the hyperbolic Dehn surgery deformations of Thurston, as pointed out by Cooper at ICERM in September 2013.

Virtually abelian groups satisfy (NS) clearly. Since finite-volume hyperbolic $n$-orbifolds satisfy (IE) and (NA) (see P.151 of [134] for example), strongly tame properly convex orbifolds admitting complete hyperbolic structures end fundamental groups generated by finite order elements will satisfy the premise. Hence, $2h_{1,1}$ and the double of the simplex orbifold discussed in Section 3.5 do also.

Since Coxeter orbifolds satisfy the above properties, we obtain a simple case:

**Corollary 3.3** Let $\mathcal{O}$ be a strongly tame Coxeter $n$-dimensional orbifold, $n \geq 3$, with only $R$-ends, admitting a finite-volume complete hyperbolic structure. Then

$$S\text{Def}_{\mathcal{E},u,\text{lh}}(\mathcal{O})$$

is homeomorphic to a union of components of

$$\text{rep}_{\mathcal{E},u,\text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1,\mathbb{R}))$$

which is also a union of components of

$$\text{rep}_{\mathcal{E}}(\pi_1(\mathcal{O}), \text{PGL}(n+1,\mathbb{R}))$$

Finally,

$$S\text{Def}_{\mathcal{E},u,\text{lh}}(\mathcal{O}) = S\text{Def}_{\mathcal{E}}(\mathcal{O}).$$

**3.5 Two specific examples**

The example of S. Tillmann is an orbifold on a 3-sphere with singularity consisting of two unknotted circles linking each other only once under a projection to a 2-plane and a segment connecting the circles (looking like a linked handcuff) with vertices removed and all arcs given as local groups the cyclic groups of order three. (See Figure 1.3.) This is one of the simplest hyperbolic orbifolds in the list of Heard, Hodgson, Martelli, and Petronio [104] labeled $2h_{1,1}$. The orbifold admits a complete hyperbolic structure since we can start from a complete hyperbolic tetrahedron with four dihedral angles equal to $\pi/6$ and two equal to $2\pi/3$ at a pair of opposite edge $e_1$ and $e_2$. Then we glue two faces adjacent to $e_i$ by an isometry fixing $e_i$ for $i = 1, 2$. The end orbifolds are two 2-spheres with three cone points of orders equal to 3 respectively. These end orbifolds always have induced convex real projective structures in dimension 2, and real projective structures on them have to be convex. Each of these is either the quotient of a properly convex open triangle or a complete affine plane as we saw in Lemma 3.1.
3.5 Two specific examples

Porti and Tillmann [150] found a two-dimensional solution set from the complete hyperbolic structure by explicit computations. Their main questions are the preservation of convexity and realizability as convex real projective structures on the orbifold. Corollary 3.2 answers this since their deformation space identifies with $S\text{Def}_{\mathcal{E}, u, lh}(\mathcal{O}) = S\text{Def}_{\mathcal{E}}(\mathcal{O})$.

Another main example can be obtained by doubling a complete hyperbolic Coxeter orbifold based on a convex polytope. We take a double $D_T$ of the reflection orbifold based on a convex tetrahedron with orders all equal to 3. This also admits a complete hyperbolic structure since we can take the two tetrahedra to be the regular complete hyperbolic tetrahedra and glue them by hyperbolic isometries. The end orbifolds are four 2-spheres with three singular points of orders 3. Topologically, this is a 3-sphere with four points removed and six edges connecting them all given order 3 cyclic groups as local groups.

**Theorem 3.9** Let $\mathcal{O}$ denote the hyperbolic 3-orbifold $D_T$. We assign the $\mathcal{E}$-type to each end. Then $S\text{Def}_{\mathcal{E}}(\mathcal{O})$ equals $S\text{Def}_{\mathcal{E}, u, lh}(\mathcal{O})$ and hol maps $S\text{Def}_{\mathcal{E}}(\mathcal{O})$ as an onto-map to a component of characters

$$\text{rep}_{\mathcal{E}}(\pi_1(\mathcal{O}), \text{PGL}(4, \mathbb{R}))$$

containing a hyperbolic representation which is also a component of

$$\text{rep}_{\mathcal{E}, u, lh}(\pi_1(\mathcal{O}), \text{PGL}(4, \mathbb{R})).$$
In this case, the component is a cell of dimension 4.

Proof A solvable subgroup of $\text{PSO}(n, 1)$ fixes a point of the boundary of the Klein ball model $B$. Since $\pi_1(O)$ is not elementary, a finite-index subgroup of $\pi_1(O)$ has only trivial normal solvable subgroups. The end orbifolds have zero Euler characteristics, and all the singularities are of order 3. For each end $E$, $\pi_1(E)$ is virtually abelian. Hence, $\pi_1(E)$ satisfies (NS).

By Corollary 3.2, $\text{SDef}_{E}(O)$ equals $\text{SDef}_{E,u,\text{lb}}(O)$. Each of the ends has to be either horospherical or lens-shaped or totally geodesic radial type. Let $\partial_E O$ denote the union of end orbifolds of $O$ and for the other triangle corresponding to the link of $T$, $\pi_{n+1}$ is identified to a properly convex domain in $\mathbb{S}$, and $\text{SDef}_{E}(O)$ is the orbifold obtained from doubling a tetrahedron with edge orders $i = 1, 2, 3, 4$, corresponding to four ends, each of which gives us triangulations into two triangles. We can derive from the result of Goldman [94] and Choi-Goldman [60] that given projective invariants $\rho_{1}(O), \rho_{2}(O), \tau(O), \sigma_{1}(O), \sigma_{2}(O)$ for each of the two triangles satisfying $\rho_{1}(O) \rho_{2}(O) \rho_{3}(O) = \sigma_{1}(O) \sigma_{2}(O)$, we can determine the structure on $S_{3,3,3}^{(i)}$ for $i = 1, 2, 3, 4$ completely.

For each of these with a convex real projective structure and divided into two geodesic triangles, we compute respective invariants $\rho_{1}(i), \rho_{2}(i), \rho_{3}(i), \sigma_{1}(i), \sigma_{2}(i)$ for one of the triangles corresponding to the link of $T_1$: 

\[
s + s + s = t + 1, s + s = t + 1, s + s = t + 1,
t \left( s + s + 1 \right) = \frac{1}{t} \left( s + s + 1 \right) \left( s + s + 1 \right)
\] 

and for the other triangle corresponding to the link of $T_2$ the respective invariants are
3.5 Two specific examples

\[
\frac{1}{s_i^2} (s_i^2 + s_i \tau + 1), \quad \frac{1}{s_i^2} (s_i^2 + s_i \tau + 1), \quad \frac{1}{s_i^2} (s_i^2 + s_i \tau + 1), \quad \frac{1}{s_i^2} (s_i^2 + s_i \tau + 1),
\]

\[
\frac{1}{s_j^2} (s_j^2 + s_j \tau + 1), \quad \frac{1}{s_j^2} (s_j^2 + s_j \tau + 1) \left( s_i^2 + s_i \tau + 1 \right) \quad (3.2)
\]

where \( s_i, t_i, i = 1, 2, 3, 4 \), are Goldman parameters and \( \tau = 2 \cos 2\pi/3 \). (See [43].)

Since \( \partial E \) is a disjoint union of four spheres with singularities \((3, 3, 3)\), \( C\text{Def}(\partial E) \) is parameterized by \( s_i, t_i \) and hence is a cell of dimension 8. (This can be proved similarly to [61].)

The set \( J \) is given by projective invariants of the \((3, 3, 3)\) boundary orbifolds satisfying some equations. By the method of [53] developed by the author, we obtain the equations that \( J \) satisfies. These are

\[
s_i^2 + s_i \tau + 1 = s_j^2 + s_j \tau + 1, \quad i, j = 1, 2, 3, 4
\]

\[
\frac{1}{s_i^2} (s_i^2 + s_i \tau + 1) = \frac{1}{s_j^2} (s_j^2 + s_j \tau + 1), \quad i, j = 1, 2, 3, 4
\]

\[
t_1t_2t_3t_4 \prod_{i=1}^{4} (s_i^2 + s_i \tau + 1) = \frac{1}{t_1t_2t_3t_4} \prod_{i=1}^{4} (s_i^2 + s_i \tau + 1)^2
\]

\[
\prod_{i=1}^{4} \frac{t_i}{s_i} (s_i^2 + s_i \tau + 1) = \prod_{i=1}^{4} \frac{1}{s_i^2t_i} (s_i^2 + s_i \tau + 1)^2.
\]

The first and second lines of equations are from matching the cross ratios \( \rho^{(i)}_j \) with \( \rho^{(j)}_i \) for any pair \( i, j \) corresponding to an edge connecting the \( i \)-th vertex to the \( j \)-vertex in \( D_T \) and four faces containing the edge. (See (5) of [53]). The third and fourth lines of equations are from the equations matching the products \( \prod_{i=1}^{4} \sigma_1^{(i)} = \prod_{i=1}^{4} \sigma_2^{(i)} \) for \( T_1 \) and \( T_2 \) respectively.

The equation is solvable:

\[
s_1 = s_2 = s_3 = s_4 = s, t_1t_2t_3t_4 = C(s) \text{ for a constant } C(s) > 0 \text{ depending on } s.
\]

Thus \( J \) is contained in the solution subspace \( C \), a 4-dimensional cell in \( C\text{Def}(\partial E) \).

Conversely, given an element of \( C \), we can assign invariants at each edge of the tetrahedron and the Goldman \( \sigma \)-invariants at the vertices if the invariants satisfy the equations. This is given by starting from the first convex tetrahedron and gluing one by one using the projective invariants (see [53] and [47]): Let the first one by always be the standard tetrahedron with vertices

\[
[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], \text{ and } [0, 0, 0, 1]
\]

and we let \( T_2 \) a fixed adjacent tetrahedron with vertices

\[
[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0] \text{ and } [2, 2, 2, -1].
\]
Then projective invariants will determine all other tetrahedron triangulating $\tilde{\mathcal{O}}$. Given any deck transformation $\gamma$, $T_1$ and $\gamma(T_1)$ will be connected by a sequence of tetrahedrons related by adjacency, and their pasting maps are wholly determined by the projective invariants, where cross-ratios do not equal 0. Therefore, as long as the projective invariants are bounded, the holonomy transformations of the generators are bounded. Corollary 3.3 shows that these corresponds to elements of $\text{SDef}_{\tilde{\mathcal{O}}, u, lh}(\mathcal{O}) = \text{SDef}_{\tilde{\mathcal{O}}}(\mathcal{O})$. (This method was spoken about in our talk in Melbourne, May 18, 2009 [47].) Hence, we showed that $\text{SDef}_{\tilde{\mathcal{O}}, u, lh}(\mathcal{O})$ is parameterized by the solution set $C$. Thus $J = C$ since each element of $C$ gives us an element of $\text{SDef}_{\tilde{\mathcal{O}}}(\mathcal{O})$.

The dimension is one higher than that of the deformation space of the reflection 3-orbifold based on the tetrahedron. Thus we have examples not arising from reflection ones here as well. See the Mathematica files [44] for a different explicit method of solutions. Also, see [46] to see how to draw Figure 3.1.

We remark that the above theorem can be generalized to orders $\geq 3$ with hyperideal ends with similar computations. See [44] for examples to modify orders and so on.
Part II

The classification of radial and totally geodesic ends.
The purpose of this part is to understand the structures of ends of real projective $n$-dimensional orbifolds for $n \geq 2$. In particular, we consider the radial or totally geodesic ends. Hyperbolic manifolds with cusps and hyperideal ends are examples. For this, we will study the natural conditions on eigenvalues of holonomy representations of ends when these ends are manageably understandable. This is the most technical part of the monograph containing a large number of results useful in other two parts.

We begin the study of radial ends in Chapter 4. We will divide the class of radial ends into the class of complete affine radial ends, the class of properly convex ends, and the class of convex but not complete affine ends. We give some examples of radial ends. We will also classify the complete affine ends in this chapter.

In Chapter 5, we study the theory of affine actions. This is the major technical section in this part. We consider the case when there is a discrete affine action of a group $\Gamma$ acting cocompactly on a properly convex domain $\Omega$ in the boundary of the affine subspace $\mathbb{A}^n$ in $\mathbb{RP}^n$ or $\mathbb{S}^n$. We study the convex domain $U$ in an affine space $\mathbb{A}^n$ whose closure meets with $\text{bd}\mathbb{A}^n$ in $\Omega$. We can find a domain $U$ having asymptotic hyperspaces at each point of $\text{bd}\Omega$ if and only if $\Gamma$ satisfies the uniform middle eigenvalue condition with respect to $\text{bd}\mathbb{A}^n$. To prove, we study the flow on the affine bundle over the unit tangent space over $\Omega$ generalizing parts of the work of Goldman-Labourie-Margulis on complete flat Lorentz 3-manifolds [96]. We end with showing that a T-end has a CA-lens neighborhood if it satisfies the uniform middle eigenvalue condition.

In Chapter 6, we study the properly convex R-end theory. Tubular actions and the dual theory of affine actions are discussed. We show that distanced actions and asymptotically nice actions are dual. We explain that the uniform middle eigenvalue condition implies the existence of the distanced action. The main result here is the characterization of R-ends whose end holonomy groups satisfy uniform middle eigenvalue conditions. That is, they are generalized lens-shaped R-ends. We also discuss some important properties of lens-shaped R-ends. Finally, we show that lens-shaped T-ends and lens-shaped R-ends are dual. We end with discussing the properties of T-ends as obtained by this duality.

In Chapter 7, we investigate the applications of the radial end theory such as the stability condition. We discuss the expansion and shrinking of the end neighborhoods. Finally, we will prove Theorem 1.2, the strong irreducibility of strongly tame properly convex orbifolds with generalized-lens shaped ends or horospherical $\mathcal{A}$- or $\mathcal{I}$-ends.

In major technical Chapter 8, we discuss the R-ends that are NPNC. First, we show that the end holonomy group for an NPNC-end $E$ will have an exact sequence

$$1 \to N \to h(\pi_1(\tilde{E})) \to N_K \to 1$$

where $N_K$ is in the projective automorphism group $\text{Aut}(K)$ of a properly convex compact set $K$, $N$ is the normal subgroup of elements mapping to the trivial automorphism of $K$, and $K''/N_K$ is compact. We show that $\Sigma \tilde{K}$ is foliated by complete affine subspaces of dimension $\geq 1$. We explain that an NPNC-end satisfying
the transverse weak middle eigenvalue condition for NPNC-ends is a quasi-joined R-end under some natural conditions. A quasi-joined end is an end with an end-neighborhood covered by the join of a properly convex action and a horoball action twisted by translations (see Definition 8.1.)
Chapter 4
Introduction to the theory of convex radial ends: classifying complete affine ends

In Section 4.1, we will discuss the convex radial ends of orbifolds, covering most elementary aspects of the theory. For a properly convex real projective orbifold, the space of rays for each R-end gives us a closed real projective orbifold of dimension \( n - 1 \). The orbifold is convex. The universal cover can be a complete affine subspace (CA) or a properly convex domain (PC) or a convex domain that is neither (NPNC). We discuss objects associated with R-ends, and examples of ends; horospherical ones, totally geodesic ones, and bendings of ends to obtain more general examples of ends. In Section 4.2, we discuss complete affine ends. First, we explain the weak middle eigenvalue condition. We state the main result of the chapter Theorem 4.1, which characterizes the complete affine ends. We will prove it by the results in Chapter 8 and the results of subsequent sections: In Theorem 4.2, we show that pre-horospherical ends are horospherical ends. In Theorem 4.3, we show that a complete affine end falls into one of the two classes, one of which is a cuspidal and the other one is more complicated with two norms of eigenvalues. Theorem 4.1 will show that the second case is a quasi-join using results in Chapter 8. In Section 4.2.3, we concentrate on the ends with the holonomy of only unit norm eigenvalues showing that they have to be cuspidal.

4.1 R-Ends

In this section, we begin by explaining the R-ends. We classify them into three classes: complete affine ends, properly convex ends, and nonproperly convex and not complete affine ends. We also introduce T-ends.

Recall that an R-p-end \( \tilde{E} \) is convex if \( \tilde{\Sigma} \) is convex. Since \( \tilde{\Sigma} \) is a convex open domain and hence is contractible by Proposition 2.5, it always lifts to \( S^n \) as an embedding, a convex R-end is either

(i): complete affine (CA),
(ii): properly convex (PC), or
(iii): convex but not properly convex and not complete affine (NPNC).
4.1.1 Examples of ends

We will present some examples here, which we will fully justify later.

Recall the Klein model of hyperbolic geometry: It is a pair $(B, \text{Aut}(B))$ where $B$ is the interior of an ellipsoid in $\mathbb{R}P^n$ or $S^n$ and $\text{Aut}(B)$ is the group of projective automorphisms of $B$. Now, $B$ has a Hilbert metric which in this case is the hyperbolic metric times a constant. Then $\text{Aut}(B)$ is the group of isometries of $B$. (See Section 2.1.5.)

From hyperbolic manifolds, we obtain some examples of ends. Let $M$ be a complete hyperbolic manifold with cusps. $M$ is a quotient space of the interior $B$ of an ellipsoid in $\mathbb{R}P^n$ or $S^n$ under the action of a discrete subgroup $\Gamma$ of $\text{Aut}(B)$. Then some horoballs are $p$-end neighborhoods of the horospherical $R$-ends.

We generalize Definition 3.6. Suppose that a noncompact strongly tame convex real projective orbifold $M$ has totally geodesic embedded surfaces $S_1, \ldots, S_m$ homotopic to the ends. Let $M$ be covered by a properly convex domain $\tilde{M}$ in an affine subspace of $S^n$.

- We remove the outside of $S_j$s to obtain a properly convex real projective orbifold $M'$ with totally geodesic boundary. Suppose that each $S_j$ can be considered a lens-shaped $T$-end.
- Each $S_j$ corresponds to a disjoint union of totally geodesic domains $\bigcup_{j \in J} \tilde{S}_{i,j}$ in $\tilde{M}$ for a collection $J$. For each $\tilde{S}_{i,j} \subset \tilde{M}$, a group $\Gamma_{i,j}$ acts on it where $\tilde{S}_{i,j}/\Gamma_{i,j}$ is a closed orbifold projectively diffeomorphic to $S_i$.
- Suppose that $\Gamma_{i,j}$ fixes a point $p_{i,j}$ outside $\tilde{M}$.
- Hence, we form the cone $M_{i,j} := \{p_{i,j}\} \ast \tilde{S}_{i,j}$.
- We obtain the quotient $M_{i,j}/\Gamma_{i,j} - \{p_{i,j}\}$ and identify $\tilde{S}_{i,j}/\Gamma_{i,j}$ to $S_i,j$ in $M'$ to obtain the examples of real projective manifolds with $R$-ends.
- $\{\{p_{i,j}\}\ast \tilde{S}_{i,j}\}$ is an $R$-p-end neighborhood and the end is a totally geodesic $R$-end.

The result is convex by Lemma 10.1 since we can think of $S_j$ as an ideal boundary component of $M'$ and that of $M_{i,j}/\Gamma_{i,j} - \{p_{i,j}\}$. This orbifold is called the hyperideal extension of the convex real projective orbifold as a convex real projective orbifold. When $M$ is hyperbolic, each $S_j$ is lens-shaped by Proposition 4.1. Hence, the hyperideal extensions of hyperbolic orbifolds are properly convex.

We will fully generalize the following in Chapter 6. We remark that Proposition 4.1 also follows from Lemma 3.1. However, we used more elementary results to prove it here.

**Proposition 4.1** Suppose that $M$ is a strongly tame convex real projective orbifold. Let $\tilde{E}$ be an $R$-p-end of $M$. Suppose that

- the $p$-end holonomy group of $\pi_1(\tilde{E})$
4.1 R-Ends

is generated by the homotopy classes of finite orders or
is simple or
satisfies the unit middle eigenvalue condition

and

• \( \tilde{E} \) has a \( \pi_1(\tilde{E}) \)-invariant \( n-1 \)-dimensional totally geodesic properly convex domain \( D \) in a p-end neighborhood and not containing the p-end vertex in the closure of \( D \).

Then the R-p-end \( \tilde{E} \) is lens-shaped.

**Proof** Let \( \tilde{M} \) be the universal cover of \( M \) in \( S^n \). \( \tilde{E} \) is an R-p-end of \( \tilde{M} \), and \( \tilde{E} \) has a \( \pi_1(\tilde{E}) \)-invariant \( n-1 \)-dimensional totally geodesic properly convex domain \( D \). Since \( D \) projects to \( \tilde{\Sigma}_{\tilde{E}} \), \( D \) is transverse to radial lines from \( v_{\tilde{E}} \).

Under the first assumption, since the end holonomy group \( \Gamma_{\tilde{E}} \) is generated by elements of finite order, the eigenvalues of the generators corresponding to the p-end vertex \( v_{\tilde{E}} \) equal 1 and hence every element of the end holonomy group has 1 as the eigenvalue at \( v_{\tilde{E}} \).

Now assume that the the end holonomy groups fix the p-end vertices with eigenvalues equal to 1.

Then the p-end neighborhood \( U \) can be chosen to be the open cone over the totally geodesic domain with vertex \( v_{\tilde{E}} \). Now, \( U \) is projectively diffeomorphic to the interior of a properly convex cone in an affine subspace \( A^n \). The end holonomy group acts on \( U \) as a discrete linear group of determinant 1. The theory of convex cones applies, and using the level sets of the Koszul-Vinberg function, we obtain a one-sided convex neighborhood \( N \) in \( U \) with smooth boundary (see Lemmas 4.1.5 and 4.1.6 of Goldman [91]). Let \( F \) be a fundamental domain of \( N \) with a compact closure in \( \tilde{\Theta} \).

We obtain a one-sided neighborhood in the other side as follows: We take \( R(N) \) for by a reflection \( R \) fixing each point of the hyperspace containing \( \tilde{\Sigma} \) and the p-end vertex. Then we choose a diagonalizable transformation \( D \) fixing the p-end vertex and every point of \( \tilde{\Sigma} \) so that the image \( D \circ R(F) \) is in \( \tilde{\Theta} \). It follows that \( D \circ R(N) \subset \tilde{\Theta} \) as well. Thus, \( N \cup D \circ R(N) \) is the CA-lens we needed. The interior of the cone \( \{v_{\tilde{E}}\}*(N \cup D \circ R(N)) \) is the lens-cone neighborhood for \( \tilde{E} \).

A more specific example is below. Let \( S_{3,3,3} \) denote the 2-orbifold with base space homeomorphic to a 2-sphere and three cone-points of order 3. The 3-orbifolds satisfying the following properties are the example of Porti-Tillman [150] or the hyperbolic Coxeter 3-orbifolds based on an ideal 3-polytopes of dihedral angles \( \pi/3 \). (See Choi-Hodgson-Lee [63].)

The following is more specific version of Lemma 3.1. We give a much more elementary proof not depending on the full theory of this monograph.

**Proposition 4.2** Let \( \Theta \) be a strongly tame convex real projective 3-orbifold with R-ends where each end orbifold is diffeomorphic to a sphere \( S_{3,3,3} \) or a disk with three silvered edges and three corner-reflectors of orders 3,3,3. Assume that the holonomy group of \( \pi_1(\Theta) \) is strongly irreducible. Then the orbifold has only lens-shaped R-ends or horospherical R-ends.
Proof. Again, it is sufficient to prove this for the case $\hat{\sigma} \subset S^3$. Let $\tilde{E}$ be an R-end corresponding to an R-end whose end orbifold is diffeomorphic to $S_{3,3,3}$. It is sufficient to consider only $S_{3,3,3}$ since it double-covers the disk orbifold. Since $\Gamma_{\tilde{E}}$ is generated by finite order elements fixing a p-end vertex $v_{E}$, every holonomy element has the eigenvalue equal to 1 at $v_{E}$. Take a finite-index free abelian group $A$ of rank two in $\Gamma_{\tilde{E}}$. Since $\Sigma_{E}$ is convex, a convex projective torus $T^2$ covers $\Sigma_{E}$ finitely. Therefore, $\tilde{\Sigma}_{E}$ is projectively diffeomorphic either to

- a complete affine subspace or
- the interior of a properly convex triangle or
- a half-space

by the classification of convex tori by Nagano-Yagi [148] found in many places including [91] and [16] and Proposition 2.12. Since there exists a holonomy automorphism of order 3 fixing a point of $\tilde{\Sigma}_{\tilde{E}}$, it cannot be a quotient of a half-space with a distinguished foliation by lines. Thus, the end orbifold admits a complete affine structure or is a quotient of a properly convex triangle.

Suppose that $\tilde{\Sigma}_{E}$ has a complete affine structure. Since $\lambda_{v_{E}}(g) = 1$ for all $g \in \Gamma_{\tilde{E}}$, the only possibility from Theorem 4.3 is when $\Gamma_{\tilde{E}}$ is virtually nilpotent and we have a horospherical p-end for $\tilde{E}$.

Suppose that $\tilde{\Sigma}_{E}$ has a properly convex open triangle $T'$ as its universal cover. A acts with an element $g'$ with the largest eigenvalue $> 1$ and the smallest eigenvalue $< 1$ as a transformation in $\mathrm{SL}_+(4, \mathbb{R})$ the group of projective automorphisms of $\mathbb{S}^3_{\tilde{E}}$.

As an element of $\mathrm{SL}_+(4, \mathbb{R})$, we have $\lambda_{v_{E}}(g') = 1$ and the product of the remaining eigenvalues is 1, the corresponding the largest and smallest eigenvalues are $> 1$ and $< 1$. Thus, an element of $\mathrm{SL}_+(4, \mathbb{R})$, $g'$ fixes $v_1$ and $v_2$ other than $v_{E}$ in directions of vertices of $T'$. Since $\Gamma_{\tilde{E}}$ has an order three element exchanging the vertices of $T'$, there are three fixed points of an element of $A$ different from $v_{E}, v_{E} -$. By commutativity, there is a properly convex compact triangle $T \subset S^3$ with these three fixed points where $A$ acts on. Hence, $A$ is diagonalizable over the reals.

We can make any vertex of $T$ to be an attracting fixed point of an element of $A$. Each element $g \in \Gamma_{\tilde{E}}$ conjugates elements of $A$ to $A$. Therefore $g$ sends the attracting fixed points of elements of $A$ to those of elements of $A$. Hence $g(T) = T$ for all $g \in \Gamma_{\tilde{E}}$.

Each point of the edge $E$ of $\mathrm{Cl}(T)$ is an accumulation point of an orbit of $A$ by taking a sequence $g_i$, so that the sequence of the largest norm of eigenvalues $\lambda_1(g_i)$ and the sequence of second largest norm of the eigenvalue $\lambda_2(g_i)$ are going to $+\infty$ while the sequence $\log |\lambda_1(g_i)/\lambda_2(g_i)|$ is bounded. Since $\lambda_{v_{E}} = 1$, writing every vector as a linear combination of vectors in the direction of the four vectors, this follows. Hence $\mathrm{bd}T \subset \mathrm{bd}\hat{\sigma}$ and $T \subset \mathrm{Cl}(\sigma)$.

If $T^o \cap \mathrm{bd}\sigma \neq \emptyset$, then $T \subset \mathrm{bd}\sigma$ by Lemma 2.18. Then each segment from $v_{E}$ ending in $\mathrm{bd}\sigma$ has the direction in $\mathrm{Cl}(\Sigma_{E}) = T'$. It must end at a point of $T$. Hence, $\sigma = (T * v_{E})^o$, an open tetrahedron $\sigma$. Since the holonomy group acts on it, we can take a finite-index group fixing each vertex of $\sigma$. Thus, the holonomy group is virtually reducible. This is a contradiction.
Therefore, $T \subset \mathcal{O}$ as $T \cap \partial \mathcal{O} = \emptyset$. We have a totally geodesic R-end, and by Proposition 4.1, the end is lens-shaped. (See also [41].)

The following construction is called “bending” and was investigated by Johnson and Millson [112]. These give us examples of R-ends that are not totally geodesic R-ends. See Ballas and Marquis [9] for other examples.

**Example 4.1 (Bending)** Let $\mathcal{O}$ have the usual assumptions. We will concentrate on an end and not take into consideration of the rest of the orbifold. Certainly, the deformation given here may not extend to the rest. (If the totally geodesic hypersurface exists on the orbifold, the bending does extend to the rest.)

Suppose that $\mathcal{O}$ is an oriented hyperbolic manifold with a hyperideal end $E$. Then $E$ is a totally geodesic R-end with an R-p-end $\tilde{E}$. Let the associated orbifold $\Sigma_E$ for $E$ of $\mathcal{O}$ be a closed 2-orbifold and let $c$ be a two-sided simple closed geodesic in $\Sigma_E$. Suppose that $E$ has an open end neighborhood $U$ in $\mathcal{O}$ diffeomorphic to $\Sigma_E \times (0,1)$ with totally geodesic boundary $\partial U \cap \partial \mathcal{O}$ diffeomorphic to $\Sigma_E$. Let $\tilde{U}$ be a p-end neighborhood in $\mathcal{O}$ corresponding to $\tilde{E}$ bounded by $\Sigma_E$ covering $E$. Then $U$ has a radial foliation whose leaves lift to radial lines in $\tilde{U}$ from $v_E$.

Let $A$ be an annulus in $U$ diffeomorphic to $c \times (0,1)$, foliated by leaves of the radial foliation of $U$. Now a lift $\tilde{c}$ of $c$ is in an embedded disk $A'$, covering $A$. Let $g_c$ be the deck transformation corresponding to $\tilde{c}$ and $c$. Suppose that $g_c$ is orientation-preserving. Since $g_c$ is a hyperbolic isometry of the Klein model, the holonomy $g_c$ is conjugate to a diagonal matrix with entries $\lambda, \lambda^{-1}, 1, 1$, where $\lambda > 1$ and the last 1 corresponds to the vertex $v_E$. We take an element $k_b$ of $\text{SL}_+(4,\mathbb{R})$ of form in this system of coordinates

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & b & 1
\end{pmatrix}
$$

where $b \in \mathbb{R}$, $k_b$ commutes with $g_c$. Let us just work on the end $E$. We can “bend” $E$ by $k_b$.

Now, $k_b$ induces a diffeomorphism $\hat{k}_b$ of an open neighborhood of $A$ in $U$ to another one of $A$ since $k_b$ commutes with $g_c$. We can find tubular neighborhoods $N_1$ of $A$ in $U$ and $N_2$ of $A$. We choose $N_1$ and $N_2$ so that they are diffeomorphic by a projective map $\hat{k}_b$. Then we obtain two copies $A_1$ and $A_2$ of $A$ by completing $U - A$.

Give orientations on $A$ and $U$. Let $N_{1-}$ denote the left component of $N_1 - A$ and let $N_{2+}$ denote the right component of $N_2 - A$.

We take a disjoint union $(U - A) \sqcup N_{1-} \sqcup N_{2+}$ and

- identify the projectively diffeomorphic copy of $N_{1-}$ in $N_1$ with $N_{1-} - A$ by the identity map and
- identify the projectively diffeomorphic copy of $N_{2+}$ in $N_2$ with $N_{2+} - A$ by the identity also.

We glue back $N_1$ and $N_2$ by the real projective diffeomorphism $\hat{k}_b$ of a neighborhood of $N_1$ to that of $N_2$. Then $N_1 - (N_{1-} \cup A)$ is identified with $N_{2+}$ and $N_2 - (N_{2+} \cup A)$ is identified with $N_{1-}$. We obtain a new manifold.
For sufficiently small $b$, we see that the end is still lens-shaped. and it is not a totally geodesic R-end. (This follows since the condition of being a lens-shaped R-end is an open condition. See Section 11.2.)

For the same $c$, let $k_s$ be given by
\[
\begin{pmatrix}
s & 0 & 0 \\
0 & s & 0 \\
0 & 0 & s \\
0 & 0 & 1/s^3
\end{pmatrix}
\]  
(4.2)

where $s \in \mathbb{R}_+$. These give us bendings of the second type. For $s$ sufficiently close to 1, the property of being lens-shaped is preserved and being a totally geodesic R-end. (However, these will be understood by cohomology.)

If $s \lambda < 1$ for the maximal eigenvalue $\lambda$ of a closed curve $c_1$ meeting $c$ odd number of times, we have that the holonomy along $c_1$ has the attracting fixed point at $v_\tilde{E}$. This implies that we no longer have lens-shaped R-ends if we have started with a lens-shaped R-end.

### 4.2 Characterization of complete R-ends

The results here overlap with the results of Crampon-Marquis [74] and Cooper-Long-Tillman [73]. However, the results are of a different direction than theirs and were originally conceived before their papers appeared. We also make use of Crampon-Marquis [75].

Let $\tilde{O}$ denote a convex domain in $\mathbb{S}^n$ and covering a strongly tame orbifold $O$ with a holonomy homomorphism $h: \pi_1(O) \to \text{SL}_\pm(n+1, \mathbb{R})$. Let $\tilde{E}$ be an R-p-end. A middle eigenvalue condition for a p-end holonomy group $h(\pi_1(\tilde{E}))$ with respect to $v_\tilde{E}$ or the R-p-end structure holds if for each $g \in h(\pi_1(\tilde{E})) - \{I\}$ the largest norm $\tilde{\lambda}_1(g)$ of eigenvalues of $g$ is strictly larger than the eigenvalue $\lambda_{v_\tilde{E}}(g)$ associated with p-end vertex $v_\tilde{E}$.

Given an element $g \in h(\pi_1(\tilde{E}))$, let $\left(\tilde{\lambda}_1(g), \ldots, \tilde{\lambda}_{n+1}(g)\right)$ be the $(n+1)$-tuple of the eigenvalues listed with multiplicity given by the characteristic polynomial of $g$ where we repeat each eigenvalue with the multiplicity given by the characteristic polynomial. The multiplicity of a norm of an eigenvalue of $g$ is the number of times the norm occurs among the $(n+1)$-tuples of norms
\[
\left(|\tilde{\lambda}_1(g)|, \ldots, |\tilde{\lambda}_{n+1}(g)|\right).
\]

**Definition 4.1** Let $\tilde{E}$ be a p-end with the holonomy group $h(\pi_1(\tilde{E}))$. A weak middle eigenvalue condition for an R-p-end $\tilde{E}$ holds provided for each $g \in h(\pi_1(\tilde{E}))$ the following holds:
• whenever \( \lambda_{\nu_{E}}(g) \) has the largest norm of all norms of eigenvalues \( h(g) \), \( \lambda_{\nu_{E}}(g) \)
must have multiplicity \( \geq 2 \).

We note that these definitions depend on the choice of the p-end vertices; however, they are well defined once the radial structures are assigned.

Recall the parabolic subgroup of the isometry group \( \text{Aut}(\mathbb{B}) \) of the hyperbolic space \( \mathbb{B} \) for an \((i_0 + 1)\)-dimensional Klein model \( \mathbb{B} \subset S^{n+1} \) fixing a point \( p \) in the boundary of \( \mathbb{B} \). Such a discrete subgroup of a parabolic subgroup group is isomorphic to extensions of a lattice in \( \mathbb{R}^{n} \) and is Zariski closed by the Bieberbach theorem.

Let \( E \) be an \( i_0 \)-dimensional ellipsoid containing the point \( v \) in a subspace \( P \) of dimension \( i_0 + 1 \) in \( S^{n} \). Let \( \text{Aut}(P) \) denote the group of projective automorphisms of \( P \), and let \( \text{SL}_\pm(n+1,\mathbb{R})_{P} \) the subgroup of \( \text{SL}_\pm(n+1,\mathbb{R}) \) acting on \( P \). Let \( r_P : \text{SL}_\pm(n+1,\mathbb{R})_{P} \to \text{Aut}(P) \) denote the restriction homomorphism \( g \to g|_P \). An \( i_0 \)-dimensional partial cusp subgroup is one mapping under \( R_P \) isomorphically to a cusp subgroup of \( \text{Aut}(P) \) acting on \( E \setminus \{v\} \), fixing \( v \).

An \( i_0 \)-dimensional cusp group is a finite extension of a projective conjugate of a discrete cocompact subgroup of a group of an \( i_0 \)-dimensional partial cusp subgroup.

If the horospherical neighborhood with the p-end vertex \( v \) has the p-end holonomy group that is a discrete \((n-1)\)-dimensional cusp group, then we call the p-end to be cusp-shaped or horospherical. (See Theorem 4.2.)

### 4.2.1 Main results

Our main result classifies CA R-p-ends. We need some facts of NPNC-ends that will be explained in Section 8.2.

Given a horospherical R-p-end, the p-end holonomy group \( \Gamma_v \) acts on a p-end neighborhood \( U \) and \( \Gamma_v \) is a subgroup of an \((n-1)\)-dimensional cusp group \( \mathcal{H}_v \).

By Lemma 2.5,

\[
V := \bigcap_{g \in \mathcal{H}_v} g(U)
\]

contains a \( \mathcal{H}_v \)-invariant p-end neighborhood. Hence, \( V \) is a horospherical p-end neighborhood of \( E \). Thus, a horospherical R-end is pre-horospherical. Conversely, a pre-horospherical R-end is a horospherical R-end by Theorem 4.2 under some assumption on \( \mathcal{O} \) itself. (See Definition 1.2.)

First, we clarify by Theorem 4.2:
**Corollary 4.1** Let $\mathcal{O}$ be a strongly tame properly convex real projective $n$-orbifold. Let $E$ be an $R$-end of its universal cover $\tilde{\mathcal{O}}$. Then $E$ is a pre-horospherical $R$-end if and only if $\tilde{E}$ is a horospherical $R$-end. □

The following classifies the complete affine ends where we need some results from Chapter 8. Since these have virtually abelian holonomy groups by Theorem 4.2, they are classified in [7]. Readers may defer the reading of the proof after reading Chapter 8:

**Theorem 4.1** Let $\mathcal{O}$ be a strongly tame properly convex real projective $n$-orbifold. Let $\tilde{E}$ be an $R$-end of its universal cover $\tilde{\mathcal{O}} \subset \mathbb{S}^n$ (resp. $\subset \mathbb{R}P^n$). Let $\Gamma_{\tilde{E}}$ denote the end holonomy group. Then $\tilde{E}$ is a complete affine $R$-end if and only if $\tilde{E}$ is a horospherical $R$-end or an NPNC-end with fibers of dimension $n - 2$ with the virtually abelian end-fundamental group by altering the $p$-end vertex. Furthermore, if $\tilde{E}$ is a complete affine $R$-end and $\Gamma_{\tilde{E}}$ satisfies the weak middle eigenvalue condition, then $\tilde{E}$ is a horospherical $R$-end.

**Proof** Again, we assume that $\Omega$ is a domain of $\mathbb{S}^n$. Theorem 4.3 is the forward direction. Corollary 8.5 implies the second case above.

Now, we prove the converse using the notation and results of Chapter 8. (The reader may need to defer reading the proof to after reading Chapter 8.) Since a horospherical $R$-end is pre-horospherical, Theorem 4.2 implies the half of the converse. Given a $p$-NPNC-end $E$ with fibers of dimension $n - 2$, $\tilde{\Sigma}_E$ is projectively isomorphic to an affine half-space. Using the notation of Proposition 8.3, $K''$ is zero-dimensional and the end holonomy group $\Gamma_{\tilde{E}}$ acts on $K'' \ast \{v\}$ for an end vertex $v$. There is a foliation in $\tilde{\Sigma}_E$ by complete affine spaces of dimension $n - 2$ parallel to each other. The space of leaves is projectively diffeomorphic to the interior of $K'' \ast v'$ for a point $v'$. Let $U$ be the $p$-end neighborhood for $\tilde{E}$. Then $\text{bd}U$ is in one-to-one correspondence with $\tilde{\Sigma}_E$ by radial rays from $v$. Hence, $\text{bd}U$ has an induced foliation. Each leaf in $\text{bd}U$ lies in an open hemisphere of dimension $n - 1$. (See (8.2) in Section 8.2.) Also, $\Gamma_{\tilde{E}}$ acts on an open hemisphere $H_{v'}^{n-1}$ of dimension $(n - 1)$ with boundary a great sphere $\mathbb{S}^{n-2}$ containing $v$ in the direction of $v'$.

Now we switch the $p$-end vertex to a singleton $\{k''\} = K''$ from $v$. Then $H_{v'}^{n-1}$ corresponds to a complete affine space $A_{k''}^{n-1}$. Each leaf projects to an ellipsoid in $A_{k''}^{n-1}$ with a common ideal point $v$ corresponding to the direction of $k''v$ oriented away from $k''$. The ellipsoids are tangent to a common hyperspace in $\mathbb{S}_{v'}^{n-1}$. Hence, they are parallel paraboloids in an affine subspace $A_{k''}^{n-1}$. The uniform positive translation condition gives us that the union of the parallel paraboloids is $A_{k''}^{n-1}$. Hence, $\tilde{E}$ is a complete $R$-end with $k''$ as the vertex. The last statement follows by Corollary 8.6. □

[3]
4.2.2 Proofs of main results

**Theorem 4.2 (Horosphere)** Let $\mathcal{O}$ be a strongly tame convex real projective $n$-orbifold, $n \geq 2$. Let $\tilde{E}$ be a pre-horospherical $R$-end of its universal cover $\tilde{\mathcal{O}}$, $\tilde{\mathcal{O}} \subset \mathbb{S}^n$ (resp. $\subset \mathbb{R}P^n$), and $\Gamma_{\tilde{E}}$ denote the $p$-end holonomy group. Then the following hold:

(i) The subspace $\tilde{\Sigma}_E = \mathbb{R}^n_1(\tilde{\mathcal{O}}) \subset \mathbb{S}^{n-1}_E$ of directions of lines segments from the end-point $v_\tilde{E}$ in bd$\mathcal{O}$ into a $p$-end neighborhood of $\tilde{E}$ forms a complete affine subspace of dimension $n - 1$.

(ii) The norms of eigenvalues of $g \in \Gamma_{\tilde{E}}$ are all 1.

(iii) $\Gamma_{\tilde{E}}$ virtually is in a conjugate of an abelian parabolic or cusp subgroup of $\text{SO}(n,1)$ (resp. $\text{PO}(n,1)$) of rank $n - 1$ in $\text{SL}_+(n+1,\mathbb{R})$ (resp. $\text{PGL}(n+1,\mathbb{R})$). And hence $\tilde{E}$ is cusp-shaped.

(iv) For any compact set $\mathcal{O}' \subset \mathcal{O}$, $\mathcal{O}'$ contains a horospherical end neighborhood disjoint from $\mathcal{O}'$.

(v) A $p$-end vertex of a horospherical $p$-end cannot be an endpoint of a segment in bd$\mathcal{O}$.

**Proof** We will prove for the case $\tilde{\mathcal{O}} \subset \mathbb{S}^n$. Let $U$ be a pre-horoball $p$-end neighborhood with the $p$-end vertex $v_\tilde{E}$, closed in $\tilde{\mathcal{O}}$. The space of great segments from the $p$-end vertex passing $U$ forms a convex subset $\Sigma_E$ of a complete affine subspace $\mathbb{R}^{n-1} \subset \mathbb{S}^{n-1}_E$ by Proposition 2.12. The space covers an end orbifold $\Sigma_E$ with the discrete group $\pi_1(E)$ acting as a discrete subgroup $\Gamma'$ of the projective automorphisms so that $\Sigma_E/\Gamma'$ is projectively isomorphic to $\Sigma_E$.

(i) By Proposition 2.12, one of the following three happens:

- $\Sigma_E$ is properly convex.
- $\Sigma_E$ is foliated by complete affine subspaces of dimension $i_0$, $1 \leq i_0 < n - 1$, with the common boundary sphere of dimension $i_0 - 1$, the space of the leaves forms a properly open convex subset $K^0$ of $\mathbb{S}^{n-i_0-1}$, and $\Gamma_{\tilde{E}}$ acts on $K^0$ cocompactly but not necessarily discretely.
- $\Sigma_E$ is a complete affine subspace.

We aim to show that the first two cases do not occur.

Suppose that we are in the second case and $1 \leq i_0 \leq n - 2$. This implies that $\Sigma_E$ is foliated by complete affine subspaces of dimension $i_0 \leq n - 2$.

Since $\Gamma_{\tilde{E}}$ acts on a properly convex subset $K$ of dimension $\geq 1$, an element $g$ has a norm of an eigenvalue $> 1$ and a norm of eigenvalue $< 1$ as a projective automorphism on $\mathbb{S}^{n-i_0-1}$ by Proposition 1.1 of [18]. Hence, we obtain the largest norm of eigenvalues and the smallest one of $g$ in $\text{Aut}(\mathbb{S}^n)$ both different from 1. By Lemma 2.10, $g$ is positive bi-semi-proximal. Therefore, let $\lambda_1(g) > 1$ be the greatest norm of the eigenvalues of $g$ and $\lambda_2(g) < 1$ be the smallest norm of the eigenvalues of $g$ as an element of $\text{SL}_+(n+1,\mathbb{R})$. Let $\lambda_{v_\tilde{E}}(g) > 0$ be the eigenvalue of $g$ associated with $v_\tilde{E}$. The possibilities for $g$ are as follows.
\[ \lambda_1(g) = \lambda_{\nu_E}(g) > \lambda_2(g), \]
\[ \lambda_1(g) > \lambda_{\nu_E}(g) > \lambda_2(g), \]
\[ \lambda_1(g) > \lambda_2(g) = \lambda_{\nu_E}(g). \]

In all cases, at least one of the largest norm or the smallest norm is different from \( \lambda_{\nu_E}(g) \). By Lemma 2.10, this norm is realized by a positive eigenvalue. We take \( g^n(x) \) for a generic point \( x \in U \). As \( n \to \infty \) or \( n \to -\infty \), the sequence \( \{ g^n(x) \} \) limits to a point \( x_\infty \) in \( \text{Cl}(U) \) distinct from \( v_E \). Also, \( g \) fixes a point \( x_\infty \), and \( x_\infty \) has a different positive eigenvalue from \( \lambda_{\nu_E}(g) \). As \( x_\infty \not\in U \), it should be \( x_\infty = v_E \) by the definition of the pre-horospheres. This is a contradiction.

The first possibility is also shown not to occur similarly. Thus, \( \tilde{\Sigma}_E \) is a complete affine subspace.

(ii) If \( g \in \Gamma_E \) has a norm of eigenvalue different from 1, then we can apply the second and the third paragraphs above to obtain a contradiction. We obtain \( \tilde{\lambda}_j(g) = 1 \) for each norm \( \lambda_j(g) \) of eigenvalues of \( g \) for every \( g \in \Gamma_E \).

(iii) Since \( \tilde{\Sigma}_E \) is a complete affine subspace, \( \tilde{\Sigma}_E/\Gamma_E \) is a complete affine orbifold with the norms of eigenvalues of holonomy matrices all equal to 1 where \( \Gamma'_E \) denotes the affine transformation group corresponding to \( \Gamma_E \). (By D. Fried [84], this implies that \( \pi_1(\tilde{E}) \) is virtually nilpotent.) Again by Selberg Theorem 2.3, we can find a torsion-free subgroup \( \Gamma'_E \) of finite-index. Then \( \Gamma'_E \) is in a cusp group by Proposition 7.21 of [74] (related to Theorem 1.6 of [74]). By the proposition, we see that \( \Gamma'_E \) is in a conjugate of a parabolic subgroup of \( \text{SO}(n, 1) \) and hence acts on an \( (n - 1) \)-dimensional ellipsoid fixing a unique point. Since a horosphere has a Euclidean metric invariant under the group action, the image group is in a Euclidean isometry group. Hence, the group is virtually abelian by the Bieberbach theorem.

Actually, there is a one-dimensional family of such ellipsoids containing the fixed point where \( \Gamma'_E \) acts on.

Let \( U \) denote the domain bounded by the closure of the ellipsoid. There exist finite elements \( g_1, \ldots, g_n \) representing cosets of \( \Gamma'_E/\Gamma_E \). If \( g_i(U) \) is a proper subset of \( U \), the \( g_i^n(U) \) is so and hence \( g_i^n \) is not in \( \Gamma'_E \) for any \( n \). This is a contradiction. Hence \( \Gamma_E \) acts on \( U \) also. By same reasoning, \( \Gamma_E \) on every ellipsoid in a one-dimensional parameter space containing a unique fixed point, and an ellipsoid gives us a horosphere in the interior of a horoball. Hence, \( \Gamma_E \) is a cusp group.

(iv) We can choose an exiting sequence of \( p \)-end horoball neighborhoods \( U_i \) where a cusp group acts, We can consider the hyperbolic spaces to understand this.

(v) Suppose that \( bd\tilde{\Theta} \) contains a segment \( s \) ending at the \( p \)-end vertex \( v_E \). Then \( s \) is on an invariant hyperspace of \( \Gamma_E \). Now conjugating \( \Gamma_E \) into an \( (n - 1) \)-dimensional parabolic or cusp subgroup \( P \) of \( \text{SO}(n, 1) \) fixing \( (1, -1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \) by say an element \( h \in \text{SL}_+(n + 1, \mathbb{R}) \). By simple computations using the matrix forms of \( \Gamma_E \), we can find a sequence \( \{ g_i \}, g_i \in h \Gamma_E h^{-1} \subset P \) so that \( \{ g_i(h(s)) \} \) geometrically converges to a great segment. Thus, for \( h^{-1}g_i h \in \Gamma_E \), the sequence \( \{ h^{-1}g_i h(s) \} \) geometrically converges to a great segment in \( \text{Cl}(\tilde{\Theta}) \). This contradicts the proper convexity of \( \tilde{\Theta} \).

We will now show the converse of Theorem 4.2.
4.2 Characterization of complete R-ends

The second case will be studied later in Corollary 8.5. We will show the end \( E \) to be an NPNC-end with fiber dimension \( n-2 \) when we choose another point as the new p-end vertex for \( E \). Clearly, this case is not horospherical. (See Crampon-Marquis [74] for a similar proof.)

For the following note that Marquis classified ends for 2-manifolds [137] into cuspidal, hyperbolic, or quasi-hyperbolic ends. Since convex 2-orbifolds are convex 2-manifolds virtually, we are done for \( n=2 \).

**Theorem 4.3 (Complete affine)** Let \( \mathcal{O} \) be a strongly tame properly convex n-orbifold for \( n \geq 3 \). Suppose that \( \tilde{E} \) is a complete-affine R-p-end of its universal cover \( \tilde{\mathcal{O}} \) in \( S^n \) (resp. in \( \mathbb{RP}^n \)). Let \( v_{\tilde{E}} \in S^n \) (resp. \( \in \mathbb{RP}^n \)) be the p-end vertex with the p-end holonomy group \( \Gamma_{\tilde{E}} \).

(i) we have following two exclusive alternatives:

- \( \Gamma_{\tilde{E}} \) is virtually unipotent where all norms of eigenvalues of elements equal 1,
  or
- \( \Gamma_{\tilde{E}} \) is virtually abelian where
  - each \( g \in \Gamma_{\tilde{E}} \) has at most two norms of the eigenvalues,
  - at least one \( g \in \Gamma_{\tilde{E}} \) has two norms, and
  - if \( g \in \Gamma_{\tilde{E}} \) has two distinct norms of the eigenvalues, the norm of \( \lambda_{v_{\tilde{E}}}(g) \) has a multiplicity one.
- \( \Gamma_{\tilde{E}} \) acts as a virtually unipotent group on the complete affine space \( \tilde{\Sigma}_{\tilde{E}} \).

(ii) In the first case, \( \Gamma_{\tilde{E}} \) is horospherical, i.e., cuspidal.

**Proof** We first prove for the \( S^n \)-version. Using Theorem 2.3, we may choose a torsion-free finite-index subgroup. We may assume without loss of generality that \( \Gamma \) is torsion-free since torsion elements have only 1 as the eigenvalues of norms and we only need to prove the theorem for a finite-index subgroup. Hence, \( \Gamma \) does not fix a point in \( \tilde{\Sigma}_{\tilde{E}} \).

(i) Since \( E \) is complete affine, \( \tilde{\Sigma}_{\tilde{E}} \subset S^{n-1}_{v_{\tilde{E}}} \) is identifiable with an affine subspace \( \mathbb{A}^{n-1} \) of \( \mathbb{A}^{n-1} \). \( \Gamma_{\tilde{E}} \) induces \( \Gamma^{\prime}_{\tilde{E}} \) in \( \text{Aff}(\mathbb{A}^{n-1}) \) that are of form \( x \mapsto Mx+b \) where \( M \) is a linear map \( \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1} \) and \( b \) is a vector in \( \mathbb{R}^{n-1} \). For each \( \gamma \in \Gamma_{\tilde{E}} \),

- let \( \gamma_{\mathbb{A}^{n-1}} \) denote this affine transformation,
- we denote by \( \hat{L}(\gamma_{\mathbb{A}^{n-1}}) \) the linear part of the affine transformation \( \gamma_{\mathbb{A}^{n-1}} \), and
- let \( \nu(\gamma_{\mathbb{A}^{n-1}}) \) denote the translation vector.

A *relative eigenvalue* is an eigenvalue of \( \hat{L}(\gamma_{\mathbb{A}^{n-1}}) \).

At least one eigenvalue of \( \hat{L}(\gamma_{\mathbb{A}^{n-1}}) \) is 1 since \( \gamma \) acts without fixed point on \( \mathbb{R}^{n-1} \). (See [122].) Now, \( \hat{L}(\gamma_{\mathbb{A}^{n-1}}) \) has a maximal invariant vector subspace \( A \) of \( \mathbb{R}^{n-1} \), where all norms of the eigenvalues are 1.

Suppose that \( A \) is a proper \( \gamma \)-invariant vector subspace of \( \mathbb{R}^{n-1} \). Then \( \gamma_{\mathbb{A}^{n-1}} \) acts on the affine space \( \mathbb{R}^{n-1}/A \) as an affine transformation with the linear parts without a norm of eigenvalue equal to 1. Hence, \( \gamma_{\mathbb{A}^{n-1}} \) has a fixed point in \( \mathbb{R}^{n-1}/A \), and \( \gamma_{\mathbb{A}^{n-1}} \) acts on an affine subspace \( A’ \) parallel to \( A \).
A subspace $H$ containing $\nu_E$ corresponds to the direction of $A'$ from $\nu_{E'}$. The union of segments with endpoints $\nu_E, \nu_{E-}$ in the directions in $A' \subset S^{n-1}$ is in an open hemisphere of dimension $< n$. Let $H^+$ denote this space where $bdH^+ \ni \nu_E$ holds. Since $\Gamma_E$ acts on $A'$, it follows that $\Gamma_E$ acts on $H^+$. Then $\gamma$ has at most two eigenvalues associated with $H^+$ one of which is $\lambda_n(\gamma)$ and the other is to be denoted $\lambda_+(\gamma)$. Since $\gamma$ fixes $\nu_E$ and there is an eigenvector in the span of $H^+$ associated with $\lambda_+(\gamma)$, $\gamma$ has the matrix form

$$\gamma = \begin{pmatrix}
\lambda_+(\gamma)\hat{L}(\gamma_{R^{n-1}}) & \lambda_+(\gamma)v(\gamma_{R^{n-1}}) \\
0 & \lambda_+(\gamma) \\
* & * \\
& \lambda_{n_E}(\gamma)
\end{pmatrix}$$

where we have

$$\lambda_+(\gamma)^n \det(\hat{L}(\gamma_{R^{n-1}}))\lambda_{n_E}(\gamma) = \pm 1.$$ 

(Note $\lambda_{n_E}(\gamma^{-1}) = \lambda_{n_E}(\gamma)^{-1}$ and $\lambda_+(\gamma^{-1}) = \lambda_+(\gamma)^{-1}$.)

We will show that $\hat{L}(\gamma_{R^{n-1}})$ for every $\gamma \in \Gamma_E$ is unit-norm-eigenvalued below. There are following possibilities for each $\gamma \in \Gamma_E$:

(a) $\lambda_1(\gamma) > \lambda_+(\gamma)$ and $\lambda_1(\gamma) > \lambda_{n_E}(\gamma)$.

(b) $\lambda_1(\gamma) = \lambda_+(\gamma) = \lambda_{n_E}(\gamma)$.

(c) $\lambda_1(\gamma) = \lambda_+(\gamma), \lambda_1(\gamma) > \lambda_{n_E}(\gamma)$.

(d) $\lambda_1(\gamma) > \lambda_+(\gamma), \lambda_1(\gamma) = \lambda_{n_E}(\gamma)$.

Suppose that $\gamma$ satisfies (b). The relative eigenvalues of $\gamma$ on $\mathbb{R}^{n-1}$ are all $\leq 1$. Either $\gamma$ is unit-norm-eigenvalued or we can take $\gamma^{-1}$ and we are in case (a).

Suppose that $\gamma$ satisfies (a). There exists a projective subspace $S$ of dimension $\geq 0$ where the points are associated with eigenvalues with the norm $\lambda_1(\gamma)$ where $\lambda_1(\gamma) > \lambda_+(\gamma), \lambda_{n_E}(\gamma)$.

Let $S'$ be the subspace spanned by $H$ and $S$. Let $U$ be a p-end neighborhood of $\hat{E}$. Since the space of directions of $U$ is $\mathbb{R}^{n-1}$, we have $U \cap S' \neq \emptyset$. We can choose two generic points $y_1$ and $y_2$ of $U \cap S' - H$ so that $y_1, y_2$ meets $H$ in its interior.

Then we can choose a subsequence $\{m_i\}, \{m_j\} \to \infty$, so that $\gamma_{m_i}(y_1) \to f$ and $\gamma_{m_j}(y_2) \to f_-$ as $i \to +\infty$ unto relabeling $y_1$ and $y_2$ for a pair of antipodal points $f, f_- \in S$. This implies $f, f_- \in Cl(\hat{E})$, and $\hat{E}$ is not properly convex, which is a contradiction. Hence, (a) cannot be true.

We showed that if any $\gamma \in \Gamma_E$ satisfies (a) or (b), then $\gamma$ is unit-norm-eigenvalued. If $\gamma$ satisfies (c), then

$$\lambda_1(\gamma) = \lambda_+(\gamma) \geq \lambda_1(\gamma) \geq \lambda_{n_E}(\gamma)$$

(4.3)

for all other norms of eigenvalues $\lambda_i(\gamma)$; Otherwise, $\gamma^{-1}$ satisfies (a), which cannot happen.

Similarly if $\gamma$ satisfies (d), then we have

$$\lambda_1(\gamma) = \lambda_{n_E}(\gamma) \geq \lambda_i(\gamma) \geq \lambda_+(\gamma)$$

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for all other norms of eigenvalues $\lambda_i(\gamma)$. We conclude that either $\gamma$ is unit-norm-
eigenvalued or satisfies (4.3) or (4.4).

There is a homomorphism

$$\lambda_{e_E} : \Gamma_{\tilde{E}} \to \mathbb{R}_+$$

given by $g \mapsto \lambda_{e_E}(g)$.

This gives us an exact sequence

$$1 \to N \to \Gamma_{\tilde{E}} \to R \to 1 \quad (4.5)$$

where $R$ is a finitely generated subgroup of $\mathbb{R}_+$, an abelian group. For an element $g \in N$, $\lambda_{e_E}(g) = 1$. Since the relative eigenvalue corresponding to $L_{(g_{\mathbb{R}^{n-1}})}|A$ is 1, a matrix form shows that $\lambda_+(g) = 1$ for $g \in N$. (4.3) and (4.4) and the conclusion of the above paragraph show that $g$ is unit-norm-eigenvalued. Thus, $N$ is therefore virtually nilpotent by Theorem 2.6. (See Fried [84]). Taking a finite cover again, we may assume that $N$ is nilpotent.

Since $R$ is a finitely generated abelian group, $\Gamma_{\tilde{E}}$ is solvable by (4.5). Since $\Sigma_{\tilde{E}} = \mathbb{R}^{n-1}$ is complete affine, Proposition S of Goldman and Hirsch [95] implies

$$\det(g_{\mathbb{R}^{n-1}}) = 1$$

for all $g \in \Gamma_{\tilde{E}}$.

If $\gamma$ satisfies (c), then all norms of eigenvalues of $\gamma$ except for $\lambda_{e_E}(\gamma)$ equal $\lambda_+(\gamma)$ since otherwise by (4.3), norms of relative eigenvalues $\lambda_i(\gamma)/\lambda_+(\gamma)$ are $\leq 1$, and the above determinant is less than 1. Similarly, if $\gamma$ satisfies (d), then similarly all norms of eigenvalues of $\gamma$ except for $\lambda_{e_E}(\gamma)$ equals $\lambda_+(\gamma)$.

Therefore, only (b), (c), and (d) hold and $g_{\mathbb{R}^{n-1}}$ is unit-norm-eigenvalued for all $g \in \Gamma_{\tilde{E}}$.

By Theorem 2.6, $\Gamma_{\tilde{E}}|\mathbb{R}^{n-1}$ is an orthopotent group and hence is virtually unipo-
tent by Theorem 3 of Fried [84].

Now we go back to $\Gamma_{\tilde{E}}$. Suppose that every $\gamma$ is orthopotent. Then we have the first case of (i). If not, then the second case of (i) holds.

(ii) This follows by Lemma 4.1. 

[ST]

4.2.3 Unit-norm eigenvalued actions on ends

Here, we will collect useful results on unit-normed actions resulting in Proposition 4.4 and Lemma 4.1. The following lemma answers a question of Porti. Note that this was also proved by Theorem 5.7 in Cooper-Long-Tillman [73] using the duality theory of ends. Here we do not need to use duality.

**Lemma 4.1** Assume that $\mathcal{O}$ is a properly convex real projective orbifold with an end $E$ with the universal cover $\tilde{\mathcal{O}}$ in $\mathbb{S}^n$ (resp. $\mathbb{R}^n$). Suppose that $E$ is a convex end with a corresponding $p$-end $\tilde{E}$. Suppose that eigenvalues of elements of $\Gamma_{\tilde{E}}$ have unit norms only. Then $\Gamma_{\tilde{E}}$ is conjugate to a subgroup of a parabolic subgroup in

\[\text{(S'T)}\]
SO(n, 1) (resp. PO(n, 1)), and a finite-index subgroup of $\Gamma_{\mathfrak{E}}$ is nilpotent and $\mathfrak{E}$ is horospherical, i.e., cuspidal.

**Proof** We will assume first $\mathfrak{E}$ is horospherical, i.e., cuspidal. Suppose that a closed connected projective group $G$ acts properly and cocompactly on a convex domain $\Omega$ in $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$). Then $G$ acts transitively on $\Omega$. (Benoist [21]).

Suppose that $\Gamma$ is a uniform lattice in a closed connected group $G$ acting on a convex domain $\Omega$ in $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$). Suppose that $\Gamma$ acts properly and cocompactly on $\Omega$. Then $G$ acts transitively on $\Omega$.

**Proof** For the second item, we claim that $G$ acts properly also. Let $\hat{F}$ be the fundamental domain of $G$ with $\Gamma$ action. Let $x \in \Omega$. Let $F'$ be the image $F'(x) := \{ g(x) \mid g \in F \}$ in $\Omega$. This is a compact set. Define

$$ \Gamma_{F'} := \{ g \in \Gamma \mid g(F(x)) \cap F(x) \neq \emptyset \}. $$

Then $\Gamma_{F'}$ is finite by the properness of the action of $\Gamma$. Since an element of $G$ is a product of an element $g'$ of $\Gamma$ and $f \in F$, and $g'f(x) = x$, it follows that $g'F(x) \cap F(x) \neq \emptyset$ and $g' \in \Gamma_{F'}$. Hence the stabilizer $G_x$ is a subset of $\Gamma_{F'}F$, and $G_x$ is compact. $G$ becomes a Riemannian isometry group with respect to a metric on $\Omega$. The second part follows from the first part since $G$ must act properly and cocompactly. 

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Proposition 4.3 Let $N$ be a discrete group or an $n-1$-dimensional connected Lie group where all the elements have only eigenvalues of unit norms acting on a convex $n-1$-domain $\Omega$ in $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$) projectively and properly and cocompactly. Then $\Omega$ is a complete affine space. If $N$ is a connected Lie group, then $N$ is a simply-connected orthopotent solvable group.

Proof Again, we first prove for the $\mathbb{S}^n$-version. First consider when $N$ is discrete. By Theorem 2.6, $N$ is an orthopotent Lie group. Theorem 2.13 proved that $\Omega$ is a complete affine space.

Now, consider the case when $N$ is a connected Lie group. By Lemma 4.2, $N$ acts transitively on $\Omega$. $N$ has an $N$-invariant metric on $\Omega$ by the properness of the action. Consider an orbit map $N \to N(x)$ for $x \in \Omega$. If a stabilizer of a point $x$ of $\Omega$ contains a group of dimension $\geq 1$, then $\dim N > \dim \Omega$. The stabilizer is a finite group. Hence, $N$ covers $\Omega$ finitely. Since $\Omega$ is contractible, the orbit map is a diffeomorphism. Hence, $N$ is contractible.

By Theorem 2.6, $N$ is a solvable Lie group.

Now, $\Omega$ cannot be properly convex: Otherwise, By Fait 1.5 of [21], $N$ either acts irreducibly on $\Omega$ or $\Omega$ is a join of domain $\Omega_1, \ldots, \Omega_n$ where $N$ acts irreducibly on each $\Omega_i$. Since a solvable group never acts irreducibly unless the domain is 0-dimensional by the Lie-Kolchin theorem, $\Omega$ is a simplex or a point. (See Theorem 17.6 of [108].) Then $N$ has to be diagonalizable and this is a contradiction to the unit-norm-eigenvalued property since $N$ acts cocompactly unless $n-1 = 0$. If $n-1 = 0$, the conclusion is true.

Now suppose that $\Omega$ is not properly convex but not complete affine. Then $\Omega$ is foliated by $i_0$-dimensional complete affine spaces for $i_0 < n$. The space of affine leaves is a properly convex domain $K$ by discussions on R-ends in Section 4.1. Hence, $N$ acts on $K$. The stabilizer $N_1$ is $i_0$-dimensional since the $N$-action is simply transitive. Hence, $N/N_1$ acts on a properly convex set $K^0$ satisfying the premises. Again, this is a contradiction.

Hence, $\Omega$ is complete affine. [S^n]

Given a subgroup $G$ of an algebraic Lie group, a syndetic hull $\mathcal{S}(G)$ of $G$ is a solvable Lie group with finitely many components so that $\mathcal{S}(G)/G$ is compact. (See Fried and Goldman [86] and D. Witte [166].)

Lemma 4.3 Let $N$ be a closed orthopotent Lie group in $\text{SL}_{\pm}(n, \mathbb{R})$ acting on $\mathbb{R}^n$ inducing a proper action on an $n-1$-dimensional affine space $\mathbb{A}^{n-1}$ that is the upper half-space of $\mathbb{R}^n$ quotient by the scalar multiplications. Suppose that $N$ acts cocompactly on $\mathbb{A}^{n-1}$. Then there is a connected group $N_u$ with the following properties:

- $N/N \cap N_u$ and $N_u/N \cap N_u$ are compact.
- $N_u$ is homeomorphic to a cell,
- $N_u$ acts simply transitively on $\mathbb{A}^{n-1}$,
- $N_u$ is the unipotent subgroup in $\text{SL}_{\pm}(n, \mathbb{R})$ of dimension $n-1$ of $N$ normalized by $N$. 

Since $N$ is orthopotent, there is a flag of vector subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{R}^n$ preserved by $N$ where $N$ acts as an orthogonal group on $V_{i+1}/V_i$ for each $i = 1, \ldots, m - 1$. Here, $\mathbb{A}^{n-1}$ is parallel to the vector subspace $V_{m-1}$ of dimension $n-1$. (See Chapter 2 of Berger [26].)

Hence, there is a homomorphism $N \to \bigoplus_{i=0}^{m-1} O(V_{i+1}/V_i)$. Let $N'_u$ denote the kernel. Then $N'_u$ is a unipotent group with compact $N/N'_u$.

We define $N_u$ to be the Zariski closure of $N'_u$ in $SL_+(n, \mathbb{R})$. Now, $N_u$ is a unipotent Lie group, and $N_u/N'_u$ is compact by Malcev [133].

Since $N_u$ also acts on $\mathbb{A}^{n-1}$, $N_u/N'_u$ is compact, and $N'_u$ acts properly, it follows that $N_u$ acts properly on $\mathbb{A}^{n-1}$. By Lemma 4.2, $N_u$ acts transitively on $\mathbb{A}^{n-1}$. The action has trivial stabilizer since $N_u$ is unipotent. This implies $N_u$ is homeomorphic to $\mathbb{A}^{n-1}$.

**Proposition 4.4** Let $U$ is in an open domain in $S^n$ (resp. $\mathbb{R}P^n$) radially foliated from a point $p \in bdU$ with smooth $bdU - \{p\}$. Suppose $U$ is in a properly convex domain, and let $N$ be an $n-1$-dimensional a connected Lie group with only unit norm eigenvalues acting on $U$ and fixing $p$. Suppose that it acts on $R_p(U)$ properly and cocompactly. Then $U$ is the interior of an ellipsoid and $N$ is a unipotent cusp group and acts transitively and freely on $bdU - \{p\}$

**Proof** We first assume $U \subset S^n$. By Proposition 4.3, $R_p(U) = \mathbb{A}$ is complete affine. $N$ acts on $R_p(U)$ as a unipotent Lie group. Thus, $N$ is a simply-connected unit-norm-eigenvalue solvable Lie group by Proposition 4.3.

By Lemma 4.3, there is a unipotent group $N_u$ where $N/N \cap N_u$ and $N_u/N \cap N_u$ are compact. Since $N_u$ is isomorphic to a unipotent subgroup, and $N \cap N_u$ is a lattice in $N$ and one in $N_u$.

It follows that each element of geodesic in $N$ passing an element of $N \cap N_u$ is also unipotent being an exponential of a nilpotent element. The compactness of $N_u/N \cap N_u$ implies that these tangent vectors form a dense set in the tangent space the identity at $N_u$ as we can see from the central series extension by free abelian groups. It follows that $N \cap N_u = N_u$ and so $N_u \subset N$. Since they have the same dimensions and are connected, $N_u = N$.

We will now show that $U$ is the interior of an ellipsoid. We identify $p$ with $[1,0,\ldots,0]$. Let $W$ denote the hyperspace in $S^n$ containing $p$ sharply supporting $U$. Here, $W$ corresponds to a supporting hyperspace in $S^{n-1}_p$ of the set of directions of an open hemisphere $R_p(U)$ and hence is unique supporting hyperplane at $p$ and, thus, $N$-invariant. Also, $W \cap Cl(U)$ is a properly convex subset of $W$.

Let $y$ be a point of $U$. Suppose that $N$ contains a sequence $\{g_i\}$ so that

$$\{g_i(y)\} \to x_0 \in W \cap Cl(\partial) \text{ and } x_0 \neq p;$$

that is, $x_0$ in the boundary direction of $A$ from $p$. Let $U_1 = Cl(U) \cap W$. Let $V$ be the smallest subspace containing $p$ and $U_1$. The dimension of $V$ is $\geq 1$ as it contains $x_0$ and $p$.

Again the unipotent group $N$ acts on $V$. Now, $V$ is divided into disjoint open hemispheres of various dimensions where $N$ acts on: By Theorem 3.5.3 of [160],
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Suppose that \( \dim V = n - 1 \) for contradiction. Then \( H \cap U_1 \) is not empty since otherwise, we would have a smaller dimensional \( V \). Let \( h_v \) be the component of \( H \) meeting \( U_1 \). Since \( N \) is unipotent, \( h_v \) has an \( N \)-invariant metric by Theorem 3 of Fried [84].

We claim that the orbit of the action of \( N \) is of dimension \( n - 1 \) and hence locally transitive on \( H \): If not, then a one-parameter subgroup \( N' \) fixes a point of \( h_v \). This group acts trivially on \( h_v \) since the unipotent group contains a trivial orthogonal subgroup. Since \( N' \) is not trivial, it acts as a group of nontrivial translations on the affine subspace \( H' \). We obtain that \( N'(U) \) is not properly convex. This is absurd. Hence, an orbit of \( N \) is open in \( h_v \), and \( N \) acts locally simply-transitively without fixed points.

Since \( N \) has trivial stabilizers on \( h_v \), there is an \( N \)-invariant Riemannian metric on \( h_v \). The orbit of \( N \) in \( h_v \) is closed since \( h_v \) has an \( N \)-invariant metric, and \( N \) is closed in the isometry group of \( h_v \). Thus, \( N \) acts transitively on \( h_v \) since \( \dim N = \dim h_v \).

Hence, the orbit \( N(y) \) of \( N \) for \( y \in H \cap U_1 \) contains a component of \( H \). This contradicts the assumption that \( \text{Cl}(U) \) is properly convex (compare with arguments in [74].)

Suppose that the dimension of \( (V) \) is \( \leq n - 2 \). Let \( J \) be a subspace of dimension 1 bigger than \( \dim V \) and containing \( V \) and meeting \( U \). Let \( J_h \) denote the subspace of \( \mathbb{A}^{n-1} \) corresponding to the directions in \( J \). Then \( J_h \) is sent to disjoint subspaces or to itself under \( N \). Since \( N \) acts on \( \mathbb{A} \) transitively, a nilpotent subgroup \( N_J \) of \( N \) acts on \( J_h \) transitively. Hence,

\[
\dim N_J = \dim J_h = \dim V,
\]

and we are in a situation immediately above. The orbit \( N_J(y) \) for a limit point \( y \in H \) contains a component of \( V - V_{k-1} \) as above. Thus, \( N_J(y) \) contains the same component, an affine subspace. As above, we have a contradiction to the proper convexity since the above argument applies to \( N_J \).

Therefore, points such as \( x_0 \in W \cap \text{bd}(\mathring{\Theta}) - \{p\} \) do not exist. Hence for any sequence of elements \( g_i \in \Gamma_E \), we have \( \{g_i(y)\} \to p \). Hence,

\[
\text{bd}U = (\text{bd}U \cap \mathring{\Theta}) \cup \{p\}.
\]

Clearly, \( \text{bd}U \) is homeomorphic to an \( (n - 1) \)-sphere.

Since \( U \) is radial, this means that \( U \) is a pre-horospherical p-end neighborhood. (See Definition 1.2.) Since \( N \) acts transitively on a complete affine space \( R_p(U) \), and there is a 1 to 1 radial correspondence of \( R_p(U) \) and \( \text{bd}U - \{p\} \), it acts so on \( \text{bd}U - p \). Since \( N \) is unipotent and acts transitively on \( \text{bd}U - \{p\} \), Lemma 7.12 of [74] shows that \( U \) is bounded by an ellipsoid. Choose \( x \in U \), then \( N(x) \subset U \) is an horospherical p-end neighborhood also. Since \( \text{Aut}(U) \) is the group of hyperbolic isometry group of \( U \) with the Hilbert metric, it follows that \( N \) is the cusp group. [394?T]
Chapter 5
The affine action on a properly convex domain whose boundary is in the ideal boundary

In this chapter, we will show the asymptotic niceness of the affine actions when the affine group $\Gamma$ acts on a convex domain $\Omega$ in $\mathbb{A}^n$ and a properly convex domain in the ideal boundary of $\mathbb{A}^n$. We will find a properly convex domain in $\mathbb{A}^n$ with boundary in $\Omega$. The main tools will be Anosov flows on the affine bundles over the unit tangent bundles as in Goldman-Labourie-Margulis [96]. We will introduce a flat bundle and decompose it in an Anosov-type manner. Then we will find an invariant section. We will prove the asymptotic niceness using the sections. In Section 5.1, we will define asymptotic niceness and flow decomposition of the vector bundles over $\mathbb{U}\Omega/\Gamma$. In Section 5.2, we begin with a strictly convex domain $\Omega$ with a hyperbolic $\Gamma$ and the main result Theorem 5.1. We define proximal flows and decompose the vector bundle flows into contracting and repelling and neutral subbundles. In Section 5.2.2, we show that contracting and expansion properties of the contracting and repelling subbundles, with a somewhat technical argument involving pulling-back. However, the neutral subbundles here are more of a generalized type than what they had. We obtain the neutralized sections as Goldman-Labourie-Margulis did. We will prove the main result for strictly convex $\Omega$ at the end of this section using the neutralized sections to obtain asymptotic hyperspaces. In Section 5.3, we will generalize these results to the case when $\Omega$ is not necessarily strictly convex. They are Theorems 5.2 and 5.3. A basic technique here is to make the unit tangent bundle larger to an augmented unit tangent bundle by blowing up using the compact sets of hyperspaces at the endpoints of geodesics. The strategy to prove the second main result is analogous to the strictly convex case for $\Omega$. In Section 5.4, we discuss the lens condition for T-ends obtained by the uniform middle eigenvalue condition. We will end by finding strictly convex smooth hypersurfaces approximating any convex boundary components for these types of domains. Except for Section 5.4, we will work only in $\mathbb{S}^n$ for simplicity.
5.1 Affine actions

Let $\Gamma$ be an affine group acting on the affine subspace $\mathbb{A}^n$ with boundary $\text{bd}\mathbb{A}^n = S^n_{\infty}$ in $S^n$. $\mathbb{A}^n$ is an open $n$-hemisphere. Let $U$ be a properly convex invariant $\Gamma$-invariant domain with the property in $\mathbb{A}^n$:

$$\text{Cl}(U) \cap \text{bd}\mathbb{A}^n = \text{Cl}(\Omega) \subset \text{bd}\mathbb{A}^n$$

for a properly convex open domain $\Omega$. To begin with, we assume only that $\Omega$ is properly convex. We also assume that $\Omega/\Gamma$ is a closed orbifold. The action of $\Gamma$ on $S^n$ or $\mathbb{R}P^n$ is said to be a properly convex affine action. Also, $(\Gamma, U, \Omega)$ is said to be a properly convex affine triple.

A sharply supporting hyperspace $P$ at $x \in \partial\text{Cl}(\Omega)$ is asymptotic to $U$ if there are no other sharply supporting hyperplane $P'$ at $x$ so that $P' \cap \mathbb{A}^n$ separates $U$ and $P \cap \mathbb{A}^n$. In this case, we say that hyperspace $P$ is asymptotic to $U$. We will use the abbreviation $AS$-hyperspace to indicate for asymptotic sharply supporting hyperspace.

A subspace $U$ of $\mathbb{R}^n$ is expanding under a linear map $L$ if $\|L(u)\| \geq C\|u\|$ for every $u \in \mathbb{R}^n$ for a fixed norm $\|\cdot\|$ of $\mathbb{R}^n$ and $C > 1$.

A subspace $U$ of $\mathbb{R}^n$ is contracting under a linear map $L$ if $\|L(u)\| \leq C\|u\|$ for every $u \in \mathbb{R}^n$ for a fixed norm $\|\cdot\|$ of $\mathbb{R}^n$ and $0 < C < 1$.

The expanding condition is equivalent to the condition that all the norms of eigenvalues of $L|U$ are strictly larger than 1. (See Corollary 1.2.3 of Katok and Hasselblatt [118].)

In this section, we will work with $S^n$ only, while the $\mathbb{R}P^n$ versions of the results follows from the results here in an obvious manner by Results in Section 2.1.7 and then projecting back to $\mathbb{R}P^n$.

For each element of $g \in \Gamma$,

$$h(g) = \left( \begin{array}{cc} \frac{1}{\lambda_E(g)^{1/n}} \hat{h}(g) & b_g \\ 0 & \lambda_E(g) \end{array} \right)$$

(5.1)
where \( b \) is an \( n \times 1 \)-vector and \( \hat{h}(g) \) is an \( n \times n \)-matrix of determinant \( \pm 1 \) and \( \lambda_E(g) > 0 \). In the affine coordinates, it is of the form

\[
x \mapsto x + \frac{1}{\lambda_E(g)^{1 + \frac{1}{\pi}}} \hat{h}(g)x + \frac{1}{\lambda_E(g)} b.
\]

(5.2)

Let \( \lambda_1(g) \) denote the maximal norm of the eigenvalue of \( g, g \in \Gamma \). If there exists a uniform constant \( C > 1 \) so that

\[
v^{-1} \text{length}_\Omega(g) \leq \log \frac{\lambda_1(g)}{\lambda_E(g)} \leq \text{Length}_\Omega(g), \quad g \in \Gamma_E - \{1\},
\]

then \( \Gamma \) is said to satisfy the uniform middle eigenvalue condition with respect to the boundary hyperspace.

This implies that \( \lambda_1(g)/\lambda_E(g) > 1 \) and \( \lambda_n(g)/\lambda_E(g) < 1 \).

(5.4)

5.1.1 Flow setup

The following flow setup will be applicable in the following. A slight modification is required later in Section 5.3.

We generalize the work of Goldman-Labourie-Margulis [96] using Anosov flows: We assume that \( \Gamma \) has a properly convex affine action with the triple \((\Gamma, U, \Omega)\) for \( U \subset \mathbb{A}^n \). Since \( \Omega \) is properly convex, \( \Omega \) has a Hilbert metric. Let \( T\Omega \) denote the tangent space of \( \Omega \). Let \( \mathbb{U}\Omega \) denote the unit tangent bundle over \( \Omega \). This has a smooth structure as a quotient space of \( T\Omega - O/\sim \) where

- \( O \) is the image of the zero-section, and
- \( v \sim w \) if \( v \) and \( w \) are over the same point of \( \Omega \) and \( v = sw \) for a real number \( s > 0 \).

Let \( \mathbb{A}^n \) be the \( n \)-dimensional affine subspace. Let \( h : \Gamma \to \text{Aff}(\mathbb{A}^n) \) denote the representation as described in (5.2). We form the product \( \mathbb{U}\Omega \times \mathbb{A}^n \) that is an affine bundle over \( \mathbb{U}\Omega \). We take the quotient \( \mathbb{A} := \mathbb{U}\Omega \times \mathbb{A}^n \) by the diagonal action

\[
g(x, u) = (g(x), h(g)u) \quad \text{for} \quad g \in \Gamma, x \in \mathbb{U}\Omega, u \in \mathbb{A}^n.
\]
We denote the quotient by $A$ which fibers over the smooth orbifold $U\Omega/\Gamma$ with fiber $A^n$.

Let $V^n$ be the vector space associated with $A^n$. Then we can form $\tilde{V} := U\Omega \times V^n$ and take the quotient under the diagonal action:

$$g(x, u) = (g(x), L(h)u)$$

where $L$ is the homomorphism taking the linear part of $g$. We denote by $V$ the fiber bundle over $U\Omega/\Gamma$ with fiber $V^n$.

There exists a flow $\hat{\Phi}_t : U\Omega/\Gamma \rightarrow U\Omega/\Gamma$ for $t \in \mathbb{R}$ given by sending $v$ to the unit tangent vector to at $\alpha(t)$ where $\alpha$ is a geodesic tangent to $v$ with $\alpha(0)$ equal to the base point of $v$. This flow is induced from the geodesic flow $\tilde{\hat{\Phi}}_t : U\Omega \rightarrow U\Omega$.

We define a flow on $\tilde{\Phi}_t : \tilde{A} \rightarrow \tilde{A}$ by considering a unit-speed geodesic flow line $l$ in $U\Omega$ and considering $l \times A^n$ and acting trivially on the second factor as we go from $v$ to $\tilde{\hat{\Phi}}_t(v)$ for each $t$. This induces a flow $L(\Phi)_t : V \rightarrow V$. (This generalizes the flow on [96].)

We let $\|\cdot\|_{\text{fiber}}$ denote some metric on these bundles over $U\Sigma/\Gamma$ defined as a fiberwise inner product: We chose a cover of $\Omega/\Gamma$ by compact sets $K_i$ and choosing a metric over $K_i \times A^n$ and use the partition of unity. This induces a fiberwise metric on $V$ as well. Pulling the metric back to $\tilde{A}$ and $\tilde{V}$, we obtain a fiberwise metric to be denoted by $\|\cdot\|_{\text{fiber}}$.

We recall the trivial product structure. $U\Omega \times A^n$ is a flat $A^n$-bundle over $U\Omega$ with a flat affine connection $\nabla^A$, and $U\Omega \times V^n$ has a flat linear connection $\nabla^V$. The above action preserves the connections. We have a flat affine connection $\nabla^A$ on the bundle $A$ over $U\Sigma$ and a flat linear connection $\nabla^V$ on the bundle $V$ over $U\Sigma$.

**Remark 5.1** In [96], the authors uses the term "recurrent geodesic". A geodesic is recurrent if it accumulates to compact subsets in both directions. They work in a compact subsurface where geodesics are recurrent in both directions. In our work, since $\Omega/\Gamma$ is a closed orbifold, every geodesic is recurrent in their sense. Hence, their theory generalizes here.

### 5.2 The proximal flow.

We will start with the case when $\Gamma$ is hyperbolic and hence when $\Omega$ must be strictly convex with $\partial \Omega$ being $C^1$ by Theorem 1.1 of [22].
Theorem 5.1 We assume that $\Gamma$ is a hyperbolic group with a properly convex affine action with the triple $(\Gamma, U, \Omega)$. Suppose that $\Gamma$ satisfies the uniform middle-eigenvalue condition with respect to $\partial \Lambda^n$. Then

- $\Gamma$ is asymptotically nice with the properly convex open domain $U$, and
- if any open set $U'$ so that $(\Gamma, U', \Omega)$ is a properly convex triple, then the AS-hyperspace at each point of $\partial \Omega$ is the same as that of $U$.

In the case when the linear part of the affine maps are unimodular, Theorem 8.2.1 of Labourie [126] shows that such a domain $U$ exists but without showing the asymptotic niceness of the group. Also, when the linear part of $\Gamma$ is a geometrically finite Kleinian group in $\text{SO}(n, 1)$, Barbot showed this result in Theorem 4.25 of [11] in the context of globally hyperbolic Lorentzian spacetimes. We believe our theory also generalize to the case when $L(\Gamma)$ is convex cocompact. Fried also found a solution using cocyles [85] with informal notes in the same context but in the dual picture of R-ends as in Chapter 6.

The hyperbolicity of $\Gamma$ shows that $\Omega$ is strictly convex by Benoist [22]. We will generalize the theorem to Theorem 5.2 without the hyperbolicity condition of $\Gamma$. Furthermore, we will show that the middle eigenvalue condition actually implies the existence of the properly convex domain $U$ in Theorem 5.2. Also, the uniqueness of the set of asymptotic hyperspaces is given by Theorem 5.3.

The reason for presenting weaker Theorem 5.1 is to convey the basic idea of the proof of the generalized theorem.

5.2.1 The decomposition of the flow

We are assuming that $\Gamma$ is hyperbolic. Since $\Sigma := \Omega / \Gamma$ is a closed strictly convex real projective orbifold, $U\Sigma := U\Omega / \Gamma$ is a compact smooth orbifold again. A geodesic flow on $U\Omega / \Gamma$ is Anosov and hence topologically mixing. Hence, the flow is nonwandering everywhere. (See [20].) $\Gamma$ acts irreducibly on $\Omega$, and $\partial \Omega$ is $C^1$. Denote by $\Pi: U\Omega \to \Omega$ the projection to the base points.

We can identify $\partial \Lambda^n = S(V^n) = S^{n-1}$ where $g$ acts by $\mathcal{L}(g) \in \text{GL}(n, \mathbb{R})$. We give a decomposition of $\hat{V}$ into three parts $\hat{V}_+, \hat{V}_0, \hat{V}_-$:

- For each vector $u \in \overbar{U}\Omega$, we find the maximal oriented geodesic $l$ ending at the backward endpoint $\partial_- l$ and the forward endpoint $\partial_+ l \in \partial \Omega$. They correspond to the 1-dimensional vector subspaces $\hat{V}_+(u)$ and $\hat{V}_-(u) \subset V$.
- Recall that $\partial \Omega$ is $C^1$ since $\Omega$ is strictly convex (see [22]). There exists a unique pair of sharply supporting hyperspheres $H_+$ and $H_-$ in $\partial \Lambda^n$ at each of $\partial_+ l$ and $\partial_- l$. We denote by $H_0 = H_+ \cap H_-$. It is a codimension 2 great sphere in $\partial \Lambda^n$ and corresponds to a vector subspace $\hat{V}_0(u)$ of codimension-two in $\hat{V}$.
- For each vector $u$, we find the decomposition of $V$ as $\hat{V} = \hat{V}_+(u) \oplus \hat{V}_0(u) \oplus \hat{V}_-(u)$ and hence we can form the subbundles $\hat{V}_+, \hat{V}_0, \hat{V}_-$ over $U\Omega$ where

$$\hat{V} = \hat{V}_+ \oplus \hat{V}_0 \oplus \hat{V}_-.$$
The map $\mathcal{U}\Omega \to \text{bd}\Omega$ by sending a vector to the endpoint of the geodesic tangent to it is $C^1$. The map $\text{bd}\Omega \to \mathcal{H}$ sending a boundary point to its sharply supporting hyperspace in the space $\mathcal{H}$ of hyperspaces in $\text{bd}\mathbb{A}^n$ is continuous. Hence $\tilde{V}_+, \tilde{V}_0$, and $\tilde{V}_-$ are continuous bundles. Since the action preserves the decomposition of $\tilde{V}$, $V$ also decomposes as
\[ V = V_+ \oplus V_0 \oplus V_- . \tag{5.5} \]

For each complete geodesic $l$ in $\Omega$, let $I$ denote the set of unit vectors on $l$ in one of the two directions. On $I$, we have a decomposition
\[ \tilde{V}|l = \tilde{V}_+|l \oplus \tilde{V}_0|l \oplus \tilde{V}_-|l \] of form
\[ l \times \tilde{V}_+(u), l \times \tilde{V}_0(u), l \times \tilde{V}_-(u) \] for a vector $u$ tangent to $l$

where we recall:
- $\tilde{V}_+(u)$ is the space of vectors in the direction of the backward endpoint of $l$.
- $\tilde{V}_-(u)$ is the space of vectors in the direction of the forward endpoint of $l$.
- $\tilde{V}_0(u)$ is the space vectors in directions of $H_0 = H_+ \cap H_- \cap \partial l$.

That is, these bundles are constant bundles along $l$.

Suppose that $g \in \Gamma$ acts on a complete geodesic $l$ with a unit vector $u$ in the direction of the action of $g$. Then $\tilde{V}_-(u)$ and $\tilde{V}_+(u)$ corresponding to endpoints of $l$ are respectively eigenspaces of the largest norm $\lambda_1(g)$ of the eigenvalues and the smallest norm $\lambda_n(g)$ of the eigenvalues of the linear part $L(g)$ of $g$. Hence
- on $\tilde{V}_-(u)$, $g$ acts by expending by
  \[ \frac{\lambda_1(g)}{\lambda_E(g)} > 1 , \tag{5.6} \]
and
- on $\tilde{V}_+(u)$, $g$ acts by contracting by
  \[ \frac{\lambda_n(g)}{\lambda_E(g)} < 1 . \tag{5.7} \]

There exists a flow $\tilde{\Phi}_t : \mathcal{U}\Omega \to \mathcal{U}\Omega$ for $t \in \mathbb{R}$ given by sending $v$ to the unit tangent vector to at $\alpha(t)$ where $\alpha$ is a geodesic tangent to $v$ with $\alpha(0)$ equal to the base point of $v$.

We define a flow on $\tilde{\Phi}_t : \mathcal{A} \to \mathcal{A}$ by considering a unit-speed geodesic-flow line $l$ in $\mathcal{U}\Omega$ and considering $l \times \mathbb{A}^n$ and acting trivially on the second factor as we go from $v$ to $\tilde{\Phi}_t(v)$ (See remarks in the beginning of Section 3.3 and equations in Section 4.1 of [96]). Each flow line in $\mathcal{U}\Sigma$ lifts to a flow line on $A$ from every point in it. This induces a flow $\Phi_t : A \to A$.

We defined a flow on $\tilde{\Phi}_t : \mathcal{V} \to \mathcal{V}$ by considering a unit-speed geodesic-flow line $l$ in $\mathcal{U}\Omega$ and and considering $l \times \mathcal{V}$ and acting trivially on the second factor as we go from $v$ to $\tilde{\Phi}_t(v)$ for each $t$. (This generalizes the flow on [96].) Also, $L(\tilde{\Phi}_t)$
5.2 The proximal flow.

preserves $V_+, V_0,$ and $V_-$ since on the line $l$, the endpoint $\partial_\pm l$ does not change. Again, this induces a flow

$$L(\Phi)_t : V \to V, V_+ \to V_+, V_0 \to V_0, V_- \to V_-.$$

We let $\|\cdot\|_S$ denote some metric on these bundles over $\mathbb{U} \Sigma / \Gamma$ defined as a fiber-wise inner product: We chose a cover of $\Omega / \Gamma$ by compact sets $K_i$ and choosing a metric over $K_i \times \mathbb{A}^n$ and use the partition of unity. This induces a fiberwise metric on $V$ as well. Pulling the metric back to $\mathbb{A}$ and $\mathbb{V}$, we obtain a fiberwise metrics to be denoted by $\|\cdot\|_S$.

By the uniform middle-eigenvalue condition, $V$ satisfies the following properties for $u \in U \Omega / \Gamma$:

- the flat linear connection $\nabla^V$ on $V$ is bounded with respect to $\|\cdot\|_{\text{fiber}}$.
- hyperbolicity: There exists constants $C, k > 0$ so that

$$\|L(\Phi)_t(v)\|_{\text{fiber}} \geq \frac{1}{C} \exp(kt) \|v\|_{\text{fiber}} \text{ as } t \to \infty \quad (5.8)$$

for $v \in V_+$ and

$$\|L(\Phi)_t(v)\|_{\text{fiber}} \leq C \exp(-kt) \|v\|_{\text{fiber}} \text{ as } t \to \infty \quad (5.9)$$

for $v \in V_-$.

Using Proposition 5.1, we prove this property by taking $C$ sufficiently large according to $t_1$, which is a standard technique.

5.2.2 The proof of the proximal property.

We may assume that $\Gamma$ has no finite order elements by taking a finite index group using Theorem 2.3. Also, by Benoist [22], elements of $\Gamma$ are positive bi-proximal. (See Theorem 2.7.)

We can apply this to $V_-$ and $V_+$ by possibly reversing the direction of the flow. The Anosov property follows from the following proposition.

Let $V_{-1}$ denote the subset of $V_-$ of the unit length under $\|\cdot\|_{\text{fiber}}$.

**Proposition 5.1** Let $\Omega / \Gamma$ be a closed strictly convex real projective orbifold with hyperbolic fundamental group $\Gamma$. Then there exists a constant $t_1$ so that

$$\|L(\Phi)_t(v)\|_{\text{fiber}} \leq \tilde{C} \|v\|_{\text{fiber}}, v \in V_- \text{ and } \|L(\Phi)^{-t}(v)\|_{\text{fiber}} \leq \tilde{C} \|v\|_{\text{fiber}}, v \in V_+$$

for $t \geq t_1$ and a uniform $\tilde{C}, 0 < \tilde{C} < 1$. 
Proof It is sufficient to prove the first part of the inequalities since we can substitute \( t \rightarrow -t \) and switching \( V_+ \) with \( V_- \) as the direction of the vector changed to the opposite one.

By following Lemma 5.2, the uniform convergence implies that for given \( 0 < \varepsilon < 1 \), for every vector \( v \) in \( V_- \), there exists a uniform \( T \) so that for \( t > T \), \( \mathcal{L}(\Phi_t)(v) \) is in an \( \varepsilon \)-neighborhood \( U_\varepsilon(S_0) \) of the image \( S_0 \) of the zero section. Hence, we obtain that \( \mathcal{L}(\Phi_t) \) is uniformly contracting near \( S_0 \), which implies the result. \( \square \)

The line bundle \( V_- \) lifts to \( \tilde{V}_- \) where each unit vector \( u \) on \( \Omega \) one associates the line \( V_-u \) corresponding to the starting point in \( \text{bd}\Omega \) of the oriented geodesic \( l \) tangent to it. \( \tilde{V}_- \) equals \( I \times V_-u \). \( \mathcal{L}(\Phi_t) \) lifts to a parallel translation or constant flow \( \mathcal{L}(\tilde{\Phi}_t) \) of form

\[
(u,v) \rightarrow (\tilde{\Phi}_t(u),v).
\]

Fig. 5.1 The figure for Lemma 5.2. Here \( y_i, y_j \) denote the images under \( \Pi_\Omega \) the named points in the proof of Lemma 5.2.

**Lemma 5.2** Suppose that \( \Omega \) is strictly convex with \( \partial \Omega \) being \( C^1 \), and \( \Gamma \) acts properly discontinuously and cocompactly on \( \Omega \) satisfying the uniform middle eigenvalue condition. Then \( \{ \| \mathcal{L}(\Phi_t)|V_- \|_{\text{fiber}} \} \rightarrow 0 \) uniformly as \( t \rightarrow \infty \).

**Proof** Let \( F \) be a fundamental domain of \( U\Omega \) under \( \Gamma \). It is sufficient to prove this for \( \mathcal{L}(\Phi_t) \) on the fibers of over \( F \) of \( U\Omega \) with a fiberwise metric \( \| \|_{\text{fiber}} \).

We choose an arbitrary sequence \( \{ x_i \}, \{ x_i \} \rightarrow x \) in \( F \). For each \( i \), let \( v_{-i} \) be a Euclidean unit vector in \( V_{-i} := \tilde{V}_-(x_i) \) for the unit vector \( x_i \in U\Omega \). That is, \( v_{-i} \) is
in the 1-dimensional subspace in \(\mathbb{R}^n\), corresponding to the backward endpoint of the geodesic \(l_i\) in \(\Omega\) determined by \(x_i\) in \(\text{bd}\Omega\) and in a direction of \(\text{Cl}(\Omega)\).

We will show that

\[
\{| \mathcal{L}(\Phi_{t_i})(x_i, v_{-i}) \}_\text{fiber} \to 0 \text{ for any sequence } t_i \to \infty,
\]

which is sufficient to prove the uniform convergence to 0 by the compactness of \(V_{-1}\).

It is sufficient to show that any sequence of \(\{t_i\} \to \infty\) has a subsequence \(\{t_j\}\) such that

\[
\{| \mathcal{L}(\Phi_{t_j})(x_i, v_{-i}) \}_\text{fiber} \to 0.
\]

This follows since if the uniform convergence did not hold, then we can easily find a sequence without such subsequences.

Let \(y_i := \Phi_{t_i}(x_i)\) for the lift of the flow \(\hat{\Phi}\). By construction, we recall that each \(\Pi_i(\Omega(y_i))\) is in the geodesic \(l_i\). Since we have the sequence of vectors \(\{x_i\} \to x, x_i, x \in F\), we obtain that \(\{l_i\}\) geometrically converges to a line \(l_\infty\) passing \(\Pi_i(\Omega)\) in \(\Omega\). Let \(y_+\) and \(y_-\) be the endpoints of \(l_\infty\) where \(\{\Pi_i(\Omega)\} \to y_-\). Hence,

\[
\{| (v_{+,i}) \} \to y_+, \{| (v_{-,i}) \} \to y_-
\]

Find a deck transformation \(g_i\) so that \(g_i(y_i) \in F\) and \(g_i\) acts on the line bundle \(\tilde{\mathcal{V}}_-\), by the linearization of the matrix of form of (5.1):

\[
\mathcal{L}(g_i) : \tilde{\mathcal{V}}_- \to \tilde{\mathcal{V}}_- \text{ with } (y_i, v) \mapsto (g_i(y_i), \mathcal{L}(g_i)(v)) \text{ where } \mathcal{L}(g_i) = \frac{1}{\lambda_E(g_i)^{1+n}} \hat{h}(g_i) : \tilde{\mathcal{V}}_-(y_i) = \tilde{\mathcal{V}}_-(x_i) \to \tilde{\mathcal{V}}_-(g_i(y_i)). \tag{5.10}
\]

(Goal) We will show \(\{ (g_i(y_i), \mathcal{L}(g_i)(v_{-,i})) \} \to 0\) under \(\| \cdot \|_{\text{fiber}}\). This will complete the proof since \(g_i\) acts as isometries on \(\tilde{\mathcal{V}}_-\) with \(\| \cdot \|_{\text{fiber}}\).

Also, we may assume that \(\{g_i\}\) is a sequence of mutually distinct elements up to a choice of subsequences since \(g_i^{-1}(F)\) contains \(y_i\) and \(y_i\) forms an unbounded sequence.

Since \(g_i(l_i) \cap F \neq \emptyset\), we choose a subsequence of \(g_i\) and relabel it \(g_i\) so that \(\{g_i(l_i)\}\) converges to a nontrivial line \(l_\infty\) in \(\Omega\).

Remark 5.2 We need to only prove the following for a finite index group of \(\Gamma\) since the contracting properties are invariant under finite regular covering maps. Hence, we may assume that each element is proximal or semi-proximal by Theorem 2.7.

By our choice of \(l_i, y_i, g_i\) as above, and Remark 5.2, we may assume without loss of generality that each \(g_i\) is positive bi-proximal since \(\Omega\) is strictly convex.

We choose a subsequence of \(\{g_i\}\) so that the sequences \(\{a_i\}\) and \(\{r_i\}\) are convergent for the attracting fixed point \(a_i \in \text{Cl}(\Omega)\) and the repelling fixed point \(r_i \in \text{Cl}(\Omega)\) of each \(g_i\). Then
\{a_i\} \rightarrow a_* \text{ and } \{r_i\} \rightarrow r_* \text{ for } a_*, r_* \in \partial \Omega.

(See Figure 5.1.)

Suppose that \(a_* = r_*\). Then we choose an element \(g \in \Gamma\) so that \(g(a_*) \neq r_*\) and replace the sequence by \(\{gg_i\}\) and replace \(F\) by \(F \cup g(F)\). The above uniform convergence condition still holds. Then for the new attracting fixed points \(a'_i\) of \(gg_i\), we have \(\{a'_i\} \rightarrow g(a_*)\) and the sequence \(\{r'_i\}\) of repelling fixed point \(r'_i\) of \(gg_i\) converges to \(r_*\) also by Lemma 6.3. Hence, we may assume without loss of generality that

\[a_* \neq r_*\]

by replacing our sequence \(g_i\).

**Lemma 5.3** Let \(\Gamma\) act properly discontinuously on a strictly convex domain \(\Omega\). Assume that \(g_i\) is a sequence of distinct positive bi-proximal elements of \(\Gamma\). Suppose that the sequence of attracting fixed point \(a_i\) and that of repelling fixed point \(r_i\) of \(g_i\) form sequences converging to distinct pair of points. Then

\[
\{\text{length}_{\Omega}(g_i)\} \rightarrow \infty \tag{5.11}
\]

**Proof** Since \(\{a_i\} \rightarrow a_*\) and \(\{r_i\} \rightarrow r_*\), the segment \(\overline{a_i r_i}\) passes a fixed compact domain \(U\) in \(\Omega\) for sufficiently large \(i\). Suppose that \(\text{length}_{\Omega}(g_i) < C\) for a constant \(C\). Then \(g_i(U)\) passes \(\overline{a_i r_i}\) for each \(i\). Hence, \(g_i(U)\) is a subset a ball of radius \(2L + C\). Since \(\{g_i\}\) form a sequence of mutually distinct elements, this contradicts the proper discontinuity of the action of \(\Gamma\).

Now, Lemma 5.4 shows that for every compact \(K \subset \text{Cl}(\Omega) - \{r_*\}\),

\[
\{g_i(K)\} \rightarrow \{a_*\} \tag{5.12}
\]

uniformly.

Suppose that both \(y_+, y_- \neq r_*\). Then \(\{g_i(l_i)\}\) converges to a singleton \(\{a_*\}\) by (5.12) and this cannot be since \(l_* \subset \Omega\). If

\[r_* = y_+ \text{ and } y_- \in \partial \Omega - \{r_*\},\]

then \(\{g_i(y_i)\} \rightarrow a_*\) by (5.12) again. Since \(g_i(y_i) \in F\), this is a contradiction. Therefore

\[r_* = y_- \text{ and } y_+ \in \partial \Omega - \{r_*\}.

Let \(d_i = \langle \mathbf{v}_{+i} \rangle\) denote the other endpoint of \(l_i\) from \(\langle \mathbf{v}_{-i} \rangle\).

- Since \(\langle \mathbf{v}_{-i} \rangle\) \(\rightarrow y_-\) and \(\{l_i\}\) converges to a nontrivial line \(l_*\), it follows that \(\{d_i = \langle \mathbf{v}_{+i} \rangle\}\) is in a compact set in \(\partial \Omega - \{r_*\}\), and \(\{d_i\} \rightarrow y_+\).
- Then \(\{g_i(d_i)\} \rightarrow a_*\) as \(\{d_i\}\) is in a compact set in \(\partial \Omega - \{r_*\}\).
- Thus, \(\{g_i(\langle \mathbf{v}_{-i} \rangle)\}\) \(\rightarrow y' \in \partial \Omega\) where \(a_* \neq y'\) holds since \(\{g_i(l_i)\}\) converges to a nontrivial line \(l_*\) in \(\Omega\) as we said shortly above.
5.2 The proximal flow.

Also, \( g_i \) has an invariant great sphere \( S^{n-2}_i \subset \text{bd} \) \( A^n \) containing the attracting fixed point \( a_i \) and sharply supporting \( \Omega \) at \( a_i \). Thus, \( r_i \) is uniformly bounded at a distance from \( S^{n-2}_i \) since \( \{ r_i \} \to y_- = r_* \) and \( \{ a_i \} \to a_* \) with \( S^{n-2}_i \) geometrically converging to a sharply supporting sphere \( S^{n-2}_* \) at \( a_* \).

Let \( \| \cdot \|_E \) denote the standard Euclidean metric of \( \mathbb{R}^n \).

- Since \( \{ \Pi_\Omega (y_i) \} \to y_- \), \( \Pi_\Omega (y_i) \) is also uniformly bounded away from \( a_i \) and the tangent sphere \( S^{n-1}_i \) at \( a_i \).
- Since \( \{ (v \cdot i) \} \to y_- \), the vector \( v \cdot i \) has the component \( v \cdot p_i \) parallel to \( r_i \) and the component \( v \cdot S_i \) in the direction of \( S^{n-2}_i \) where \( v \cdot i = v \cdot p_i + v \cdot S_i \).
- Since \( \{ r_i \} \to r_* = y_- \) and \( \{ (v \cdot i) \} \to y_- \), we obtain \( \| v \cdot S_i \|_E \to 0 \) and that

\[
\frac{1}{C} < \| v \cdot p_i \|_E < C
\]

for some constant \( C > 1 \).

- \( g_i \) acts by preserving the directions of \( S^{n-2}_i \) and \( r_i \).

Since \( \{ g_i((v \cdot i)) \} \) converging to \( y', y' \in \text{bd} \), is bounded away from \( S^{n-2}_i \) uniformly, we obtain that

- considering the homogeneous coordinates

\[
( (\mathcal{L}(g_i)(v^\phi) : \mathcal{L}(g_i)(v^p) ) ,
\]

we obtain that the Euclidean norm of

\[
\frac{\mathcal{L}(g_i)(v^p)}{\| \mathcal{L}(g_i)(v^p) \|_E}
\]

is bounded above uniformly.

Since \( r_i \) is a repelling fixed point of \( g_i \) and \( \| v^p \|_E \) is uniformly bounded above, \( \{ \mathcal{L}(g_i)(v^p) \} \to 0 \) by (5.3) and (5.11).

\[
\{ \mathcal{L}(g_i)(v^p) \} \to 0 \text{ implies } \{ \mathcal{L}(g_i)(v^\phi) \} \to 0
\]

for \( \| \cdot \|_E \). Hence, we obtain

\[
\| \mathcal{L}(g_i)(v \cdot i) \|_E \to 0. \quad (5.13)
\]

Recall that \( \mathcal{L}(\tilde{\Phi}) \) is the identity map on the second factor of \( U \Omega \times V^n \).

\[
g_i(\mathcal{L}(\tilde{\Phi})_t (y_i, v \cdot i)) = (g_i(y_i), \mathcal{L}(g_i)(v \cdot i)) \in F \times V_-
\]

is a vector over the compact fundamental domain \( F \) of \( U \Omega \).

\[
( g_i(y_i), \mathcal{L}(g_i)(v \cdot i))
\]

is a vector over the compact fundamental domain \( F \) of \( U \Omega \).
\{ \| \mathcal{L}(g_i)(\Phi_i)(x_i, v_{-j}) \|_E \} \to 0 \text{ implies } \{ \| \mathcal{L}(g_i)(\Phi_i)(x_i, v_{-j}) \|_{\text{fiber}} \} \to 0 \]

since \( g_i(x_i) \in F \) and \( \| \cdot \|_{\text{fiber}} \) and \( \| \cdot \|_E \) are compatible over points \( F \). Since \( g_i \) is an isometry of \( \| \cdot \|_{\text{fiber}} \), we conclude that \( \{ \| \mathcal{L}(\Phi_i)(x, v_{-j}) \|_{\text{fiber}} \} \to 0 \): \( \square \)

**Lemma 5.4** Let \( \Omega \) be strictly convex and \( C^1 \). We choose a subsequence \( \{g_i\} \) of positive-bi-proximal elements of \( \Gamma \) so that the sequences \( \{a_i\} \) and \( \{r_i\} \) are convergent for the attracting fixed point \( a_i \in \text{Cl}(\Omega) \) and the repelling fixed point \( r_i \in \text{Cl}(\Omega) \) of each \( g_i \). Suppose that

\[
\{a_i\} \to a_s \text{ and } \{r_i\} \to r_s \text{ for } a_s, r_s \in \text{bd}\Omega, a_s \neq r_s.
\]

Suppose that \( g_i \) is an unbounded sequence. Then for every compact \( K \subset \text{Cl}(\Omega) - \{r_s\} \),

\[
\{g_i(K)\} \to \{a_s\}
\]

uniformly.

**Proof** Each \( g_i \) acts on an \((n - 3)\)-dimensional subspace \( W_{g_i} \in S^{n-1}_\infty \) disjoint from \( \Omega \). Here, \( W_{g_i} \) is the intersection of two sharply supporting hyperspaces of \( \Omega \) at \( a_i \) and \( r_i \). The set \( \{W_{g_i}\} \) is precompact by our condition. By the \( C^1 \)-property, we may assume that \( \{W_{g_i}\} \to W_s \) for an \( n - 3 \)-dimensional subspace \( W_s \) that is the intersection of two hyperspaces supported at \( a_s \) and \( r_s \). Also, \( W_{g_i} \cap \text{Cl}(\Omega) = \emptyset \) by this property.

Let \( \eta_i \) denote the complete geodesic connecting \( a_i \) and \( r_i \). Let \( \eta_{\infty} \) denote the one connecting \( a_s \) and \( r_s \). Since \( W_s \) is the intersection of two sharply supporting hyperspaces of \( \Omega \) at \( a_s \) and \( r_s \), \( \eta_{\infty} \) has endpoints \( a_s, r_s \), and \( \Omega \) is strictly convex, it follows \( \langle \eta_{\infty} \rangle \cap W_s = \emptyset \).

We call \( P_i \cap \Omega \) for the \( n - 2 \)-dimensional subspace \( P_i \) containing \( W_{g_i} \) a slice of \( g_i \). The closure of a component of \( \Omega \) with a slice of \( g_i \) removed is called a half-space of \( g_i \).

Let \( H_i \) denote the half-space of \( g_i \) containing \( K \). Since \( \{\text{Cl}(\eta_i)\} \) and \( \{W_{g_i}\} \) are geometrically convergent respectively, and \( \langle \eta_{\infty} \rangle \cap W_s = \emptyset \), it follows that \( \{g_i(P_i)\} \) geometrically converges to a hyperspace containing \( W_s \) passing \( a_s \). Therefore, one deduces easily that \( \{g_i(H_i)\} \to \{a_s\} \) geometrically. Since \( K \subset H_i \), the lemma follows. \( \square \)

### 5.2.3 The neutralized section.

We denote by \( \Gamma(V) \) the space of sections \( U\Sigma \to V \) and by \( \Gamma(A) \) the space of sections \( U\Sigma \to A \). Recall from [96] the one parameter-group of operators \( D\Phi_{t,s} \) on \( \Gamma(V) \) and \( \Phi_{t,s} \) on \( \Gamma(A) \). In our terminology \( D\Phi_{t,s} = \mathcal{L}(\Phi_t) \). Recall Lemma 8.3 of [96] also. We denote by \( \phi \) the vector field generated by this flow on \( U\Sigma \).

A section \( s : U\Sigma \to A \) is neutralized if

\[
\nabla^A_{\phi} s \in V_0.
\]
Lemma 5.5 If \( \psi \in \Gamma(A) \), and
\[
 t \mapsto D\Phi_t\ast(\psi)
\]
is a path in \( \Gamma(V) \) that is differentiable at \( t = 0 \), then
\[
 \frac{d}{dt} \bigg|_{t=0} (D\Phi_t)\ast(\psi) = \nabla^A_{\phi}(\psi).
\]

Recall that \( \mathbb{U}\Sigma \) is a recurrent set under the geodesic flow.

Lemma 5.6 A neutralized section \( s_0 : \mathbb{U}\Sigma \to A \) exists. This lifts to a map \( \tilde{s}_0 : \mathbb{U}\Omega \to \tilde{A} \) so that \( \tilde{s}_0 \circ \gamma = \gamma \circ \tilde{s}_0 \) for each \( \gamma \) in \( \Gamma \) acting on \( \tilde{A} = \mathbb{U}\Omega \times \mathbb{A}^n \).

Proof Let \( s \) be a continuous section \( \mathbb{U}\Sigma \to A \). We construct \( \nabla^A_{\phi} \) by projecting the values of \( \nabla \) to \( V_\pm \) and \( \nabla^A_{\phi} \) by projecting the values of \( \nabla \) to \( V_0 \). We decompose
\[
 \nabla^A_{\phi}(s) = \nabla^A_{\phi^+}(s) + \nabla^A_{\phi^0}(s) + \nabla^A_{\phi^-}(s) \in V
\]
so that \( \nabla^A_{\phi^+}(s) \in V_\pm \) and \( \nabla^A_{\phi^0}(s) \in V_0 \) hold. This can be done since along the vector field \( \phi \), \( V_\pm \) and \( V_0 \) are constant bundles. By the uniform convergence property of (5.8) and (5.9), the following integrals converge to smooth functions over \( \mathbb{U}\Sigma \). Again
\[
 s_0 = s + \int_0^\infty (D\Phi_t)\ast(\nabla^A_{\phi^-}(s))dt - \int_0^\infty (D\Phi_{-t})\ast(\nabla^A_{\phi^-}(s))dt
\]
is a continuous section and \( \nabla^A_{\phi}(s_0) = \nabla^A_{\phi^0}(s_0) \in V_0 \) as shown in Lemma 8.4 of [96].

Since \( \mathbb{U}\Sigma \) is connected, there exists a fundamental domain \( F \) so that we can lift \( s_0 \) to \( \tilde{s}_0 \) defined on \( F \) mapping to \( A \). We can extend \( \tilde{s}_0 \) to \( \mathbb{U}\Omega \to \mathbb{U}\Omega \times \mathbb{A}^n \). \( \square \)

Let \( N_2(\mathbb{A}^n) \) denote the space of codimension two affine subspaces of \( \mathbb{A}^n \). We denote by \( G(\Omega) \) the space of maximal oriented geodesics in \( \Omega \). We use the quotient topology on both spaces. There exists a natural action of \( \Gamma \) on both spaces.

For each element \( g \in \Gamma \setminus \{I\} \), we define \( N_2(g) \). Now, \( g \) acts on \( \mathbb{B}\mathbb{A}^n \) with invariant subspaces corresponding to invariant subspaces of the linear part \( \mathcal{L}(g) \) of \( g \). Since \( g \) and \( g^{-1} \) are positive proximal,

- a unique fixed point in \( \mathbb{B}\mathbb{A}^n \) corresponds to the largest norm eigenvector, an attracting fixed point in \( \mathbb{B}\mathbb{A}^n \), and
- a unique fixed point in \( \mathbb{B}\mathbb{A}^n \) corresponds to the smallest norm eigenvector, a repelling fixed point.

By [20] or [17]. There exists an \( \mathcal{L}(g) \)-invariant vector subspace \( V_0^g \) complementary to the sum of the subspace generated by these eigenvectors. (This space equals \( V_0(u) \) for the unit tangent vector \( u \) tangent to the unique maximal geodesic \( \mathcal{l}_g \) in \( \Omega \) on which \( g \) acts.) It corresponds to a \( g \)-invariant subspace \( M(g) \) of codimension two in \( \mathbb{B}\mathbb{A}^n \).
Let $\mathcal{E}$ be the geodesic in $\mathbb{U}\Sigma$ that is $g$-invariant for $g \in \Gamma$. $\tilde{s}_0(\mathcal{E})$ lies on a fixed affine subspace parallel to $V\mathbb{g}^0$ by the neutrality, i.e., Lemma 5.6. There exists a unique affine subspace $N_2(g)$ of codimension two in $\mathbb{A}^n$ containing $\tilde{s}_0(\mathcal{E})$. Immediate properties are $N_2(g) = N_2(g^m), m \in \mathbb{Z} - \{0\}$ and that $g$ acts on $N_2(g)$.

**Definition 5.2** We define $S'(\text{bd}\Omega)$ the space of hyperspaces $P$ meeting $\mathbb{A}^n$ where $P \cap \text{bd}\mathbb{A}^n$ is a sharply supporting hyperspace in $\text{bd}\mathbb{A}^n$ to $\Omega$. We denote by $S'(\text{bd}\Omega)$ the space of pairs $(x,H)$ where $H \in S'(\text{bd}\Omega)$, and $x$ is in the boundary of $H$ and in $\text{bd}\Omega$.

Define $\Delta$ to be the diagonal set of $\text{bd}\Omega \times \text{bd}\Omega$. Denote by $\Lambda^* = \text{bd}\Omega \times \text{bd}\Omega - \Delta$.

Let $G(\Omega)$ denote the space of maximal oriented geodesics in $\Omega$. $G(\Omega)$ is in a one-to-one correspondence with $\Lambda^*$ by the map taking the maximal oriented geodesic to the ordered pair of its endpoints.

**Lemma 5.7** $G(\Omega)$ is a connected subspace.

**Proof** We obtain the proof by generalizing Lemma 1.3 of [96] using a bi-proximal element of $\Gamma$. [22]. Or one can use the fact that $\mathbb{U}\Omega$ is connected.

**Proposition 5.2**

- There exists a continuous function $\tilde{s} : \Omega \to N_2(\mathbb{A}^n)$ equivariant with respect to $\Gamma$-actions.
- Given $g \in \Gamma$ and for the unique unit-speed geodesic $1_g$ in $\Omega\Omega$ lying over a geodesic $l_g$ where $g$ acts on, $\tilde{s}(l_g) = N_2(g)$.
- This gives a continuous map

$$\tilde{s} : G(\Omega) = \text{bd}\Omega \times \text{bd}\Omega - \Delta \to N_2(\mathbb{A}^n)$$

again equivariant with respect to the $\Gamma$-actions. There exists a continuous function

$$\tau : \Lambda^* \to S'(\text{bd}\Omega).$$

**Proof** Given a vector $u \in \mathbb{U}\Omega$, we find $\tilde{s}_0(u)$. There exists a lift $\tilde{\phi} : \mathbb{U}\Omega \to \mathbb{U}\Omega$ of the geodesic flow $\phi$. Then $\tilde{s}_0(\tilde{\phi}(u))$ is in an affine subspace $H^{n-2}$ parallel to $V\mathbb{g}$ for $u$ by the neutrality condition (5.15). We define $\tilde{s}(u)$ to be this $H^{n-2}$.

For any unit vector $u'$ on the maximal (oriented) geodesic in $\Omega$ determined by $u$, we obtain $\tilde{s}(u') = H^{n-2}$. Hence, this determines the continuous map $\tilde{s} : G(\Omega) \to N_2(\mathbb{A}^n)$. The $\Gamma$-equivariance comes from that of $\tilde{s}_0$.

For $g \in \Gamma$, $u$ and $g(u)$ lie on the lift $1'_g$ of the $g$-invariant geodesic $1_g$ in $\mathbb{U}\Omega$ provided $u$ is tangent to $1_g$. Since $g(\tilde{s}_0(u)) = \tilde{s}_0(g(u))$ by equivariance, $g(\tilde{s}_0(u))$ lies on $\tilde{s}(u) = \tilde{s}(g(u))$ in $\Omega\Omega$ by the third paragraph before the proposition. We conclude $g(\tilde{s}(1'_g)) = \tilde{s}(1'_g)$, which shows $N_2(g) = \tilde{s}(1'_g)$.

The map $\tilde{s}$ is defined since $\text{bd}\Omega \times \text{bd}\Omega - \Delta$ is in one-to-one correspondence with the space $G(\Omega)$. The map $\tau$ is defined by taking for each pair $(x,y) \in \Lambda^*$

- we take the geodesic $l$ with endpoints $x$ and $y$, and
- taking the hyperspace containing $\tilde{s}(l)$ and its boundary containing $x$. 

□
5.2 The proximal flow.

5.2.4 The asymptotic niceness.

We denote by $h(x,y)$ the hyperspace part in $\tau(x,y) = (x,h(x,y))$.

Lemma 5.8 Let $U$ be a $\Gamma E$-invariant properly convex open domain in $\mathbb{R}^n$ so that $\partial U \cap \partial \mathbb{A}^n = \text{Cl}(\Omega)$. Suppose that $x$ and $y$ are attracting and repelling fixed points of an element $g$ of $\Gamma$ in $\partial U$. Then $h(x,y)$ is disjoint from $U$.

Proof Suppose not. $h'(x,y) := h(x,y) \cap \mathbb{A}^n$ is a $g$-invariant open hemisphere, and $x$ is an attracting fixed point of $g$ in it. (We can choose $g^{-1}$ if necessary.) Then $U \cap h(x,y)$ is a $g$-invariant properly convex open domain containing $x$ in its boundary.

Suppose first that $h'(x,y)$ has a limit point $z$ of $g^{-n}(u)$ for some point $u \in h'(x,y) \cap U$.

Here, $y$ is the smallest eigenvalue of the linear part of $g$ so that $y$ is the attracting fixed point of a component of $\mathbb{A}^n - h'(x,y)$ containing $U$ for $g^{-1}$. The antipodal point $y_-$ is the attracting fixed point of the other component $\mathbb{A}^n - h'(x,y)$. Take a ball $B$ in $U$ with a center $u$ in the convex set $U \cap h(x,y)$ Then $\{g^{-n}(u)\}$ converges to $z$ as $n \to \infty$. Let $u_1, u_2$ be two nearby points in $B$ so that $\overline{u_1u_2}$ is separated by $h'(x,y)$ and $\overline{u_1u_2} \cap h'(x,y) = u$. Then $\{g^{-n}(\overline{u_1u_2})\}$ geometrically converges to $\overline{xy} \cup \overline{y\infty}$ for some sequence $n_i$. Hence $\text{Cl}(U)$ cannot be properly convex.

If the above assumption does not hold, then an orbit $g^{-n}(u)$ for $u \in U \cap h'(x,y)$ has a limit point only in the boundary of $h'(x,y)$. Since $g$ is biproximal, $x$ is the repelling fixed point of $h'(x,y)$ under $g^{-1}$. Hence, a limit point $y'$ is never $x$ or $x_-$.

Since $y'$ is a limit point, $y' \in \text{Cl}(U)$. It follows $y' \in \text{Cl}(\Omega)$. Now, $x, y' \in \text{Cl}(\Omega)$ implies $\overline{xy'} \subset \partial \mathbb{A}^n \subset \text{Cl}(\Omega)$. Finally, $\overline{xy'} \subset \partial h'(x,y)$ for the sharply supporting subspace $\partial h'(x,y)$ of $\text{Cl}(\Omega)$ violates the strict convexity of $\Omega$. (See Definition 1.6 and Benoist [20].)

In Theorem 7.2, we will obtain that this also gives us strict lens p-end neighborhoods. We will generalize this to Theorem 7.2 where we won’t even need the existence of the properly convex domain $U$.

Lemma 5.9 Let $\Gamma$ satisfy the conditions as above. Let $\Gamma$ acts on strictly convex domain $\Omega$ with $\partial \Omega$ being $C^1$ in a cocompact manner. Let $(x,y) \in \partial \Omega$. Then

- $\tau(x,y)$ does not depend on $y$ and is unique for each $x$.
- $h'(x,y) := h(x,y) \cap \partial \mathbb{A}^n$ contains $\overline{\text{S}(xy)}$ but is independent of $y$.
- $h(x,y) = h(x)$ and $h(x)$ is the AS-hyperspace to $U$.
- $h'(x,y)$ is never a hemisphere in $\partial \mathbb{A}^n$ for every $(x,y) \in \mathbb{A}^n$.
- The map $\tau': \partial \Omega \to S(\partial \Omega)$ induced from $\tau$ is continuous.

Proof We claim that for any $x, y$ in $\partial \Omega$, $h'(x,y)$ is disjoint from $U$: By Theorem 1.1 of Benoist [20], the geodesic flow on $\Omega/\Gamma$ is Anosov, and hence the set of closed geodesics in $\Omega/\Gamma$ is dense in the space of geodesics by the basic property of the Anosov flow. Since the fixed points are in $\partial \Omega$, we can find sequences $\{x_i\}, \{x_i\} \to x$ and $\{y_i\}, \{y_i\} \to y$ where $x_i$ and $y_i$ are fixed points of an element $g_i \in \Gamma$ for each $i$. 


If \( h'(x, y) \cap U \neq \emptyset \), then \( h'((x_i, y_i)) \cap U \neq \emptyset \) for \( i \) sufficiently large by the continuity of the map \( \tau \) from Proposition 5.2. This is a contradiction by Lemma 5.8.

Also \( \text{bd}\mathbb{A}^n \) does not contain \( h'(x, y) \) since \( h'(x, y) \) contains the \( \bar{x}(\bar{y}) \) while \( y \) is chosen \( y \neq x \).

Let \( H(x, y) \) denote the half-space in \( \mathbb{A}^n \) bounded by \( h'(x, y) \) containing \( U \). \( \partial H(x, y') \) is sharply supporting \( \text{bd}\mathcal{O} \) and hence is independent of \( y' \) as \( \text{bd}\mathcal{O} \) is \( C^1 \). So, we have

\[
H(x, y) \subset H(x, y') \text{ or } H(x, y) \supset H(x, y').
\]

For each \( x \), we define

\[
H(x) := \bigcap_{y \in \text{bd}\mathcal{O} - \{x\}} H(x, y).
\]

Define \( h(x) \) as the hyperspace so that \( H(x) \) is a component of \( \mathbb{A}^n - h(x) \).

Let \( h'(x) = h(x) \cap \mathbb{A}^n \). Now, \( U' := \bigcap_{x \in \text{bd}\mathcal{O}} H(x) \) contains \( U \) by the above disjointedness. \( \text{bd}\mathcal{O} \) has at least \( n \) points in general position with their supporting hyperspaces in \( \text{bd}\mathbb{A}^n \). Since the corresponding supporting hemispheres and the hemisphere \( \text{Cl}(\mathbb{A}^n) \) are in general position, \( U' \) is properly convex. Let \( U'' \) be the properly convex open domain

\[
\bigcap_{x \in \text{bd}\mathcal{O}} (\mathbb{A}^n - \text{Cl}(H(x))).
\]

It has the boundary \( \mathcal{A}(\text{Cl}(\mathcal{O})) \) in \( \text{bd}\mathbb{A}^n \) for the antipodal map \( \mathcal{A} \). Since the antipodal set of \( \text{bd}\mathcal{O} \) has at least \( n \) points in general position and corresponding supporting hemispheres, \( U'' \) is a properly convex domain. Note that \( U' \cap U'' = \emptyset \).

If for some \( x, y \), \( h(x, y) \) is different from \( h(x) \) where \( x \) is a fixed point of some element of \( \Gamma \), then \( h'((x, y)) \cap U'' \neq \emptyset \). This is a contradiction by Lemma 5.8 where \( U \) is replaced by \( U'' \) and \( \mathcal{O} \) is replaced by \( \mathcal{A}(\mathcal{O}) \). Thus, we obtain \( h(x, y) = h(x) \) for all \( y \in \text{bd}\mathcal{O} - \{x\} \).

We show the continuity of \( x \mapsto h(x) \) at a fixed point \( x \) of \( g \in \Gamma \). Let \( \{x_i\}, x_i \in \text{bd}\mathcal{O} \) be a sequence converging to \( x \in \text{bd}\mathcal{O} \). Then choose \( y_i \in \text{bd}\mathcal{O} \) for each \( i \) so that \( \{y_i\} \to y \) and we have \( \{h(x_i) = h(x_i, y_i)\} \) converges to \( h(x, y) = h(x) \) by the continuity of \( \tau \) by Proposition 5.2.

Now, for general point \( x \in \text{bd}\mathcal{O} \), we claim that \( h(x_i) \to h(x) \) for any sequence \( \{x_i\}, x_i \in \text{bd}\mathcal{O} \) converging to \( x \). We claim that the hemisphere \( \{H(x_i)\} \to H(x) \): We may assume without loss of generality that \( H(x_i) \to H_0 \) geometrically by choosing a subsequence if necessary. Since \( \mathbb{S}^n - H(x_i) \) is disjoint from \( U' \), it follows that \( \mathbb{S}^n - H_0 \) is disjoint from \( U' \). Then it follows that \( H(x) \subset H_0 \). Conversely, \( H(x_i) \) is disjoint from \( U'' \). Similarly, we see that \( H_0 \subset H(x) \). Hence, \( H(x) = H_0 \). This implies \( \{h(x_i)\} \to h(x) \).

Since both \( h(x, y) \) and \( h(x) \) are continuous under the variable \( x \), the density of fixed points implies that \( h(x, y) = h(x) \) for all \( y \in \text{bd}\mathcal{O} - \{x\} \) for all \( x \in \text{bd}\mathcal{O} \).

Finally, we verify that \( h(x) \) is an AS-hyperspace to \( U \). Suppose not. Then we can find a different function \( \text{bd}\mathcal{O} \to S(\text{bd}\mathcal{O}) \) of form \( x \mapsto (x, h'(x)) \). Let \( H'(x) \) denote the hemisphere bounded by \( h'(x) \) containing \( \mathcal{O} \). We again construct
5.3 Generalization to nonstrictly convex domains

\[ U''' = \bigcap_{x \in \partial \Omega} (\mathbb{A}^n - \text{Cl}(H'(x))). \]

Using \( U' \) and \( U''' \), we again show that \( h' \) is continuous by the arguments immediately above.

For a fixed point \( x \) in \( \partial \Omega \), we must have \( h(x) = h'(x) \) since otherwise \( h'(x) \) meets \( U' \) or \( U'' \). Then by the density of fixed points, we obtain \( h'(x) = h(x) \).

**Proof (Proof of Theorem 5.1)** For each point \( x \in \partial \Omega \), an \((n-1)\)-dimensional hemisphere \( h(x) \) passes \( \mathbb{A}^n \) with \( \partial h(x) \subset \partial \mathbb{A}^n \) sharply supporting \( \Omega \) by Lemma 5.9. Then a hemisphere \( H(x) \subset \mathbb{A}^n \) is bounded by \( h(x) \) and contains \( \Omega \). The properly convex open domain \( \bigcap_{x \in \partial \Omega} H(x) \) contains \( U \) since \( U \subset H(x) \) for each \( x \in \partial \Omega \). Since \( \partial \Omega \) is \( C^1 \) and strictly convex, the uniqueness of \( h(x) \) in the proof of Lemma 5.9 gives us the unique AS-hyperspace.

Let \( U_1 \) be another properly convex open domain with the same properties as \( U \). Then each \( h(x), x \in \partial \Omega \) is disjoint from \( U_1 \) by Lemma 5.8. Hence, \( U_1 \subset U'' \) for the set \( U'' \) we constructed in the proof of Lemma 5.9. We can construct a \( \Gamma \)-invariant set of AS-hyperspaces of \( \partial \Omega \) so that we choose one closest to \( U_1 \) for each \( x, x \in \partial \Omega \). Suppose that this set is different from the \( \Gamma \)-invariant set of AS-hyperplanes constructed by Lemma 5.9. We can show that the corresponding function \( \tau': \mathbb{A}^n \to S(\partial \Omega) \) is continuous as we did in the last part of the proof of Lemma 5.9. Hence, \( \tau'(x) \neq \tau(x) \) for some fixed point \( x \) of \( g \) in \( \Gamma \) by the density of the fixed points in \( \mathbb{A}^n \). Then a new asymptotic hyperspace for a fixed point meets \( U'' \) constructed in the proof of Lemma 5.9. This is again contradiction by Lemma 5.8. \( \square \)

5.3 Generalization to nonstrictly convex domains

5.3.1 Main argument

Now, we drop the condition of hyperbolicity on \( \Gamma \). Hence, \( \Omega, \Omega \subset \partial \mathbb{A}^n \), is not necessarily strictly convex. Also, \( \Omega \) is allowed to be the interior of a strict join. Here, we don’t assume that \( \Gamma \) is not necessarily hyperbolic, and hence, it is more general. Also, we obtain an asymptotically nice properly convex domain \( U \) in \( \mathbb{A}^n \) where \( \Gamma \) acts properly on.

**Theorem 5.2** Let \( \Gamma \) have an affine action on the affine subspace \( \mathbb{A}^n, \mathbb{A}^n \subset \mathbb{S}^n \), acting on a properly convex domain \( \Omega \) in \( \partial \mathbb{A}^n \). Suppose that \( \Omega / \Gamma \) is a closed \( n - 1 \)-dimensional orbifold, and suppose that \( \Gamma \) satisfies the uniform middle-eigenvalue condition. Then \( \Gamma \) is asymptotically nice with respect to a properly convex open domain \( U \), and each sharply supporting subspace \( Q \) of \( \Omega \) in \( \mathbb{S}^{n-1}_\infty \) is contained in a unique AS-hyperspace to \( U \) transverse to \( \partial \mathbb{A}^n \).

The proof is analogous to Theorem 5.1. Now \( \Omega \) is not strictly convex and hence for each point of \( \partial \Omega \) there might be more than one sharply supporting hyperspace.
in $\partial \mathbb{A}^n$. We generalize $\mathbb{U}\Omega$ to the augmented unit tangent bundle

$$\mathbb{U}^{Ag}\Omega := \{ (x, H_a, H_b) \mid x \in \mathbb{U}\Omega \text{ is a direction vector at a point of a maximal oriented geodesic } l_x \text{ in } \Omega, \]

$$H_a \text{ is a sharply supporting hyperspace in } \partial \mathbb{A}^n \text{ at the starting point of } l_x, \]

$$H_b \text{ is a sharply supporting hyperspace in } \partial \mathbb{A}^n \text{ at the ending point of } l_x \}.$$ 

Here, we regard $x$ as a based vector and hence has information on where it is on $l$ and $H_a$ and $H_b$ is given orientations so that $\Omega$ is in the interior direction to them. This is not a manifold but a locally compact Hausdorff space and is a metrizable space. Since the set of sharply supporting hyperspaces of $\Omega$ at a point of $\partial \mathbb{A}^n$ is compact, $\mathbb{U}^{Ag}\Omega / \Gamma$ is a compact Hausdorff space fibering over $\Omega / \Gamma$ with compact fibers. The obvious metric is induced from $\Omega$ and the space $\mathbb{S}^n$ of oriented hyperspaces in $\mathbb{S}^n$.

We also write $\Pi^{Ag} : \mathbb{U}^{Ag}\Omega \to \Omega$ the obvious projection $\langle x, H_a, H_b \rangle = \Pi\Omega(x)$.

From Section 2.5, we recall the augmented boundary $\partial \mathbb{A}^n\Omega$. We define

$$\mathbb{A}^{Ag} = \partial \mathbb{A}^n\Omega \times \partial \mathbb{A}^n\Omega - (\Pi^{Ag} \times \Pi^{Ag})^{-1}(\mathbb{A}^{Ag})$$

where $\mathbb{A}^{Ag}$ is defined as the closed subset

$$\{(x, y) \mid x, y \in \partial \mathbb{A}^n, x = y \text{ or } \langle x, y \rangle \subset \partial \mathbb{A}^n\}.$$ 

Define $G^{Ag}(\Omega)$ denote the set of oriented maximal geodesics in $\Omega$ with endpoints augmented with the sharply supporting hyperspace at each endpoint. The elements are called augmented geodesics. There is a one to one and onto correspondence between $\mathbb{A}^{Ag}$ and $G^{Ag}(\Omega)$. We denote by $\langle x, h_1 \rangle \langle y, h_2 \rangle$ the complete geodesic in $\Omega$ with endpoints $x, y$ and sharply supporting hyperspaces $h_1$ at $x$ and $h_2$ at $y$.

**Lemma 5.10** $G^{Ag}(\Omega)$ is a connected subspace.

**Proof** We can use the fact that $\mathbb{U}^{Ag}\Omega$ is connected. \[\square\]

Now, we follow Section 5.1.1 and define $\mathbb{A} = \mathbb{U}^{Ag}\Omega \times \mathbb{A}^n, \mathbb{V} = \mathbb{U}^{Ag}\Omega \times V^n, \mathbb{V}$ by $\mathbb{U}^{Ag}\Omega \times \mathbb{A}^n / \Gamma$ and $V := \mathbb{U}^{Ag}\Omega \times V^n / \Gamma$ and corresponding subbundles $\mathbb{V}_+, \mathbb{V}_-, \mathbb{V}_0, V_+, V_-$, and $V_0$. We define the flows $\Phi, \Phi_+, \Phi_-, \mathfrak{L}(\Phi), \mathfrak{L}(\Phi_+), \mathfrak{L}(\Phi_-)$ by replacing $\mathbb{U}\Omega$ by $\mathbb{U}^{Ag}\Omega$ and geodesics by augmented geodesics and so on in an obvious way.

For each point $x = (x, H_a, H_b)$ of $\mathbb{U}^{Ag}\Omega$,

- we define $\mathbb{V}_+(x)$ to be the space of vectors in the direction of the backward endpoint of $l_x$,
- $\mathbb{V}_-(x)$ that for the forward endpoint of $l_x$,
- $\mathbb{V}_0(x)$ to be the space of vectors in directions of $H_a \cap H_b$.

For each $x \in \mathbb{U}^{Ag}\Omega$,

$$V^n = \mathbb{V}_+(x) \oplus \mathbb{V}_0(x) \oplus \mathbb{V}_-(x).$$
This gives us a decomposition, $V = V_+ \oplus V_0 \oplus V_-$, and $V = V_+ \oplus V_0 \oplus V_-$. Clearly, $V_+$ and $V_-$ are $C^0$-line bundles since the beginning and the endpoints depend continuously on points of $\mathbb{L}^\Lambda g \Omega$. Also, $V_0$ is the vector subspace of $\mathbb{R}^n$ whose directions of nonzero vectors form $H_0 \cap H_\rho$. Since $(H_\alpha, H_\beta)$ depends continuously on points of $\mathbb{L}^\Lambda g \Omega$, we obtain that $V_0$ is a continuous bundle on $\mathbb{L}^\Lambda g \Omega$.

Obviously, the geodesic flows exist on $\mathbb{L}^\Lambda g \Omega$ using the ordinary geodesic flow with respect to the geodesics and not considering the augmented boundary.

There exists constants $C, k > 0$ so that

$$\| \mathcal{L}(\Phi_t)(v) \|_{\text{fiber}} \geq \frac{1}{C} \exp(kt) \| v \|_{\text{fiber}} \quad \text{as } t \to \infty$$

(5.17)

for $v \in V_+$ and

$$\| \mathcal{L}(\Phi_t)(v) \|_{\text{fiber}} \leq C \exp(-kt) \| v \|_{\text{fiber}} \quad \text{as } t \to \infty$$

(5.18)

for $v \in V_-$. We prove this by proving $\{\| \mathcal{L}(\Phi_t)(v) \|_{V_-} \} \to 0$ uniformly as $t \to \infty$ i.e., Proposition 5.1 under the more general conditions that $\Omega$ is properly convex but not necessarily strictly convex. We generalize Lemma 5.2. We will repeat the strategy of the proof since it is important to check. However, the proof follows the same philosophy with some technical differences.

We first need

**Lemma 5.11** Let $g_j$ be a sequence of elements of $\Gamma$. Suppose that $a_n(h(g_j)) \to 0$ as $j \to \infty$. Then $a_{n+1}(g_j)/\lambda_{\nu_{\varepsilon}}(g_j) \to 0$ as $j \to \infty$ provided the sequence of word-length of $g_j$ goes to the infinity.

**Proof** Suppose that length$_\Omega(g_j)$ is bounded. Then we can conjugate $g_j$ by an element $k_j$ so that $k_j g_j k_j^{-1}$ sends an element of a fundamental domain $F$ of $\Omega$ to a point of a bounded distance from it. Hence, there are only finitely element $h_1, \ldots, h_l$. Hence $g_j = k_j^{-1} h_i k_j$ for $i_j = 1, \ldots, l$. Then $\lambda_{\nu_{\varepsilon}}(g_j) = \lambda_{\nu_{\varepsilon}}(h_i)$ taking finitely many values.

Since $a_{n+1}(g_j) = a_n(h(g_j))/\lambda_{\nu_{\varepsilon}}(g_j)^{1+\frac{1}{\varepsilon}}$, the conclusion follows in this condition.

It is now sufficient to consider when the sequence length$_\Omega(g_j)$ is converging to infinity. Then the uniform middle eigenvalue condition implies that

$$\exp(-C \text{length}_\Omega(g_j)) \leq \lambda_{n+1}(g_j)/\lambda_{\nu_{\varepsilon}}(g_j) \leq \exp(-C^{-1} \text{length}_\Omega(g_j))$$

implies that $\lambda_{n+1}(g_j)/\lambda_{\nu_{\varepsilon}}(g_j) \to 0$. Let $|s|_E$ denote the length of a segment in $\mathbb{R}^{n+1}$ under the fixed Euclidean metric. We have $\lambda_{n+1}(g_j) \geq a_{n+1}(g_j)$ since we can consider segments in a eigendirection of $g_j$ and we have at least that amount of shrinking of a segment and

$$a_{n+1}(g) = \min \left\{ \frac{\| g(s) \|_E}{|s|_E} \mid s \text{ is a segment in } \mathbb{R}^{n+1} \right\}.$$
Hence, the result follows. (This is related to [34], [27] and [29].) \hfill \Box

**Lemma 5.12** Assume that $\Omega$ is properly convex and $\Gamma$ acts properly discontinuously satisfying the uniform middle eigenvalue condition with respect to $\partial \mathbb{R}^n$. Then \( \| \Phi_t \|_{\text{fiber}} \rightarrow 0 \) uniformly as $t \rightarrow \infty$.

**Proof** We proceed as in the proof of Lemma 5.2. It is sufficient to prove the uniform convergence to 0 by the compactness of $\mathcal{V}_-$. Let $F$ be a fundamental domain of $\mathbb{R}^n$. It is sufficient to prove this for $\mathcal{L}(\partial \Phi_t)$ on the fibers of over $F$ of $\mathbb{R}^n$ with a fiberwise metric $\| \cdot \|_{\text{fiber}}$.

We choose an arbitrary sequence $\{ x_i \} \subset \mathbb{R}^n$, \( \{ x_i \} \rightarrow x \) in $F$ where $a_i, b_i$ are the backward and forward point of the maximal oriented geodesic passing $x_i$ in $\Omega$. For each $i$, let $v_{-i}$ be a Euclidean unit vector in $\mathbb{R}^n$. Let $\mathcal{V}_-$ be a fundamental domain of $\mathbb{R}^n$. That is, $\mathcal{V}_-$ is in the 1-dimensional subspace in $\mathbb{R}^n$, corresponding to the forward endpoint of the geodesic determined by $x_i$ in $\Omega$.

Let $x_i$ be as above converging to $x$ in $F$. Here, $\| \cdot \|_{\text{fiber}}$ is an endpoint of $l_i$ in the direction given by $x_i$. For this, we just need to show that any sequence of $\{ t_i \} \rightarrow \infty$ has a subsequence $\{ t_j \}$ so that \( \| \mathcal{L}(\Phi_{t_j})(x_i, v_{-j}) \|_{\text{fiber}} \rightarrow 0 \). This follows since if the uniform convergence did not hold, then we can easily find a sequence without such subsequences.

Let $y_i := \Phi_{t_i}(x_i)$ for the lift of the flow $\Phi$. By construction, we recall that each $\Pi^\mathbb{R}_{\partial \Phi_t}(y_i)$ is in the geodesic $l_i$. Since we have the sequence $\{ x_i \} \rightarrow x$, we obtain that $\{ l_i \}$ is in the 1-dimensional subspace in $\mathbb{R}^n$, and it is bounded by the forward endpoint of the geodesic determined by $x_i$ in $\mathbb{R}^n$. Let $y_+$ and $y_-$ be the endpoints of $l_i$ such that $\Pi^\mathbb{R}_{\partial \Phi_t}(y_i) \rightarrow y_-$. Hence,

\[
\| \mathcal{V}_{+i} \| \rightarrow y_+, \| \mathcal{V}_{-i} \| \rightarrow y_-
\]

(See Figure 5.1 for the similar situation.)

Find a deck transformation $g_i$ so that $g_i(y_i) \in F$, and $g_i$ acts on the line bundle $\mathcal{V}_-$ by the linearization of the matrix of form (5.1):

\[
g_i : \mathcal{V}_- \rightarrow \mathcal{V}_-
\]

\[\langle y_i, v \rangle \rightarrow \langle g_i(y_i), \mathcal{L}(g_i)(v) \rangle\]

\[
\mathcal{L}(g_i) := \frac{1}{\lambda^E(g_i)^2 + \frac{1}{\pi}} \hat{h}(g_i) : \mathcal{V}_-(y_i) = V_-(x_i) \rightarrow \mathcal{V}_-(g_i(y_i))
\]

(5.19)

We will show $\{ g_i(y_i), \mathcal{L}(g_i)(v_{-j}) \} \rightarrow 0$ under $\| \cdot \|_{\text{fiber}}$. This will complete the proof since $g_i$ acts as isometries on $\mathcal{V}_-$ with $\| \cdot \|_{\text{fiber}}$.

Since $g_i(y_i) \in F$ for every $i$, we obtain

\[g_i(l_i) \cap F \neq \emptyset.
\]

Since $g_i(l_i) \cap F \neq \emptyset$, we choose a subsequence of $g_i$ and relabel it $g_i$ so that $\{ \Pi^\mathbb{R}_{\partial \Phi_t}(g_i(l_i)) \}$ converges to a nontrivial line $\hat{l}_\infty$ in $\Omega$. 

...
Remark 5.2 shows that we may assume without loss of generality that each element of $\Gamma$ is positive bi-semi-proximal.

We recall facts from Section 2.3.3. Given a generalized convergence sequence $g_i$, we obtain an endomorphism $g_\infty$ in $M_n(\mathbb{R})$ so that $\{g_i\} \to g_\infty \in S(M_n(\mathbb{R}))$.

Recall

$$A_+(\{g_i\}) := S(\text{Im} g_\infty) \cap \text{Cl}(\Omega) \quad \text{and} \quad N_+(\{g_i\}) := S(\ker g_\infty) \cap \text{Cl}(\Omega).$$

We have $A_+(\{g_i\}), N_+(\{g_i\}) \subset \text{bd}\Omega$ are both nonempty by Theorem 2.10.

Up to a choice of subsequence, Theorem 2.8 implies that for any compact subset $K$ of $\text{Cl}(\Omega) - N_+(\{g_i\})$, there is a convex compact subset $K_*$ in $A_+$,

$$\{g_i(K)\} \to K_* \subset A_+.$$  \hfill (5.20)

Suppose that $y_- \in \text{Cl}(\Omega) - N_+(\{g_i\})$. Then $\{g_i(y_i)\} \to \hat{y} \in A_+(\{g_i\})$ since $y_i$ are in a compact subset of $\text{Cl}(\Omega) - N_+(\{g_i\})$ and (5.20). This is a contradiction since $g_i(y_i) \in F$. Hence, $y_- \in N_+(\{g_i\})$.

Let $d_i = \langle v_{+,i} \rangle$ denote the other endpoint of $l_i$ than $\langle v_{-,i} \rangle$ as above. Let $d_\infty$ denote the limit of $d_i$ in $\text{bd}\Omega$. We deduce as above up to a choice of a subsequence:

- Since $\{\langle v_{-,i} \rangle\} \to y_-, y_- \in N_+(\{g_i\})$ and $\{l_i\}$ converges to a nontrivial line $l_\infty \subset \Omega$ and $N_+(\{g_i\})$ is compact convex in $\text{bd}\Omega$, it follows that $\{d_i\}$ is in a compact set in $\text{bd}\Omega - N_+(\{g_i\})$.  

Fig. 5.2 The figure for Lemma 5.12.
Then \( \{ g_i(d_i) \} \to a_* \in A_\ast(\{ g_i \}) \) by (5.20), since \( \{ d_i \} \) is in a compact set in \( \text{bd}\Omega - N_\ast(\{ g_i \}) \).

Thus, \( \{ g_i((v_\ldots)) \} \to y' \in \text{bd}\Omega - A_\ast(g_i) \) holds since \( \{ g_i(l_i) \} \) converges to a nontrivial line in \( \Omega \).

Let \( m_\ast \) be obtained for \( \{ g_i \} \) as in Theorem 2.9. Recall that \( \| \cdot \|_E \) denote the standard Euclidean metric of \( \mathbb{R}^n \). Write \( g_i = k_i D_i \tilde{k}_i^{-1} \) for \( k_i, \tilde{k}_i \in O(n, \mathbb{R}) \), and \( D_i \) is a positive diagonal matrix of determinant \( \pm 1 \) with nonincreasing diagonal entries.

Since \( \{ (v_\ldots) \} \to y_\ldots \), the vector \( v_\ldots \) has the component \( v^E_i \) parallel to \( N^E(g_i) = \hat{k}_i(\mathbb{S}([m_\ast + 1, n])) \) and the component \( v^M_i \) in the orthogonal complement \( (N^p(g_i))^\perp = \hat{k}_i(\mathbb{S}([1, m_\ast])) \) where \( v_\ldots = v^E_i + v^M_i \). We may require \( \| v_\ldots \|_E = 1 \). (We remark \( \{ \| v^E_i \|_E \} \to 1 \) and \( \{ \| v^M_i \|_E \} \to 0 \) since \( v_\ldots \) converges to a point of \( N_\ast(\{ g_i \}) \).

- \( \{ \| s \mathcal{L}(g_i)(v^E_i) \|_E \} \to 0 \) by Theorem 2.9 and Lemma 5.11 since the sequence of the word length of \( g_i \) goes to infinity while \( g_i \) moves a point far away to a point of the fundamental domain.

- Since \( \{ g_i((v_\ldots)) \} \) converges to \( y', y' \in \text{bd}\Omega - A_\ast(g_i) \), \( \{ g_i((v_\ldots)) \} \) is uniformly bounded away from \( A_\ast(\{ g_i \}) \).

Because of the orthogonal decomposition \( \hat{k}_i(\mathbb{S}([m_\ast + 1, n])) \) and \( \hat{k}_i(\mathbb{S}([1, m_\ast])) \), and the fact that \( g_i = k_i D_i \tilde{k}_i^{-1} \), and \( \{ \| s \mathcal{L}(g_i) \|_E \} \to \| g_{\ast} \|_E \) in \( \mathbb{S}(M_n(\mathbb{R})) \), it follows that \( \{ \mathcal{L}(g_i)(v^E_i) \} \) either converges to zero, or \( \{ \| \mathcal{L}(g_i)(v^E_i) \|_E \} \) converges to \( A_\ast(\{ g_i \}) \) by Theorem 2.9. We have

\[
\{ \mathcal{L}(g_i)(v^E_i) \} \to 0 \implies \{ \mathcal{L}(g_i)(v^E_i) \} \to 0
\]

for \( \| \cdot \|_E \) since otherwise \( \{ \| \mathcal{L}(g_i)(v^E_i) \|_E \} \) converges to a point of \( A_\ast(\{ g_i \}) \subset F_\ast(\{ g_i \}) \) and hence \( \{ g_i((v_\ldots)) \} \) cannot be converging to \( y' \).

Hence, we obtain \( \{ \mathcal{L}(g_i)(v_\ldots) \} \to 0 \) under \( \| \cdot \|_E \). Now, we can deduce the result as in the final part of the proof of Lemma 5.2.

Now, we find as in Section 5.2.3 the neutralized section \( s : \mathbb{U}^A\Sigma \to A \) with \( \nabla^A s \in \mathbb{V}_0 \).

Since we are looking at \( \mathbb{U}^A\Omega \), the section \( s : \mathbb{U}^A\Omega \to N_2(A^n) \), we need to look at the boundary point and a sharply supporting hyperspace at the point and find the affine subspace of dimension \( n - 2 \) in \( \mathbb{R}^n \), generalizing Proposition 5.2. We generalize Definition 5.2:

**Definition 5.3** We denote by \( S^A(\text{bd}A^n) \) the space of pairs \( ((x, H \cap \text{bd}A^n), H) \) where \( H \in S'(\text{bd}A^n) \), and \( x \) is in the boundary of \( H \) and \( (x, H \cap \text{bd}A^n) \in \text{bd}A^n \).

**Proposition 5.3**
- There exists a continuous function \( \hat{s} : \mathbb{U}^A\Omega \to N_2(A^n) \) equivariant with respect to \( \Gamma \)-actions.
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- Given $g \in \Gamma$ and for the unique unit-speed geodesic $l_g$ in $\mathbb{U}^A \Omega$ lying over an augmented geodesic $l_g$ where $g$ acts on, $\delta(l_g) = \{ N_z(g) \}$. 
- This gives a continuous map
  
  $$s^A_g : \text{bd}^A \Omega \times \text{bd}^A \Omega - (\Pi^A \times \Pi^A)^{-1}(\Delta^A) \rightarrow N_z(\mathbb{A}^n)$$

  again equivariant with respect to the $\Gamma$-actions. There exists a continuous function

  $$\tau^A : \Lambda^A \rightarrow S^A(\text{bd} \Omega).$$

  **Proof** The proof is entirely similar to that of Proposition 5.2 but using a straightforward generalization of Lemma 5.6.

  We generalize Proposition 5.9. We will define $\tau' : \mathbb{U}^A \Omega \rightarrow S^A(\text{bd} \Omega)$ as a composition of $\tau^A$ with the map from $\mathbb{U}^A \Omega$ to $\Lambda^A$. This is a continuous map. Here, we don’t assume that $\Gamma$ acts on a properly convex domain in $\mathbb{A}^n$ with boundary $\Omega$. Hence, it is more general and we need a different proof. We just need that the orbit closures are compact.

**Lemma 5.13** Let an affine group $\Gamma$ acts on an affine subspace $\mathbb{A}^n$ on a properly convex domain $\Omega$ in the boundary of an affine subspace $\mathbb{A}^n$. Let $\Gamma$ acts on a properly convex domain $\Omega$ with a cocompact and Hausdorff quotient and satisfies the uniform middle eigenvalue condition with respect to $\text{bd} \mathbb{A}^n$. Let $((x, h_1), (y, h_2)) \in \Lambda^A$. Then

  - $\tau^A((x, h_1), (y, h_2))$ does not depend on $(y, h_2)$ and is unique for each $(x, h_1)$.
  - $h((x, h_1), (y, h_2))$ contains $s^A((x, h_1), (y, h_2))$ but is independent of $(y, h_2)$.
  - $h((x, h_1), (y, h_2))$ is never a hemisphere in $\text{bd} \mathbb{A}^n$ for every $((x, h_1), (y, h_2)) \in \Lambda^A$.
  - $\tau^A$ induces a map $\tau^A : \text{bd}^A \Omega \rightarrow S^A(\text{bd} \Omega)$ that is continuous.
  - There exists an asymptotically nice convex $\Gamma$-invariant open domain $U$ in $\mathbb{A}^n$ with $\text{bd} U \cap \partial \mathbb{A}^n = \text{Cl}(\Omega)$. For every $(x, h_1) \in \text{bd}^A \Omega$, $\tau(x, h_1)$ is an AS-hyperplane of $U$.

  **Proof** Let $l_1$ be an augmented geodesic in $\mathbb{U}^A \Omega$ with endpoints $(x, h_1)$ and $(\xi, h_2)$ oriented towards $x$. Consider a connected subspace $\mathcal{L}_{(x, h_1)}$ of $\mathbb{U}^A \Omega$ of points of maximal augmented geodesics in $\Omega$ ending at $(x, h_1)$. The space of geodesic leaves in $\mathcal{L}_{(x, h_1)}$ is in one-to-one correspondence with $\text{bd}^A \Omega - \Pi^A^{-1}(K_x)$ for the maximal flat $K_x$ in $\text{bd} \Omega$ containing $x$. We will show that $\tau^A$ is locally constant on $\mathcal{L}_{(x, h_1)}$ showing that it is constant.

  Let $\tilde{l}_1$ denote the lift of $l_1$ in $\mathbb{U}^A \Omega$. Let $S$ be a compact neighborhood in $\mathcal{L}_{(x, h_1)}$ of a point $y$ of $\tilde{l}_1$ transverse to $\tilde{l}_1$. Any two rays of geodesic flow $\Phi : S \times \mathbb{R} \rightarrow \mathbb{U}^A \Omega$ are asymptotic on $\mathcal{L}_{(x, h_1)}$ by Lemma 1.2.

  Let $y \in \tilde{l}_1$. Consider another point $y' \in S \subset \mathbb{U}^A \Omega$ with with endpoints $x$ and $\xi'$ in a sharply supporting hyperplane $h_2'$. where

  $$((x, h_1), (\xi, h_2)), ((x, h_1), (\xi', h_2')) \in \Lambda^A.$$
Choose a fixed fundamental domain $F$ of $\hat{U}\Omega$. Let $\{y_i = \Phi_i(y)\}, y_i \in \tilde{n}$, be a sequence whose projection under $\Pi_\Omega$ converges to $x$. We use a deck transformation $g_i$ so that $g_i(y_i) \in F$. Then $g_i(\tau^{Ag}(l_1)) = \tau^{Ag}(g_i(l_1))$ is a hyperspace containing $g_i(x)$ and $g_i(l_1)$ and $\tilde{s}(g_i(l_1))$.

Let $v_+$ denote a vector in the direction of the end of $l_1$ other than $x$. Equation (5.17) shows that $\{\|\mathcal{L}(\Phi_i)\|_{v_+} \} \rightarrow \infty$ as $i \rightarrow \infty$. Since $g_i$ is isometry under $\|\cdot\|_{\text{fiber}}$, and $\Phi_i(y) = y_i$ and $g_i(y_i)$, it follows that the $\mathcal{V}_+$-component of $g_i(y_i, v_+)$ satisfies

$$\{\|\mathcal{L}(g_i)\|_{v_+} \} \rightarrow \infty. \tag{5.21}$$

Since $g_i(y_i) \in F$ and under the Euclidean norm since over a compact set $F$ the metrics are compatible by a uniform constant, we obtain $\{\|\mathcal{L}(g_i)\|_E \} \rightarrow \infty$.

Since the affine hyperplanes in $\tau^{Ag}((x, h_1), (z, h_2))$ and $\tau^{Ag}((x, h_1), (z', h_2'))$ contain $x$ and $h_1$ in their boundary, they restrict to parallel affine hyperplanes in $\mathcal{A}$. Suppose that the affine hyperspace part of $\tau^{Ag}(((x, h_1), (z, h_2))$ differs from one of $\tau^{Ag}((x, h_1), (z', h_2'))$ by a translation by a constant times $v_+$. This implies that the sequence of the Euclidean distances between the respective affine hyperspaces corresponding to

$$g_i(\tau^{Ag}((x, h_1), (z, h_2)))$$

and $g_i(\tau^{Ag}((x, h_1), (z', h_2')))$$

goes to infinity as $i \rightarrow \infty$.

Now consider $\Phi(S \times [t_i, t_i + 1]) \subset U^{Ag}\Omega$, and we have obtained $g_i$ so that $g_i(\Phi(S \times [t_i, t_i + 1]))$ is in a fixed compact subset $\tilde{P}$ of $U^{Ag}\Omega$ by the uniform boundedness of $\Phi(S \times [t_i, t_i + 1])$ shown in the second paragraph of this proof. There is a map $E : U^{Ag}\Omega \rightarrow \Lambda^{Ag}$ given by sending the vector in $U^{Ag}\Omega$ to the ordered pair of endpoints and supporting hyperspaces of the geodesic passing the vector. Since $\tilde{s}$ is continuous, $\tau^{Ag} \circ E(\tilde{P})$ is uniformly bounded. The above paragraph shows that the sequence of the diameters of $\tau^{Ag} \circ E(g_i(S \times [t_i, t_i + 1]))$ can become arbitrarily large. This is a contradiction. Hence, $\tau^{Ag}_i$ is constant on $\mathcal{L}(v, h_1)$.

This proves the first two items. The fourth item follows since $\tau^{Ag}$ is an induced map. The image of $\tau^{Ag}$ is compact since $bd\Omega$ is compact. This implies the third item.

Define $H(x, h_1)$ to be the open $n$-dimensional hemisphere in $\mathbb{S}^n$ bounded by the great sphere containing the affine hyperspace $\tau^{Ag}(x, h_1)$ and containing $\Omega$. We define

$$U := \bigcap_{(x, h_1) \in bd^{Ag}\Omega} H(x, h_1) \cap \mathcal{A}^n.$$  

Now, we show that the affine hyperspace part of $\tau^{Ag}(x, h_1)$ is an AS-hyperspace for $U$: Suppose that for $(x, h_1) \in bd^{Ag}\Omega$, the AS-hyperspace $Q$ with $Q \cap bd\mathcal{A}^n = h_1$ and $Q \neq \tau^{Ag}(x, h_1)$. Then the hemisphere $H_Q$ bounded by $Q$ contains $U$. By definition, $H_Q \subset H(x, h_1)$. Then again we choose a segment $l_1$ ending at $x$. Then we choose sequences $g_i$ as above in the proof before (5.21). This shows as above the sequence of the Euclidean distances between the respective affine hyperspace parts of
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\[ g_i(\tau^{Ag}(x, h_1)) \text{ and } g_i(Q) \]

goes to infinity. Proposition 5.3 shows that \( \{g_i(\tau^{Ag}(x, h_1))\} \) is in the image of \( \tau^{Ag} \), a compact set. The set of supporting hyperplanes of \( U \) is bounded away from \( \text{bd} \mathbb{A}^n \) since they have to be between those of the image of \( \tau^{Ag} \) and \( U \). Since \( g_i(Q) \) is still a supporting hyperplane of \( U \), the equation is a contradiction. □

Proof (Proof of Theorem 5.2) First, we obtain a properly convex domain where \( \Gamma \) acts. Since \( \text{bd}^{Ag} \Omega \) is compact, the image under \( \tau^{Ag} \) is compact. Then \( U := \bigcap_{(x,h) \in \text{bd}^{Ag} \Omega} H_{(x,h)}^n \cap \mathbb{A}^n \) contains \( \Omega \). This is an open set since the compact set of \( H_{(x,h)} \), \( (x, h) \in \text{bd}^{Ag} \Omega \) has a lower bound on angles with \( \text{bd} \mathbb{A}^n \). Thus, \( U \) is asymptotically nice. Now, the proof is identical with that of Theorem 5.1 with Lemma 5.13 replacing Lemma 5.9.

5.3.2 Uniqueness of AS-hyperplanes

Finally, we end with some uniqueness properties.

Theorem 5.3 Let \( (\Gamma, U, D) \) be a properly convex affine triple. Suppose that \( \Gamma \) satisfies the uniform middle-eigenvalue condition. Then the set of AS-planes for \( U \) containing all sharply supporting hyperspaces of \( \Omega \) in \( \text{bd} \mathbb{A}^n \) is independent of the choice of \( U \).

Proof Let \( U' \) be a properly convex domain with the same properties as \( U \) stated in the premise. For each \( x, x \in \text{bd} \Omega \), and \( h \) be a supporting hyperspace of \( \Omega \) in \( \text{bd} \mathbb{A}^n \). Let \( S_{(x,h)} \) be the AS-hyperspace in \( \mathbb{S}^n \) for \( U \) so that \( S_{(x,h)} \cap \text{bd} \mathbb{A}^n = h \). Again for each \( (x, h) \), we let \( S'_{(x,h)} \) is the AS-hyperspace in \( \mathbb{S}^n \) for \( U' \) so that \( S'_{(x,h)} \cap \text{bd} \mathbb{A}^n = h \).

Since \( U \) and \( U' \) are both asymptotically nice, the sets of AS-hyperplanes are compact. For each \( (x, h) \), \( S_{(x,h)} \) and \( S'_{(x,h)} \) differ by a uniformly bounded distance in \( \mathbb{S}^{n+} \).

Suppose that \( S_{(x,h_1)} \) is different from \( S'_{(x,h_1)} \) for some \( x, h_1 \in \text{bd} \Omega \). Now, we follow the argument in the proof of Lemma 5.13. We again obtain a sequence \( g_i \in \Gamma \) so that

\[ g_i(S_{(x,h_1)} \cap \mathbb{A}^n) \text{ and } g_i(S'_{(x,h_1)} \cap \mathbb{A}^n) \]

are parallel affine planes, and the sequence of their Euclidean distances are going to \( \infty \) as \( i \to \infty \). By compactness, we know both sequences \( \{g_i(S_{(x,h_1)} \cap \mathbb{A}^n)\} \) and \( \{g_i(S'_{(x,h_1)} \cap \mathbb{A}^n)\} \) respectively converge to two hyperplanes up to a choice of a subsequence. This means that their Euclidean distances are uniformly bounded. Again (5.21) contradicts this. □

We question whether any properly convex open domain \( U \) with above properties must be always asymptotically nice. We proved this when \( \Gamma \) is hyperbolic in Theorem 5.1.
Remark 5.3 Theorems 5.1 and 5.2 also generalize to the case when $\Gamma$ acts on $\Omega$ as convex cocompact group: i.e., there is a convex domain $C \subset \Omega$ so that $C/\Gamma$ is compact but not necessarily closed. We work on the set of geodesics in $C$ only and the set $\Lambda$ of endpoints of these. In this case the limit set $\Lambda$ may be disconnected. The definition such as asymptotic niceness should be restricted to points of $\Lambda$ only. Here we do not need the connectedness of Lemmas 5.7 and 5.10 to be generalized to this case. However, the proofs indicated there will work.

5.4 Lens type $T$-ends

5.4.1 Existence of lens-neighborhood

The following is a consequence of Theorems 5.1 and 5.2.

**Theorem 5.4** Let $\Gamma$ be a discrete group in $\text{SL}_\pm(n+1, \mathbb{R})$ (resp. $\text{PGL}(n+1, \mathbb{R})$) acting on a properly convex domain $\Omega$ cocompactly and properly, $\Omega \subset \text{bd} \mathbb{A}^n \subset S^n$ (resp. $\subset \mathbb{R}P^n$), so that $\Omega/\Gamma$ is a closed $n$-orbifold.

- Suppose that $\Omega$ has a $\Gamma$-invariant open domain $U$ forming a neighborhood of $\Omega$ in $\mathbb{A}^n$.
- Suppose that $\Gamma$ satisfies the uniform middle eigenvalue condition with respect to $\text{bd} \mathbb{A}^n$.
- Let $P$ be the hyperspace containing $\Omega$.

Then $\Gamma$ acts on a properly convex domain $L$ in $S^n$ (resp. in $\mathbb{R}P^n$) with strictly convex boundary $\partial L$ such that 

$$ \Omega \subset L \subset U, \partial L \subset S^n - P \text{ (resp. } \subset \mathbb{R}P^n - P). $$

Moreover, $L$ satisfies $\text{bd} \partial L \subset P$ and $L/\Gamma$ is a lens-orbifold.

**Proof** We prove for $S^n$ first. We will just prove for the general case since the case when $\Omega$ is strictly convex and $C^1$, the augmented boundary is given as the set of all $(x, h)$ where $x$ is in $\text{bd} \Omega$ and $h$ is the unique supporting hyperspace of $\Omega$ at $x$. Assume without loss of generality that $U$ is an asymptotically nice open domain for $\Gamma$.

For each $(x, h)$ in the augmented boundary of $\Omega$, define a half-space $H(x, h) \subset \mathbb{A}^n$ bounded by $\tau A^g(x, h)$ and containing $\Omega$ in the boundary. For each $H(x, h)$, $(x, h) \in \text{bd} \mathbb{A}^g \Omega$, the proofs of Theorems 5.1 and 5.2, an open $n$-hemisphere $H'(x, h) \subset S^n$ satisfies $H'(x, h) \cap \mathbb{A}^n = H(x, h)$. Then we define

$$ V := \bigcap_{(x, h) \in \text{bd} \mathbb{A}^g \Omega} H'(x, h) \subset S^n $$

is a convex open domain containing $\Omega$ as in the proof of Lemma 5.9.
Lemma 5.14 Let $\Gamma$ be a discrete group in $\text{SL}_+ (n+1, \mathbb{R})$ acting on a properly convex domain $\Omega$. $\Omega \subset \text{bd} \mathbb{A}^n$, so that $\Omega / \Gamma$ is a closed $n$-orbifold. Suppose that $\Gamma$ satisfies
the uniform middle eigenvalue condition with respect to \( \bd \Lambda^n \) and acts on a properly convex domain \( V \) in \( S^n \) so that \( \Cl(V) \cap \bd \Lambda^n = \Cl(\Omega) \) holds. Suppose that \( \gamma_i \) is a sequence of mutually distinct elements of \( \Gamma \) acting on \( \Omega \). Let \( J \) be a compact subset of \( V \). Then \( \{ g_i(J) \} \) can accumulate only to a subset of \( \Cl(\Omega) \).

**Proof** Since \( \Omega / \Gamma \) is compact, \( h(g_i)_i \) is an unbounded sequence of elements in \( \SL_+(n, \R) \). Recalling (5.1), we write the elements of \( g_i \) as

\[
\begin{pmatrix}
\frac{1}{\hat{\lambda}_E(g_i)} \hat{h}(g_i) & b_{g_i} \\
0 & \hat{\lambda}_E(g_i)
\end{pmatrix}
\]

(5.22)

where \( b_{g_i} \) is an \( n \times 1 \)-vector and \( h(g_i) \) is an \( n \times n \)-matrix of determinant \( \pm 1 \) and \( \hat{\lambda}_E(g_i) > 0 \). Let \( m(h(g_i)) \) denote the maximal modulus of the entry of the matrix of \( h(g_i) \) in \( \SL_+(n, \R) \). We may assume without loss of generality \( \{ h(g_i), m(h(g_i)) \} \to g_{n-1, \infty} \) in \( M_n(\R) \). A matrix analysis easily tells us \( \lambda_1(h(g_i)) \leq nm(h(g_i)) \) since the later term bounds the amount of stretching of norms of vectors under the action of \( h(g_i) \). By Lemma 2.12, dividing by \( \lambda_1(h(g_i)) \), we have

\[
\hat{h}(g_i)/\lambda_1(h(g_i)) \to Ag_{n-1, \infty}
\]

for \( A \geq 1/n \), or the sequence is unbounded in \( M_n(\R) \).

Now, we let \( \{ g_i(\| g_i \|) \to \| g_\infty \| \} \) where \( g_\infty \) is obtained as a limit of

\[
\begin{pmatrix}
\frac{1}{\hat{\lambda}_1(h(g_i))} \hat{h}(g_i) & \frac{\hat{\lambda}_E(g_i)^{1/n}}{\hat{\lambda}_1(h(g_i))} b_{g_i} \\
0 & \frac{\hat{\lambda}_E(g_i)^{1/n} \hat{\lambda}_E(g_i)}{\hat{\lambda}_1(h(g_i))}
\end{pmatrix}
\]

(5.23)

By the uniform middle eigenvalue condition, we obtain that \( \{ g_i(\| g_i \|) \to \| g_\infty \| \} \) where \( g_\infty \) is of form

\[
\begin{pmatrix}
g_{n-1, \infty} b \\
0 & 0
\end{pmatrix}
\]

(5.24)

by rescaling if necessary. Now, \( R_\iota(\{ g_i \}) \) is a subset of \( S^1_{n-1} \) since the lower row is zero. Hence, \( \{ g_i(J) \} \geq \iota \) geometrically converges to \( g_\infty(J) \subset S^1_{n-1} \). Since \( \Gamma \) acts on \( \Cl(V) \) and \( \Cl(V) \cap \bd \Lambda^n = \Cl(\Omega) \), we obtain \( g_\infty(J) \subset \Cl(\Omega) \). \( \square \)

**Lemma 5.15** Let \( \Gamma \) be a discrete group of projective automorphisms of a properly convex domain \( V \) and a domain \( \Omega \subset V \) of dimension \( n-1 \). Assume that \( \Omega / \Gamma \) is a closed \( n \)-orbifold. Suppose that \( \Gamma \) satisfies the uniform middle eigenvalue condition with respect to the hyperspace containing \( \Omega \). Let \( P \) be a subspace of \( S^n \) so that \( P \cap \Cl(\Omega) = \emptyset \) and \( P \cap V \neq \emptyset \). Then \( \{ g(P) \cap V \mid g \in \Gamma \} \) is a locally finite collection of closed sets in \( V \).

**Proof** Suppose not. Then there exists a sequence \( \{ x_i \} \), \( x_i \in \Gamma \cap V \) and \( \{ g_i \} \), \( g_i \in \Gamma \) so that \( \{ x_i \} \to x_\infty \in V \) and \( \{ g_i \} \) is a sequence of mutually distinct elements.
We have \( x_i \in F \) for a compact set \( F \subset V \). Then Lemma 5.14 applies. \( \{ g_i^{-1}(F) \} \) accumulates to \( \text{bd}\Omega \). This means that \( g_i^{-1}(x_i) \) accumulates to \( \text{bd}\Omega \). Since \( g_i^{-1}(x_i) \in P \cap V \), and \( P \cap \text{Cl}(\Omega) = \emptyset \), this is a contradiction. \( \square \)

### 5.4.2 Approximating a convex hypersurface by strictly convex hypersurfaces

**Theorem 5.5** We assume that \( \Gamma \) is a projective group with a properly convex affine action with the triple \((\Gamma, U, D)\) for \( U \subset \mathbb{A}^n \subset \mathbb{S}^n \). Assume the following:

- \( U \) is an asymptotically nice properly convex domain closed in \( \mathbb{A}^n \),
- the manifold boundary \( \partial U = \text{bd}U \cap \mathbb{A}^n \) is in an asymptotically nice properly convex open domain \( V' \) where \( \Gamma \) acts on, and \( \text{Cl}(U) \cap \mathbb{A}^n \subset V' \),
- \( \text{Cl}(V') \cap \text{bd}\mathbb{A}^n = D \), and
- \( \partial U/\Gamma \) is a compact convex hypersurface.

Then there exists an asymptotically nice properly convex domain \( V \) closed in \( V' \) containing \( U \) so that \( \partial V/\Gamma \) is a compact hypersurface with strictly convex smooth boundary. Furthermore, \( \partial V/\Gamma \) can be chosen to be arbitrarily close to \( \partial U/\Gamma \) in \( V/\Gamma \) with any complete Riemannian metric on \( V'/\Gamma \).

**Proof** Let \( V'' \) be a properly convex domain so that \( \text{Cl}(V'') \subset U \) and \((\Gamma, V'', D)\) is a properly convex affine triple. We can construct such a domain by the proof of Theorem 5.4 not including the part showing that \( \partial L \) is smooth. (One has to be careful that we do not use the smoothness of \( \partial L \) where the proof uses this theorem.)

Let \( \mathcal{P} \) denote the set of hyperspaces sharply supporting \( U \) at \( \partial U \). Let \( P \) be in \( \mathcal{P} \). The dual \( P^* \) of \( P \) in \( \mathbb{S}^n \) is a point of the properly convex domain \( V'^* \) by (2.26). Hence, the set \( \mathcal{P}^* \) of dual points corresponding to elements of \( \mathcal{P} \) is a properly embedded hypersurface in the interior of \( \text{Cl}(V'^*) \) by Proposition 2.19 applied to \( \text{Cl}(U) \) and the hyperspace \( \partial U \) and its dual \( \mathcal{P}^* \).

Also, \( \partial U/\Gamma \) is a compact orbifold by Proposition 2.19(iv). By the above duality, \( \mathcal{P}^*/\Gamma^* \) is a compact orbifold, and so is \( \mathcal{P}/\Gamma \). There is a fundamental domain \( F \) of \( \mathcal{P} \) under \( \Gamma \).

Any sequence \( g_i(P) \) for \( P \in F \) and unbounded sequence of \( \{ g_i \} \), \( g_i \in \Gamma \), has accumulation points only in the supporting hyperplane of \( V' \) since \( \Gamma \) acts properly in the interior of \( \text{Cl}(V'^*) \). By Theorem 5.3, these are supporting hyperplanes of \( V' \) since these are supporting hyperplanes of \( V'' \). Hence for each point of \( \partial U \), there is a neighborhood \( N \) whose closure is in \( V'' \).

Hence, \( \mathcal{P} \) is a locally compact collection of elements in \( V'^* \) with accumulations only in the set of hyperplanes sharply supporting \( V' \).

For any \( \varepsilon > 0 \), there exists a compact set \( K' \) so that elements of \( \mathcal{P} - K' \) are \( \varepsilon \cdot \text{d}_{H} \)-close to the hyperspaces asymptotic to \( V' \) by the above paragraph. Hence, Lemma 5.16 implies the result. \( \square \)

**Theorem 5.6** Assume the following:
• $\Gamma$ is a projective discrete group acting properly on a properly convex domain $U$ closed in $\mathbb{A}^n \subset \mathbb{S}^n$.
• $\{p\} = \text{Cl}(U) - U$ is a singleton,
• $\Gamma$ is a cusp group,
• the manifold boundary $\partial U$ is in a properly convex open domain $V'$ where $\Gamma$ acts on,
• $U$ is closed in $V'$, and
• $\partial U / \Gamma$ is a compact convex hypersurface.

Then there exists an properly convex domain $V$ closed in $V'$ containing $U$ so that $\partial V / \Gamma$ is a compact hypersurface with strictly convex smooth boundary. Furthermore, $\partial V / \Gamma$ can be chosen to be arbitrarily close to $\partial U / \Gamma$ in $V / \Gamma$ with any complete Riemannian metric on $V' / \Gamma$.

**Proof** Let $\mathbb{A}^n$ be the affine space bounded by a hyperspace tangent to $\text{Cl}(U)$ at $p$. Let $\mathcal{P}$ denote the set of hyperspaces sharply supporting $U$. There is a unique supporting hyperplane to $U$ at $p$ only to which $\mathcal{P}$ can accumulate. This follows since a cusp group acts on $\mathcal{P}$ and by duality. Then the analogous proof as that of Theorem 5.5 using Lemma 5.16 is applicable exactly. \(\square\)

**Lemma 5.16** We assume that $\Gamma$ is a projective group with a properly convex affine action with the triple $(\Gamma, U, D)$ for $U \subset \mathbb{A}^n \subset \mathbb{S}^n$. Suppose that $U$ is a properly convex domain in an affine space $\mathbb{A}^n$ in $\mathbb{S}^n$ with $\partial U = \text{bd}U \cap \mathbb{A}^n$ an embedded hypersurface in it. Let $\mathcal{P}$ denote the set of sharply supporting hyperspaces of $U$ meeting $\partial U$. Each hyperspace $P$ in $\mathcal{P}$ has a component $H_P$ of $\mathbb{A}^n$ not meeting $U$. Assume the following:

• A group $G$ of affine transformations of $\mathbb{A}^n$ acts on $U$ so that $\partial U / G$ is a compact hypersurface.
• $U$ is in a convex open domain $V'$ where $\Gamma$ acts on,
• Each point of $\partial U$ has a neighborhood $N \subset V'$ so that $N - \text{Cl}(U)$ has only compact set of hyperspaces $P$ in $\mathcal{P}$ with $H_P \cap N \neq \emptyset$.

Then there exists an properly convex domain $V$ closed in $V'$ containing $U$ so that $\partial V / \Gamma$ is a compact hypersurface with strictly convex smooth boundary. Furthermore, $\partial V / \Gamma$ can be chosen to be arbitrarily close to $\partial U / \Gamma$ in $V / \Gamma$ with any complete Riemannian metric on $V' / \Gamma$.

**Proof** For each $P \in \mathcal{P}$, we choose a 1-form $u_P \in \mathbb{R}^{n*}$ determined by the linear subspace parallel to $P$ in $\mathbb{A}^n$. Recall the cone of $C(D) \subset \mathbb{R}^n$ of the domain $D$ in $\text{bd} \mathbb{A}^n$ and its dual $C(D)^* \subset \mathbb{R}^{n*}$. We choose so that $u_P$ is on a level hypersurface at level 1 of a Koszul-Vinberg function for the dual cone $C(D)^*$ of the convex cone $C(D)$ in $\mathbb{R}^n$ where $\mathcal{L}(\Gamma)^*$-acts on by the homomorphism $g \mapsto g^{-1}$. (See Chapter 4 of [91].)

We define an affine function $f_P$ obtained by adding a constant $-u_P(x_0)$ for any $x_0 \in P$ to $u_P$ so that $f_P^{-1}(0) = P$ and $f_P > 0$ on the component $H_P$ of $\mathbb{A}^n - P$ disjoint from $U^n$. It follows that if $P' = g(P)$ for $P \in \mathcal{P}$ and $g \in \Gamma$, then $f_{P'} \circ g^{-1} = f_P$ since
5.4 Lens type T-ends

\[ u_{p'} = u_p \circ g^{-1} \] by the above paragraph and \( u_{p'}(g(x_0)) = u_p(x_0) \).

We define a smooth function

\[ g(t) = t^2 \exp(1/t^2) \] for \( t > 0 \), and \( g(t) = 0 \) for \( t \leq 0 \).

We let \( g_P = g \circ f_p \). Then by the premise, \( g_P(x) \) for each \( x \in V' \) is nonzero for only compact subset of \( \mathcal{P} \). Since \( \Gamma^* \)-action on \( \mathcal{P}^* \subset V'^* \) is properly discontinuous, \( \mathcal{P}^* / \Gamma^* \) is a topological orbifold. It has local chart where each chart is associated with a finite group action. There is a \( \Gamma \)-invariant measure \( d\mu \) on \( \mathcal{P} \) compatible with a positive continuous function times a volume on each chart of \( \mathcal{P} \). We define a smooth function

\[ \chi_U : V' \to \mathbb{R} \text{ by } \int_{P \in \mathcal{P}} g_P d\mu. \]

Hence, \( \chi_U \) is well-defined in \( V' \) by the above paragraph. Moreover,

\[ \chi_U^{-1}(0) = \bigcap_{P \in \mathcal{P}} (\mathbb{R}^n - H_P) = Cl(U) \cap \mathbb{R}^n. \]

The proof of Proposition 2.1 of Ghomi [89] implies that \( \chi \) is strictly convex on \( V' - U \). By our definition, \( \chi_U \) is \( \Gamma \)-invariant.

We give an arbitrary Riemannian metric \( \mu' \) on \( V'/\Gamma \). There exists a neighborhood \( N \) of \( Cl(U) \cap V'/\Gamma \) in \( V'/\Gamma \) where \( \chi_U \) has a nonzero differential in \( N - Cl(U)/\Gamma \) as we can see from the integral \( \int_{P \in \mathcal{P}} Dg_P d\mu \) where \( Dg_P \) are in a properly convex cone \( C_\mathcal{P} \) in \( \mathbb{R}^n \) spanned by \( \{ u_P | P \in \mathcal{P} \} \) for each point of \( V' - Cl(U) \). Then as \( \epsilon \to 0 \),

\[ \Sigma_\epsilon := \{ \chi_U^{-1}(\epsilon) \}/\Gamma \to \partial U/\Gamma \]

generically since \( N \) has a compact closure and the gradient vectors are uniformly bounded with respect to \( \mu' \) and are zero only at points of \( U/\Gamma \) in the closure and hence we can isotopy \( \Sigma_\epsilon \) along the gradient vector field to as close to \( \partial U/\Gamma \) as we wish. (See Batyrev [13] and Ben-Tal [14] also.) Furthermore, since \( \chi_U \) is strictly convex, \( \chi_U^{-1}(\epsilon) \) is a strictly convex smooth hypersurface on which \( \Gamma \) acts.
Chapter 6
Properly convex radial ends and totally geodesic ends: lens properties

We will consider properly convex ends in this chapter. In Section 6.1, we define the uniform middle eigenvalue conditions for R-ends and T-ends. We state the main results of this chapter Theorem 6.1: the equivalence of these conditions with the generalized lens conditions for R-ends or T-ends. The generalized lens conditions often improve to lens conditions, as shown in Theorem 6.2. In Section 6.2, we start to study the R-end theory. First, we discuss the holonomy representation spaces. Tubular actions and the dual theory of affine actions are discussed. We show that distanced actions and asymptotically nice actions are dual. Hence, the uniform middle eigenvalue condition implies the distanced action deduced from the dual theory in Chapter 5. In Section 6.3, we prove the main results. In Section 6.3.1, we estimate the largest norm $\lambda_1(g)$ of eigenvalues in terms of word length. In Section 6.3.2, we study orbits under the action with the uniform middle eigenvalue conditions. In Section 6.3.3, we prove a minor extension of Koszul’s openness for bounded manifolds, well-known to many people. In Section 6.3.4, we show how to prove the strictness of the boundary of lenses and prove our main result Theorem 6.1 using the orbit results and the Koszul’s openness. In Section 6.3.5, we now prove major Theorem 6.2. In Section 6.4, we show that the lens-shaped ends have concave neighborhoods, and we discuss the properties of lens-shaped ends in Theorems 6.7 and 6.8. If the generalized lens-shaped end is virtually factorizable, it can be made into a lens-shaped totally-geodesic R-end, which is a surprising result. In Section 6.5, we obtain the duality between the lens-shaped T-ends and generalized lens-shaped R-ends.

The main reason that we are studying the lens-shaped ends is to use them in studying the deformations preserving the convexity properties. These objects are useful in trying to understand this phenomenon. We also remark that sometimes a lens-shaped $p$-end neighborhood may not exist for an $R$-$p$-end within a given convex real projective orbifold. However, a generalized lens-shaped $p$-end neighborhood may exist for the $R$-$p$-end.
6.1 Main results

Let $O$ be a strongly tame convex real projective orbifold and let $\tilde{O}$ be a convex domain in $\mathbb{S}^n$ covering $O$. Let $h : \pi_1(O) \to \text{SL}_{\pm}(n+1, \mathbb{R})$ denote the holonomy homomorphism with its image $\Gamma$. We will take $\mathbb{S}^n$ as the default place where the statements take place in this chapter. However, reader can easily modify these to $\mathbb{R}P^n$-versions by Proposition 2.13 results in Section 2.1.7.

Definition 6.1 Suppose that $\tilde{E}$ is an R-end of generalized lens-type. Then $\tilde{E}$ have a p-end-neighborhood that is projectively diffeomorphic to the interior of $\{p\}^*L - \{p\}$ under $\text{dev}$ where $\{p\}^*L$ is a generalized lens-cone over a generalized lens $L$ where $\partial(\{p\}^*L - \{p\}) = \partial_+L$ for a boundary component $\partial_+L$ of $L$, and let $h(\pi_1(\tilde{E}))$ acts on $L$ properly and cocompactly. A concave pseudo-end-neighborhood of $\tilde{E}$ is the open pseudo-end-neighborhood in $\tilde{O}$ projectively diffeomorphic to $\{p\}^*L - \{p\} - L$.

6.1.1 Uniform middle eigenvalue conditions

The following applies to both R-ends and T-ends. Let $\tilde{E}$ be a p-end and $\Gamma_{\tilde{E}}$ the associated p-end holonomy group. We say that $\tilde{E}$ is non-virtually-factorizable if any finite index subgroup has a finite center or $\Gamma_{\tilde{E}}$ is virtually center-free; otherwise, $\tilde{E}$ is virtually factorizable by Theorem 1.1 of [21]. (See Section 2.4.4.)

Let $\tilde{\Sigma}_{\tilde{E}}$ denote the universal cover of the end orbifold $\Sigma_{\tilde{E}}$ associated with $\tilde{E}$. We recall Proposition 2.15 (Theorem 1.1 of Benoist [23]). If $\Gamma_{\tilde{E}}$ is virtually factorizable, then $\Gamma_{\tilde{E}}$ satisfies the following condition:

- $\text{Cl}(\tilde{\Sigma}_{\tilde{E}}) = K_1 \ast \cdots \ast K_k$ where each $K_i$ is properly convex or is a singleton.
- Let $G_i$ be the restriction of the $K_i$-stabilizing subgroup of $\Gamma_{\tilde{E}}$ to $K_i$. Then $G_i$ acts on $K_i^0$ cocompactly. (Here $K_i$ can be a singleton, and $G_i$ a trivial group.)
- A finite index subgroup $G'$ of $\Gamma_{\tilde{E}}$ is isomorphic to a cocompact subgroup of $\mathbb{Z}^{k-1} \times G_1 \times \cdots \times G_k$.
- The center $\mathbb{Z}^{k-1}$ of $G'$ is a subgroup acting trivially on each $K_i$.

Note that there are examples of discrete groups of form $\Gamma_{\tilde{E}}$ where $G_i$ are non-discrete. (See also Example 5.5.3 of [146] as pointed out by M. Kapovich.)

We will use simply $\mathbb{Z}^{k-1}$ to represent the corresponding group on $\Gamma_{\tilde{E}}$. Here, $\mathbb{Z}^{k-1}$ is called a virtual center of $\Gamma_{\tilde{E}}$.

Let $\Gamma$ be generated by finitely many elements $g_1, \ldots, g_m$. Let $w(g)$ denote the minimum word length of $g \in G$ written as words of $g_1, \ldots, g_m$. The conjugate word length $\text{cwl}(g)$ of $g \in \pi_1(\tilde{E})$ is

$$\min\{w(cgc^{-1})|c \in \pi_1(\tilde{E})\}.$$
Let $d_K$ denote the Hilbert metric of the interior $K^\circ$ of a properly convex domain $K$ in $\mathbb{R}P^n$ or $\mathbb{S}^n$. Suppose that a projective automorphism group $\Gamma$ acts on $K$ properly. Let $\text{length}_K(g)$ denote the infimum of $\{d_K(x, g(x)) | x \in K^\circ\}$, compatible with $\text{cwl}(g)$.

**Definition 6.2** Let $v_\tilde{E}$ be a p-end vertex of an R-p-end $\tilde{E}$. Let $K := \text{Cl}(\tilde{\Sigma})$. The p-end holonomy group $\Gamma_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition with respect to $v_\tilde{E} or the R-p-end structure if the following hold:

- each $g \in \Gamma_{\tilde{E}}$ satisfies for a uniform $C > 1$ independent of $g$

$$C^{-1}\text{length}_K(g) \leq \log \left( \frac{\lambda_1(g)}{\lambda_{v_\tilde{E}}(g)} \right) \leq C\text{length}_K(g),$$

for the largest norm $\lambda_1(g)$ of the eigenvalues of $g$ and the eigenvalue $\lambda_{v_\tilde{E}}(g)$ of $g$ at $v_\tilde{E}$.

Of course, we choose the matrix of $g$ so that $\lambda_{v_\tilde{E}}(g) > 0$. See Remark 1.1 as we are looking for the lifting of $g$ that acts on p-end neighborhood. We can replace $\text{length}_K(g)$ with $\text{cwl}(g)$ for properly convex ends. We remark that the condition does depend on the choice of $v_\tilde{E}$; however, the radial end structures will determine the end vertices.

The definition of course applies to the case when $\Gamma_{\tilde{E}}$ has the finite-index subgroup with the above properties.

We recall a dual definition identical with the definition in Section 5.1 but adopted to T-p-ends.

**Definition 6.3** Suppose that $\tilde{E}$ is a properly convex T-p-end. Suppose that the ideal boundary component $\tilde{\Sigma}$ of the T-p-end is properly convex. Let $K := \text{Cl}(\tilde{\Sigma})$. Let $g^* : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the dual transformation of $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. The p-end holonomy group $\Gamma_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition with respect to $\tilde{\Sigma}_{\tilde{E}} or the T-p-end structure

- if each $g \in \Gamma_{\tilde{E}}$ satisfies for a uniform $C > 1$ independent of $g$

$$C^{-1}\text{length}_K(g) \leq \log \left( \frac{\lambda_1(g)}{\lambda_{K^*}(g^*)} \right) \leq C\text{length}_K(g),$$

for the largest norm $\lambda_1(g)$ of the eigenvalues of $g$ and the eigenvalue $\lambda_{K^*}(g^*)$ of $g^*$ in the vector in the direction of $K^*$, the point dual to the hyperspace containing $K$.

Again, the condition depends on the choice of the hyperspace containing $\tilde{\Sigma}_{\tilde{E}}$, i.e., the T-p-end structure. (We again lift $g$ so that $\lambda_{K^*}(g) > 0$.)

Here $\Gamma_{\tilde{E}}$ will act on a properly convex domain $K''$ of lower dimension, and we will apply the definition here. This condition is similar to the Anosov condition studied by Guichard and Wienhard [103], and the results also seem similar. We do not use their theories. They also use word length instead. One may look at the paper of Kassel-Potrie [117] to understand the relationship between eigenvalues and
singular values. We use the eigenvalues to obtain conjugacy invariant conditions which is needed in proving the converse part of Theorem 6.1. Our main tools to understand these questions are in Chapter 5 which we will use here.

We will see that the condition is an open condition; and hence a “structurally stable one.” (See Corollary 7.1.)

6.1.2 Lens and the uniform middle eigenvalue condition

As holonomy groups, the condition for being a generalized lens R-p-end and one for being a lens R-p-end are equivalent. For the following, we are not concerned with a lens-cone being in \( \widehat{\mathcal{O}} \).

**Theorem 6.1 (Lens holonomy)** Let \( \widehat{E} \) be an R-p-end of a strongly tame convex real projective orbifold. Then the holonomy group \( h(\pi_1(\widehat{E})) \) satisfies the uniform middle eigenvalue condition for the R-p-end vertex \( v_{\widehat{E}} \) if and only if it acts on a lens-cone with vertex \( v_{\widehat{E}} \) and its lens properly and cocompactly. Moreover, in this case, the lens-cone exists in the union of great segments with the vertex \( v_{\widehat{E}} \) in the directions of in the direction of a properly convex domain \( \Omega \subset S^{n-1}_{\widehat{E}} \) where \( h(\pi_1(\widehat{E})) \) acts properly discontinuously.

For the following, we are concerned with a lens-cone being in \( \widehat{\mathcal{O}} \).

**Theorem 6.2 (Actual lens-cone)** Let \( \mathcal{O} \) be a strongly tame convex real projective orbifold.

- Let \( E \) be a properly convex R-p-end.
  - The p-end holonomy group satisfies the uniform middle-eigenvalue condition if and only if \( E \) is a generalized lens-shaped R-p-end.
- Assume that the holonomy group of \( \mathcal{O} \) is strongly irreducible, and \( \mathcal{O} \) is properly convex. If \( \mathcal{O} \) satisfies the triangle condition (see Definition 6.7) or \( \widehat{E} \) is virtually factorizable or is a totally geodesic R-end, then we can replace the term “generalized lens-shaped” to “lens-shaped” in the above statement.

This is repeated as Theorem 6.6. We will prove the analogous result for totally geodesic ends in Theorem 6.9.

Notice that there is no condition on \( \mathcal{O} \) to be properly convex.

Another main result is on the duality of lens-shaped ends: Recalling from Section 2.5.1, we have \( \mathbb{RP}^n = \mathbb{P}(\mathbb{R}^{n+1}) \) the dual real projective space of \( \mathbb{RP}^n \). Recall also \( S^{*n} = S(\mathbb{R}^{n+1}) \) as the dual spherical projective space of \( S^n \). In Section 6.2, we define the projective dual domain \( \Omega^* \) in \( \mathbb{RP}^n \) to a properly convex domain \( \Omega \) in \( \mathbb{RP}^n \) where the dual group \( \Gamma^* \) to \( \Gamma \) acts on. Vinberg showed that there is a duality diffeomorphism between \( \Omega/\Gamma \) and \( \Omega^*/\Gamma^* \). The ends of \( \mathcal{O} \) and \( \mathcal{O}^* \) are in a one-to-one correspondence. Horospherical ends are dual to themselves, i.e., “self-dual types”, and properly convex R-ends and T-ends are dual to one another. (See Proposition
6.11.) We will see that generalized lens-shaped properly convex R-ends are always dual to lens-shaped T-ends by Corollary 6.3. We mention that Fried also solved this question when the linear holonomy is in $SO(2, 1)$ [85]. Also, we found out later that the dual consideration of Barbot’s work on the existence of globally hyperbolic spacetimes for geometrically finite linear holonomy in $SO(n, 1)$ also solves this problem in the setting of finding Cauchy hyperspaces in flat Lorentz spaces. (See Theorem 4.25 of [11].)

6.2 The end theory

In this section, we discuss the properties of lens-shaped radial and totally geodesic ends and their duality also.

6.2.1 The holonomy homomorphisms of the end fundamental groups: the tubes.

We will discuss for $S^n$ only here but the obvious $\mathbb{RP}^n$-version exists for the theory. Let $\tilde{E}$ be an $\mathcal{R}$-p-end of $\partial^1$. Let $\text{SL}_{\pm}(n+1, \mathbb{R})_{\tilde{v}_E}$ be the subgroup of $\text{SL}_{\pm}(n+1, \mathbb{R})$ fixing a point $\tilde{v}_E \in \mathbb{S}^n$. This group can be understood as follows by letting $\tilde{v}_E = [0, \ldots, 0, 1]$ as a group of matrices: For $g \in \text{SL}_{\pm}(n+1, \mathbb{R})_{\tilde{v}_E}$, we have

$$
\begin{pmatrix}
\frac{1}{\lambda_{v_\tilde{E}}(g)^n} \hat{h}(g) \\
0 \\
v_g \\
\lambda_{v_\tilde{E}}(g)
\end{pmatrix}
$$

(6.3)

where $\hat{h}(g) \in \text{SL}_{\pm}(n, \mathbb{R})$, $v \in \mathbb{R}^n$, $\lambda_{v_\tilde{E}}(g) \in \mathbb{R}_+$, is the so-called linear part of $h$. Here, $\lambda_{v_\tilde{E}} : g \mapsto \lambda_{v_\tilde{E}}(g)$ for $g \in \text{SL}_{\pm}(n+1, \mathbb{R})_{\tilde{v}_E}$ is a homomorphism so it is trivial in the commutator group $[\Gamma_{\tilde{E}}, \Gamma_{\tilde{E}}]$. There is a group homomorphism

$$
\mathcal{L} : \text{SL}_{\pm}(n+1, \mathbb{R})_{\tilde{v}_E} \rightarrow \text{SL}_{\pm}(n, \mathbb{R}) \times \mathbb{R}_+
$$

(6.4)

$$
g \mapsto (\hat{h}(g), \lambda_{v_\tilde{E}}(g))
$$

with the kernel equal to $\mathbb{R}^n$, a dual space to $\mathbb{R}^n$. Thus, we obtain a diffeomorphism

$$
\text{SL}_{\pm}(n+1, \mathbb{R})_{\tilde{v}_E} \rightarrow \text{SL}_{\pm}(n, \mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}_+.
$$

We note the multiplication rules
\((A, v, \lambda)(B, w, \mu) = (AB, \frac{1}{\mu^{1/n}} vB + \lambda w, \lambda \mu)\).  \hspace{1cm} (6.5)

We denote by \(\mathcal{L} : \text{SL}_\pm(n + 1, \mathbb{R})_{\nu_E} \to \text{SL}_\pm(n, \mathbb{R})\) the further projection to \(\text{SL}_\pm(n, \mathbb{R})\).

Let \(\Sigma_{\tilde{E}}\) be the end \((n - 1)\)-orbifold. Given a representation \(\hat{h} : \pi_1(\Sigma_{\tilde{E}}) \to \text{SL}_\pm(n, \mathbb{R})\) and a homomorphism \(\lambda_{\nu_{\tilde{E}}} : \pi_1(\Sigma_{\tilde{E}}) \to \mathbb{R}_+\),

we denote by \(R^n_{\hat{h}, \lambda_{\nu_{\tilde{E}}}}\) the \(\mathbb{R}\)-module with the \(\pi_1(\Sigma_{\tilde{E}})\)-action given by

\[g \cdot v = \frac{1}{\lambda_{\nu_{\tilde{E}}}(g)^{1/n}} \hat{h}(g)(v)\]

And we denote by \(R^n_{\hat{h}, \lambda_{\nu_{\tilde{E}}}}^*\) the dual vector space with the right dual action given by

\[g \cdot v = \frac{1}{\lambda_{\nu_{\tilde{E}}}(g)^{1/n}} \hat{h}(g)^*(v)\]

Let \(H^1(\pi_1(\tilde{E}), \mathbb{R}^n_{\hat{h}, \lambda_{\nu_{\tilde{E}}}})\) denote the cohomology space of 1-cocycles

\[\Gamma \ni g \mapsto \nu(g) \in \mathbb{R}^n_{\hat{h}, \lambda_{\nu_{\tilde{E}}}}\]

As \(\text{Hom}(\pi_1(\Sigma_{\tilde{E}}), \mathbb{R}_+)\) equals \(H^1(\pi_1(\Sigma_{\tilde{E}}), \mathbb{R})\), we obtain:

**Theorem 6.3** Let \(\mathcal{O}\) be a strongly tame properly convex real projective orbifold, and let \(\tilde{\mathcal{O}}\) be its universal cover. Let \(\Sigma_{\tilde{E}}\) be the end orbifold associated with an \(R\)-p-end \(\tilde{E}\) of \(\tilde{\mathcal{O}}\). Then the space of representations

\[\text{Hom}(\pi_1(\Sigma_{\tilde{E}}), \text{SL}_\pm(n + 1, \mathbb{R})_{\nu_{\tilde{E}}})/\text{SL}_\pm(n + 1, \mathbb{R})_{\nu_{\tilde{E}}}\]

is the fiber space over

\[\text{Hom}(\pi_1(\Sigma_{\tilde{E}}), \text{SL}_\pm(n, \mathbb{R}))/\text{SL}_\pm(n, \mathbb{R}) \times H^1(\pi_1(\Sigma_{\tilde{E}}), \mathbb{R})\]

with the fiber isomorphic to \(H^1(\pi_1(\Sigma_{\tilde{E}}), \mathbb{R}^n_{\hat{h}, \lambda_{\nu_{\tilde{E}}}})\) for each \((\hat{h}, \lambda)\).

On a Zariski open subset of \(\text{Hom}(\pi_1(\Sigma_{\tilde{E}}), \text{SL}_\pm(n, \mathbb{R}))/\text{SL}_\pm(n, \mathbb{R})\), the dimensions of the fibers are constant (see Johnson-Millson [112]). A similar idea is given by Mess [140]. In fact, the dualizing these matrices gives us a representation to \(\text{Aff}(\mathbb{A}^n)\). (See Chapter 5.) In particular if we restrict ourselves to linear parts to be in \(\text{SO}(n, 1)\), then we are exactly in the cases studied by Mess. (The concept of the duality is explained in Section 6.2.3)
6.2.2 Tubular actions.

Let us give a pair of antipodal points \( v \) and \( v_\circ \). If a group \( \Gamma \) of projective automorphisms fixes a pair of fixed points \( v \) and \( v_\circ \), then \( \Gamma \) is said to be \textit{tubular}. There is a projection \( \Pi_v : S^n - \{v, v_\circ\} \to S^{n-1}_v \) given by sending every great segment with endpoints \( v \) and \( v_\circ \) to a point of the sphere of directions at \( v \).

A \textit{tube} in \( S^n \) (resp. in \( \mathbb{R}P^n \)) is the closure of the inverse image \( \Pi^{-1}_v(\Omega) \) of a domain \( \Omega \) in \( S^{n-1}_v \) (resp. in \( \mathbb{R}P^{n-1}_v \)). We often denote the closure in \( S^n \) by \( \mathcal{T}_v(\Omega) \), and we call it a \textit{tube domain}. Given an \( \mathbb{R}\)-p-end \( \hat{E} \) of \( \partial \), let \( v := v_E \). The \textit{end domain} is \( R_v(\partial) \). If an \( \mathbb{R}\)-p-end \( \hat{E} \) has the end domain \( \Sigma_E = R_v(\partial) \), the group \( \hat{h}(\pi_1(\hat{E})) \) acts on \( \mathcal{T}_v(\Omega) \).

The image of the tube domain \( \mathcal{T}_v(\Omega) \) in \( \mathbb{R}P^n \) is still called a \textit{tube domain} and denoted by \( \mathcal{T}[v](\Omega) \) where \([v]\) is the image of \( v \).

We will now discuss for the \( S^n \)-version but the \( \mathbb{R}P^n \) version is obviously clearly obtained from this by a minor modification.

Letting \( v \) have the coordinates \([0, \ldots, 0, 1]\), we obtain the matrix of \( g \) of \( \pi_1(\hat{E}) \) of form

\[
\begin{pmatrix}
\frac{1}{\lambda_n(g)} \tilde{h}(g) & 0 \\
{b}_g & \lambda_n(g)
\end{pmatrix}
\]

where \( b_g \) is an \( n \times 1 \)-vector and \( \tilde{h}(g) \) is an \( n \times n \)-matrix of determinant \( \pm 1 \) and \( \lambda_n(g) \) is a positive constant.

Note that the representation \( \tilde{h} : \pi_1(\hat{E}) \to SL_+(n, \mathbb{R}) \) is given by \( g \mapsto \tilde{h}(g) \). Here we have \( \lambda_n(g) > 0 \). If \( \Sigma_E \) is properly convex, then the convex tubular domain and the action is said to be \textit{properly tubular}.

6.2.3 Affine actions dual to tubular actions.

Let \( S^{n-1}_v \) in \( S^n = S(\mathbb{R}^{n+1}) \) be a great sphere of dimension \( n - 1 \). A component of a component of the complement of \( S^{n-1} \) can be identified with an affine space \( \mathbb{A}^n \).

The subgroup of projective automorphisms preserving \( S^{n-1} \) and the components equals the affine group \( \text{Aff}(\mathbb{A}^n) \).

By duality, a great \((n-1)\)-sphere \( S^{n-1}_v \) corresponds to a point \( v_{S^{n-1}} \). Thus, for a group \( \Gamma \) in \( \text{Aff}(\mathbb{A}^n) \), the dual groups \( \Gamma^* \) acts on \( S^{n*} := S(\mathbb{R}^{n+1,*}) \) fixing \( v_{S^{n-1}} \). (See Proposition 2.19 also.)

Let \( S^{n*}_{S^n} \) denote a hyperspace in \( S^n \). Suppose that \( \Gamma \) acts on a properly convex open domain \( U \) where \( \Omega := \text{bd}U \cap S^{n*}_{S^n} \) is a properly convex domain. We recall that \( \Gamma \) has a properly convex affine action. Let us recall some facts from Section 2.5.

- A great \((n-2)\)-sphere \( P \subset S^n \) is dual to a great circle \( P^* \) in \( S^{n*} \) given as the set of hyperspheres containing \( P \).
- The great sphere \( S^{n-1}_{S^n} \subset S^n \) with an orientation is dual to a point \( v \in S^{n*} \) and it with an opposite orientation is dual to \( v_\circ \in S^{n*} \).
An oriented hyperspace $P \subset S^{n-1}_\infty$ of dimension $n - 2$ is dual to an oriented great circle passing $v$ and $v_-$, giving us an element $P^\dag$ of the linking sphere $S^{n-1}_v$ of rays from $v$ in $S^n$.

The space $S$ of oriented hyperspaces in $S^{n-1}_\infty$ equals $S^{n-1}_v$. Thus, there is a projective isomorphism

$$\mathcal{J}_2 : S = S^{n-1}_v \ni P \leftrightarrow P^\dag \in S^{n-1}_v.$$  

For the following, let’s use the terminology that an oriented hyperspace $V$ in $S^i$ supports an open submanifold $A$ if it bounds an open $i$-hemisphere $H$ in the right orientation containing $A$.

**Proposition 6.1** Suppose that $\Gamma \subset SL_\pm(n + 1, \mathbb{R})$ acts on a properly convex open domain $\Omega \subset S^{n-1}_\infty$ cocompactly. Then the dual group $\Gamma^*$ acts on a properly tubular domain $B$ with vertices $v := v_{S^n-1}$ and $v_- := v_{S^n-1}^-$ dual to $S^{n-1}_\infty$. Moreover, the domain $\Omega$ and domain $\mathcal{R}_\infty(B)$ in the linking sphere $S^{n-1}_\infty$ from $v$ in the directions of $B^\ast$ are projectively diffeomorphic to a pair of dual domains in $S^{n-1}_\infty$ respectively.

**Proof** Given $\Omega \subset S^{n-1}_\infty$, we obtain the properly convex open dual domain $\Omega^*$ in $S^{n-1}_\infty$. An oriented $n - 2$-hemisphere sharply supporting $\Omega$ in $S^{n-1}_\infty$ corresponds to a point of $bd \Omega^*$ and vice versa. (See Section 2.5.) An oriented great $n - 1$-sphere in $S^n$ supporting $\Omega$ but not containing $\Omega$ meets a great $n - 2$-sphere $P$ in $S^{n-1}_\infty$ supporting $\Omega$. The dual $P^\ast$ of $P$ is the set of hyperspaces containing $P$, a great circle in $S^n$. The set of oriented great $n - 1$-spheres containing $P$ supporting $\Omega$ but not containing $\Omega$ forms a pencil; in this case, a great open segment $I_P$ in $S^n$ with endpoints $v$ and $v_-$. Let $P^\dag \in S^{n-1}_\infty$ denote the dual of $P$ in $S^{n-1}_\infty$. Then $P^\dag := \mathcal{J}_2(P^\ast)$ is the direction of $P^\ast$ at $v$ as we can see from the projective isomorphism $\mathcal{J}_2$. Recall from the beginning of Section 2.5.1 $P$ supports $\Omega$ if and only if $P^\dag \in \Omega^*$. Hence, there is a homeomorphism

$$I_P := \{Q|Q \text{ is an oriented great } n - 1 \text{-sphere supporting } \Omega, Q \cap S^{n-1}_\infty = P\} \rightarrow S_{P^\ast} = \{p|p \text{ is a point of a great open segment in } P^\ast \text{ with endpoints } v, v_- \text{ where the direction } P^\dag = \mathcal{J}_2(P^\ast), P^\dag \in \Omega^*\}. \quad (6.7)$$

The set $B$ of oriented hyperspaces supporting $\Omega$ possibly containing $\Omega$ meets an oriented $(n - 2)$-hyperspace in $S^{n-1}_\infty$ supporting $\Omega$. Denote by $\alpha_x$ the great segment with vertices $v$ and $v_-$ in the direction of $x \in S^{n-1}_\infty$. Thus, we obtain

$$B^\ast = \bigcup_{P \in \Omega} S_{P^\ast} = \bigcup_{x \in \mathcal{J}_2(\Omega^*)} \alpha_x \subset S^n.$$

Let $\mathcal{J}(\Omega^*)$ denote the union of open great segments with endpoints $v$ and $v_-$ in direction of $\Omega^*$. Thus, $B^\ast = \mathcal{J}(\Omega^*)$. Thus, there is a homeomorphism

$$I := \{Q|Q \text{ is an oriented great } n - 1 \text{-sphere sharply supporting } \Omega\} \rightarrow S = \{p|p \in S_{P^\ast}, P^\dag \in bd \Omega^*\} = bd B^\ast - \{v, v_-\}. \quad (6.8)$$
Also, \( R_v(B^*) = \mathcal{R}_x(\Omega^*) \) by the above equation. Thus, \( \Gamma \) acts on \( \Omega \) if and only if \( \Gamma^* \) acts on \( I \) if and only if \( \Gamma^* \) acts on \( S \) if and only if \( \Gamma^* \) acts on \( B^* \) and on \( \Omega^* \). \( \square \)

### 6.2.4 Distanced tubular actions and asymptotically nice affine actions.

The approach is similar to what we did in Chapter 5 but is in the dual setting. Hence, we can think of our approach as a generalization of this work.

**Definition 6.4**

Radial action \ A properly tubular action of \( \Gamma \) is said to be **distanced** if a \( \Gamma \)-invariant tubular domain contains a properly convex compact \( \Gamma \)-invariant subset disjoint from the vertices of the tubes.

Affine action \ We recall from Chapter 5. A properly convex affine action of \( \Gamma \) is said to be **asymptotically nice** if \( \Gamma \) acts on a properly convex open domain \( U' \) in \( \mathbb{A}^n \) with boundary in \( \Omega \subset S_{n-1}^\infty \), and \( \Gamma \) acts on a compact subset

\[
J := \{ H | H \text{ is an AS-hyperspace passing } x \in \text{bd} \Omega, H \not\subset S_{n-1}^\infty \}
\]

where we require that every sharply supporting \((n-2)\)-dimensional space of \( \Omega \) in \( S_{n-1}^\infty \) is contained in at least one of the element of \( J \).

The following is a simple consequence of the homeomorphism given by equation (6.8).

**Proposition 6.2** Let \( \Gamma \) and \( \Gamma^* \) be dual groups where \( \Gamma \) has an affine action on \( \mathbb{A}^n \) and \( \Gamma^* \) is tubular with the vertex \( v = v_{S_{n-1}^\infty} \) dual to the boundary \( S_{n-1}^\infty \) of \( \mathbb{A}^n \). Let \( \Gamma = (\Gamma^*)^\ast \) acts on a convex open domain \( \Omega \) with a closed \( n \)-orbifold \( \Omega / \Gamma \). Then \( \Gamma \) acts asymptotically nicely if and only if \( \Gamma^* \) acts on a properly tubular domain \( B \) and is distanced.

**Proof** From the definition of asymptotic niceness, we can do the following: for each point \( x \) and a sharply supporting hyperspace \( P \) of bd\( \Omega \) passing \( x \) in \( S_{n-1}^\infty \), we choose a great \( n-1 \)-sphere in \( S^n \) sharply supporting \( \Omega \) at \( x \) containing \( P \) and uniformly bounded at a distance in the \( d_H \)-sense from \( S_{n-1}^\infty \). This forms a compact \( \Gamma \)-invariant set \( J \) of hyperspaces.

The dual points of the supporting hyperspaces passing points of bd\( \Omega \) are points on bd\( B \) for a tube domain \( B \) with vertex \( v \) dual to \( S_{n-1}^\infty \) by (6.8) in the proof of Proposition 6.1. Since the hyperspaces in \( J \) sharply supporting \( U \) at \( x \in \text{bd} \Omega \), are bounded at a distance from \( S_{n-1}^\infty \) in the \( d_H \)-sense, the dual points are uniformly bounded at a distance from the vertices \( v \) and \( v_{-} \). We take the closure of the set of hyperspaces in the dual space of \( S^n \). Let us call this compact set \( K \). Let \( \Omega^x \subset S_{n-1}^\infty \) be the dual domain of \( \Omega \). Then for every point of bd\( \Omega^x \), we have a point of \( K \) in the corresponding great segment from \( v \) to \( v_{-} \). Then \( K \) is uniformly bounded at a distance from \( v \) and \( v_{-} \) in the \( d \)-sense. The convex hull of \( K \) in \( \text{Cl}(\Theta) \) is a compact convex set bounded at a uniform distance from \( v \) and \( v_{-} \) since the tube domain is
proportionally convex. Since $K$ is $\Gamma^*$-invariant, so is the convex hull in $\text{Cl} (\omega)$. Therefore, $\Gamma^*$ acts on $B$ as a distanced action.

The converse is also very simple to prove by (6.8) in the proof of Proposition 6.1. □

**Theorem 6.4** Let $\Gamma$ have a nontrivial properly convex tubular action at vertex $v = v_{0n-1}$ on $S^n$ (resp. in $\mathbb{R}P^n$) and acts on a properly convex tube $B$ and satisfies the uniform middle-eigenvalue conditions with respect to $v_{0n-1}$. We assume that $\Gamma$ acts on a convex open domain $\Omega \subset S^n$ where $B = \mathcal{T}_v(\Omega)$ and $\Omega / \Gamma$ is a closed $n$-orbifold. Then $\Gamma$ is distanced inside the tube $B$, and $B$ contains a distanced $\Gamma$-invariant compact set $K$. Finally, we can choose the distanced set $K$ to be in a hypersphere disjoint from $v, v_{-}$ when $\Gamma$ is virtually factorizable.

**Proof** We will again prove for $S^n$. Let $\Omega$ denote the convex domain in $S^n$ corresponding to $B$. By Theorems 5.1 and 5.2, $\Gamma^*$ is asymptotically nice. Proposition 6.2 implies the result.

Now, we prove the final part to show the total geodesic property of virtually factorizable ends: Suppose that $\Gamma$ acts virtually reducibly on $S_n$ on a properly convex domain $\Omega$. Then $\Gamma$ is virtually isomorphic to a cocompact subgroup of $\mathbb{Z}_{l_0} \times \Gamma_{l_0} \times \cdots \times \Gamma_{l_0}$ where $\Gamma_{l_0}$ is irreducible by Proposition 2.15. Also, $\Gamma$ acts on $K := K_1 \ast \cdots \ast K_{l_0} \subset S^n$ where $K_i$ denotes the properly convex compact set in $S^n$ where $\Gamma_i$ acts on for each $i$. $\Gamma_i$ acts trivially on $K_j$ for $j \neq i$. Here, $K_i$ is 0-dimensional for $i = s+1, \ldots, l_0$ for $s+1 \leq l_0$. Let $B_i$ be the convex tube with vertices $v$ and $v_{-}$ corresponding to $K_i$. Each $\Gamma_i$ for $i = 1, \ldots, s$ acts on a nontrivial tube $B_i$ with vertices $v$ and $v_{-}$ in a subspace.

For each $i, s+1 \leq i \leq r$, $B_i$ is a great segment with endpoints $v$ and $v_{-}$. A point $p_i$ corresponds to $B_i$ in $S^n$.

The virtual center isomorphic to $\mathbb{Z}_{l_0}$ is in the group $\Gamma$ by Proposition 2.15. Recall that a nontrivial element $g$ of the virtual center acts trivially on the subspace $K_i$ of $S^n$; that is, $g$ has only one associated eigenvalue in points of $K_i$. There exists a nontrivial element $g$ of the virtual center with the largest norm eigenvalue in $K_i$ for the induced $g$-action on $S^n$ since the action of $\Gamma$ on $\Omega$ is cocompact. By the middle eigenvalue condition, for each $i$, we can find $g$ in the center so that $g$ has a hyperspace $K_i^j \subset B_i$ with largest norm eigenvalues. The convex hull of $K_i^j \cup \cdots \cup K_{l_0}^j$

in $\text{Cl}(B)$ is a distanced $\Gamma$-invariant compact convex set. For $(\xi_1, \ldots, \xi_{l_0}) \in \mathbb{R}^+_{l_0}$, we define
6.2 The end theory

\[ \zeta(\zeta_1, \ldots, \zeta_l) := \begin{pmatrix} \zeta_1I_{n_1} + 1 & 0 & \cdots & 0 \\ 0 & \zeta_2I_{n_2} + 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta_lI_{n_l} + 1 \end{pmatrix}, \quad \sum_{j=1}^l \zeta_j I_{n_j} + 1 = 1, \]

(6.9)

using the coordinates where each \( K_i \) corresponds to a block.

Now, we consider the general case. The element \( x \) of \( K'' \subset S^{n-1}_\Gamma \) has coordinates

\[ ((\lambda_1, \ldots, \lambda_l, x_1, \ldots, x_l)), \quad \text{where} \quad \sum_{i=1}^l \lambda_i = 1, \quad x = \left( \sum_{i=1}^l \lambda_i x_i \right) \]

for \( x_i \) is a unit vector in the direction of \( K''_i \) for \( i = 1, \ldots, l_0 \).

Let \( \mathcal{Z}(G) \) for any subgroup \( G \) of \( \text{SL}_\pm(n + 1, \mathbb{R}) \) denote the Zariski closure in \( \text{SL}_\pm(n + 1, \mathbb{R}) \).

Let \( \Gamma' \) denote the finite index normal subgroup acting on each of \( K_i \) in \( \Gamma \). We take the Zariski closure of \( \Gamma' \). It is isomorphic to

\[ \mathbb{R}^{l_0-1} \times \mathcal{Z}(\Gamma_1) \times \cdots \times \mathcal{Z}(\Gamma_{l_0}) \]

where \( \mathcal{Z}(\Gamma_i) \) is the Zariski closure of \( \Gamma_i \), easily derivable from Theorem 1.1 of Benoist [21] for our setting. The elements of \( \mathbb{R}^{l_0} \) commute with elements of \( \Gamma_i \) and hence with \( \Gamma' \).

There is a linear map \( Z : \mathbb{Z}^{l_0-1} \rightarrow \mathbb{R}^{l_0} \) so that an isomorphism \( \mathbb{Z}^{l_0-1} \rightarrow \Gamma' \) is represented by \( \zeta \circ \exp \circ Z \).

Let \( \log \lambda_1 : \mathbb{Z}^{l_0-1} \rightarrow \mathbb{R} \) denote a map given by taking the log of the largest norm eigenvalue \( \lambda \) on \( \mathbb{Z}^{l_0-1} \rightarrow \mathbb{R} \) given by taking the log of the smallest norm \( \log \lambda \), where \( \log \lambda \) extends to piecewise linear functions on \( \mathbb{R}^{l_0-1} \) that are linear over cones with origin as the vertex.

\[ \log \lambda_1 \text{ has only nonnegative values and} \quad \log \lambda \text{ has nonpositive values.} \]

The uniform middle eigenvalue condition is equivalent to the condition that \( \log \lambda_1 > \log \lambda \) holds over \( \mathbb{R}^{l_0} - \{0\} \).

Let \( B_i \) denote the tube \( \mathcal{T}(K_i) \). We choose an element \( g \) of the center having largest norm eigenvalue at points of \( K_i \) as an automorphism of \( S^{n-1}_\Gamma \). \( g \) acts on \( B_i \). By the uniform middle eigenvalue conditions, \( g \) fixes an subspace \( \mathcal{T}_i \) equal to \( B_i \cap P_t \) in the span of \( B_i \) corresponding to the largest norm eigenvalue of \( g \) as an element of \( \text{SL}_\pm(n + 1, \mathbb{R}) \). By commutativity, the center also acts on \( \mathcal{T}_i \). Hence, the center acts on the join \( \mathcal{T}_1 \cdots \mathcal{T}_l \) equals \( \mathcal{T} \cap P \) for a hyperspace \( P \). By commutativity, \( \Gamma' \) acts on \( B_i \) also.

Suppose that for some \( g \in \Gamma - \Gamma', g(P) \neq P \). Then \( g(P) \cap B_j \) has a point \( x \) closer to \( v \) or \( v_- \) than \( P \cap B_j \) for some \( j \). Assume that it is closer to \( v \) without loss of generality. We find a sequence \( \{K_i\} \) so that \( \lambda_i \) is the largest eigenvalue at points of \( B_i \) and \( \lambda_i(g_i) \rightarrow \infty \). Since \( \lambda_i(g_i^{-1}) = \lambda_i(g_i)^{-1} \) and \( \lambda_i^{-1}(g_i^{-1}) = \lambda_i(g_i)^{-1} \), we obtain that \( \{g_i^{-1}(x)\} \rightarrow v \) as \( i \rightarrow \infty \). Then we obtain that \( g_i(g(P)) \cap \mathcal{T} \) is not distanced. This contradicts the first paragraph of the proof.
6.3 The characterization of lens-shaped representations

The main purpose of this section is to characterize the lens-shaped representations in terms of eigenvalues, a major result of this monograph.

First, we prove the eigenvalue estimation in terms of lengths for non-virtually-factorizable and hyperbolic ends. We show that the uniform middle-eigenvalue condition implies the existence of limit sets. This proves Theorem 6.1. Finally, we prove the equivalence of the lens condition and the uniform middle-eigenvalue condition in Theorem 6.6 for both R-ends and T-ends under very general conditions. That is, we prove Theorem 6.2.

Techniques here are somewhat related to the work of Guichard-Wienhard [103] and Benoist [18].

6.3.1 The eigenvalue estimations

Let \( O \) be a strongly tame real projective orbifold and \( \hat{O} \) be the universal cover in \( S^n \). Let \( \hat{E} \) be a properly convex R-p-end of \( \hat{O} \), and let \( v_{\hat{E}} \) be the p-end vertex. Let

\[
h : \pi_1(\hat{E}) \to SL_{\pm}(n+1, \mathbb{R})_{v_{\hat{E}}}
\]

be a homomorphism and suppose that \( \pi_1(\hat{E}) \) is hyperbolic.

In this article, we assume that \( h \) satisfies the middle eigenvalue condition. We denote by the norms of eigenvalues of \( g \) by

\[
\hat{\lambda}_1(g), \ldots, \hat{\lambda}_n(g), \hat{\lambda}_{v_{\hat{E}}}(g), \text{ where } \lambda_1(g) \cdots \lambda_n(g) \lambda_{v_{\hat{E}}}(g) = \pm 1, \text{ and } \hat{\lambda}_1(g) = \ldots = \hat{\lambda}_n(g), \tag{6.10}
\]

where we allow repetitions.

Recall the linear part homomorphism \( \mathcal{L}_1 \) from the beginning of Section 6.2. We denote by \( \hat{h} : \pi_1(\hat{E}) \to SL(n, \mathbb{R}) \) the homomorphism \( \mathcal{L}_1 \circ h \). Since \( \hat{h} \) is a holonomy of a closed convex real projective \((n-1)\)-orbifold, and \( \Sigma_{\hat{E}} \) is assumed to be properly convex, \( \hat{h}(\pi_1(\hat{E})) \) divides a properly convex domain \( \Sigma_{\hat{E}} \) in \( S^{n-1}_{\mathbb{E}} \).

We denote by \( \tilde{\lambda}_1(g), \ldots, \tilde{\lambda}_n(g) \) the norms of eigenvalues of \( \hat{h}(g) \) so that

\[
\tilde{\lambda}_1(g) \geq \ldots \geq \tilde{\lambda}_n(g), \tag{6.11}
\]

hold. These are called the relative norms of eigenvalues of \( g \). We have \( \lambda_i(g) = \tilde{\lambda}_i(g)/\lambda_{v_{\hat{E}}}(g)^{1/n} \) for \( i = 1, \ldots, n \).

For nontorsion elements, eigenvalues corresponding to

\[
\lambda_1(g), \tilde{\lambda}_1(g), \lambda_n(g), \tilde{\lambda}_n(g), \lambda_{v_{\hat{E}}}(g)
\]
are all positive since the nontorsion elements are positive semi-proximal by Benoist [22]. We define
\[
\text{length}(g) := \log \left( \frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_n(g)} \right) = \log \left( \frac{\hat{\lambda}_1(g)}{\hat{\lambda}_n(g)} \right). 
\]
This equals the infimum of the Hilbert metric lengths of the associated closed curves in \( \tilde{\Sigma}_E/\hat{h}(\pi_1(\tilde{E})) \) as first shown by Kuiper. (See [17] for example.)

We recall the notions in Section 2.3.2. (See [17] and [18] also.) When \( \Gamma \) acts on a properly convex domain cocompactly and properly, every nonelliptic element is positive bi-semiproximal by Theorem 2.7. (See [23] also). Since \( \tilde{\Sigma}_E \) is properly convex, all infinite order elements of \( \hat{h}(\pi_1(\tilde{E})) \) are positive bi-semiproximal and a finite-index subgroup has only positive bi-semiproximal elements and the identity.

When \( \pi_1(\tilde{E}) \) is hyperbolic, all infinite order elements of \( \hat{h}(\pi_1(\tilde{E})) \) are positive biproximal and a finite index subgroup has only positive biproximal elements and the identity.

Assume that \( \Gamma_\tilde{E} \) is hyperbolic. Suppose that \( g \in \Gamma_\tilde{E} \) is proximal. We define
\[
\alpha_g := \frac{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_n(g)}{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_{n-1}(g)}, \quad \beta_g := \frac{\log \hat{\lambda}_1(g) - \log \hat{\lambda}_n(g)}{\log \hat{\lambda}_1(g) - \log \hat{\lambda}_2(g)},
\]
and denote by \( \Gamma^p_\tilde{E} \) the set of proximal elements. We define
\[
\beta_{\Gamma_\tilde{E}} := \sup_{g \in \Gamma^p_\tilde{E}} \beta_g, \quad \alpha_{\Gamma_\tilde{E}} := \inf_{g \in \Gamma^p_\tilde{E}} \alpha_g.
\]

Proposition 20 of Guichard [102] shows that we have
\[
1 < \alpha_{\Gamma_\tilde{E}} \leq \alpha_{\Gamma} \leq 2 \leq \beta_{\Gamma} \leq \beta_{\Gamma_\tilde{E}} < \infty
\]
for constants \( \alpha_{\Gamma_\tilde{E}} \) and \( \beta_{\Gamma_\tilde{E}} \) depending only on \( \tilde{\Sigma}_E \) since \( \tilde{\Sigma}_E \) is properly and strictly convex.

Here, it follows that \( \alpha_{\Gamma^p_\tilde{E}}, \beta_{\Gamma^p_\tilde{E}} \) depends on \( \hat{h} \), and they form positive-valued functions on the union of components of
\[
\text{Hom}(\pi_1(\tilde{E}), \text{SL}_\pm(n + 1, \mathbb{R}))/\text{SL}_\pm(n + 1, \mathbb{R})
\]
consisting of convex divisible representations with the algebraic convergence topology as given by Benoist [23].

**Theorem 6.5** Let \( \tilde{O} \) be a strongly tame convex real projective orbifold. Let \( \tilde{E} \) be a properly convex \( \tilde{O} \)-end of the universal cover \( \tilde{O}, \tilde{O} \subset \mathbb{S}^n, n \geq 2. \) Let \( \Gamma_\tilde{E} \) be a hyperbolic group. Then
\[
\frac{1}{n} \left( 1 + \frac{n-2}{\beta_{\Gamma_\tilde{E}}} \right) \text{length}(g) \leq \log \tilde{\lambda}_1(g) \leq \frac{1}{n} \left( 1 + \frac{n-2}{\alpha_{\Gamma_\tilde{E}}} \right) \text{length}(g)
\]
for every nonelliptic element \( g \in \hat{h}(\pi_1(E)) \).

**Proof** Since there is a positive bi-proximal subgroup of finite index, we concentrate on positive bi-proximal elements only. We obtain from above that

\[
\frac{\log \tilde{\lambda}_1(g)}{\log \tilde{\lambda}_2(g)} \leq \beta \Gamma \tilde{E}.
\]

We deduce that

\[
\tilde{\lambda}_1(g) \geq \tilde{\lambda}_2(g) = \left( \frac{\lambda_1(g)}{\lambda_n(g)} \right)^{1/\beta \Gamma \tilde{E}} = \exp \left( \frac{\text{length}(g)}{\frac{\alpha}{\beta} \Gamma \tilde{E}} \right).
\]

(6.14)

Since we have \( \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \) for \( i \geq 2 \), we obtain

\[
\tilde{\lambda}_1(g) \geq \left( \frac{\lambda_1(g)}{\lambda_n(g)} \right)^{1/\beta \Gamma \tilde{E}} \geq \left( \frac{\lambda_1(g)}{\lambda_n(g)} \right)^{n-2} \frac{n-1}{\beta \Gamma \tilde{E}}.
\]

(6.15)

We obtain

\[
\log \tilde{\lambda}_1(g) \geq \frac{1}{n} \left( 1 + \frac{n-2}{\alpha \Gamma \tilde{E}} \right) \text{length}(g).
\]

(6.16)

By similar reasoning, we also obtain

\[
\log \tilde{\lambda}_1(g) \leq \frac{1}{n} \left( 1 + \frac{n-2}{\alpha \Gamma \tilde{E}} \right) \text{length}(g).
\]

\( \square \)

**Remark 6.1** Under the assumption of Theorem 6.5, if we do not assume that \( \pi_1(E) \) is hyperbolic, then we obtain

\[
\frac{1}{n} \text{length}(g) \leq \log \tilde{\lambda}_1(g) \leq \frac{n-1}{n} \text{length}(g)
\]

(6.17)

for every semiproximal element \( g \in \hat{h}(\pi_1(E)) \).

**Proof** Let \( \tilde{\lambda}_i(g) \) denote the norms of \( \tilde{h}(g) \) for \( i = 1, 2, \ldots, n \).

\[
\log \tilde{\lambda}_1(g) \geq \ldots \geq \log \tilde{\lambda}_n(g), \log \tilde{\lambda}_1(g) + \cdots + \log \tilde{\lambda}_n(g) = 0
\]

hold. We deduce
6.3 The characterization of lens-shaped representations

\[
\log \tilde{\lambda}_n(g) = - \log \tilde{\lambda}_1 - \cdots - \log \tilde{\lambda}_{n-1}(g) \\
\geq -(n-1) \log \tilde{\lambda}_1 \\
\log \tilde{\lambda}_1(g) \geq -\frac{1}{n-1} \log \tilde{\lambda}_n(g) \\
\left(1 + \frac{1}{n-1}\right) \log \tilde{\lambda}_1(g) \geq \frac{1}{n-1} \log \frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_n(g)} \\
\log \tilde{\lambda}_1(g) \geq \frac{1}{n} \text{length}(g). \tag{6.18}
\]

We also deduce

\[
- \log \tilde{\lambda}_1(g) = \log \tilde{\lambda}_2(g) + \cdots + \log \tilde{\lambda}_n(g) \\
\geq (n-1) \log \tilde{\lambda}_n(g) \\
-(n-1) \log \tilde{\lambda}_n(g) \geq \log \tilde{\lambda}_1(g) \\
(n-1) \log \frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_n(g)} \geq n \log \tilde{\lambda}_1(g) \\
\frac{n-1}{n} \text{length}(g) \geq \log \tilde{\lambda}_1(g). \quad \square
\]

**Remark 6.2** We cannot show that the middle-eigenvalue condition implies the uniform middle-eigenvalue condition. This could be false. For example, we could obtain a sequence of elements \(g_i \in \Gamma\) so that \(\{\lambda_1(g_i)/\lambda_v E(g_i)\} \to 1\) while \(\Gamma\) satisfies the middle-eigenvalue condition. Certainly, we could have an element \(g\) where \(\lambda_1(g) = \lambda_v E(g)\). However, even if there is no such element, we might still have a counter-example. For example, suppose that we might have

\[
\begin{cases}
\log \frac{\lambda_1(g)}{\lambda_v E(g)} \\
\text{length}(g)
\end{cases} \to 0.
\]

This could happen by changing \(\lambda_v E\) considered as a homomorphism \(\pi_1(\Sigma E) \to \mathbb{R}_+\). Such assignments are not really understood globally but see Benoist [17]. Also, an analogous phenomenon seems to happen with the Margulis space-time and diffused Margulis invariants as investigated by Charette, Drumm, Goldman, Labourie, and Margulis recently. See [96])

### 6.3.2 The uniform middle-eigenvalue conditions and the orbits.

Let \(\tilde{E}\) be a properly convex \(R\)-p-end of the universal cover \(\tilde{\mathcal{O}}\) of a strongly tame properly convex real projective orbifold \(\mathcal{O}\). Assume that \(\Gamma_{\tilde{E}}\) satisfies the uniform
middle-eigenvalue condition. There exists a $\Gamma_E$-invariant compact set to be denoted $L_E$ distanced from $\{v_E, v_{E-}\}$ by Theorem 6.4. For the corresponding tube $\mathcal{T}_E(\hat{E})$, $L_E \cap \text{bd} \mathcal{T}_E(\hat{E})$ is a compact subset distanced from $\{v_E, v_{E-}\}$. Let $\mathcal{H}(L)$ be the convex hull of $L$ in the tube $\mathcal{T}_E(\hat{E})$ obtained by Theorem 6.4. Then $\mathcal{H}(L)$ is a $\Gamma_E$-invariant distanced subset of $\mathcal{T}_E(\hat{E})$.

We of course are doing everything in $S^d$. But $\mathbb{RP}^d$-versions are fairly clear to obtain.

**Definition 6.5** We define the limit set $A_E$ of a properly convex R-end $E$ to be the compact $\Gamma_E$-invariant subset of $\partial \mathcal{T}_E(\hat{E}) - \{v_E, v_{E-}\}$.

Following Corollary 6.1 shows that the limit set is well-defined. Compare also to Definition 7.1.

The following main result of this subsection shows that $L_E$ is characterized.

**Corollary 6.1** A distanced compact $\Gamma_E$-invariant set $C$ in $\partial \mathcal{T}_E(\hat{E})$ which meets every great segment $\partial \mathcal{T}_E(\hat{E})$ in its interior exists and is unique. Also, it satisfies $\mathcal{H}(C) \cap \mathcal{T}_E(\hat{E}) = C$.

**Proof** Proposition 6.3 will show that $C$ is independent of the choice and meets each great segment for any distanced compact convex set $L$ in $\mathcal{T}_E(\hat{E})$ at a unique point.

Since $\mathcal{H}(C) \cap \text{bd} \mathcal{T}_E(\hat{E})$ contains $C$ and is also a $\Gamma_E$-invariant distanced compact set, the uniqueness part of Proposition 6.3 shows that it equals $C$. □

Also, $A_E \cap \text{bd} \mathcal{T}_E(\hat{E})$ contains all attracting and repelling fixed points of $E \in \Gamma_E$ by the $\Gamma_E$-invariance and the middle-eigenvalue condition.

By Corollary 6.1, we obtain a distanced compact $\Gamma_E$-invariant set $C$ in $\partial \mathcal{T}_E(\hat{E}) - \{v_E, v_{E-}\}$ satisfying $\mathcal{H}(C) \cap \partial \mathcal{T}_E(\hat{E}) = C$ and meeting every great line in $\partial \mathcal{T}_E(\hat{E})$ is said to be a limit set for $E$. It always exists. Theorem 6.4 shows that a distanced compact $\Gamma_E$-invariant set $C$ exists. Then $\mathcal{H}(C) \cap \partial \mathcal{T}_E(\hat{E})$ is a limit set for $E$.

### 6.3.2.1 Hyperbolic groups

We first consider when $\Gamma_E$ is hyperbolic.

**Lemma 6.1** Let $\mathcal{O}$ be a strongly tame convex real projective orbifold. Let $E$ be a properly convex R-end. Assume that $\Gamma_E$ is hyperbolic and satisfies the uniform middle eigenvalue conditions.

- Suppose that $\gamma$ is a sequence of elements of $\Gamma_E$ acting on $\mathcal{T}_E(\hat{E})$.
- The sequence of attracting fixed points $a_i$ and the sequence of repelling fixed points $b_i$ are so that $\{a_i\} \to a_\infty$ and $\{b_i\} \to b_\infty$ where $a_\infty, b_\infty$ are not in $\{v_E, v_{E-}\}$.
- Suppose that the sequence $\{\lambda_i\}$ of eigenvalues where $\lambda_i$ corresponds to $a_i$ converges to $+\infty$. 
Then the point \( a_\infty \) in \( \partial \mathcal{K}_E(\widetilde{\Sigma}_E) - \{ v_E, v_{E^-} \} \) is the geometric limit of \( \{ \gamma(K) \} \) for any compact subset \( K \subset M \).

**Proof** We may assume without loss of generality that \( a_\infty \neq b_\infty \) since otherwise we replace \( \{ g_i \} \) with \( \{ gg_i \} \) where \( g(a_\infty) \neq b_\infty \). Proving for this case implies the general cases.

Let \( k_i \) be the inverse of the factor

\[
\min \left\{ \frac{\lambda_1(\gamma)}{\lambda_2(\gamma)}, \frac{\lambda_1(\gamma)}{\lambda_{\nu_E}(\gamma)^{\frac{1}{\nu_E}}} = \frac{\lambda_1(\gamma)}{\lambda_{\nu_E}(\gamma)} \right\}.
\]

Then \( \{ k_i \} \to 0 \) by the uniform middle eigenvalue condition and (6.14).

There exists a totally geodesic sphere \( S_i^{n-1} \) sharply supporting \( \mathcal{K}_E(\widetilde{\Sigma}_E) \) at \( b_i \). \( a_i \) is uniformly bounded away from \( S_i^{n-1} \) for \( i \) sufficiently large. \( S_i^{n-1} \) bounds an open hemisphere \( H_i \) containing \( a_i \) where \( a_i \) is the attracting fixed point by Corollary 1.2.3 of [118] or by Proposition 2.8 so that for a Euclidean metric \( d_{E,i} \), \( \gamma|H_i : H_i \to H_i \) we have

\[
d_{E,i}(\gamma(x), \gamma(y)) \leq k_i d_{E,i}(x,y), x, y \in H_i.
\]

Note that \( \{ \text{Cl}(H_i) \} \) converges geometrically to \( \text{Cl}(H) \) for an open hemisphere containing \( a \) in the interior.

Actually, we can choose a Euclidean metric \( d_{E,i} \) on \( H_i^n \) so that \( \{ d_{E,i}|J \times J \} \) is uniformly convergent for any compact subset \( J \) of \( H_\infty \). Hence there exists a uniform positive constant \( C' \) so that

\[
d(a_i, K) < C' d_{E,i}(a_i, K).
\]

provided \( a_i, K \subset J \) and sufficiently large \( i \).

Since \( \Gamma_E \) is hyperbolic, the domain \( \Omega \) corresponding to \( \mathcal{K}_E(\widetilde{\Sigma}_E) \) in \( S_i^{n-1} \) is strictly convex. For any compact subset \( K \) of \( M \), the equation \( K \subset M \) is equivalent to

\[
K \cap \text{Cl}(\bigcup_{i=1}^\infty b_i v_E \cup b_i v_{E^-}) = \emptyset.
\]

Since the boundary sphere \( \text{bd}H_\infty \) meets \( \mathcal{K}_E(\widetilde{\Sigma}_E) \) in this set only by the strict convexity of \( \Omega \), we obtain \( K \cap \text{bd}H_\infty = \emptyset \). And \( K \subset H_\infty \) since \( \mathcal{K}_E(\widetilde{\Sigma}_E) \subset \text{Cl}(H_\infty) \).

We have \( d(K, \text{bd}H_\infty) > \epsilon_0 \) for \( \epsilon_0 > 0 \). Thus, the distance \( d(K, \text{bd}H_\infty) \) is uniformly bounded by a constant \( \delta \). \( d(K, \text{bd}H_\infty) > \delta \) implies that \( d_{E,i}(a_i, K) \leq C/\delta \) for a positive constant \( C > 0 \) Acting by \( g_i \), we obtain \( d_{E,i}(g_i(K), a_i) \leq k_i C/\delta \) by (6.19), which implies \( d(g_i(K), a_i) \leq C' k_i C/\delta \) by (6.20). Since \( \{ k_i \} \to 0 \), the fact that \( \{ a_i \} \to a \) implies that \( \{ g_i(K) \} \) geometrically converges to \( a \). \( \square \)

**Lemma 6.2** Let \( \Theta \) be a strongly tame convex real projective orbifold. Let \( \tilde{E} \) be a properly convex \( R \)-\( p \)-end. Assume that \( \Gamma_{\tilde{E}} \) is hyperbolic, and satisfies the uniform middle eigenvalue conditions. Suppose that \( \{ \gamma \} \) is sequence of elements of \( \Gamma_{\tilde{E}} \) acting on \( \mathcal{K}_{\tilde{E}}(\widetilde{\Sigma}_{\tilde{E}}) \) and forms a convergence sequence acting on \( S_{\tilde{E}}^{n-1} \). Then given any
limit set $L_E$ for $E$, it contains the geometric limit of any subsequence of $\{\gamma(\Omega)\}$ for any compact subset $K \subset \mathcal{F}_E^0$. Furthermore,

$$A_s(\{\gamma\}) \cap \mathcal{F}_E(\tilde{E}_E), R_s(\{\gamma\}) \cap \mathcal{F}_E(\tilde{E}_E) \subset L_E.$$  

**Proof** Let $z \in \mathcal{F}_E^0$. Let \((z')\) denote the element in $\Sigma_E$ corresponding to the ray from $v_E$ to $z$. Let $\{\gamma\}$ be any sequence in $\Gamma_E^\infty$ so that the corresponding sequence $\{\gamma((z))\}$ in $\Sigma_E \subset \mathbb{S}^{n-1}$ converges to a point $z'$ in $\text{bd} \Sigma_E \subset \mathbb{S}^{n-1}$.

Clearly, a fixed point of $g \in \Gamma_E^\infty \setminus \{1\}$ in $\text{bd} \mathcal{F}_E(\tilde{E}_E) \setminus \{v_E, v_E^-\}$ is in $L_E$ since $g$ has at most one fixed point on each open segment in the boundary. For the attracting fixed points $a_i$ and $r_i$ of $\gamma_i$, we can assume that

$$\{a_i\} \to a, \{r_i\} \to r \text{ for } a_i, r_i \in L_E$$

where $a, r \in L_E$ by the closedness of $L_E$. Assume $a \neq r$ first. By Lemma 6.1, we have $\{\gamma((z))\} \to a$ and hence the limit $z_\infty = a$.

However, it could be that $a = r$. In this case, we choose $\gamma_0 \in \Gamma_E^\infty$ so that $\gamma_0(a) \neq r$. Then $\gamma_0\gamma_1$ has the attracting fixed point $a'_i$ so that we obtain $\{a'_i\} \to \gamma_0(a)$ and repelling fixed points $r'_i$ so that $\{r'_i\} \to r$ holds by Lemma 6.3. This implies the first part.

Then as above $\{\gamma_0\gamma((z))\} \to \gamma_0(a)$ and we need to multiply by $\gamma_0^{-1}$ now to show $\{\gamma((z))\} \to a$. Thus, the limit set is contained in $L_E$. \qed

**Lemma 6.3** Let $\{g_i\}$ be a sequence of projective automorphisms acting on a strictly convex domain $\Omega$ in $\mathbb{S}^n$. Suppose that the sequence of attracting fixed points $\{a_i \in \text{bd} \Omega\} \to a$ and the sequence of repelling fixed points $\{r_i \in \text{bd} \Omega\} \to r$. Assume that the corresponding sequence of eigenvalues of $a_i$ limits to $+\infty$ and that of $r_i$ limits to 0. Let $g$ be any projective automorphism of $\Omega$. Then $\{g g_i\}$ has the sequence of attracting fixed points $\{a'_i\}$ converging to $g(a)$ and the sequence of repelling fixed points converging to $r$.

**Proof** Recall that $g$ is a quasi-isometry. Given $\varepsilon > 0$ and a compact ball $B$ disjoint from a ball around $r$, we obtain that $g g_i(B)$ is in a ball of radius $\varepsilon$ of $g(a)$ for sufficiently large $i$. For a choice of $B$ and sufficiently large $i$, we obtain $g g_i(B) \subset B^0$. Since $g g_i(B) \subset B^0$, we obtain

$$(g g_i)^n(B) \subset (g g_i)^m(B)^0 \text{ for } n > m$$

by induction. There exists an attracting fixed point $a'_i$ of $g_i$ in $g g_i(B)$. Since the sequence of the diameters of sets of form $g g_i(B)$ is converging to 0, we obtain that $\{a'_i\} \to g(a)$.

Also, given $\varepsilon > 0$ and a compact ball $B$ disjoint from a ball around $g(a)$, $g_i^{-1} g^{-1}(B)$ is in the ball of radius $\varepsilon$ of $r$. Similarly to above, we obtain the needed conclusion. \qed
6.3.2.2 Non-hyperbolic groups

Now, we generalize to not necessarily hyperbolic $\Gamma_E$. A $\Gamma_E$-invariant distanced set $L_E$ contains the attracting fixed set $A_i$ and the repelling fixed set $R_i$ of any $g \in \Gamma_E$ by invariance and sequence arguments.

**Lemma 6.4** Let $\mathcal{O}$ be a strongly tame convex real projective orbifold. Let $E$ be a properly convex R-p-end. Assume that $\Gamma_E$ is non-hyperbolic and or virtually-factorizable and satisfies the uniform middle eigenvalue conditions with respect to $v_E$. Suppose that $\{\gamma\}$ is a generalized convergence sequence of elements of $\Gamma_E$ acting on $\mathcal{T}_E(\Sigma_E)$. Let $L_E$ be a limit set for $\bar{E}$. Then $L_E$ contains the geometric limit of any subsequence of $\{\gamma(K)\}$ for any compact subset $K \subset \mathcal{T}_E(\Sigma_E)^\circ$. Furthermore,

$$A_+((\gamma)) \cap \mathcal{T}_E(\Sigma_E), R_+((\gamma)) \cap \mathcal{T}_E(\Sigma_E) \subset L_E.$$  

**Proof** Let $L = \mathcal{C}(L_E) \cap \mathcal{T}_E(\Sigma_E)$. Then $L$ is a convex set uniformly bounded away from $v_E$ and its antipode by a geometric consideration.

Given any sequence $g_i$, we can extract a convergence sequence $\{g_i\}$ with a convergence limit $g_\infty$.

Suppose that $L^0 = \emptyset$. Then $L$ is a convex domain on a hyperspace $P$ disjoint from $v_E$. We use a coordinate system where each $\gamma \in \Gamma$ is of form (6.6) where $b_\gamma = 0$. Dividing $g_i$ by $\lambda_1(g_i)$ and taking a limit, we obtain that $g_\infty$ equals

$$\left( \frac{\hat{g}_\infty}{0} \right)$$

by the uniform middle eigenvalue condition and Lemma 2.12. Hence $A_+((g_i)) \subset P$.

By Theorem 2.10,

$$A_+((g_i)) \subset P \cap \partial \mathcal{T}_E(\Sigma_E) = L_E.$$  

The remainders are simple to show.

Suppose that $L^0$ is not empty. Then $L^0 \cap N_+(\{g_i\}) = \emptyset$ by Lemma 2.15. Given any convergence sequence $\{g_i\}$, $g_i \in \Gamma$ converging to $g_\infty$, the sequence $g_i(x)$ for $x \in L$ converges to a point of $A_+((g_i))$.

By Lemma 2.12, $v_E \in N_+(\{g_i\})$ since $\{\lambda_\gamma E(g_i) / \lambda(g_i)\} \to 0$ by the uniform middle eigenvalue condition. Dividing $g_i$ by $\lambda_1(g_i)$ and taking a limit, we obtain that $g_\infty$ equals

$$\left( \frac{\hat{g}_\infty}{\hat{b}} \right)$$

by the uniform middle eigenvalue condition and Lemma 2.12 dualizing the proof of Lemma 5.14. Here $\hat{g}_\infty$ is not zero since otherwise we have uniform convergence to $v_E$ or $v_{E_-}$ for any compact set disjoint from $\{v_E, v_{E_-}\}$ while $L$ is invariant set, which is absurd. Since $\hat{g}_\infty \neq 0$, the image of $g_\infty$ is now a subspace of the same dimension as $A_+((\hat{g}_i))$. Actually, it is graph over $A_+((\hat{g}_i))$ where the vertical direction is given by the direction to $v_E$ for a linear function given by $\hat{b}$. 

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Since $\mathbf{G}_E$ acts on $L$, $g_\omega(x) \in \text{Cl}(L) \cap \text{bd} \mathcal{R}_E(\hat{\Sigma}_E)$. Hence, $g_\omega(L) = A_\ast \{ \{ g_l \} \} \subset L_E$. Using $\{ g_l^{-1} \}$, we obtain $R_\ast(\{ g_l \}) \subset L_E$.

Any element of $x \in \mathcal{R}_E(\hat{\Sigma}_E)$ satisfies $x = \{ (v_x) \}, v_x = v_L + cv_E$ for a constant $c > 0$ and a vector $v_L$ in the direction of a point of $L$ and a vector $v_E$ in direction of $v_E$. Then
\[
g_\omega(\{ (v_x) \}) = \{ (g_\omega(v_L) + cg_\omega(v_E)) \}.
\]
Since $cg_\omega(v_E) = 0$ from (6.22), we obtained that $g_\omega(\mathcal{R}_E(\hat{\Sigma}_E)) = g_\omega(L)$. Since
\[
g_\omega(\mathcal{R}_E(\hat{\Sigma}_E)) \subset A_\ast(\{ \{ g_l \} \}) \cap \mathcal{R}_E(\hat{\Sigma}_E),
\]
and $g_\omega(L) = A_\ast(\{ \{ g_l \} \})$, we obtain the result. The final statement is also proved by taking the sequence $g_l^{-1}$.

For the following, $\mathbf{G}_E$ can be virtually factorizable. By following Proposition 6.3, $A_\ast E$ is well-defined independent of the choice of $K$.

**Proposition 6.3** Let $\emptyset$ be a strongly tame convex real projective orbifold. Let $\hat{E}$ be a properly convex $R$-end. Assume that $\mathbf{G}_E$ satisfies the uniform middle eigenvalue condition with respect to the $R$-end structure. Let $v_E$ be the $R$-end vertex and $z \in \mathcal{R}_E(\hat{\Sigma}_E)$.$^n$. Let $L_E$ be a limit set for $\hat{E}$, and let $L$ be the closure of $\mathcal{C}_E(L_E)$. Then the following properties are satisfied:

(i) $L_E$ contains all the limit points of orbits of each compact subset of $\mathcal{R}_E(\hat{\Sigma}_E)^n$. $L_E$ contains all attracting fixed sets of elements of $\mathbf{G}_E$. If $\mathbf{G}_E$ is hyperbolic, then the set of attracting fixed point is dense in the set.

(ii) For each segment $s$ in $\partial \mathcal{R}_E(\hat{\Sigma}_E)$ with an endpoint $v_E$, the great segment containing $s$ meets $L_E$ at a unique point other than $v_E, v_{E-}$. That is, there is a one-to-one correspondence between $\text{bd} \hat{\Sigma}_E$ and $L_E$.

(iii) $L_E$ is homeomorphic to $S^{n-2}$.

(iv) For any $\mathbf{G}_E$-distanced compact set $L'$ in $\partial \mathcal{R}_E(\hat{\Sigma}_E) - \{ v_E, v_{E-} \}$ meeting every great segment in $\mathcal{R}_E(\hat{\Sigma}_E)$, we have $L_E = L'$. (uniqueness)

**Proof** We will first prove (i),(ii),(iii) for various cases and then prove (iv) all together:

(A) Consider first when $\mathbf{G}_E$ is not virtually factorizable and hyperbolic. Proposition 6.2 proves (i) here. Let $L$ be the closure of $\mathcal{C}_E(L_E)$, which is $\mathbf{G}_E$-invariant. Let $K' = L \cap \text{bd} \mathcal{R}_E(\hat{\Sigma}_E) - \{ v_E, v_{E-} \}$. Clearly $L_E \subset K'$.

Since $\mathbf{G}_E$ is hyperbolic, any point $y$ of $\text{bd} \hat{\Sigma}_E \subset S^{n-1}$ is a limit point of some sequence $\{ g_l(x) \}$ for $x \in \hat{\Sigma}_E$ by [22]. Thus, at least one point in the segment $l_y$ in the direction of $y, y \in S_{v_E}^{n-1}$ with endpoints $v_E$ and $v_{E-}$ is a limit point of some subsequence of $\{ g_l(x) \}$ by Lemma 6.1. Thus, $l_y \cap L_E \neq \emptyset$, and $l_y \cap K' \neq \emptyset$.

Let us choose a convenient Euclidean metric $\| \cdot \|_E$ for $\mathbb{R}^{n+1}$. We claim that $l_y \cap K'$ is unique: Suppose not. Let $z$ and $z'$ be the two points of $l_y \cap K'$. We choose a line $l$ in $\hat{\Sigma}_E$ ending at $y$. Let $y_z$ be the sequence of points on $l$ covering to $y$. We choose $g_i$ as in the proof of Lemma 5.2 so that $g_i(y_z) \in F$ for a compact fundamental domain $F$ of $\hat{\Sigma}_E$. We assume that $\{ g_i \}$ is a convergence sequence by choosing a subsequence.
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if necessary. (Here, \( \langle \mathbf{v}_- \rangle \) is fixed to be a single point \( y = \langle \mathbf{v}_- \rangle \).) Given the other endpoint \( z \) of \( l \), we have

\[
\{ g_i(z) \} \rightarrow a_*\{ \{ g_i \} \}
\]

for an attractor of \( a_*\{ \{ g_i \} \} \) of \( \{ g_i \} \). This follows by the same reasoning the proof of Lemma 5.2. This means that \( \{ g_i(y) \} \) is uniformly bounded away from \( a_*\{ \{ g_i \} \} \) since \( g_i(l) \) passes \( F \) with \( \{ g_i(z) \} \) converging to \( a_*\{ \{ g_i \} \} \). Since \( \{ g_i(y) \} \) is bounded away from \( a_*\{ \{ g_i \} \} \) uniformly, as in the proof of Lemma 5.2 using (5.10) and similarly to proving the conclusion of the lemma, we obtain

\[
\left\{ \frac{1}{\lambda_{\mathcal{E}}(g_i)^{1/2}} \hat{h}(g_i)(\mathbf{v}_-) \right\} \rightarrow 0 \quad (6.23)
\]

in the Euclidean metric. To explain more, we write \( \mathbf{v}_- \) as a sum of \( \mathbf{v}_p + \mathbf{v}_S \) as there.

The rest is analogous.

Let \( \mathbf{v}_E \) denote the unit vector in the direction of \( \mathbf{v}_E \). We consider \( \mathbb{R}^n \) to be a complementary subspace to this vector under the norm \( \| \cdot \| \). We write the vector for \( z \) as \( \mathbf{v}_z = \lambda \mathbf{v}_- + \mathbf{v}_E \) and the vector for \( z' \) as \( \mathbf{v}_z' = \lambda' \mathbf{v}_- + \mathbf{v}_E \). Then

\[
g_i(\mathbf{v}_z) = \lambda \frac{1}{\lambda_{\mathcal{E}}(g_i)^{1/2}} \hat{h}(g_i)(\mathbf{v}_-)(\mathbf{v}_-)^\top + (\lambda \mathbf{b}_{gi} \cdot \mathbf{v}_- + \lambda_{\mathcal{E}}(g_i)\mathbf{v}_E)\mathbf{v}_E.
\]

by (6.3). Let us denote

\[
c_i := \left\| \frac{1}{\lambda_{\mathcal{E}}(g_i)^{1/2}} \hat{h}(g_i)(\mathbf{v}_-) \right\|_E.
\]

Since the direction of \( g_i(\mathbf{v}_z) \) is bounded away from \( \mathbf{v}_E \),

\[
\left| \frac{\lambda \mathbf{b}_{gi} \cdot \mathbf{v}_-}{c_i} + \frac{\lambda_{\mathcal{E}}(g_i)}{c_i} \right|
\]

is uniformly bounded. By (6.23), we obtain

\[
\left\{ \left| \frac{\lambda_{\mathcal{E}}(g_i)}{c_i} \right| \right\} \rightarrow \infty.
\]

Hence,

\[
\left\{ \left| \frac{\mathbf{b}_{gi} \cdot \mathbf{v}_-}{c_i} \right| \right\} \rightarrow \infty \text{ as } i \rightarrow \infty.
\]

We also have

\[
g_i(\mathbf{v}_{z'}) = \lambda' \frac{1}{\lambda_{\mathcal{E}}(g_i)^{1/2}} \hat{h}(g_i)(\mathbf{v}_-)(\mathbf{v}_-)^\top + (\lambda' \mathbf{b}_{gi} \cdot \mathbf{v}_- + \lambda_{\mathcal{E}}(g_i)\mathbf{v}_E)\mathbf{v}_E.
\]
Since $\lambda' \neq \lambda$, and $\left\{ \frac{b_{g_i} \cdot v_-}{c_i} \right\} \to \infty$

$$
\left\{ \lambda b_{g_i} \cdot v_- + \frac{\lambda v_{\hat{\Sigma}}(g_i) v_E}{c_i} \right\} = \left( \lambda - \lambda \right) \frac{b_{g_i} \cdot v_-}{c_i} + \frac{\lambda v_{\hat{\Sigma}}(g_i) v_E}{c_i}
$$

cannot be uniformly bounded. This implies that $g_i(z')$ converges to $v_E$ or $v_{E-}$. Since $z' \in K'$ and $K'$ is $\Gamma_E$-invariant, this is a contradiction.

By Lemma 6.1, $L_E$ meets every great segment in $\mathcal{T}$. Thus, $K' \cap l_y = L_E \cap l_y$ for every $y$ in $bd\hat{\Sigma}_E$. Thus, $K' = L_E$, and (i) and (ii) hold for $L_E$.

(iii) Since $L_E$ is closed and compact and bounded away from $v_E, v_{E-}$, the section $s : bd\Omega_E \to bd\mathcal{R}_E(\hat{\Sigma}_E)$ is continuous. If not, we can contradict (ii) by taking two sequences converging to distinct points in a great segment from $v_E$ to its antipode.

(B) Now suppose that $\Gamma_E$ is not virtually factorizable and is not hyperbolic.

Lemma 6.4 proves that the orbits limit to $L_E$ only. An attracting fixed sets in $\mathcal{T}_{E}(\hat{\Sigma}_E)$ is in $L_E$ as in case (A).

First suppose that a great segment $\eta$ in $\mathcal{T}_{E}(\hat{\Sigma}_E)$ with endpoints $v_E$ and $v_{E-}$ corresponds to an element $y$ of $bd\hat{\Sigma}_E$. Now we take a line $l$ in $\hat{\Sigma}_E$ as in the hyperbolic case. Then (6.23) holds as above using Lemma 5.12 instead of Lemma 5.2. The identical argument will show that $\eta'$ meets with $L_E$ at a unique point. This proves (i) and (ii). (iii) follows as above.

(C) Suppose that $\Gamma_E$ is virtually factorizable. We follow the proof of Theorem 6.4. Now, the space of open great segments with an endpoint $v_E$ in $\mathcal{T}_{E}(\hat{\Sigma}_E)^\circ$ corresponds to a properly convex domain $\Omega$ that is the interior of the strict join $K_1 \ast \cdots \ast K_l$. Then a totally geodesic $\Gamma_E$-invariant hyperspace $H$ is disjoint from $\{v_E, v_{E-}\}$ by the proof of Theorem 6.4. Here, we may regard $K_i \subset H$ for each $i = 1, \ldots, l$. Then consider any sequence $g_i$ so that $\{g_i(x)\} \to x_0$ for a point $x \in \mathcal{T}_{E}(\hat{\Sigma}_E)^\circ$ and $x_0 \in \mathcal{T}_{E}(\hat{\Sigma}_E)$. Let $x'$ denote the corresponding point of $\hat{\Sigma}_E$ for $x$. Then $\{g_i(x')\}$ converges to a point $y \in S_{E}^{1}$. Let $x \in \mathbb{R}^{n+1}$ be the vector in the direction of $x$. We write

$$
x = x_E + x_H
$$

where $x_H$ is in the direction of $H$ and $x_E$ is in the direction of $v_E$. By the uniform middle eigenvalue condition and estimating the size of vectors, we obtain the same situation as in (6.21) and $\{g_i(x)\} \to x_0$ for $x \in L_E$ and $x_0 \in H$. Hence, $x_0 \in H \cap L_E$. Thus, every limit point of an orbit of $x$ is in $H$.

If there is a point $y$ in $L_E - H$, then there is a strict join $K_1 \ast \cdots \ast K_l \ast \{v_E\}$ for a proper collection containing $y$. As in the proof of Theorem 6.4, by Proposition 2.15, we can find a sequence $g_i$, virtually central $g_i \in \Gamma_E$, so that $\{g_i|K_1 \ast \cdots \ast K_l\}$ converges to the identity and the maximal norm $\tilde{\Lambda}_i(g_i)$ of the eigenvalue associated with $K_1 \ast \cdots \ast K_l$ and $\tilde{\Lambda}_E(g_i)$ satisfy $\{\tilde{\Lambda}_i(g_i)/\tilde{\Lambda}_E(g_i)\} \to 0$ by the uniform middle eigenvalue condition choosing the maximal norm of $g_i$ to be in the complementary domains. Hence $\{g_i(y)\} \to v_E$. Again this is a contradiction. Hence, we obtain $L_E \subset H$. (i), (ii), (iii) follow easily now.

(iv) Suppose that we have another distanced set $L'$. We take a convex hull of $L_E \cup L'$ and apply the same reasoning as above. □
6.3.3 An extension of Koszul’s openness

Here, we state and prove a well-known minor modification of Koszul’s openness result.

A radial affine connection is an affine connection on \( \mathbb{R}^{n+1} - \{O\} \) invariant under a scalar dilatation \( S_t : v \to tv \) for every \( t > 0 \).

**Proposition 6.4 (Koszul)** Let \( M \) be a properly convex real projective compact \( n \)-orbifold with strictly convex boundary. Let

\[
h : \pi_1(M) \to \text{PGL}(n+1, \mathbb{R}) \quad \text{(resp. } h : \pi_1(M) \to \text{SL}_\pm(n+1, \mathbb{R}))
\]

denote the holonomy homomorphism acting on a properly convex domain \( \Omega_h \) in \( \mathbb{R}P^n \) (resp. in \( \mathbb{R}^n \)). Assume that \( M \) is projectively diffeomorphic to \( \Omega_h/h(\pi_1(M)) \). Then there exists a neighborhood \( U \) of \( h \) in

\[
\text{Hom}(\pi_1(M), \text{PGL}(n+1, \mathbb{R})) \quad \text{(resp. } \text{Hom}(\pi_1(M), \text{SL}_\pm(n+1, \mathbb{R})))
\]

so that every \( h' \in U \) acts on a properly convex domain \( \Omega_{h'} \) so that \( \Omega_{h'}/h'(\pi_1(M)) \) is a compact properly convex real projective \( n \)-orbifold with strictly convex boundary. Also, \( \Omega_{h'}/h'(\pi_1(M)) \) is diffeomorphic to \( M \).

**Proof** We prove for \( S^n \). Let \( \Omega_h \) be a properly convex domain covering \( M \). We may modify \( M \) by pushing \( \partial M \) inward: Let \( \Omega_h' \) be the inverse image of \( M' \) in \( M \). Then \( M' \) and \( \Omega_h' \) are properly convex by Lemma 2.16.

The linear cone \( C(\Omega_h') \subset \mathbb{R}^{n+1} = \Pi^{-1}(\Omega_h') \) over \( \Omega_h' \) has a smooth strictly convex Hessian function \( V \) by Vey [161] or Vinberg [113]. Let \( C(\Omega_h') \) denote the linear cone over \( \Omega_h' \). We extend the group \( \mu(\pi_1(M)) \) by adding a transformation \( \gamma' : v \mapsto 2v \) to \( C(\Omega_h') \). For the fundamental domain \( F' \) of \( C(\Omega_h') \) under this group, the Hessian matrix of \( V \) restricted to \( F \cap C(\Omega_h') \) has a lower bound. Also, the boundary \( \partial C(\Omega_h') \) is strictly convex in any affine coordinates in any transverse subspace to the radial directions at any point.

Let \( N' \) be a compact orbifold \( C(\Omega_h')/\langle \mu(\pi_1(E)), \gamma \rangle \) with a flat affine structure. Note that \( S_t, t \in \mathbb{R}_+ \), becomes an action of a circle on \( M \). The change of representation \( h \) to \( h' : \pi_1(M) \to \text{SL}_\pm(n+1, \mathbb{R}) \) is realized by a change of holonomy representations of \( M \) and hence by a change of affine connections on \( C(\Omega_h') \). Since \( S_t \) commutes with the images of \( h \) and \( h' \), \( S_t \) still gives us a circle action on \( N' \) with a different affine connection. We may assume without loss of generality that the circle action is fixed and \( N' \) is invariant under this action.

Thus, \( N' \) is a union of \( B_1, \ldots, B_{m_0} \) that are the products of \( n \)-balls by intervals foliated by connected arcs in circles that are flow arcs of \( S_t \). We can change the affine structure on \( N' \) to one with the holonomy group \( \langle h'(\pi_1(E)), \gamma \rangle \) by local regluing \( B_1, \ldots, B_{m_0} \) as in [54]. We reglue using maps that preserve the leaves for which we need to find maps commuting with the \( \gamma \)-action. We assume that \( S_t \) still gives us a circle affine action since \( \gamma \) is not changed. We may assume that \( N' \) and \( \partial N' \) are foliated by circles that are flow curves of the circle action. The change corresponds
to a sufficiently small $C^r$-change in the affine connection for $r \geq 2$ as we can see from [54]. Now, the strict positivity of the Hessian of $V$ in the fundamental domain and the boundary convexity are preserved. Let $C(\Omega'_h)$ denote the universal cover of $N'$ with the new affine connection. Thus, $C(\Omega''_h)$ is also a properly convex affine cone by Koszul’s work [123]. Also, it is a cone over a properly convex domain $\Omega''_h$ in $\mathbb{S}^n$.

We denote by $\text{PGL}(n+1, \mathbb{R})$, the subgroup of $\text{PGL}(n+1, \mathbb{R})$ fixing a point $v, v \in \mathbb{R}^{n+1}$, and denote by $\text{SL}_+(n+1, \mathbb{R})$, the subgroup of $\text{SL}_+(n+1, \mathbb{R})$ fixing a point $v, v \in \mathbb{S}^n$. Let $\text{Hom}_C(\Gamma, \text{PGL}(n+1, \mathbb{R})_v)$ denote the space of representations acting cocompactly and discretely on a properly convex domain in $\mathbb{S}^{n-1}$. Respectively, we define $\text{Hom}_C(\pi_1(M), \text{SL}_+(n+1, \mathbb{R}))$ similarly.

**Proposition 6.5** Let $T$ be a tube domain over a properly convex domain $\Omega \subset \mathbb{R}^{p+1}$ (resp. $\subset \mathbb{S}^{n-1}$). Let $B$ be a strictly convex hypersurface bounding a properly convex domain in a tube domain $T$. Let $v$ be a vertex of $T$. $B$ meets each radial ray in $T$ from $v$ transversely. Assume that a projective group $\Gamma$ acts on $\Omega$ properly discontinuously and cocompactly. Then there exists a neighborhood of the inclusion map in

$$
\text{Hom}_C(\Gamma, \text{PGL}(n+1, \mathbb{R})_v) \ (\text{resp. } \text{Hom}_C(\pi_1(M), \text{SL}_+(n+1, \mathbb{R}))
$$

where every element $h$ acts on a strictly convex hypersurface $B_h$ in a tube domain $\mathcal{T}_h$ meeting each radial ray at a unique point and bounding a properly convex domain in $\mathcal{T}_h$.

**Proof** We assume first that $B, T \subset \mathbb{S}^n$. For sufficiently small neighborhood $V$ of $h$ in $\text{Hom}_C(\Gamma, \text{SL}_+(n+1, \mathbb{R})_v), h(\Gamma), h \in V$ acts on a properly convex domain $\Omega_h$ properly discontinuously and cocompactly by Theorem 4.1 of [54] (see Koszul [123]). A large compact subset $K$ of $\Omega$ flows to a compact subset $K_h$ by a diffeomorphism by a method of Section 5 of [54]. Let $\mathcal{T}_h$ denote the tube over $\Omega_h$. Since $B/\Gamma$ is a compact orbifold, we choose $V' \subset V$ so that for the projective connections on a compact neighborhood of $B/\Gamma$ corresponding to elements of $V'$, $B/\Gamma$ is still strictly convex and transverse to radial lines. For each $h \in V'$, we obtain an immersion to a strictly convex domain $t_h : B \rightarrow \mathcal{T}_h$ transverse to radial lines since we can think of the change of holonomy as small $C^1$-change of connections. (Or we can use the method described in Section 5 of [54].) Let $p_{\mathcal{T}_h} : \mathcal{T}_h \rightarrow \Omega_h$ denote the projection with fibers equal to the radial lines. Also, in this way of viewing as the connection change, $p_{\mathcal{T}_h} \circ t_h$ is a proper immersion to $\Omega_h$, it is a diffeomorphism to $B \rightarrow \Omega_h$. (Here again we can use Section 5 of [54].) Each point of $B$ is transverse to a radial segment from $v$. By considering the compact fundamental domains of $B$, we see that same holds for $B_h$ for $h$ sufficiently near $I$. Also, $B_h$ is strictly convex and smooth. By Lemma 2.16, the conclusion follows.
6.3.4 Convex cocompact actions of the p-end holonomy groups.

**Definition 6.6** In \( S^n \), a (resp. generalized) lens-shaped R-p-end with the p-end vertex \( v_E \) in \( S^n \) is strictly (resp. generalized) lens-shaped if we can choose a (resp. generalized) CA-lens domain \( D \) in \( S^n \) so that the interior of \( D \times v_E \) is a p-end neighborhood with the top hypersurfaces \( A \) and the bottom one \( B \) so that each great open segment in \( S^n \) from \( v_E \) in the direction of \( bd \hat{\Sigma}_E \) meets \( Cl(D) - A - B \) at a unique point. In \( \mathbb{R}P^n \), such an p-end vertex \( v_E \) is strict if its lift is one in \( S^n \).

A (resp. generalized) lens \( L \) is called **strict lens** if the following hold:

\[
\partial Cl(A) = Cl(A) - A = \partial Cl(B) = Cl(B) - B, \quad A \cup B = \partial L, \quad \text{and} \quad Cl(A) \cup Cl(B) = \partial Cl(L).
\]

Recall that in order that \( L \) is to be a lens, we assume that \( \pi_1(\hat{E}) \) acts cocompactly on \( L \). Also, \( Cl(A) - A \) must equal the limit set \( \Lambda_E \) of \( \hat{E} \) by Corollary 6.1.

Also, the images of these under \( p_{\mathbb{R}P^n} \) are called by the same names respectively.

Obviously, a lens of a lens-shaped R-p-end is strict if and only if the R-p-end is strictly lens-shaped.

In this section, we will prove Proposition 6.6 obtaining a lens.

For the following, \( \mathcal{O} \) needs not be properly convex but merely convex.

**Proposition 6.6** Let \( \mathcal{O} \) be a strongly tame convex real projective orbifold where \( \mathcal{O} \subset S^n \) (resp. \( \mathbb{R}P^n \)).

- Let \( \Gamma_E \) be the holonomy group of a properly convex R-p-end \( \hat{E} \).
- Let \( \mathcal{T}_{\nu E}(\hat{\Sigma}_E) \) be an open tube corresponding to \( R(\nu E) \).
- Suppose that \( \Gamma_E \) satisfies the uniform middle eigenvalue condition with respect to the R-p-end structure, and acts on a distanced compact convex set \( K \) in \( \mathcal{T}_{\nu E}(\hat{\Sigma}_E) \) where \( K \cap \mathcal{T}_{\nu E}(\hat{\Sigma}_E) \) lifts under \( \text{dev} \) to \( \mathcal{O} \) as an embedded subset.

Then any open p-end-neighborhood \( U \) containing a lift to \( \mathcal{O} \) of \( K \cap \mathcal{T}_{\nu E}(\hat{\Sigma}_E) \) contains a lens \( L \) and a lens-cone p-end-neighborhood \( L^* \) of \( \{v_E\} - \{v_E\} \) of the R-p-end \( \hat{E} \). We can choose the lens \( L \) in \( U \) so that \( bd L^* \cap \mathcal{T}^* = A \cup B \) for strictly convex smooth connected hypersurfaces \( A \) and \( B \). Furthermore, every lens of the cone is a strict lens.

**Proof** First suppose that \( \mathcal{O} \subset S^n \). We may assume that \( U \) embeds to a neighborhood of \( L \) under a developing map by taking \( U \) sufficiently small. We denote by \( U \) the image again. By assumption, \( U - K \) has two components since

- either \( \Gamma_E \) acts a totally geodesic hyperspace containing \( K \) and meeting the rays from \( v_E \) transversely, or
- \( K \cap \mathcal{T}_{\nu E}(\hat{\Sigma}_E) \neq \emptyset \) and \( K \cap \mathcal{T}_{\nu E}(\hat{\Sigma}_E) \) has two boundary components closer and farther away from \( v_E \).

Let \( \Lambda_E \) denote \( bd \mathcal{T}_{\nu E}(\hat{\Sigma}_E) \cap K \). Let us choose finitely many points \( z_1, \ldots, z_m \in U - K \) in the two components of \( U - K \).

Proposition 6.3 shows that the orbits of \( z_i \) for each \( i \) accumulate to points of \( \Lambda_E \) only. Hence, a totally geodesic hypersphere separates \( v_E \) with these orbit points.
and another one separates \(v_{E^-}\) and the orbit points. Define the convex hull \(C_1 := \mathcal{C}(\Gamma_E(\{z_1, \ldots, z_m\}) \cup K)\). Thus, \(C_1\) is a compact convex set disjoint from \(v_E\) and \(v_{E^-}\) and \(C_1 \cap \partial \mathcal{F}_E(\Sigma)_E = \Lambda_E\). (See Definition 2.3.)

We need the following lemma:

**Lemma 6.5** We continue to assume as in Proposition 6.6. Then we can choose \(z_1, \ldots, z_m \in U\) so that for \(C_1 := \mathcal{C}(\Gamma_E(\{z_1, \ldots, z_m\}) \cup K)\), \(\partial C_1 \cap \tilde{\partial} \) is disjoint from \(K\) and \(C_1 \subset U\).

**Proof** First, suppose that \(K\) is not in a hyperspace. Then \((\partial K \cap \mathcal{F}_E(\Sigma)_E) / \mathcal{F}_E\) is diffeomorphic to a disjoint union of two copies of \(\Sigma_E\). We can cover a compact fundamental domain of \(\partial K \cap \mathcal{F}_E(\Sigma)_E\) by the interior of \(n\)-balls in \(\tilde{\partial}\) that are convex hulls of finite sets of points in \(U\). Since \((K \cap \tilde{\partial}) / \mathcal{F}_E\) is compact, there exists a positive lower bound of \(\{d(x, \partial U) \in \mathbb{R}_+ \mid x \in K\}\). Let \(F\) denote the union of these finite sets. We can choose \(\varepsilon > 0\) so that the \(\varepsilon\)-neighborhood \(U'\) of \(K\) in \(\tilde{\partial}\) is a subset of \(U\). Moreover \(U'\) is convex by Lemma 2.3 following [73]. We let \(z_1, \ldots, z_m\) denote the points of \(F\). If we choose \(F\) to be in \(U'\), then \(C_1\) is in \(U'\) since \(U'\) is convex.

The disjointedness of \(\partial C_1\) from \(K \cap \mathcal{F}_E(\Sigma)_E\) follows since the \(\Gamma_E\)-orbits of above balls cover \(\partial K \cap \mathcal{F}_E(\Sigma)_E\).

If \(K\) is in a hyperspace, the reasoning is similar to the above. \(\square\)

We continue:

**Lemma 6.6** Let \(L\) be as above. Let \(C\) be a \(\Gamma_E\)-invariant distanced compact convex set with boundary in where \(\{C \cap \mathcal{F}_E\} / \mathcal{F}_E\) is compact. There are two connected strictly convex \(C^\infty\)-hypersurfaces \(A\) and \(B\) of \(\partial C \cap \mathcal{F}_E\) meeting every great segment in \(\mathcal{F}_E\). Suppose that \(A\) (resp. \(B\)) are disjoint from \(L\). Then \(A\) (resp. \(B\)) contains no line ending in \(\partial C\).

**Proof** It is enough to prove for \(A\). Suppose that there exists a line \(l\) in \(A\) ending at a point of \(\partial \mathcal{F}_E(\Sigma)_E\). Assume \(l \subset A\). The line \(l\) projects to a line \(l'\) in \(E\).

Let \(C_1 = C \cap \mathcal{F}_E(\Sigma)_E\). Since \(A / \mathcal{F}_E\) and \(B / \mathcal{F}_E\) are both compact, and there exists a fibration \(C_1 / \mathcal{F}_E \rightarrow A / \mathcal{F}_E\) induced from \(C_1 \rightarrow A\) using the foliation by great segments with endpoints \(v_E, v_{E^-}\).

Since \(A / \mathcal{F}_E\) is compact, we choose a compact fundamental domain \(F\) in \(A\) and choose a sequence \(\{x_i \in l\}\) whose image sequence in \(l'\) converges to the endpoint of \(l'\) in \(\partial \Sigma_E\). We choose \(\gamma_i \in \Gamma_E(\Sigma)_E\) so that \(\gamma_i(x_i) \in F\) where \(\{\gamma_i(\text{Cl}(l'))\}\) geometrically converges to a segment \(l'_o\) with both endpoints in \(\partial \Sigma_E\). Hence, \(\{\gamma_i(\text{Cl}(l))\}\) geometrically converges to a segment \(l_o\) in \(A\). We can assume that for the endpoint \(z\) of \(l\) in \(A\), \(\gamma(z)\) converges to the endpoint \(p_1\). Proposition 6.3 implies that the endpoint \(p_1\) of \(l_o\) is in \(L_E := L \cap \partial \mathcal{F}_E\). Let \(\gamma\) be the endpoint of \(l\) not equal to \(z\). Then \(\gamma \in A\). Since \(\gamma\) is not a bounded sequence, \(\gamma(t)\) converges to a point of \(\Lambda_E\). Thus, both endpoints of \(l_o\) are in \(\Lambda_E\) and hence \(l'_o \subset L\) by the convexity of \(L\). However, \(l \subset A\) implies that \(l'_o \subset A\). As \(A\) is disjoint from \(L\), this is a contradiction. The similar conclusion holds for \(B\). \(\square\)
**Proof (Proof of Proposition 6.6 continued)** We will denote by $C$ the compact convex subset $C = C \cap \mathcal{T}_V(\Sigma_\tilde{E})$ for $C$ obtained by Lemma 6.5 and satisfying Lemma 6.6. Since $A$ and analogously $B$ do not contain any geodesic ending at $\text{bd}\tilde{\mathcal{O}}$, $\text{bd}C \cap \mathcal{T}_V(\tilde{\Sigma}_E)^\circ$ is a union of compact $n - 1$-dimensional simplices meeting one another in strictly convex dihedral angles.

Proposition 6.7 completes the proof of Proposition 6.6.

For $\mathbb{RP}^n$ version, we can argue by projecting by $p_\beta$ and Proposition 2.13.

**Proposition 6.7** Assume the premise of Proposition 6.6. Suppose that a lens cone $L_1 \ast \{v_E\} - \{v_E\}$ is in a convex $p$-end neighborhood $U$ of a $p$-end neighborhood of $\tilde{E}$. Suppose that $L_1$ contains a lens $L$ in its interior where $L \ast \{v_E\} - \{v_E\}$ is again. Suppose that $L_1$ is bounded by two connected convex polyhedral hypersurfaces. Then there exists a lens $L_2$ bounded by two connected strictly convex hypersurfaces so that $L_2 \subset U$ and $L \subset L_2 \subset L_1$.

**Proof** First, assume $\mathcal{O} \subset S^\gamma$. Let us take the dual domain $U_L$ of $(L \ast \{v_E\})^\circ$. The dual $U_{L_1}$ of $(L_1 \ast \{v_E\})^\circ$ is an open subset of $U_L$ by (2.26). By Proposition 6.2, $U_L$ and $U_1$ are asymptotically nice properly convex domains. By Lemma 2.20 (iii), the hyperplanes sharply supporting $(L_1 \ast \{v_E\})^\circ$ at $v_E$ correspond to points of a totally geodesic domain $D$, $D = \text{Cl}(U_1) \cap P = \text{Cl}(U_L) \cap P$

for a $\Gamma_{E^\circ}$-invariant hyperplane $P$. Hence, $U_L$ and $U_1$ are asymptotically nice domains with respect to $D^\circ$. (See Section 5.1.)

By the premise, we have a connected convex polyhedral open subspace

$S_1 := \text{bd}(L_1 \ast \{v_E\} \cap \mathcal{F}_V(\tilde{\Sigma}_E)^\circ) \subset \text{bd}(L_1 \ast \{v_E\})$.

By Lemma 2.20 (iv), $S_1$ corresponds to a connected convex polyhedral hypersurface

$S_1^* \subset \text{bd}U_1$

by $\mathcal{T}^\circ_{U_1 \ast \{v_E\}}$. Since $S_1$ is disjoint from $L$ by the premise, it follows $S_1^* \subset \text{bd}U_1 \cap U_1$ by (2.26). Since $S_1 / \Gamma_{E^\circ}$ is compact, so is $S_1^* / \Gamma_{E^\circ}$ by Proposition 2.19. Theorem 5.4 shows that $D \cup S_1^* = \text{bd}U_2$. Hence, $\text{bd}U_1 \cap U_1 = S_1^*$.

By Theorem 5.5, we obtain an asymptotically nice closed domain $U_2$ with connected strictly convex smooth hypersurface boundary $S_2$ in $U_L$ with $U_1 \cup S_1 \subset U_2^\circ$. The dual $U_2^\circ$ of $U_2$ has a connected strictly convex smooth hypersurface boundary $S_2^\circ$ in $\mathcal{T}_V(\tilde{\Sigma}_E)^\circ$ disjoint from $(L \ast \{v_E\})^\circ$ and inside $(L_1 \ast \{v_E\})^\circ$ by (2.26). This is what we wanted.

Also, considering $(L \ast \{v_{E^-}\})^\circ$ and $(L_1 \ast \{v_{E^-}\})^\circ$, we obtain a connected strictly convex smooth hypersurface in the other component of $\mathcal{T}_V(\tilde{\Sigma}_E)^\circ - L$ in $U$. The union of the two hypersurfaces bounds a lens $L_2$ in $U$. (See Section 2.5.1.)

Let $F$ denote the compact fundamental domain of the boundary of the lens. The strictness of the lens follows from Proposition 6.3 since the boundary of the lens is a union of orbits of $F$ and the limit points are only in $A_E$.

Again Proposition 2.13 completes the proof for $\mathbb{RP}^n$. 

□
Proof (Proof of Theorem 6.1) Proposition 6.6 is the forward direction using \( \hat{\mathcal{O}} := \mathcal{R}_g(\tilde{\Sigma}_E) \).

Now, we show the converse. It is sufficient to prove for the case \( \hat{\mathcal{O}} \subset \mathbb{R}^n \). Let \( L \) be a CA-lens of the lens-cone where \( \Gamma_E \) acts cocompactly on. Let \( \mathcal{R}_g(\tilde{\Sigma}_E) \) be the tube corresponding to \( L \).

We will denote by \( h : \pi_1(\tilde{E}) \to \text{SL}_+(n+1, \mathbb{R}) \) denote the holonomy homomorphism of the end fundamental group with image \( \Gamma_E \). We assume that the image of \( h \) are matrices of form (6.3).

There is an abelianization map

\[
A : \pi_1(\tilde{E}) \to H_1(\pi_1(\tilde{E}), \mathbb{R})
\]

obtained by taking a homology class. The above map \( g \to \log \lambda_{v_E}(h(g)) \) induces homomorphism

\[
\Lambda^h : H_1(\pi_1(\tilde{E}), \mathbb{R}) \to \mathbb{R}
\]

that depends on the holonomy homomorphism \( h \).

Let us give an arbitrary Riemannian metric \( \mu \) on \( \Sigma_E \). Recall that a current is a transverse measure on a partial foliation by 1-dimensional subspaces in the compact space \( \mathbb{R} \Sigma_E \) on the transverse measure. (See [152].) These are not necessarily geodesic currents as in Bonahon [31]. The space of currents is denoted by \( \mathcal{C}(\mathbb{R} \Sigma_E) \) which is given a weak topology.

The abelianization map \( \pi_1(\Sigma_E) \to H_1(\pi_1(\tilde{E}), \mathbb{R}) \) can be understood as sending a closed curve to a current the corresponding homology class. This map extends to \( \mathcal{C}(\mathbb{R} \Sigma_E) \to H_1(\pi_1(\tilde{E}), \mathbb{R}) \). (See Proposition 1 of [152] and Theorem 14 of [78].) Also, \( \Lambda^h : \pi_1(\tilde{E}) \to \mathbb{R} \) gives rise to the continuous map \( \hat{\Lambda}^h : \mathcal{C}(\mathbb{R} \Sigma_E) \to \mathbb{R} \) which restricts to \( \Lambda^h \) on the image currents of \( \pi_1(\Sigma_E) \): \( \Lambda^h \) is given by integrating a 1-form on \( \Sigma_E \) along the closed curve representing \( \pi_1(\Sigma_E) \) since \( \text{Hom}(H_1(\pi_1(\tilde{E}), \mathbb{R}), \mathbb{R}) = H^1(\pi_1(\tilde{E}), \mathbb{R}) \). Since the integration along currents are well-defined, we are done.

Let \( \lambda^\mu_{v_E}(h(g)) \) denote the maximal norm of the eigenvalues \( h(h(g)) \) of the upper-left corner of \( h \) in (6.3). Obviously, \( \lambda^\mu_{v_E}(h(g)) \geq \lambda_{v_E}(h(g)) \) for each \( g \in \pi_1(\tilde{E}) \): If the eigenvalue of the upper-left corner matrix of \( h(g) \) is strictly smaller than \( \lambda^\mu_{v_E}(h(g)) \), Proposition 2.8 shows that the closure of \( L \) contains \( v_E \) or \( v_{E-} \) considering the orbit of \( \{g^n\} \), a contradiction.

Let \( g \in \Gamma_E \). Let \( |g| \) denote the current supported on a closed curve \( c_g \) on \( \Sigma_E \) corresponding to \( g \) lifted to \( \mathbb{R} \Sigma_E \). Define \( \text{length}_\mu(g) \) to be the infimum of the \( \mu \)-length of such closed curves corresponding to \( g \). Suppose that \( \Gamma_E \) does not satisfy

\[
C^{-1} \text{length}_\mu(g) \leq \frac{\lambda^\mu_{v_E}(h(g))}{\lambda_{v_E}(h(g))} \leq C \text{length}_\mu(g)
\]

for a uniform constant \( C > 1 \). Then there exists a sequence \( g_i \) of elements of \( \Gamma_E \) so that
6.3 The characterization of lens-shaped representations

\[ \log \left( \frac{\lambda_{\text{ul}}(h(g_i))}{\lambda_{\text{vE}}(h(g_i))} \right) \rightarrow 0 \text{ as } i \rightarrow \infty. \]

Let \([g_\infty]\) denote a limit point of \(\{[g_i]/\text{length}_{\mu}(g_i)\}\) in the space of currents on \(U\Sigma\tilde{E}\). Since \(U\Sigma\tilde{E}\) is compact, a limit point exists. We may modify \(h\) by changing the homomorphism \(g \mapsto \lambda_{\text{vE}}(h(g))\) only; that is, we only modify the \((n+1) \times (n+1)\) entry of the matrices form (6.3) with corresponding changes. Proposition 6.5 implies that the perturbed CA-lens \(L'\) is still a properly convex domain with the same tube domain whose closure does not contain \(v_{\tilde{E}}\). By considering the image of \([g_\infty]\) in \(H_1(\Sigma\tilde{E}, \mathbb{R})\), we can make a sufficiently small change of \(h\) to \(h'\) in this way so that \(\Lambda^h([g_\infty]) > \Lambda^h([g_\infty])\). From this, we obtain that

\[ \log \left( \frac{\lambda_{\text{ul}}(h'(g_i))}{\lambda_{\text{vE}}(h'(g_i))} \right) < 0 \text{ for sufficiently large } i. \]  

By (6.24), we obtain that \(\lambda_{\text{ul}}(h'(g)) \leq \lambda_{\text{vE}}(h'(g))\) for some \(g\) and that \(\lambda_{\text{vE}}(h'(g))\) at \(v_{\tilde{E}}\). Hence, we can decompose \(S^n\) into a hyperspace \(S'\) and the complementary \(\{v_{\tilde{E}}, v_{\tilde{E} -}\}\). The norms of eigenvalues associated with \(S'\) are strictly less than that of \(v_{\tilde{E}}\). Proposition 2.8 shows that the closure of \(L\) contains \(v_{\tilde{E}}\) or \(v_{\tilde{E} -}\) by considering the orbits under \([g']\). \(\square\)

6.3.5 The uniform middle-eigenvalue conditions and the lens-shaped ends.

Now, we aim to prove Theorem 6.2 restated as Theorem 6.6. A \emph{radially foliated end-neighborhood system} of \(\mathcal{O}\) is a collection of end-neighborhoods of \(\mathcal{O}\) that is radially foliated and outside a compact suborbifold of \(\mathcal{O}\) whose interior is isotopic to \(\mathcal{O}\).

\textbf{Definition 6.7} We say that a strongly tame properly convex \(\mathcal{O}\) with \(\tilde{\mathcal{O}} \subset \mathbb{S}^n\) (resp. \(\subset \mathbb{R}P^n\)) satisfies the \textit{triangle condition} if for any fixed end-neighborhood system of \(\mathcal{O}\), every triangle \(T \subset \text{Cl}(\tilde{\mathcal{O}})\), if \(\partial T \subset \text{bd} \tilde{\mathcal{O}}, T^n \subset \tilde{\mathcal{O}}\), and \(\partial T \cap \text{Cl}(U) \neq \emptyset\) for a radial p-end neighborhood \(U\), then \(\partial T\) is a subset of \(\text{Cl}(U) \cap \text{bd} \tilde{\mathcal{O}}\).

For example, by Corollary 7.7, strongly tame strict SPC-orbifolds with generalized lens-shaped or horospherical ends satisfy this condition. The converse is not necessarily true.

A \emph{minimal} \(\Gamma_{\tilde{E}}\)-invariant distanced compact set is the smallest compact \(\Gamma_{\tilde{E}}\)-invariant distanced set in \(\mathcal{T}_{\tilde{E}}\).

\textbf{Theorem 6.6} Let \(\mathcal{O}\) be a strongly tame convex real projective orbifold. Let \(\Gamma_{\tilde{E}}\) be the holonomy group of a properly convex R-end \(\tilde{E}\) and the end vertex \(v_{\tilde{E}}\). Then the following are equivalent:

\[ \log \left( \frac{\lambda_{\text{ul}}(h(g_i))}{\lambda_{\text{vE}}(h(g_i))} \right) \rightarrow 0 \text{ as } i \rightarrow \infty. \]
(i) $\hat{E}$ is a generalized lens-shaped R-end.
(ii) $\Gamma_{\hat{E}}$ satisfies the uniform middle-eigenvalue condition with respect to $v_{\hat{E}}$.

Assume that the holonomy group of $\pi_1(\mathcal{O}^\prime)$ is strongly irreducible, and $\mathcal{O}$ is properly convex. If $\mathcal{O}^\prime$ furthermore satisfies the triangle condition or, alternatively, assume that $\hat{E}$ is virtually factorizable, then the following holds:

- $\Gamma_{\hat{E}}$ is lens-shaped if and only if $\Gamma_{\hat{E}}$ satisfies the uniform middle-eigenvalue condition.

**Proof** Assume $\tilde{\mathcal{O}} \subset S^n$. (ii) $\Rightarrow$ (i): This follows from Theorem 6.1 since we can intersect the lens with $\tilde{\mathcal{O}}$ to obtain a generalized lens and a generalized lens-cone from it. (Here, of course $\pi_1(\hat{E})$ acts cocompactly on the generalized lens.)

(i) $\Rightarrow$ (ii): Let $L$ be a generalized CA-lens in the generalized lens cone $L \ast v_{\hat{E}}$. Let $B$ be the lower boundary component of $L$ in the tube $\mathcal{T}_{\hat{E}}(\Sigma_{\hat{E}})$. Since $B$ is strictly convex, the upper component of $\mathcal{T}_{\hat{E}}(\Sigma_{\hat{E}}) - B$ is a properly convex domain, which we denote by $U$. Let $l_i$ denote the maximal segment from $v_{\hat{E}}$ passing $x$ for $x \in U - L$.

We define a function $f : U - L \to \mathbb{R}$ given by $f(x)$ to be the Hilbert distance on line $l_i$ from $x$ to $L \cap l_i$. Then a level set of $f$ is always strictly convex: This follows by taking a 2-plane $P$ containing $v_{\hat{E}}$ passing $L$. Let $x, y$ be a points of $f^{-1}(c)$ for a constant $c > 0$. Let $x'$ be the point of $\text{Cl}(L) \cap l_i$ closest to $x$ and $y'$ be one of $\text{Cl}(L) \cap l_i$ closest to $y$. Let $x''$ be one of $\text{Cl}(L) \cap l_i$ furthest from $x$. Let $y''$ be one of $\text{Cl}(L) \cap l_i$ furthest from $x$. Since $f(x) = f(y)$, a cross-ratio argument shows that the lines extending $x'y', x'y''$ and $x'y''$ are concurrent outside $U \cap P$. The strict convexity of $B$ shows that $f(z) < \varepsilon$ for $z \in x'y''$.

We can approximate each level set by a convex polyhedral hypersurface in $U - L$ by taking vertices in the level set and taking the convex hull using the strict convexity of the level set. Then we can smooth it to be a strictly convex hypersurface by Proposition 6.7. Let $V$ denote the domain bounded by this and $B$. Then $V$ has strictly convex smooth boundary in $U$. Theorem 6.1 implies (ii).

The final part follows by Lemma 6.7.

[5369] S

**Lemma 6.7** Suppose that $\mathcal{O}$ is a strongly tame properly convex real projective orbifold and satisfies the triangle condition or, alternatively, assume that an R-p-end $\hat{E}$ is virtually factorizable. Suppose that the holonomy group $\Gamma$ is strongly irreducible. Then the R-p-end $\hat{E}$ is generalized lens-shaped if and only if it is lens-shaped.

**Proof** Again, we prove for $S^n$. If $\hat{E}$ is virtually factorizable, this follows by Theorem 6.8.

Suppose that $\hat{E}$ is not virtually factorizable. Now assume the triangle condition. Given a generalized CA-lens $L$, let $L^h$ denote $\text{Cl}(L) \cap \mathcal{T}_{\hat{E}}(\Sigma_{\hat{E}})$. We obtain the convex hull $M$ of $L^h$. $M$ is a subset of $\text{Cl}(L)$. The lower boundary component of $L$ is a smooth strictly convex hypersurface.

Let $M_1$ be the outer component of $\text{bd} M \cap \mathcal{T}_{\hat{E}}(\Sigma_{\hat{E}})$. Suppose that $M_1$ meets $\text{bd} \tilde{\mathcal{O}}$. $M_1$ is a union of the interior of simplices. By Lemma 2.18, either a simplex $\sigma$ in $\text{Cl}(\tilde{\mathcal{O}})$ is in $\text{bd} \tilde{\mathcal{O}}$ or its interior $\sigma^\circ$ is disjoint from it. Hence, there is a simplex $\sigma$ in
6.4 The properties of lens-shaped ends.

A trivial one-dimensional cone is an open half-space in $\mathbb{R}^1$ given by $x > 0$ or $x < 0$.

**Lemma 6.9** Let $\mathcal{O}$ be a strongly tame properly convex orbifold. Then given any end-neighborhood, there is a concave end-neighborhood in it. Furthermore, the $d_\partial$-diameter of the boundary of a concave end-neighborhood of an $R$-end $E$ is bounded by the Hilbert diameter of the end orbifold $\Sigma_E$ of $E$.

**Proof** It is sufficient to prove for the case $\mathcal{O} \subset \mathbb{S}^n$. Suppose that we have a generalized lens-cone $V$ that is a $p$-end-neighborhood equal to the interior of $L \ast v_\tilde{E}$ where $L$ is a generalized CA-lens bounded away from $v_\tilde{E}$.

Now take a $p$-end neighborhood $U'$. We assume without loss of generality that $U'$ covers a product end-neighborhood with compact boundary.

By taking smaller $U'$ if necessary, we may assume that $U'$ and $L$ are disjoint. Since $\partial U' / h(\pi_1(\tilde{E}))$ and $L / h(\pi_1(\tilde{E}))$ are compact, $\varepsilon > 0$. Let

$M_1 \cap \partial \tilde{\mathcal{O}}$. Taking the convex hull of $v_\tilde{E}$ and an edge in $\sigma$, we obtain a triangle $T$ with $\partial T \subset \partial \tilde{\mathcal{O}}$ and $T^o \subset \tilde{\mathcal{O}}$. This contradicts the triangle condition by Lemma 6.8.

Thus, $M_1 \subset \tilde{\mathcal{O}}$. By Theorem 6.6, the end satisfies the uniform middle eigenvalue condition. By Proposition 6.6, we obtain a lens-cone in $\tilde{\mathcal{O}}$. 

**Lemma 6.8** Suppose that $\mathcal{O}$ is a strongly tame properly convex real projective orbifold and satisfies the triangle condition. Then no triangle $T$ with $T^o \subset \tilde{\mathcal{O}}$, $\partial T \subset \partial \tilde{\mathcal{O}}$ has a vertex equal to an $R$-$p$-end vertex.

**Proof** Assume $\tilde{\mathcal{O}} \subset \mathbb{S}^n$. Let $v_\tilde{E}$ be a $p$-end vertex. Choose a fixed radially foliated $p$-end-neighborhood system. Suppose that a triangle $T$ with $\partial T \subset \partial \tilde{\mathcal{O}}$ contains a vertex equal to a $p$-end vertex. Let $U$ be a component of the inverse image of a radially foliated end-neighborhood in the end-neighborhood system, and be a $p$-end neighborhood of a $p$-end $\tilde{E}$ with a $p$-end vertex $v_\tilde{E}$. By the triangle condition, $\partial T \subset \text{Cl}(U) \cap \partial \tilde{\mathcal{O}}$.

Since $U$ is foliated by radial lines from $v_\tilde{E}$, we choose $U$ so that $\partial U \cap \partial \tilde{\mathcal{O}}$ covers a compact hypersurface in $\mathcal{O}$. Let $\mathcal{U}$ denote the set of segments in $\text{Cl}(U) \cap \partial \tilde{\mathcal{O}}$.

Every segment in $\mathcal{U}$ in the direction of $\Sigma_{v_\tilde{E}}$ ends in $\partial U \cap \partial \tilde{\mathcal{O}}$. Also, the segments $\mathcal{U}$ in directions of $\partial U \cap \partial \tilde{\mathcal{O}}$ are in $\partial U \cap \partial \tilde{\mathcal{O}}$ by the definition of $\Sigma_{v_\tilde{E}}$. Also, $\text{Cl}(U) \cap \partial \tilde{\mathcal{O}}$ is a union of segments in $\mathcal{U}$. Thus, $\text{Cl}(U) \cap \partial \tilde{\mathcal{O}}$ is a union of segments in directions of $\partial \Sigma_{v_\tilde{E}}$.

Since $T^o \subset \tilde{\mathcal{O}}$, each segment in $\mathcal{U}$ with interior in $T^o$ is not in directions of $\partial \Sigma_{v_\tilde{E}}$.

Let $w$ be the endpoint of the maximal extension in $\tilde{\mathcal{O}}$ of such a segment. Then $w$ is not in $\text{Cl}(U) \cap \partial \tilde{\mathcal{O}}$ by the conclusion of the above paragraph. This contradicts $\partial T \subset \text{Cl}(U) \cap \partial \tilde{\mathcal{O}}$.

The proof for $\mathbb{R}^n$ case follows by Proposition 2.13.
\[ L' := \{ x \in V | d_V(x, L) \leq \varepsilon \}. \]

Since a lower component of \( \partial L \) is strictly convex, we can show that \( L' \) can be polyhedrally approximated and smoothed to be a CA-lens by Proposition 6.7.

Clearly, \( h(\pi_1(\tilde{E})) \) acts on \( L' \).

We choose sufficiently large \( \varepsilon' \) so that \( \partial bd U \cap \tilde{O} \subset L' \), and hence \( V - L' \subset U \) form a concave p-end-neighborhood as above.

Let \( \tilde{E} \) be a p-end corresponding to \( E \). Let \( U \) be a concave p-end neighborhood of \( \tilde{E} \) that is a cone: \( U \) is the interior of \( \{ v \} + L - L \) for a generalized CA-lens \( L \) and the p-end vertex \( v \) corresponding to \( U \). Let \( T \) denote the tube with vertex \( v \) in the direction of \( L \). Then \( B := \partial U \cap \tilde{O} \) is a smooth lower boundary component of \( L \).

Any tangent hyperspace \( P \) at a point of \( B \) meets \( \partial T \) in a sphere of dimension \( n - 2 \). By convexity of \( L \) and the strict convexity \( B \), it follows that \( P \cap \partial T \) is a point. We claim that \( P \cap \partial T \subset P \cap \partial T \): We put \( T \) into an affine space \( A^n \) with vertices in \( \partial A^n \). Then \( T \) is foliated by parallel complete affine lines. Consider these as vertical lines. \( B \) is a strictly convex hypersurface meeting these vertical lines transversely. Then the property of \( P \) becomes clear now.

Thus any maximal segment in \( \tilde{O} \) tangent to \( B \) at \( x \) must end in \( \partial T \cap \partial \tilde{O} \). There is a projection \( \Pi_v : B \to \Sigma_{\tilde{E}} \) that is a diffeomorphism. Hence, the maximal segment is sent to a maximal segment of \( \Sigma_{\tilde{E}} \) under \( \Pi_v \), which forms an isometry on the segment with the Hilbert metrics. Moreover this is a Finsler isometry by considering the Finsler metric restricted to the tangent space to \( B \) at \( x \) to that of the tangent space to \( \Sigma_{\tilde{E}} \) at \( \Pi(x) \) and the Finsler metrics. The conclusion follows. \[ S^n \]

### 6.4.1 The properties for a lens-cone in non-virtually-factorizable cases

We may assume that each infinite-order \( g \in \Gamma_{\tilde{E}} \) is positive bi-semi-proximal by Theorem 2.7.

**Theorem 6.7** Let \( \tilde{O} \) be a strongly tame convex real projective \( n \)-orbifold. Let \( \tilde{E} \) be an R-p-end of \( \tilde{O} \subset S^n \) (resp. in \( \mathbb{R}P^n \)) with a generalized lens p-end-neighborhood. Let \( \nu_{\tilde{E}} \) be the p-end vertex. Assume that \( \pi_1(\tilde{E}) \) is non-virtually-factorizable. Then \( \Gamma_{\tilde{E}} \) satisfies the uniform middle eigenvalue condition with respect to \( \nu_{\til{E}} \), and there exists a generalized CA-lens \( D \) disjoint from \( \nu_{\til{E}} \) with the following properties:

1. \( \partial D - \partial D = \Lambda_{\til{E}} \) is independent of the choice of \( D \) where \( \Lambda_{\til{E}} \) is from Proposition 6.3.
2. \( D \) is strictly generalized lens-shaped.
3. Each element \( g \in \Gamma_{\til{E}} \) has an attracting fixed set in \( \partial D \) intersected with the union of some great segments from \( \nu_{\til{E}} \) in the directions in \( \partial \Sigma_{\til{E}} \).
4. The closure of the union of attracting fixed set is a subset of \( \partial D - A - B \) for the top and the bottom hypersurfaces \( A \) and \( B \). The sets are equal if \( \Gamma_{\til{E}} \) is hyperbolic.
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(ii) Assume that $l$ be a segment $l \subset \partial \overline{\mathcal{O}}$ with $l^0 \cap \text{Cl}(U) \neq \emptyset$ for any concave $p$-end-neighbourhood $U$ of $v_{\mathcal{E}}$. Then $l$ is in $\bigcup S(v_{\mathcal{E}})$ and in the closure in $\text{Cl}(V)$ of every concave or proper $p$-end-neighbourhood $V$ of $v_{\mathcal{E}}$.

- The set $S(v_{\mathcal{E}})$ of maximal segments from $v_{\mathcal{E}}$ in $\text{Cl}(V)$ is independent of a concave or proper $p$-end neighborhood $V$ (in fact it is the set of maximal segments from $v_{\mathcal{E}}$ ending in $\partial D - A - B$).

$$\bigcup S(v_{\mathcal{E}}) = \text{Cl}(V) \cap \partial \overline{\mathcal{O}}.$$

(iii) $S(g(v_{\mathcal{E}})) = g(S(v_{\mathcal{E}}))$ for $g \in \pi_1(\mathcal{E})$.

(iv) Given $g \in \pi_1(\mathcal{O})$, we have

$$\left( \bigcup S(g(v_{\mathcal{E}})) \right) \cap \bigcup S(v_{\mathcal{E}}) = \emptyset \text{ or else } \bigcup S(g(v_{\mathcal{E}})) = \bigcup S(v_{\mathcal{E}}) \text{ with } g \in \Gamma_{\mathcal{E}}. \quad (6.25)$$

(v) A concave $p$-end neighborhood is a proper $p$-end neighborhood.

(vi) Assume that $v_{\mathcal{E}}$ is the $p$-end vertex of an $R$-$p$-end $\mathcal{E}$. We can choose mutually disjoint concave $p$-end neighborhoods for every $R$-$p$-ends. Then

$$\left( \bigcup S(v_{\mathcal{E}}) \right) \cap \bigcup S(v_{\mathcal{E}}) = \emptyset \text{ or } \bigcup S(v_{\mathcal{E}}) = \bigcup S(v_{\mathcal{E}}) \text{ with } v_{\mathcal{E}} = v_{\mathcal{E}}', \mathcal{E} = \mathcal{E}'$$

for an $R$-$p$-end vertex $v_{\mathcal{E}}$. (This is a sharpening of (iv).)

**Proof** Suppose first $\mathcal{O} \subset S'$. Theorem 6.6 implies the uniform middle eigenvalue condition.

(i) Let $U_1$ be a concave end neighborhood. Since $\Gamma_{\mathcal{E}}$ acts on $U_1$, $U_1$ is a component of the complement of a generalized lens $D$ in a generalized $R$-end of form $D * \{v_{\mathcal{E}}\}$ by definition. The action on $D$ is cocompact and proper since we can use a foliation by great segments in a tube corresponding to $\mathcal{E}$.

Proposition 6.3 implies that the lens is a strict one. This implies (i).

(ii) Consider any segment $l$ in $\partial \overline{\mathcal{O}}$ with $l^0$ meeting $\text{Cl}(U_1)$ for a concave $p$-end-neighbourhood $U_1$ of $v_{\mathcal{E}}$. Here, the generalized lens $D$ has boundary components $A$ and $B$ where $B$ is also a boundary component of $U_1$ in $\mathcal{O}$. Let $T$ be the open tube corresponding to $\Sigma_{\mathcal{E}}$. Then $\mathcal{O} \subset T$ since $\Sigma_{\mathcal{E}}$ is the direction of all segments in $\mathcal{O}$ starting from $v_{\mathcal{E}}$. Let $T_1$ be a component of $\partial D - \partial_1 B$ containing $v_{\mathcal{E}}$. Then $T_1 \subset \text{Cl}(U_1) \cap \partial \overline{\mathcal{O}}$ by the definition of concave $p$-end neighborhoods. In the closure of $U_1$, an endpoint of $l$ is in $T_1$. Then $l^0 \subset \partial D$ since $l^0$ is tangent to $\partial D - \{v_{\mathcal{E}}, v_{\mathcal{E}}^{-}\}$. Any convex segment $s$ from $v_{\mathcal{E}}$ to any point of $l$ must be in $\partial D$. By the convexity of $\text{Cl}(\mathcal{O})$, we have $s \subset \text{Cl}(\mathcal{O})$. Thus, $s$ is in $\partial \overline{\mathcal{O}}$ since $\partial D \cap \text{Cl}(\mathcal{O}) \subset \partial \overline{\mathcal{O}}$. Therefore, the segment $l$ is in the union of segments in $\partial \overline{\mathcal{O}}$ from $v_{\mathcal{E}}$.

We now suppose that $l$ is a segment from $v_{\mathcal{E}}$ containing a segment $l_0$ in $\text{Cl}(U_1) \cap \partial \overline{\mathcal{O}}$ from $v_{\mathcal{E}}$, and we will show that $l$ is in $\text{Cl}(U_1) \cap \partial \overline{\mathcal{O}}$, which will be sufficient to prove (ii). $l^0$ contains a point $p$ of $\partial D - A - B$, which is a subset of $\partial \mathcal{E} (\Sigma_{\mathcal{E}}) \cap D$. Since $l \subset \text{Cl}(\mathcal{O})$, we obtain $\bigcup_{v_{\mathcal{E}} \in \mathcal{E}} g(l) \subset \text{Cl}(\mathcal{O})$, a properly convex subset. Hence, $\bigcup_{v_{\mathcal{E}} \in \mathcal{E}} g(l) - U_1$ is a distanced set, and has a distanced compact closure. Then the convex hull of the closure meets $\partial \mathcal{E} (\Sigma_{\mathcal{E}})$ in a way contradicting Proposition 6.3
(ii) where $D$ is $\Lambda_E$ in the proposition. Thus, $l'$ does not meet $\partial D - A - B$. Thus,

$$l \subset \Cl(U_1) \cap \partial \tilde{\mathcal{O}}.$$

We define $S(v_E)$ as the set of maximal segments in $\Cl(U_1) \cap \partial \tilde{\mathcal{O}}$. Such a maximal segment is also maximal in $\Cl(U) \cap \partial \tilde{\mathcal{O}}$ by the above paragraph. Hence, we can characterize $S(v_E)$ as the set of maximal segments in $\partial \tilde{\mathcal{O}}$ from $v_E$ ending at points of $\partial D - A - B$. Also, $\bigcup S(v_E) = \Cl(U_1) \cap \partial \tilde{\mathcal{O}}$.

For any other concave affine neighborhood $U_2$ of $U_1$, we have

$$U_2 = \{v_E\} \ast D_2 - D_2 - \{v_E\}$$

for a generalized CA-lens $D_2$. Since $\Cl(D_2) - \partial D_2$ equals $\Cl(D) - \partial D$, we obtain that $\Cl(U_2) \cap \partial \tilde{\mathcal{O}} = \Cl(U_1) \cap \partial \tilde{\mathcal{O}} = \bigcup S(v_E)$.

Let $U'$ be any proper p-end-neighborhood associated with $v_E$. $U_1 \subset U'$ for a concave p-end neighborhood $U_1$ by Lemma 6.9. Again, $U_1 = \{v_E\} \ast D - D - \{v_E\}$ for a generalized CA-lens $D$ where $v_E \not\in \Cl(D)$. Hence, $\Cl(U_1) \cap \partial \tilde{\mathcal{O}} \subset \Cl(U') \cap \partial \tilde{\mathcal{O}}$. Moreover, every maximal segment in $S(v_E)$ is in $\Cl(U')$.

We can form $S'(v_E)$ as the set of maximal segments from $v_E$ in $\Cl(U') \cap \partial \tilde{\mathcal{O}}$. Then no segment $l$ in $S'(v_E)$ has interior points in $\partial D - A - B$ as above. Thus,

$$S(v_E) = S'(v_E).$$

Also, since every points of $\Cl(U') \cap \partial \tilde{\mathcal{O}}$ has a segment in the direction of $\partial \mathcal{O}_E$, we obtain

$$\bigcup S(v_E) = \Cl(U') \cap \partial \tilde{\mathcal{O}}.$$

(iii) Since $g(D)$ is the generalized CA-lens for the the generalized lens neighborhood $g(U)$ of $g(v_E)$, we obtain $g(S(v_E)) = S(g(v_E))$ for any p-end vertex $v_E$.

(iv) Choose a proper p-end neighborhood $U$ of $\hat{E}$ covering an end-neighborhood of product form with compact boundary. We choose a generalized CA-lens $L$ of a generalized lens-cone so that $C_E := \{v_E\} \ast L - L - \{v_E\}$ is in $U$ by Lemma 6.9. We can choose $C_E$ to be a proper concave p-end neighborhood since can choose one in a proper p-end neighborhood. The properness of $U$ shows that

$$g(C_E) = C_E \text{ for } g \in \Gamma_E \text{, or else } g(C_E) \cap C_E = \emptyset \text{ for every } g \in \Gamma_E. \tag{6.26}$$

Let $B_L$ denote the boundary component of $L$ meeting the closure of $C_E$. Now, $\bigcup S(v_E)^0$ has an open neighborhood of form $C_E \cup \bigcup S(v_E)^0$ in $\tilde{\mathcal{O}}$ since $B_L$ is separating hypersurface in $\tilde{\mathcal{O}}$. We obtain the conclusion since the intersection of the two sets implies the intersections of the neighborhoods of the sets.

(v) Let $C_E$ be a concave p-end neighborhood $\{v_E\} \ast L - L - \{v_E\}$ for a lens $L$. We will now show that $C_E$ is a proper p-end neighborhood. Suppose for contradiction that

$$g(C_E) \cap C_E \neq \emptyset \text{ and } g(C_E) \neq C_E.$$
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Since $C_E$ is concave, each point $x$ of $\text{bd}C_E \cap \partial H$ is contained in a sharply supporting hyperspace $H$ so that

- a component $C$ of $C_E - H$ is in $C_E$
- $\text{Cl}(C) \ni v_{C_E}$ for the p-end vertex $v_{C_E}$ of $C_E$.

Similar statements hold for $g(C_E)$.

Since $g(C_E) \cap C_E \neq \emptyset$ and $g(C_E) \neq C_E$, it follows that

$$\text{bd}g(C_E) \cap C_E \neq \emptyset \text{ or } g(C_E) \cap \text{bd}C_E \neq \emptyset.$$ 

Assume the second case without the loss of generality. Let $x \in \text{bd}C_E$ in $g(C_E)$ and choose $H, C$ as above. Let $\text{Cl}(C)$ be the closure containing $v_E$ of a component $C$ of $\text{Cl}(\partial H) - H$ for a separating hyperspace $H$. $C \cap \text{bd}H$ is a union of lines in $S(v_E)$. Now, $H \cap g(C_E)$ contains an open neighborhood in $H$ of $x$.

Since $H$ contains a point of a concave p-end neighborhood $g(C_E)$ of $g(\hat{E})$, it meets a point of $\{g(v_E)\} + g(D) - g(D) - \{g(v_E)\}$ for a lens $D$ of $\hat{E}$ and a ray from $g(v_E)$ in $g(C_E)$. We deduce that $H \cap g(C_E)$ separates $g(C_E)$ into two open sets $C_1$ and $C_2$ in the direction of one of the sides of $H$ where $\text{Cl}(C_1) - H$ and $\text{Cl}(C_1) - H$ meet $g(\bigcup S(v_E))'$ at nonempty sets. One of $C_1$ and $C_2$ is in $C$ since $C$ is a component of $\partial - H$. Also, $\text{Cl}(C_1) - H$ meets the set at $(\text{Cl}(C) - H) \cap \text{bd}H \subset \bigcup S(v_E)$. Hence, this implies

$$g\left(\bigcup S(v_E)\right)' \cap S(v_E) \neq \emptyset.$$ 

By (iv), this means $g \in \pi_1(\hat{E})$. Hence, $g(C_E) = C_E$ and this is absurd. We have

$$g(C_E) \cap C_E = \emptyset \text{ or } g(C_E) = C_E \text{ for } g \in \pi_1(\partial).$$

Since $g$ acts on $C_E$ and the maximal segments in $S(v_E)$ must go to maximal segments, and the interior points of maximal segments cannot be an image of $v_E$, we must have $g(v_E) = v_E$. Hence, $g(U) \cap U \neq \emptyset$ for any proper p-end neighborhood of $\hat{E}$, and $g \in \pi_1(\hat{E})$.

(vi) Suppose that $S(v_E)' \cap S(v_E') \neq \emptyset$. Then $S(v_E)' \cup C_E$ is a neighborhood of $S(v_E)'$ for a proper concave p-end neighborhood $C_E$ of $\hat{E}$. Also, $S(v_E)' \cup C_E$ is a neighborhood of $S(v_E)'$ for a proper concave p-end neighborhood $C_E$ of $\hat{E}'$.

The above argument in (iv) applies within this situation to show that $\hat{E} = \hat{E}'$ and $v_E = v_E'$.

Proposition 2.13 implies the version for $\mathbb{R}P^n$.

[5"T]

6.4.2 The properties of lens-cones for factorizable case

Recall that a group $G$ divides an open domain $\Omega$ if $\Omega/G$ is compact. For virtually factorizable ends, we have more results. We don’t require that the quotient is Hausdorff.
Theorem 6.8 Let \( \mathcal{O} \) be a strongly tame properly convex real projective \( n \)-orbifold. Suppose that

- \( \text{Cl}(\mathcal{O}) \) is not of form \( v_E \ast D \) for a totally geodesic properly convex domain \( D \), or
- the holonomy group \( \Gamma \) is strongly irreducible.

Let \( \bar{E} \) be an \( R \)-p-end of the universal cover \( \bar{\mathcal{O}} \), \( \bar{\mathcal{O}} \subset \mathbb{S}^n \) (resp. \( \subset \mathbb{R}P^n \)) with a generalized lens \( p \)-end-neighborhood. Let \( v_E \) be the \( p \)-end vertex, and the \( p \)-end orbifold \( \Sigma_E \) of \( \bar{E} \). Suppose that the \( p \)-end holonomy group \( \Gamma_E \) is virtually factorizable. Then \( \Gamma_E \) satisfies the uniform middle eigenvalue condition with respect to \( v_E \), and the following statements hold:

(i) The \( R \)-p-end is totally geodesic. \( D_i \subset \mathbb{S}^{n-1}_v \) is projectively diffeomorphic by the projection \( \Pi_{\Sigma_E} \) to totally geodesic convex domain \( D_i' \subset \mathbb{S}^n \) (resp. in \( \mathbb{R}P^n \)) disjoint from \( v_E \). Moreover, \( \Gamma_E \) is virtually a cocompact subgroup of \( \mathbb{R}^1 \times \prod_{i=1}^{l_0} \Gamma_i \), where \( \Gamma_i \) acts on \( D_i' \) irreducibly and trivially on \( D_j' \) for \( j \neq i \), and \( \mathbb{R}^1 \times \prod_{i=1}^{l_0} \Gamma_i \) acts trivially on \( D_j' \) for every \( j = 1, \ldots, l_0 \).

(ii) The \( R \)-p-end is strictly lens-shaped, and each \( C_i' \) corresponds to a cone \( C_i' = v_E \ast D_i' \). The \( R \)-p-end has a \( p \)-end-neighborhood equal to the interior of

\[
\{v_E\} \ast D = \text{Cl}(D_1') \ast \cdots \ast \text{Cl}(D_{l_0}')
\]

where the interior of \( D \) forms the boundary of the \( p \)-end neighborhood in \( \bar{\mathcal{O}} \).

(iii) The set \( S(v_E) \) of maximal segments in \( bd \bar{\mathcal{O}} \) from \( v_E \) in the closure of a \( p \)-end-neighborhood of \( v_E \) is independent of the \( p \)-end-neighborhood.

\[
\bigcup_{i=1}^{l_0} S(v_E) = \bigcup_{i=1}^{l_0} \{v_E\} \ast \text{Cl}(D_1') \ast \cdots \ast \text{Cl}(D_{l-1}') \ast \text{Cl}(D_{l+1}') \ast \cdots \ast \text{Cl}(D_{l_0}').
\]

Finally, the statements (i), (ii), (iii), (iv), (v) and (vi) of Theorem 6.7 also hold.

Proof. Again the \( \mathbb{S}^n \)-version is enough by Proposition 2.13. Theorem 6.6 implies the uniform middle eigenvalue condition. (i) This follows by Proposition 2.15 (see Benoist [22]).

As in the proof of Theorem 6.7, Theorem 6.6 implies that \( \Gamma_E \) satisfies the uniform middle eigenvalue condition. Proposition 6.6 implies that the CA-lens is a strict one. Theorem 6.4 implies that the distanced \( \Gamma_E \)-invariant set is contained in a hyperspace \( P \) disjoint from \( v_E \).

(i) By the uniform middle eigenvalue condition, the largest norm of the eigenvalue \( \lambda_1(g) \) is strictly bigger than \( \lambda_{\Sigma_E}(g) \).

By Proposition 2.15, \( \Gamma \) is a virtually a subgroup of \( \mathbb{R}^{l_0-1} \times \Gamma_1 \times \cdots \times \Gamma_{l_0} \) with \( \mathbb{R}^{l_0-1} \) acting as a diagonalizable group, and there are subspaces \( \mathbb{S}_j \) for \( j = 1, \ldots, l_0 \), in \( \mathbb{S}^{n-1} \) where the factor groups \( \Gamma_1, \ldots, \Gamma_{l_0} \) act irreducibly by Benoist [22]. Let \( \mathbb{S}_j \), \( j = 1, \ldots, l_0 \), be the projective subspaces in general position meeting only at the \( p \)-end vertex \( v_E \) which goes to \( \bar{\mathcal{O}} \) under \( \Pi_{v_E} \). Now, \( \text{Cl}(\Sigma_E) \cap \mathbb{S}_j \) is a properly convex domain \( K_j \) by Benoist [22]. Let \( C_j \) denote the union of great segments from \( v_E \) with
6.4 The properties of lens-shaped ends.

The abelian center isomorphic to \( \mathbb{Z}^{l_0-1} \) acts as the identity on the subspace corresponding to \( C_i \) in the projective space \( \mathbb{S}^{n-1} \).

We denote by \( D'_i := C_i \cap P \). We denote by \( D = D'_1 \ast \cdots \ast D'_{l_0} \subset P \). Also, the interior of \( v_E * D \) is a p-end neighborhood of \( \bar{E} \). This proves (i).

Let \( U \) be the p-end-neighborhood of \( v_E \) obtained in (iv). \( \Gamma_{\bar{E}} \) acts on \( v_E \) and \( D'_1, \ldots, D'_{l_0} \).

Recall that the virtual center of \( \Gamma_{\bar{E}} \) isomorphic to \( \mathbb{Z}^{l_0-1} \subset \mathbb{R}^{l_0-1} \) has diagonalizable matrices acting trivially on \( \mathbb{S}_j \) for \( j = 1, \ldots, l_0 \). For all \( C_i \), every nonidentity \( g \) in the virtual center acts as nonidentity now by the uniform middle eigenvalue condition.

For each \( i \), we can find a sequence \( g_j \) in the virtual center of \( \Gamma_{\bar{E}} \) so that the premise of Proposition 2.18 are satisfied, and \( \text{Cl}(D'_i) \subset \text{Cl}(\bar{E}) \). By Proposition 2.18, (ii) follows. Therefore, we obtain

\[
v_E * \text{Cl}(D'_1) * \cdots * \text{Cl}(D'_{l-1}) * \text{Cl}(D'_{l}) = \text{bd} \bar{E} \cap \text{Cl}(U)
\]

by the middle eigenvalue conditions. (iii) follows.

(ii) We need to show \( D' \subset \bar{E} \). By Lemma 2.18, we have either \( D' \subset \bar{E} \) or \( D \subset \text{bd} \bar{E} \). In the second case, \( \text{Cl}(\bar{E}) = \{v_E\} * D \) since \( S(v_E) \subset \text{bd} \bar{E} \) and \( D \subset \text{bd} \bar{E} \). This contradicts the premise.

If \( \Gamma \) is strongly irreducible, \( \bar{E} \) cannot be a strict join by Proposition 2.17. Thus, this completes the proof.

We can prove the strictness of the lens and the final part by generalizing the proof of Theorem 6.7 to this situation. The proof statements do not change.

\[\text{[\S^nT]}\]

6.4.3 Uniqueness of vertices outside the lens

We will need this later in Chapter 11.

**Proposition 6.8** Suppose that \( h : \pi_1(E) \to \text{SL}_\pm(n+1, \mathbb{R}) \) is a holonomy representation of end fundamental group \( \pi_1(E) \) of a strongly tame convex real projective orbifold. Let \( h(\pi_1(E)) \) act on a generalized lens-cone \( \{v\} * L \) with vertex \( v \), acting on a generalized lens \( L \) properly and cocompactly, or act on a horosphere as a lattice in a cusp group. Then the following hold:

- the vertex of the lens-cone is determined up to the antipodal map.
- If the lens-cone is given an outward direction, then the vertex of any lens-cone where \( h(\pi_1(E)) \) acts on as a p-end neighborhood equals the vertex of the lens-cone and is uniquely determined.
- The vertex of the horospherical end is uniquely determined.

**Proof** The horospherical case can be understood from the horospherical action acting on a ball of a Klein model where fixed points form a pair of antipodal points.
Suppose that \( h(\pi_1(E)) \) acts on a generalized lens cone \( \{v\} \ast L \) for a generalized lens \( L \) as in the premise and a vertex \( v \). By Theorem 6.1, \( h(\pi_1(E)) \) satisfies the uniform middle eigenvalue condition with respect to \( v \). Suppose that there exists another point \( w \) fixed by \( h(\pi_1(E)) \) so that \( \{w\} \ast L' \) is a generalized lens cone for another generalized lens \( L' \) properly and cocompactly. Let \( \overline{vw} \) denote a vector tangent to \( \overline{vw} \) oriented away from \( v \). Then \( \overline{vw} \) goes to a point \( \langle \overline{vw} \rangle \) on \( \mathbb{H}^n_\omega \) which \( h(\pi_1(E)) \) fixes. Hence, \( \pi_1(E) \) acts reducibly on \( \mathbb{H}^n_\omega \), and \( h(\pi_1(E)) \) is virtually factorizable. Thus, \( h(\pi_1(E)) \) acts on a hyperspace \( S \) disjoint from \( v \) by Theorem 6.8. There is a properly convex domain \( D \) in \( S \) where \( h(\pi_1(E)) \) acts properly discontinuously. Also, \( \text{Cl}(D) = K_1 \ast \cdots \ast K_m \) for properly convex domain \( K_j \) where \( h(\pi_1(E)) \) acts irreducibly by Proposition 2.15.

Suppose that \( w \in S \). Then \( \{w\} = K_j \) or its antipode \( K_j \) for some \( j \). The uniform middle eigenvalue condition with respect to \( v \) implies that \( \pi_1(E) \) does not have the same property with respect to \( w \). Hence, \( w \not\in \Omega \).

Hence \( \pi_1(E) \) acts on a domain \( \Omega \) equal to the interior of \( K := K_1 \ast \cdots \ast K_m \) where \( K_j \) is compact and convex and a finite-index subgroup \( \Gamma_j \) of \( \pi_1(E) \) acts on each \( K_j \) irreducibly. The great segment from \( v \) containing \( w \) meets \( S \) in a point \( w' \). There exists a virtual-center diagonalizable group \( D \) acting on each \( K_j \) as the identity by Proposition 2.15 (more precisely Proposition 4.4 of [21]). Hence \( \{w'\} \) must be one of \( K_j \) or its antipode \( K_j \) since otherwise we can find an element of \( D \) not fixing \( w' \).

Since the action is cocompact on \( \Omega \), there must be an element of \( D \) acting with largest norm eigenvalue on \( K_j \).

Since \( h(\pi_1(E)) \) acts on \( \{w\} \ast L' \) with a compact set \( L' \) disjoint from \( w \). We construct a tube domain \( T \) and \( L' \cap T \) gives us a CA-lens in the tube. Hence, by Theorem 6.1 \( \lambda_v(g) \) satisfies the uniform middle eigenvalue condition. We choose \( g \in D \) with a largest norm eigenvalue at \( K_j \). Since \( v, w, w' \) are distinct points in a properly convex segment and are fixed points of \( g \), it follows that \( \lambda_1(g) = \lambda_v(g) = \lambda_v(g) \). This contradicts the uniform middle eigenvalue condition for \( v \) under \( \pi_1(E) \). Thus, we obtain \( v = w \).

\begin{proposition}
Suppose that \( h : \pi_1(E) \to \text{SL}_+(n+1, \mathbb{R}) \) is a representation
\begin{itemize}
    \item acting properly and cocompactly on a lens neighborhood of a totally geodesic \((n-1)\)-dimensional domain \( \Omega \) and \( \Omega / h(\pi_1(E)) \) is a compact orbifold or
    \item acting on a horosphere as a cocompact cusp group.
\end{itemize}
Then \( h(\pi_1(E)) \) uniquely determines the hyperplane \( P \) with one of the following properties:
\begin{itemize}
    \item \( P \) meets a lens domain \( L' \) with the property that \((L' \cap P)/h(\pi_1(E)) \) is a compact orbifold with \( L' \cap P = L'' \cap P \).
    \item \( P \) is tangent to the \( h(\pi_1(E))\)-invariant horosphere at the cusp point of the horosphere.
\end{itemize}
\end{proposition}

\textbf{Proof} The duality will prove this by Proposition 6.11 and Corollary 6.2. The vertex and the hyperspace exchanges the roles. \( \Box \)
6.5 Duality and lens-shaped T-ends

We first discuss the duality map. We show a lens-cone p-end neighborhood of an R-p-end is dual to a lens p-end neighborhood of a T-p-end. Using this we prove Theorem 6.9 dual to Theorem 6.6, i.e., Theorem 6.2.

6.5.1 Duality map.

The Vinberg duality diffeomorphism induces a one-to-one correspondence between p-ends of $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}^*$ by considering the dual relationship $\Gamma_{\tilde{E}}$ and $\Gamma_{\tilde{E}'}^*$ for each pair of p-ends $\tilde{E}$ and $\tilde{E}'$ with dual p-end holonomy groups. (See Section 2.5.)

Given a properly convex domain $\Omega$ in $\mathbb{S}^n$ (resp. $\mathbb{R}^n$), we recall the augmented boundary of $\Omega$

$$\text{bd}^{Ag} \Omega := \{(x, H) \mid x \in \text{bd} \Omega, x \in H, \text{H is an oriented sharply supporting hyperspace of } \Omega \} \subset \mathbb{S}^n \times \mathbb{S}^{n*}. \quad (6.27)$$

This is a closed subspace. Each $x \in \text{bd} \Omega$ is contained in at least one sharply supporting hyperspace oriented towards $\Omega$, and an element of $\mathbb{S}^{n*}$ represent an oriented hyperspace in $\mathbb{S}^{n*}$.

We recall a duality map.

$$\mathcal{D}_{\Omega}^{Ag} : \text{bd}^{Ag} \Omega \rightarrow \text{bd}^{Ag} \Omega^* \quad (6.28)$$

given by sending $(x, H)$ to $(H, x)$ for each $(x, H) \in \text{bd}^{Ag} \Omega$. This is a diffeomorphism since $\mathcal{D}_{\Omega}^{Ag}$ has an inverse given by switching factors by Proposition 2.19 (iii).

Fig. 6.1 The figure for Corollary 6.2. Lines passing v in the left figure correspond to points on P in the left. The line passing a point of A corresponds to a point on A'. Lines in the right figure correspond to points in the left figure.
We will need the corollary about the duality of lens-cone and lens-neighborhoods. Recall that given a properly convex domain $D$ in $\mathbb{S}^n$ or $\mathbb{RP}^n$, the dual domain is the closure of the open set given by the collection of (oriented) hyperspaces in $\mathbb{S}^n$ or $\mathbb{RP}^n$ not meeting $\text{Cl}(D)$. Let $\Omega$ be a properly convex domain. We need to recall the duality from Section 2.5.1 with the projection map
\[ \Pi_{\Omega}^{Ag} : \text{bd}^aAg_{\Omega} \to \text{bd} \Omega \]
sending each pair $(x, h)$ of a point $x, x \in \text{bd} \Omega$, and sharply supporting hyperplane $h$ at $x$.

**Corollary 6.2** The following hold in $\mathbb{S}^n$:

- Let $L$ be a lens and $\{v\} \notin \text{Cl}(L)$ so that $v \ast L$ is a properly convex lens-cone. Suppose that the smooth strictly convex boundary component $A$ of $L$ is tangent to a segment from $v$ at each point of $\partial \text{Cl}(A)$ and $\{v\} \ast L = \{v\} \ast A$. Then the following hold:
  - The dual domain of $\text{Cl}(\{v\} \ast L)$ is the closure of a component $L_1$ of $L' - P$ where $L'$ is a lens and $P$ is a hyperspace meeting $L$.
  - $A$ corresponds to a hypersurface $A' \subset \text{bd} L'$ under the duality (6.28).
  - $A' \cup D$ is the boundary of $L_1$ for a totally geodesic properly convex $(n - 1)$-dimensional compact domain $D$ dual to $R_v(\{v\} \ast L)$ where $D$ is given by $D_v^\ast(A'_L(\Pi_{\{v\} \ast L}^{-1}(v)))$.

- Conversely, we are given a lens $L'$ and $P$ is a hyperspace meeting $L_1$ not meeting the boundary of $L'$. Let $L_1$ be a component of $L' - P$ with smooth strictly convex boundary $\partial L_1$ so that $\text{bd} \partial L_1 \subset P$. Here, we assume $\partial L_1$ is an open hypersurface. Then the following hold:
  - The dual of the closure of a component $L_1$ of $L' - P$ is the closure of $v \ast L$ for a lens $L$ and $v \notin L$ so that $v \ast L$ is a properly convex lens-cone. Here, $v = D_v^\ast(P)$.
  - The outer boundary component $A$ of $L$ is tangent to a segment from $v$ at each point of $\text{bd}A$.

- In the above, the vertex denoted by $v$ corresponds to a hyperplane denoted by $P$ uniquely.

**Proof** In the proof all hyperspaces are oriented so that $L$ is in its interior direction. Let $A$ denote the boundary component of $L$ so that $\{v\} \ast L = \{v\} \ast A$. We will determine the dual domain $(\text{Cl}(\{v\} \ast L))^* \ast$ of $\{v\} \ast L$ by finding the boundary of $D$ using the duality map $D_v^\ast(A'_L \ast L)$. The set of hyperspaces sharply supporting $\text{Cl}(\{v\} \ast L)$ at $v$ forms a properly totally geodesic domain $D$ in $\mathbb{S}^n$ contained in a hyperspace $P$ dual to $v$ by Lemma 2.20. Also the set of hyperspaces sharply supporting $\text{Cl}(\{v\} \ast L)$ at points of $A$ goes to the strictly convex hypersurface $A'$ in $\text{bd}(v \ast L)^*$ by Lemma 2.20 since $D_v^\ast(A'_L)$ is a diffeomorphism. (See Remark 2.5 and Figure 6.1.) The subspace $S := \text{bd}(\{v\} \ast A) - A$ is a union of segments from $v$. The sharply supporting hyperspaces containing these segments go to points in $\partial D$. Each point of $\text{Cl}(A') - A'$ is a
limit of a sequence \( \{ p_i \} \) of points of \( A' \), corresponding to a sequence of sharply supporting hyperspheres \( \{ h_i \} \) to \( A \). The tangency condition of \( A \) at \( \partial \mathrm{Cl}(A) \) implies that the limit hypersphere contains the segment in \( S \) from \( v \). We obtain that \( \mathrm{Cl}(A') - A' \) equals the set of hyperspheres containing the segments in \( S \) from \( v \), and they go to points of \( \partial D \) with \( \text{bd} A' = \partial D \). We conclude

\[
\mathcal{D}_{\{v\}+L}(\text{bd}(\{v\} * L)) = A' \cup D.
\]

Let \( P \) be the unique hyperspace containing \( D \). Then each point of \( \text{bd} A \) goes to a sharply supporting hyperspace at a point of \( \text{bd} A' \) distinct from \( P \). Let \( L^* \) denote the dual domain of \( \mathrm{Cl}(L) \). Since \( \mathrm{Cl}(L) \subset \mathrm{Cl}(\{v\} * L) \), we obtain \( (\mathrm{Cl}(\{v\} * L))^* \subset (\mathrm{Cl}(L))^* \) by (2.26). Since \( A \subset \text{bd} L \), we obtain

\[
\mathcal{D}_{\{v\}+L}(\text{bd}(\{v\} * L)) \subset A' \cup P, \text{ and } A' \subset \text{bd} L^*.
\]

Proposition 2.19 implies that \( (\{v\} * L)^o \) is a component of \( (L^o)^* - P \) since the first domain can have boundary points in \( A' \cup P \) only and cannot have points outside the component. Hence, the dual of \( \mathrm{Cl}(\{v\} * L) \) is \( \mathrm{Cl}(L_1) \). Moreover, \( A' \subset \text{bd} L_1 \) since \( A' \) is a strictly convex hypersurface with boundary in \( P \).

The second item is proved similarly to the first. Now hyperspaces are oriented so that \( L_i^o \) is in its interior. Then \( \partial L_i \) goes to a hypersurface \( A \) in the boundary of the dual domain \( L_i^o \) of \( L_i \) under \( \mathcal{D}_{L_i} \). Again \( A \) is a smooth strictly convex boundary component. Since \( \text{bd} \partial L_1 \subset P \) and \( L_1 \) is a component of \( L' - P \), we obtain \( \text{bd} L_1 - \partial L_1 = \mathrm{Cl}(L_1) \ intersect P \). This is a totally geodesic properly convex domain \( D \) in \( P \).

Suppose that \( l \subset P \) be an \( n - 2 \)-dimensional space disjoint from \( L_i^o \). Then a space of oriented hyperspaces containing \( l \) bounding an open hemisphere containing \( L_i^o \) forms a parameter dual to a convex projective geodesic in \( \mathbb{S}^n \). An \( L_1 \)-pencil \( P_i \) with ends \( P_0, P_1 \) is a parameter satisfying

\[
P_i \cap P = P_0 \cap P, P_i \cap L_i^o = \emptyset \text{ for all } t \in [0, 1]
\]

where \( P_i \) is oriented so that it bounds a open hemisphere containing \( L_i^o \).

There is a one-to-one correspondence

\[
\{ P | P' \} \text{ is an oriented hyperspace that supports } L_i \text{ at points of } \partial D \} \leftrightarrow \{ v \} * \text{bd} A:
\]

Every supporting hyperspace \( P' \) to \( L_i \) at points of \( \partial D \) is contained in an \( L_1 \)-parameter \( P_i \) with \( P_0 = P', P_1 = P \). \( v \) is the dual to \( P \) in \( \mathbb{S}^n \). Each of the path \( P_i \) is a segment in \( \mathbb{S}^n \) with an endpoint \( v \).

Under the duality map \( \mathcal{D}_{L_1} \), the image of \( \text{bd} L_1 \) is a union of \( A \) and the union of these segments. Given any hyperspace \( P' \) disjoint from \( L_i^o \), we find a one-parameter family of hyperspaces containing \( P' \cap P \). Thus, we find an \( L_1 \)-pencil family \( P_i \) with \( P_0 = P', P_1 = P \). We can extend the \( L_1 \)-pencil so that the ending hyperspace \( P'' \) of the \( L_1 \)-pencil meets \( \partial L_1 \) tangentially or tangent to \( \partial L_1 \) and \( P'' \cap P \) is a supporting hyperspace of \( D \) in \( P \). Since the hyperspaces are disjoint from \( L_1 \), the segment is
in $L_1^t$. Since $L_1$ is a properly convex domain, we can deduce that $(\text{Cl}(L_1))^\ast$ is the closure of the cone $\{v\} \ast A$.

Let $L''$ be the dual domain of $\text{Cl}(L')$. Since $L' \supset L_1$, we obtain $L'' \subset (\text{Cl}(L_1))^\ast$ by (2.26). Since $\partial L_1 \subset \text{bd}L'$, we obtain $A \subset \text{bd}L''$ by the duality map $\mathcal{D}_{L_1}$. We obtain that $L'' \cup A \subset \text{Cl}(\{v\} \ast A)$.

Let $B$ be the image of the other boundary component $B'$ of $L'$ under $\mathcal{D}_{L_1}$. We take a sharply supporting hyperspace $P_v$ at $v \in B'$. Then $P_v \cap P$ is disjoint from $\text{Cl}(D)$ by the strict convexity of $B'$. We find an $L_1$-pencil $P_i$ of hyperspaces containing $P_v \cap P$ with $P_0 = P_v$, $P_1 = P$. This $L_1$-pencil goes into the segment from $v$ to a point of $B$ under the duality. We can extend the $L_1$-pencil so that the ending hyperspace meets $\partial L_1$ tangentially. The dual pencil is a segment from $v$ to a point of $A$. Thus, each segment from $v$ to a point of $A$ meets $B$. Thus, $L'' \cup A \cup B$ is a lens of the lens cone $\{v\} \ast A$. This completes the proof. \hfill $\Box$

### 6.5.2 The duality of T-ends and properly convex R-ends.

Let $\Sigma$ be a convex real projective strong tame orbifold with ends where $\Sigma = \hat{\Sigma}/\Gamma$ for a properly convex domain $\hat{\Sigma} \subset \mathbb{S}^n$ and a discrete projective group $\Gamma$. Let $p_{\Sigma} : \hat{\Sigma} \to \Sigma$ denote the covering map. Suppose that a p-end fundamental group $\Gamma_E$ acts on a connected hypersurface $\Sigma$ in $\hat{\Sigma}$. Then a component $U$ of $\hat{\Sigma} - \Sigma$ is a p-end neighborhood of $E$. Furthermore the following hold for a component $U'$ of $\Sigma - \hat{\Sigma}$.

- **Suppose that** $U'$ **is a horoball** so that $\text{bd}U' \cap \hat{\Sigma} = \text{bd}U' - \{p\} = \Sigma$ **for the common fixed point** $p$ **of** $\Gamma_E$. Then $p_{\Sigma}(U')$ **can be given the structure of a horospherical R-end neighborhood of** $E$ **and** $p = v_E$.
- **Suppose that** $U'$ **equals** $U_L := L \ast \{p\} - L$ **for a lens** $L$ **where**
  - $h(\pi_1(\hat{E}))$ **acts cocompactly,**
  - we have a lens-cone $L \ast \{p\}$ **for a common fixed point** $p$ **of** $\Gamma_E$, **and**
  - $\text{bd}U_L \cap \hat{\Sigma} = \Sigma$.

Then $p_{\Sigma}(U')$ **can be given the structure of a concave R-end neighborhood of** $\hat{E}$ **and** $p = v_E$.
- **Suppose that** $U'$ **equals a component** $L_1$ **of** $L - P$ **for a lens** $L$ **where**
  - $h(\pi_1(\hat{E}))$ **acts cocompactly on** $L$,
  - $h(\pi_1(\hat{E}))$ **acts on the hyperplane** $P$,
  - $L_1 \subset \hat{\Sigma}$, **and**
  - $\text{bd}L_1 \cap \hat{\Sigma} = \Sigma$.  

Note: The text is slightly fragmented and contains mathematical symbols and equations that are not properly rendered. It appears to discuss the relationship between certain geometric structures and their duals, possibly in the context of orbifolds and projective geometry.
Then $p_{\Omega}(L^*_1)$ can be given the structure of a T-end neighborhood of $\bar{E}$ and $\Sigma_{\bar{E}}$ equals $L \cap P$.

Moreover the corresponding end completions gives us a compact smooth orbifold $\mathcal{O}$ whose interior is $\mathcal{O}^*$.

**Proof** It is sufficient to prove for the case when the orbifold is orientable and without singular points since we can take a finite quotient by Theorem 2.3. Let $U$ be a proper p-end neighborhood of $\bar{E}$. We take $\partial/\Gamma_{\bar{E}}$ which is diffeomorphic to $A := (\partial U \cap \partial) / \Gamma_{\bar{E}} \times R$ where $B := (\partial U \cap \partial) / \Gamma_{\bar{E}}$ is a closed submanifold of codimension-one. We can take an exiting sequence $U_i$ of p-end neighborhoods in $U$. Then $C := \Sigma / \Gamma_{\bar{E}}$ is a closed submanifold freely homotopic to the above one. Hence, one $U_B$ of the two components $A - B$ contains $U_i$ for sufficiently large $i$. Hence, a component of the inverse image of $U_B$ which is a component of $\partial - \Sigma$ is a p-end neighborhood.

In the first case, we can choose a sufficiently small horoball $U$ inside $K^o$ and in $H$ since the sharply supporting hyperspaces at the vertex of $H$ must coincide by the invariance under $h(\pi_1(\mathcal{O}))$ by a limiting argument. By above, one of the two component of $K^o - \text{bd}U$ is a p-end neighborhood. One cannot put the outside component into $U$ by an element of $h(\pi_1(\mathcal{O}))$. Since $h(\pi_1(\bar{E}))$ acts on $\partial U \cap K^o$ properly discontinuously, $U$ is foliated by rays going to properly embedded rays in $\mathcal{O}$. Hence, $U$ is a horospherical p-end neighborhood of $\bar{E}$. The radial foliation is given by the space of segments in $U$ ending in the unique fixed point of $h(\pi_1(\bar{E}))$ in the closure of $U$.

For the second item, we show that $U_L$ is the p-end neighborhood and not the other component. One cannot put $\mathcal{O} - U_l$ into $U_l$ by an element of $h(\pi_1(\mathcal{O}))$ so that $\text{bd} \mathcal{O} - \text{Cl}(U_l)$ is mapped to $\partial \Sigma(\mathcal{O}) \cap \text{Cl}(U_l)$ since the second set is foliated by segments from $\mathcal{O}$. Lemma 1.1 implies that $U_l$ is foliated by radial lines mapping to properly imbedded lines in $\mathcal{O}$. Thus, $U_L$ is the p-end neighborhood of $\bar{E}$, and $\bar{E}$ has a radially foliated p-end neighborhood with $\mathcal{O}$ as the p-end vertex.

In the third case, let $D$ denote $\text{Cl}(L_1) \cap P$, a properly convex domain. By premise, $D^*/h(\pi_1(\bar{E}))$ is a closed orbifold of codimension-one. Then one of the two components of $\partial - \text{bd}L_1$ is a p-end neighborhood of $\bar{E}$ by above. The set $\partial - L_1$ cannot be put into $L_1$ by an element of $h(\pi_1(\mathcal{O}))$ since $D^*$ is totally geodesic and $\text{bd} \mathcal{O} - \text{Cl}(L_1)$ is not. Hence, $L^*_1$ is a p-end neighborhood of $\bar{E}$. By premise, $h(\pi_1(\bar{E}))$ acts properly on $L_1 \cup D^*$. The T-end structure is given by $(D^* \cup L_1)/h(\pi_1(\bar{E}))$ which is the completion of the end neighborhood $p(L^*_1)$ projectively diffeomorphic to $L^*_1/h(\pi_1(\bar{E}))$. (See 1.3.2.)

**Proposition 6.10** Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold. The following conditions are equivalent:

(i) A properly convex R-end $E$ of $\mathcal{O}$ satisfies the uniform middle-eigenvalue condition.

(ii) The corresponding T-end $E^*$ of $\mathcal{O}^*$ satisfies this condition with the correspondence of the vertex of the p-end $\bar{E}$ of $E$ to the hyperplane of p-end $\bar{E}^*$ is given as the unique hyperplane containing $\mathcal{O}^*(\Pi_{\mathcal{O}}^{N^o-1}(v_{\bar{E}})^*)$. 

□
**Proof**  The items (i) and (ii) are equivalent by considering (6.1) and (6.2). Proposition 2.13 implies the $\mathbb{R}P^n$-version.

We now prove the dual to Theorem 6.6. For this we do not need the triangle condition or the reducibility of the end.

**Theorem 6.9** Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold. Let $\tilde{S}_E$ be a totally geodesic ideal boundary component of a T-p-end $\tilde{E}$ of $\mathcal{O}$. Then the following conditions are equivalent:

(i) The end holonomy group of $\tilde{E}$ satisfies the uniform middle-eigenvalue condition with respect to the T-p-end structure of $\tilde{E}$.

(ii) $\tilde{S}_E$ has a CA-lens neighborhood in an ambient open manifold containing $\mathcal{O}$ with cocompact action of $\pi_1(\tilde{E})$, and hence $\tilde{E}$ has a lens-shaped p-end-neighborhood in $\mathcal{O}$.

**Proof**  We prove for the $\mathbb{S}^n$-version. Assuming (i), we can deduce the existence of a lens neighborhood from Theorem 5.4 and Lemma 6.10. Assuming (ii), we obtain a totally geodesic $(n - 1)$-dimensional properly convex domain $\tilde{S}_E$ in a subspace $\mathbb{S}^{n-1}$ on which $\Gamma_E$ acts. Let $U$ be a CA-lens-neighborhood of it on which $\Gamma_E$ acts. Then since $U$ is a neighborhood, the sharply supporting hemisphere at each point of $\text{Cl}(S_E) - S_E$ is now transverse to $\mathbb{S}^{n-1}$. Let $P$ be the hyperspace containing $\tilde{S}_E$, and let $U_1$ be the component of $U - P$. Then the dual $U_1^*$ is a lens-cone by the second part of Corollary 6.2 where $P$ corresponds to a vertex of the lens-cone. The dual $U^*$ of $U$ is a CA-lens contained in a lens-cone $U_1^*$ where $\Gamma_E$ acts on $U^*$. We apply the part (i) $\Rightarrow$ (ii) of Theorem 6.6. Proposition 2.13 implies for $\mathbb{R}P^n$.

**Proposition 6.11** Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold with R-ends or T-ends with universal covering domain $\Omega$. Let $\mathcal{O}^*$ be the dual orbifolds with a universal covering domain $\tilde{\mathcal{O}}^*$. Then $\mathcal{O}^*$ is also strongly tame and can be given R-end and T-end structures to the ends where the following hold:

- there exists a one-to-one correspondence $\mathcal{C}$ between the set of p-ends of $\mathcal{O}$ and the set of p-ends of $\mathcal{O}^*$ by sending a p-end neighborhood to a p-end neighborhood using the Vinberg diffeomorphism of Theorem 2.16.
- $\mathcal{C}$ restricts to such a one between the subset of horospherical p-ends of $\mathcal{O}$ and the subset of horospherical ones of $\mathcal{O}^*$. Also, the augmented Vinberg duality homeomorphism $\mathcal{D}_{\mathcal{O}}$ send the p-end vertex to the p-end vertex of the dual p-end.
- $\mathcal{C}$ restricts to such a one between the subset of all properly convex lens-shaped R-ends of $\mathcal{O}$ and the subset of all lens-shaped T-ends of $\mathcal{O}^*$. Also, $\mathcal{D}_{\mathcal{O}}$ gives a one to one correspondence between $\Pi_{\mathcal{O}}(\mathbb{S}(v_E))$ in $\text{bd}^{\mathcal{O}}$ to $\Pi_{\mathcal{O}}^{-1}(\text{Cl}(S_E))$ of $\text{bd}^{\mathcal{O}}$. 


• \( \mathcal{G} \) restricts to such a one between the set of lens-shaped T-p-ends of \( \hat{\mathcal{O}} \) with the set of p-ends of properly convex lens-shaped R-p-ends of \( \hat{\mathcal{O}}^* \). The ideal boundary component \( S_E \) for a T-p-end \( E \) is projectively diffeomorphic to the properly convex open domain dual to the domain \( \hat{S}_E \) for the corresponding R-p-end \( \hat{E}^* \) of \( \hat{E} \). Also, \( \mathcal{O}^* \) gives one to one correspondence between \( \Pi_{\hat{\mathcal{O}}}^{-1} (\text{Cl}(S_E)) \) in \( \text{bd}\mathcal{O}^* \) to \( \Pi_{\hat{\mathcal{O}}}^{-1} (\bigcup S(\hat{E}^*)) \) of \( \text{bd}\mathcal{O}^* \).

**Proof** We prove for the \( \mathbb{S}^n \)-version first. Let \( \hat{\mathcal{O}} \) be the universal cover of \( \mathcal{O} \). Let \( \hat{\mathcal{O}}^* \) be the dual domain. By the Vinberg duality diffeomorphism of Theorem 2.16, \( \mathcal{O}^* := \hat{\mathcal{O}}^*/\Gamma^* \) is also strongly tame for the dual group \( \Gamma^* \). The first item follows by the fact that this diffeomorphism sends p-end neighborhoods to p-end neighborhoods.

Let \( \hat{E} \) be a horospherical R-p-end with \( x \) as the end vertex. Since there is a subgroup \( \Gamma^* \) of a cusp group acting on \( \text{Cl}(\hat{\mathcal{O}}) \) with a unique fixed point, the intersection of the unique sharply supporting hyperspace \( h \) with \( \text{Cl}(\hat{\mathcal{O}}) \) at \( x \) is a singleton \( \{x\} \). (See Theorem 4.2.) The dual subgroup \( \Gamma^*_E \) is also a cusp group and acts on \( \text{Cl}(\hat{\mathcal{O}}^*) \) with \( h \) fixed. So the corresponding \( \hat{\mathcal{O}}^* \) has the dual hyperspace \( x^* \) of \( x \) as the unique intersection at \( h^* \) dual to \( h \) at \( \text{Cl}(\hat{\mathcal{O}}^*) \). There is a horosphere \( S \) where the end fundamental group \( \Gamma^*_E \) acts on. By Lemma 6.10, \( S \) bounds a horospherical p-end neighborhood of \( \hat{E} \). Hence \( x^* \) is the vertex of a horospherical end. \( \mathcal{O}^* \) acts properly on \( \text{bd}\mathcal{O}^* \) and these are unique fixed points.

An R-p-end \( \hat{E} \) of \( \mathcal{O} \) has a p-end vertex \( v_E \). \( \hat{S}_E \) is a properly convex domain in \( \mathbb{S}^{n-1}_{\hat{\mathcal{O}}} \). The space of sharply supporting hyperspaces of \( \mathcal{O} \) at \( v_{\hat{E}} \) forms a properly convex domain of dimension \( n-1 \) since they correspond to hyperspaces in \( \mathbb{S}^{n-1}_{\hat{\mathcal{O}}} \) not intersecting \( \hat{S}_E \). Under the duality map \( \mathcal{O}^* \) in Proposition 2.19, \((v_{\hat{E}}, h)\) for a sharply supporting hyperspace \( h \) is sent to \((h^*, v^*_E)\) for a point \( h^* \) and a hyperspace \( v^*_E \). Lemma 2.20 shows that \( h^* \) is a point in a properly convex \( n-1 \)-dimensional domain \( D := \text{bd}\mathcal{O}^* \cap P \) for \( P = v^*_E \), a hyperspace.

Corollary 6.2 implies the fact about \( \mathcal{O}^*_E \).

Since \( D \) is a properly convex domain with a Hilbert metric, \( \pi_1(\hat{E}) \) acts properly on \( D^o \). The \( n \)-orbifold \((\hat{\mathcal{O}}^* \cup D^o)/\pi_1(\hat{E}) \) has closed-orbifold boundary \( D^o/\pi_1(\hat{E}) \). There is a Riemannian metric on the \( n \)-orbifold so that \( D^o/\pi_1(\hat{E}) \) is totally geodesic. Using the exponential map, we obtain a tubular neighborhood of \( D^o/\pi_1(\hat{E}) \). Hence, \( \hat{\mathcal{O}} \) has a p-end neighborhood corresponding to \( \pi_1(\hat{E}) \) containing \( D^o \) in the boundary. The dual group \( \Gamma^*_E \) satisfies the uniform middle eigenvalue condition since \( \Gamma^*_E \) satisfies the condition. By Theorem 5.4 and Lemma 6.10, we can find a p-end neighborhood \( U \) in \( \hat{\mathcal{O}}^* \) bounded by a strictly convex hypersurface \( \partial U = \text{bd}\mathcal{U} \cap \hat{\mathcal{O}} \) where \( \text{Cl}(\partial U) - \partial U \subset \partial D \).

By Lemma 6.10, \( \hat{S}_E \subset \text{bd}\mathcal{O}^* \), and \( \hat{E}^* \) is a totally geodesic end with \( \hat{S}_E \), dual to \( \hat{S}_E \). This proves the third item.

The fourth item follows similarly. Take a T-p-end \( \hat{E} \). We take the ideal p-end boundary \( S_E \). The map \( \mathcal{O}^*_E \) sends \( P \) to a singleton \( P^* \) in \( \text{bd}\mathcal{O}^* \) and points of \( \text{Cl}(S_E) \) go to hyperspaces supporting \( \hat{\mathcal{O}}^* \) at \( P^* \). Since \( \Gamma^*_E \) satisfies the uniform middle eigenvalue condition with respect to \( P^* \), Theorem 6.1 shows that there exists a lens-cone.
where $\Gamma^*_E$ acts on. Also, $\Gamma^*_E$ acts on a tube domain $\mathcal{T}_p(D^o)$ for a properly convex domain $D^o$. Then $\text{Cl}(\hat{\partial}^*) \cap \text{bd} \mathcal{T}_p(D^o)$ is a $\Gamma^*_E$-invariant closed set. Also, this set is the image under $\mathcal{D}_\hat{\partial}^{Ag}$ of all hyperspaces supporting $\hat{\partial}$ at points of $\text{Cl}(\hat{\Sigma}_E)$ by Corollary 6.2. Hence, $R^o(\hat{\partial}^*) = D^o$ by convexity. Since $D^o$ is properly convex, $\Gamma^*_E$ acts properly on it. By Lemma 1.1, $P^o$ is a p-end vertex of a p-end neighborhood. There is a lens $L$ so that $U_L := P^o * L - L$ is a $\Gamma^*$-invariant. There is a boundary component $\partial L$ of this $U_L$ in $\hat{\partial}^*$. By Lemma 6.10, this implies that $U_L$ is a p-end neighborhood corresponding to $\Gamma^*_E$. Corollary 6.2 implies the fact about $\mathcal{D}_\hat{\partial}^{Ag}$.

The proof for $\mathbb{R}^m$-version follows by Proposition 2.13.

Remark 6.3 We also remark that the map induced on the limit points of p-end neighborhoods of $\Omega$ to that of $\Omega^*$ by $\mathcal{D}_\Omega^{Ag}$ is compatible with the Vinberg diffeomorphism by the continuity part of Theorem 2.17. That is the limit points of $\text{bd}^Ag \Omega$ of a p-end neighborhood of a p-end $\hat{E}$ goes to the limit points of $\text{bd}^Ag \Omega^*$ of a p-end neighborhood of a dual p-end $E^*$ of $\hat{E}$ by $\mathcal{D}_\Omega^{Ag}$.

$\mathcal{C}$ restricts to a correspondence between the lens-shaped R-ends with lens-shaped T-ends. See Corollary 6.3 for detail.

Theorems 6.6 and 6.9 and Propositions 6.11 and 6.10 imply

Corollary 6.3 Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold and let $\mathcal{O}^*$ be its dual orbifold. Then we can give the structure of R-ends and T-ends to ends of $\mathcal{O}$ and $\mathcal{O}^*$ so that dual end correspondence $\mathcal{C}$ restricts to a correspondence between the generalized lens-shaped R-ends with lens-shaped T-ends and horospherical ends to themselves. If $\mathcal{O}$ satisfies the triangle condition or every end is virtually factorizable, $\mathcal{C}$ restricts to a correspondence between the lens-shaped R-ends with lens-shaped T-ends and horospherical ends to themselves.

Corollary 6.4 Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold. Let $\hat{E}$ be a lens-shaped p-end. Then for a lens-cone p-end neighborhood $U$ of form $\{v_\hat{E}\}L - \{v_\hat{E}\}$ for lens $L$, we have the upper boundary component $A$ is tangent to radial rays from $v_\hat{E}$ at $\text{bd}A$.

Proof By Corollary 6.3, $\hat{E}$ corresponds to a T-p-end $\hat{E}^*$ of $\mathcal{O}^*$ of lens type. Now, we take the dual domain $L'_1 \setminus \{v_\hat{E}\}$ of $\hat{E}^* \setminus \{v_\hat{E}\}$ and $\text{bd} \hat{E}$. The second part of Corollary 6.2 applied to $L'_1 \setminus \{v_\hat{E}\}$ gives us result for $\hat{E}^* \setminus \{v_\hat{E}\}$. Proposition 2.13 finishes the proof.

[5^mT]
Chapter 7
Application: The openness of the lens properties, and expansion and shrinking of end neighborhoods

This chapter lists applications of the main theory of Part 2, except for Chapter 8, which are results we need in Part 3. In Section 7.1, we show that the lens-shaped property is stable under the change of holonomy representations. In Section 7.2, we will define limits sets of ends and discuss the properties. We obtain the exhaustion of \( \tilde{O} \) by a sequence of p-end-neighborhoods of \( \tilde{O} \), we show that any end-neighborhood contains horospherical or concave end-neighborhood, and we discuss on maximal concave end-neighborhoods. In Section 7.3, Corollary 7.6 shows that the closures of p-end neighborhoods are disjoint in the closures of the universal cover in \( S^n \) (resp. in \( \mathbb{R}P^n \)). We prove from this the strong irreducibility of \( \mathcal{O} \), Theorem 1.2 under the conditions (IE) and (NA).

For results in this chapter, we don’t necessarily assume that the holonomy group of \( \pi_1(\mathcal{O}) \) is strongly irreducible. Also, we will not explicitly mention Proposition 2.13 since its usage is well-established.

7.1 The openness of lens properties.

As conditions on representations of \( \pi_1(\tilde{E}) \), the condition for generalized lens-shaped ends and one for lens-shaped ends are the same. Given a holonomy group of \( \pi_1(\tilde{E}) \) acting on a generalized lens-shaped cone p-end neighborhood, the holonomy group satisfies the uniform middle eigenvalue condition by Theorem 6.6. We can find a lens-cone by choosing our orbifold to be \( \mathcal{R}_{\tilde{E}}(\Sigma_{\tilde{E}})^{o}/\pi_1(\tilde{E}) \) by Proposition 6.6.

Let

\[
\text{Hom}_{\mathcal{O}}(\pi_1(\tilde{E}), \text{SL}_{\pm}(n+1, \mathbb{R})) \quad \text{(resp. } \text{Hom}_{\mathcal{O}}(\pi_1(\tilde{E}), \text{PGL}(n+1, \mathbb{R}))\text{)}
\]

denote the space of representations of the fundamental group of an \((n-1)\)-orbifold \( \Sigma_{\tilde{E}} \).
Recall Definition 6.6 for strictly generalized lens-shaped R-ends. A (resp. generalized) lens-shaped representation for an R-end fundamental group is a representation acting on a (resp. generalized) lens-cone as a p-end neighborhood.

**Theorem 7.1** Let \( \hat{O} \) be a strongly tame properly convex real projective orbifold. Assume that the universal cover \( \hat{O} \) is a subset of \( \mathbb{S}^n \) (resp. \( \mathbb{R}P^n \)). Let \( \hat{E} \) be a properly convex R-p-end of the universal cover \( \hat{O} \). Then

(i) \( \hat{E} \) is a generalized lens-shaped R-end if and only if \( \hat{E} \) is a strictly generalized lens-shaped R-end.

(ii) The subspace of generalized lens-shaped representations of an R-end is open in

\[ \text{Hom}_E(\pi_1(\hat{E}), \text{SL}_+(n+1, \mathbb{R})) \text{ (resp. Hom}_E(\pi_1(\hat{E}), \text{PGL}(n+1, \mathbb{R})). \]

Finally, if \( \hat{O} \) is properly convex and satisfies the triangle condition or \( \hat{E} \) is virtually factorizable, then we can replace the term “generalized lens-shaped” to “lens-shaped” in each of the above statements.

**Proof** We will assume \( \hat{O} \subset \mathbb{S}^n \) first. (i) If \( \pi_1(\hat{E}) \) is non-virtually-factorizable, then the equivalence is given in Theorem 6.7 (i), and if \( \pi_1(\hat{E}) \) is virtually factorizable, then it is in Theorem 6.8 (ii). The converse is obvious.

(ii) Let \( \mu \) be a representation \( \pi_1(\hat{E}) \to \text{SL}_+(n+1, \mathbb{R}) \) associated with a generalized lens-cone. By Theorem 6.2, we obtain that \( \pi_1(\hat{E}) \) satisfies the uniform middle eigenvalue condition with respect to \( v \hat{E} \). By Theorem 6.1, we obtain a CA-lens \( K \) in \( \mathcal{R}_\mathcal{E}(\hat{E}) \) with smooth convex boundary components \( A \cup B \) since \( \mathcal{R}_\mathcal{E}(\hat{E}) \) itself satisfies the triangle condition although it is not properly convex. (Note we don’t need \( K \) to be in \( \hat{O} \) for the proof.)

\( K \cup (A \cup B) \) is a compact orbifold whose boundary is the union of two closed \( n \)-orbifold components \( A \cup B \cup (A \cup B) \). Suppose that \( \mu \) is sufficiently near \( \mu \). We may assume that \( v \hat{E} \) is fixed by conjugating \( \mu \) by a bounded projective transformation. By considering the radial segments in \( K \), we obtain a foliation by radial lines in \( \{ v \hat{E} \} \). By Proposition 6.4, applying Proposition 6.5 to the both boundary components of the CA-lens, we obtain a lens-cone in a tube domain \( \mathcal{R}_\mathcal{E}(\hat{E}) \) in general different from the original one. This implies that the sufficiently small change of holonomy keeps \( E \) to have a concave p-end neighborhood. This completes the proof of (ii).

The final statement follows by Lemma 6.7.

**Theorem 7.2** Let \( \hat{O} \) be a strongly tame properly convex real projective orbifold. Assume that the universal cover \( \hat{O} \) is a subset of \( \mathbb{S}^n \) (resp. \( \mathbb{R}P^n \)). Let \( \hat{E} \) be a T-p-end of the universal cover \( \hat{O} \). Let \( \text{Hom}_E(\pi_1(\hat{E}), \text{SL}_+(n+1, \mathbb{R})) \) (resp. \( \text{Hom}_E(\pi_1(\hat{E}), \text{PGL}(n+1, \mathbb{R})) \)) be the space of representations of the fundamental group of an \( n-1 \)-orbifold \( \Sigma_\mathcal{E} \). Then the subspace of lens-shaped representations of a T-p-end is open.

**Proof** By Theorem 6.9, the condition of the lens T-p-end is equivalent to the uniform middle eigenvalue condition for the end. Proposition 6.10 and Theorems 6.2 and 7.1 complete the proof.
Corollary 7.1 We are given a properly convex R- or T-end $\tilde{E}$ of a strongly tame convex orbifold $\tilde{O}$. Assume that $\tilde{O} \subset S^n$ (resp. $\tilde{O} \subset \mathbb{RP}^n$). Then the subset of

$$\text{Hom}_\phi(\pi_1(\tilde{E}), SL_{\pm}(n + 1, \mathbb{R}))$$

consisting of representations satisfying the uniform middle-eigenvalue condition with respect to some choices of fixed points or fixed hyperplanes of the holonomy group is open.

Proof For R-p-ends, this follows by Theorems 6.6 and 7.1. For T-p-ends, this follows by dual results: Theorem 6.9 and Theorems 7.2.

7.2 The end and the limit sets.

Definition 7.1

- Define the limit set $\Lambda(\tilde{E})$ of an R-p-end $\tilde{E}$ with a generalized p-end-neighborhood to be $\text{bd}D - \partial D$ for a generalized CA-lens $D$ of $\tilde{E}$ in $S^n$ (resp. $\mathbb{RP}^n$). This is identical with the set $\Lambda_\Gamma$ in Definition 6.5 by Corollary 6.1.
- The limit set $\Lambda(\tilde{E})$ of a lens-shaped T-p-end $\tilde{E}$ to be $\text{Cl}(\tilde{S}_E) - \tilde{S}_E$ for the ideal boundary component $\tilde{S}_E$ of $\tilde{E}$.
- The limit set of a horospherical end is the set of the end vertex.

The definition does depend on whether we work on $S^n$ or $\mathbb{RP}^n$. However, by Proposition 2.13, there are always straightforward one-to-one correspondences. We remark that this may not equal to the closure of the union of the attracting fixed set for some cases.

Corollary 7.2 Let $\tilde{O}$ be a strongly tame convex real projective n-orbifold where $\tilde{O} \subset S^n$ (resp. $\subset \mathbb{RP}^n$). Let $U$ be a p-end-neighborhood of $\tilde{E}$ where $\tilde{E}$ is a lens-shaped T-p-end or a generalized lens-shaped or lens-shaped or horospherical R-p-end. Then $\text{Cl}(U) \cap \text{bd} \tilde{O}$ equals $\text{Cl}(\tilde{S}_E)$ or $\text{Cl}(\bigcup \text{S}(v_E))$ or $\{v_E\}$ depending on whether $\tilde{E}$ is a lens-shaped T-p-end or a generalized lens-shaped or horospherical R-p-end. Furthermore, this set is independent of the choice of $U$ and so is the limit set $\Lambda(\tilde{E})$ of $\tilde{E}$.

Proof We first assume $\tilde{O} \subset S^n$. Let $\tilde{E}$ be a generalized lens-shaped R-p-end. Then by Theorem 6.6, $\tilde{E}$ satisfies the uniform middle eigenvalue condition. Suppose that $\pi_1(\tilde{E})$ is not virtually factorizable. Let $L^b$ denote $\partial \mathcal{V}_E(\tilde{S}_E) \cap L$ for a distanced compact convex set $L$ where $\Gamma_E$ acts on. We have $L^b = \Lambda(\tilde{E})$ by Proposition 6.3. Since $S(v_E)$ is an $h(\pi_1(\tilde{E}))$-invariant set, and the convex hull of $\text{bd} \bigcup S(v_E)$ is a distanced compact convex set by the proper convexity of $\tilde{S}_E$, Theorems 6.7 and 6.8 show that the limit set is determined by the set $L^b$ in $\bigcup S(v_E)$, and $\text{Cl}(U) \cap \text{bd} \tilde{O} = \bigcup S(v_E)$.
Suppose now that $\pi(E)$ is virtually factorizable. Then by Theorem 6.8, $E$ is a totally geodesic $R$-$p$-end. Proposition 6.3 and Theorem 6.8 again imply the result.

Let $\tilde{E}$ be a $T$-$p$-end. Theorems 6.9 and 5.4 imply

$$\text{Cl}(A) - A \subset \text{Cl}(\tilde{S}_{\tilde{E}})$$

for a CA-lens neighborhood $L$ by the strictness of the lens. Thus, $\text{Cl}(U) \cap \text{bd} \tilde{O}$ equals $\text{Cl}(\tilde{S}_{\tilde{E}})$.

For horospherical ones, we simply use the definition to obtain the result. [ST]

**Definition 7.2** An **SPC-structure** or a stable properly-convex real projective structure on an $n$-orbifold is a convex real projective structure so that the orbifold has a stable irreducible holonomy group. That is, it is projectively diffeomorphic to a quotient orbifold of a properly convex domain in $S^n$ (resp. in $\mathbb{RP}^n$) by a discrete group of projective automorphisms that is stable and irreducible.

**Definition 7.3** Suppose that $\mathcal{O}$ has an SPC-structure. Let $\tilde{U}$ be the inverse image in $\tilde{\mathcal{O}}$ (resp. in $\mathbb{RP}^n$) of the union $U$ of some choice of a collection of mutually disjoint end neighborhoods of $\mathcal{O}$. If every straight arc in the boundary of the domain $\tilde{\mathcal{O}}$ and every non-$C^1$-point is contained in the closure of a component of $\tilde{U}$ for some choice of $U$, then $\mathcal{O}$ is said to be strictly convex with respect to the collection of the ends. And $\mathcal{O}$ is also said to have a strict SPC-structure with respect to the collection of ends.

Proposition 2.13 shows that this definition is equivalent to Definition 1.6. Corollary 7.3 shows the independence of the definition with respect to the choice of the end-neighborhoods when the ends are generalized lens-type $R$-end or lens-shaped $T$-ends. We conjecture that this holds also for the ends of four types given by Ballas-Cooper-Leitner [7].

**Corollary 7.3** Suppose that $\mathcal{O}$ is a strongly tame strictly SPC-orbifold with generalized lens-shaped $R$-ends or lens-shaped $T$-ends or horospherical ends. Let $\tilde{\mathcal{O}}$ is a properly convex domain in $\mathbb{RP}^n$ (resp. in $S^n$) covering $\mathcal{O}$. Choose any disjoint collection of end neighborhoods in $\mathcal{O}$. Let $U$ denote their union. Let $p_\mathcal{O} : \tilde{\mathcal{O}} \to \mathcal{O}$ denote the universal cover. Then any segment or a non-$C^1$-point of $\text{bd} \tilde{O}$ is contained in the closure of a component of $p_\mathcal{O}^{-1}(U)$ for any choice of $U$.

**Proof** We first assume $\tilde{\mathcal{O}} \subset S^n$. By the definition of a strict SPC-orbifold, any segment or a non-$C^1$-point has to be in the closure of a $p$-end neighborhood. Corollary 7.2 proves the claim. [ST]
7.2 The end and the limit sets.

7.2.1 Convex hulls of ends.

We will sharpen Corollary 7.2 and the convex hull part in Lemma 7.2. Again, these sets are all defined in $S^n$ and we define the corresponding objects for $\mathbb{RP}^n$ by their images under $\mathbb{RP}^n$ by Proposition 2.13.

One can associate a convex hull $\mathcal{CH}(\tilde{E})$ of a $p$-end $\tilde{E}$ of $\tilde{O}$ as follows:

- For horospherical $p$-ends, the convex hull of each is defined to be the set of the end vertex actually.
- The convex hull of a lens-shaped totally geodesic $p$-end $\tilde{E}$ is the closure $\text{Cl}(S_{\tilde{E}})$, the totally geodesic ideal boundary component $S_{\tilde{E}}$ corresponding to $\tilde{E}$.
- For a generalized lens-shaped $p$-end $\tilde{E}$, the convex hull of $\tilde{E}$ is the convex hull of $\bigcup S(v_{\tilde{E}})$ in $\text{Cl}(\tilde{O})$; that is, $I(\tilde{E}) = \mathcal{CH}(\bigcup S(v_{\tilde{E}}))$.

The first two equal $\text{Cl}(U) \cap \text{bd} \tilde{O}$ for any $p$-end neighborhood $U$ of $\tilde{E}$ by Corollary 7.2. Corollary 7.2 and Proposition 7.1 imply that the convex hull of an end is well-defined. We can also characterize as the intersection $I(\tilde{E}) = \bigcap_{U \in \mathcal{U}} \mathcal{CH}(\text{Cl}(U_1))$ for the collection $\mathcal{U}$ of $p$-end neighborhoods $U_1$ of $v_{\tilde{E}}$ by Proposition 7.1.

We define $\partial I(\tilde{E})$ as the set of endpoints of maximal rays from $v_{\tilde{E}}$ ending at $\partial I(\tilde{E})$ and in the directions of $\Sigma_{\tilde{E}}$. Since the convex hull of $\bigcup S(v_{\tilde{E}})$ is a subset of the tube with a vertex $v_{\tilde{E}}$ in the directions of elements of $\text{Cl}(\Pi_{v_{\tilde{E}}}(R_{v_{\tilde{E}}}(\Sigma_{\tilde{E}})))$, we obtain

$$\partial I(\tilde{E}) = \partial_3 I(\tilde{E}) \cup \bigcup S(v_{\tilde{E}}).$$

(7.1)

A topological orbifold is one where we are allowed to use continuous maps as charts. We say that two topological orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ are homeomorphic if there is a homeomorphism of the base spaces and that is induced from a homeomorphism $f : M_1 \to M_2$ of respective very good manifold covers $M_1$ and $M_2$ where $f$ induces the isomorphism of the deck transformation group of $M_1 \to \mathcal{O}_1$ to that of $M_2 \to \mathcal{O}_2$.

**Proposition 7.1** Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold with radial ends or lens-shaped totally geodesic ends and satisfy (IE) and (NA). Let $\tilde{E}$ be a generalized lens-shaped R-p-end and $v_{\tilde{E}}$ an associated p-end vertex. Let $I(\tilde{E})$ be the convex hull of $\tilde{E}$.

(i) Suppose that $\tilde{E}$ is a lens-shaped radial p-end. Then $\partial_3 I(\tilde{E}) = \partial I(\tilde{E}) \cap \tilde{O}$, and $\partial_3 I(\tilde{E})$ is contained in the CA-lens in a lens-shaped p-end neighborhood.

(ii) $I(\tilde{E})$ contains any concave p-end-neighborhood of $\tilde{E}$ and
\[ I(\tilde{E}) = \mathcal{C}(\text{Cl}(U)) \]
\[ I(\tilde{E}) \cap \tilde{\partial} = \mathcal{C}(\text{Cl}(U)) \cap \tilde{\partial} \]

for a concave p-end neighborhood \( U \) of \( \tilde{E} \). Thus, \( I(\tilde{E}) \) has a nonempty interior:

(iii) Each segment from \( v_{\tilde{E}} \) maximal in \( \tilde{\partial} \) meets the set \( \partial_\delta I(\tilde{E}) \) exactly once and \( \partial_\delta I(\tilde{E})/\Gamma_{\tilde{E}} \) is homeomorphic to \( \Sigma_{\tilde{E}} \) for very good covers.

(iv) There exists a nonempty interior of the convex hull \( I(\tilde{E}) \) of \( \tilde{E} \) where \( \Gamma_{\tilde{E}} \) acts so that \( I(\tilde{E})/\Gamma_{\tilde{E}} \) is homeomorphic to the end orbifold times an interval.

**Proof** Assume first that \( \tilde{\partial} \subset \partial^s \). (i) Suppose that \( \tilde{E} \) is lens-shaped. We define \( S_1 \) as the set of 1-simplices with endpoints in segments in \( \bigcup S(v_{\tilde{E}}) \) and we inductively define \( S_i \) to be the set of \( i \)-simplices with faces in \( S_{i-1} \). Then

\[ I(\tilde{E}) = \bigcup_{\sigma \in S_1 \cup S_2 \cup \cdots \cup S_m} \sigma. \]

**Lemma 7.1** If \( \sigma \in S_i \) meets \( \partial_\delta I(\tilde{E}) \), then \( \sigma^\alpha \subset \partial_\delta I(\tilde{E}) \) and the vertices of \( \sigma \) are endpoints of maximal segments in \( S(v_{\tilde{E}}) \).

**Proof** Suppose that \( x \in \partial_\delta I(\tilde{E}) \) and \( x \in \sigma^\alpha \) for a simplex \( \sigma \in S_i \) for minimal \( i \). The vertices are in \( \bigcup S(v_{\tilde{E}}) \). If at least one vertex \( v_1 \) is in the interior point of a segment in \( S(v_{\tilde{E}}) \), then by taking points in the neighborhood of \( v_1 \) in \( \bigcup S(v_{\tilde{E}}) \), we can deduce that \( \sigma^\alpha \) is not in the boundary of the convex hull. Moreover, \( \sigma^\alpha \) is a subset of \( \partial_\delta I_{\tilde{E}} \) by Lemma 2.18. Hence, the vertices are endpoints of the maximal segments in \( S(v_{\tilde{E}}) \).

Suppose that a point of \( \sigma^\alpha \) is in a segment in \( S(v_{\tilde{E}}) \). Then an interior point of \( \Pi_{v_{\tilde{E}}}(\sigma) \) meets the boundary of \( \text{Cl}(R_{v_{\tilde{E}}}(\tilde{\partial})) \). By Lemma 2.18, \( \Pi_{v_{\tilde{E}}}(\sigma) \subset \text{bd}(R_{v_{\tilde{E}}}(\tilde{\partial})) \). Thus, \( \sigma \) is in a union of segments from \( v_{\tilde{E}} \) in the directions of \( \text{bd}(R_{v_{\tilde{E}}}(\tilde{\partial})) \). By Theorems 6.7 and 6.8, such segments are contained in \( \bigcup S(v_{\tilde{E}}) \). We obtain \( \sigma \subset \bigcup S(v_{\tilde{E}}) \). This is a contradiction. By (7.1), \( \sigma^\alpha \subset \partial_\delta I_{\tilde{E}} \).

**Proof (Proof of Proposition 7.1 continued)** Since any point of \( \partial_\delta I(\tilde{E}) \) is in some simplex \( \sigma, \sigma \in S_i \), we obtain that \( \partial_\delta I(\tilde{E}) \) is the union

\[ \bigcup_{\sigma \in S_1 \cup S_2 \cup \cdots \cup S_m, \sigma^\alpha \subset \partial_\delta I(\tilde{E})} \sigma^\alpha \]

by Lemma 7.1.

Suppose that \( \sigma \in S_i \) with \( \sigma^\alpha \subset \partial_\delta I(\tilde{E}) \). Then each of its vertices must be in an endpoint of a segment in \( S(v_{\tilde{E}}) \) by Lemma 7.1. By Theorems 6.7 and 6.8, the endpoints of the segments in \( S(v_{\tilde{E}}) \) are in \( \Lambda(\tilde{E}) \). Hence, \( \sigma^\alpha \) is contained in a CA-lens-shaped domain \( L \) as the vertices of \( \sigma \) is in \( \text{bd}L - \partial L = \Lambda(\tilde{E}) \) by the convexity of \( L \).

Thus, each point of \( \partial_\delta I(\tilde{E}) \) is in \( L' \subset \tilde{E} \). Hence \( \partial_\delta I(\tilde{E}) \subset \text{bd}I(\tilde{E}) \cap \tilde{\partial} \). Conversely, a point of \( \partial I(\tilde{E}) \cap \tilde{\partial} \) is an endpoint of a maximal segment in a direction of \( \tilde{E}_\tilde{E} \). By (7.1), we obtain \( \partial_\delta I(\tilde{E}) = \partial I(\tilde{E}) \cap \tilde{\partial} \).
(ii) Since \( I(\tilde{E}) \) contains the segments in \( S(v_{\tilde{E}}) \) and is convex, and so does a concave p-end neighborhood \( U \), we obtain \( \text{bd} U \subset I(\tilde{E}) \); Otherwise, let \( x \) be a point of \( \text{bd} U \cap \text{bd} I(\tilde{E}) \cap \partial \tilde{E} \) where some neighborhood in \( \text{bd} U \) is not in \( I(\tilde{E}) \). Then since \( \text{bd} U \) is a union of a strictly convex hypersurface \( \text{bd} U \cap \partial \tilde{E} \) and \( S(v_{\tilde{E}}) \), each sharply supporting hyperspace at \( x \) of the convex set \( \text{bd} U \cap \partial \tilde{E} \) meets a segment in \( S(v_{\tilde{E}}) \) in its interior; consider the lens \( L \) so that one of the boundary components is \( \text{bd} U \). The supporting hyperspace at the boundary component cannot meet the closure of \( L \) in other points by the strict convexity.

This is a contradiction since \( x \) must be then in \( I(\tilde{E})^p \). Thus, \( U \subset I(\tilde{E}) \). Thus,

\[
\mathcal{CH}(\text{Cl}(U)) \subset I(\tilde{E}).
\]

Conversely, since \( \text{Cl}(U) \supset \bigcup S(v_{\tilde{E}}) \) by Theorems 6.7 and 6.8, we obtain that

\[
\mathcal{CH}(\text{Cl}(U)) \supset I(\tilde{E}).
\]

(iii) We again use Proposition 2.3. It is sufficient to prove the result by taking a very good cover of \( \partial \tilde{E} \) and considering the corresponding end to \( \tilde{E} \). Each point of it meets a maximal segment from \( v_{\tilde{E}} \) in the end but not in \( S(v_{\tilde{E}}) \) at exactly one point since a maximal segment must leave the lens cone eventually. Thus \( \partial_S I(\tilde{E}) \) is homeomorphic to an \( (n-1) \)-cell and the result follows.

(iv) This follows from (iii) since we can use rays from \( x \) meeting \( \text{bd} I(\tilde{E}) \cap \partial \tilde{E} \) at unique points and use them as leaves of a fibration.

![Fig. 7.1 The structure of a lens-shaped p-end.](image-url)
7.2.2 Expansion of lens or horospherical p-end-neighborhoods.

**Lemma 7.2** Let \( \mathcal{O} \) have a strongly tame properly convex real projective structure \( \mu \).

- Let \( U_1 \) be a p-end neighborhood of a horospherical or a lens-shaped R-p-end \( \hat{E} \) with the p-end vertex \( v_E \); or
- Let \( U_1 \) be a lens-shaped p-end neighborhood of a T-p-end \( \hat{E} \).

Let \( \Gamma_E \) denote the p-end holonomy group corresponding to \( \hat{E} \). Then we can construct a sequence of lens-cone or lens p-end neighborhoods \( U_i, i = 1, 2, \ldots, \), satisfying \( U_i \subset U_j \subset \hat{O} \) for every pair \( i, j, i > j \) where the following hold:

- Given a compact subset of \( \hat{O} \), there exists an integer \( i_0 \) such that \( U_i \) for \( i > i_0 \) contains it.
- The Hausdorff distance between \( U_i \) and \( \hat{O} \) can be made as small as possible, i.e.,
  \[ \forall \varepsilon > 0, \exists J, J > 0, \text{ so that } d_H(U_i, \hat{O}) < \varepsilon \text{ for } i > J. \]
- There exists a sequence of convex open p-end neighborhoods \( U_i \) of \( \hat{E} \) in \( \hat{O} \) so that \( (U_i - U_j) / \Gamma_E \) for a fixed \( j \) and \( i > j \) is diffeomorphic to a product of an open interval with the end orbifold.
- We can choose \( U_i \) so that \( \text{bd}U_i \cap \hat{O} \) is smoothly embedded and strictly convex with \( \text{Cl}(\text{bd}U_i) - \hat{O} \subset \Sigma(E) \).

**Proof** Suppose that \( \hat{O} \subset \mathbb{S}^n \) first. Suppose that \( \hat{E} \) is a lens-shaped R-end first. Let \( U_1 \) be a lens-cone. Take a union of finitely many geodesic leaves \( L \) from \( v_E \) in \( \hat{O} \) of \( d_{\hat{E}} \)-length \( t \) outside the lens-cone \( U_1 \) and take the convex hull of \( U_1 \) and \( \Gamma_E(L) \) in \( \hat{O} \). Denote the result by \( \Omega \). Thus, the endpoints of \( L \) not equal to \( v_E \) are in \( \hat{O} \).

We claim that

- \( \text{bd}\Omega \cap \hat{O} \) is a connected \((n - 1)\)-cell,
- \( (\text{bd}\Omega \cap \hat{O}) / \Gamma_E \) is a compact \((n - 1)\)-orbifold diffeomorphic to \( \Sigma_E \), and
- \( \text{bd}U_1 \cap \hat{O} \) bounds a compact orbifold diffeomorphic to the product of a closed interval with \( (\text{bd}\Omega \cap \hat{O}) / \Gamma_E \):

First, each leaf of \( g(l), g \in \Gamma_E \) for \( l \) in \( L \) is so that any converging subsequence of \( \{g_l(l)\}, g_l \in \Gamma_E \), converges to a segment in \( S(v_E) \) for a sequence \( \{g_l\} \) of mutually distinct elements. This follows since a limit is a segment in \( \text{bd}\hat{O} \) with an endpoint \( v_E \) and must belong to \( S(v_E) \) by Theorems 6.7 and 6.8.

Let \( S_1 \) be the set of segments with endpoints in \( \Gamma_E(L) \cup S(v_E) \). We define inductively \( S_i \) to be the set of simplices with sides in \( S_{i-1} \). Then the convex hull of \( \Gamma_E(L) \) in \( \text{Cl}(\hat{O}) \) is a union of \( S_1 \cup \cdots \cup S_m \).

We claim that for each maximal segment \( s \) in \( \text{Cl}(\hat{O}) \) from \( v_E \) not in \( S(v_E) \), \( s^o \) meets \( \text{bd}\Omega \cap \hat{O} \) at a unique point: Suppose not. Then let \( v' \) be its other endpoint of \( s \) in \( \text{bd}\hat{O} \) with \( s^o \cap \text{bd}\Omega \cap \hat{O} = \emptyset \). Thus, \( v' \in \text{bd}\Omega \).

Now, \( v' \) is contained in the interior of a simplex \( \sigma \) in \( S_i \) for some \( i \). Since \( \sigma^o \cap \text{bd}\hat{O} \neq \emptyset, \sigma \subset \text{bd}\hat{O} \) by Lemma 2.18. Since the endpoints \( \Gamma_E(L) \) are in \( \hat{O} \), the only possibility is that the vertices of \( \sigma \) are in \( \bigcup S(v_E) \). Also, \( \sigma^o \) is transverse to radial
rays since otherwise \( v' \) is not in \( \text{bd} \tilde{\sigma} \). Thus, \( \sigma'' \) projects to an open simplex of same dimension in \( \tilde{S}_E \). Since \( U_1 \) is convex and contains \( \bigcup S(v_E) \) in its boundary, \( \sigma \) is in the lens-cone \( \text{Cl}(U_1) \). Since a lens-cone has boundary a union of a strictly convex open hypersurface \( A \) and \( \bigcup S(v_E) \), and \( \sigma'' \) cannot meet \( A \) tangentially, it follows that \( \sigma'' \) is in the interior of the lens-cone. and no interior point of \( \sigma \) is in \( \text{bd} \tilde{\sigma} \), a contradiction. Therefore, each maximal segment \( s \) from \( v_E \) meets the boundary \( \text{bd} \Omega_1 \cap \tilde{\sigma} \) exactly once.

As in Lemma 6.6, \( \text{bd} \Omega_1 \cap \tilde{\sigma} \) contains no segment ending in \( \text{bd} \tilde{\sigma} \). The strictness of convexity of \( \text{bd} \Omega_1 \cap \tilde{\sigma} \) follows by smoothing as in the proof of Proposition 6.6. By taking sufficiently many leaves for \( L \) with sufficiently large \( d_\theta \)-lengths \( t_i \), we can show that any compact subset is inside \( \Omega_i \). Choose some sequence \( \{ t_i \} \) so that \( \{ t_i \} \to \infty \) as \( i \to \infty \). Now, let \( U_i := \Omega_{t_i} \). From this, the final item follows. The first three items now follow if \( \tilde{E} \) is an R-end.

Suppose now that \( \tilde{E} \) is horospherical and \( U_1 \) is a horospherical \( p \)-end neighborhood. We can smooth the boundary to be strictly convex. \( \Gamma_{\tilde{E}} \) is in a parabolic or cusp subgroup of a conjugate of \( \text{SO}(n, 1) \) by Theorem 4.3. By taking \( L \) sufficiently densely, we can choose similarly to above a sequence \( \Omega_i \) of polyhedral convex horospherical open sets at \( v_E \) so that eventually any compact subset of \( \tilde{\sigma} \) is in it for sufficiently large \( i \). Theorem 5.6 gives us a smooth strictly convex horospherical \( p \)-end neighborhood \( U_i \).

Suppose now that \( \tilde{E} \) is totally geodesic. Now we use the dual domain \( \tilde{\sigma}^* \) and the group \( \Gamma_{\tilde{E}} \). Let \( v_E \), denote the vertex dual to the hyperspace containing \( \tilde{S}_E \). By the diffeomorphism induced by great segments with the common endpoint \( v_E \), we obtain an orbifold homeomorphism

\[
(\text{bd} \tilde{\sigma}^* - \bigcup S(v_E)) / \Gamma_{\tilde{E}}^* \cong \Sigma_{\tilde{E}} / \Gamma_{\tilde{E}}^*.
\]

a compact orbifold. Then we obtain \( U_i \) containing \( \tilde{\sigma}^* \) in \( \tilde{\mathcal{R}}_{\tilde{E}}(\tilde{S}_E) \) by taking finitely many hyperspace \( F_i \) disjoint from \( \tilde{\sigma}^* \) but meeting \( \tilde{\mathcal{R}}_{\tilde{E}}(\tilde{S}_E) \). Let \( H_i \) be the open hemisphere containing \( \tilde{\sigma}^* \) bounded by \( F_i \). Then we form \( U_1 := \bigcap_{g \in \Gamma_{\tilde{E}}} g(H_i) \). By taking more hyperspaces, we obtain a sequence

\[
U_1 \supset U_2 \supset \cdots \supset U_i \supset U_{i+1} \supset \cdots \supset \tilde{\sigma}^*
\]

so that \( \text{Cl}(U_{i+1}) \subset U_i \) and

\[
\bigcap_i \text{Cl}(U_i) = \text{Cl}(\tilde{\sigma}^*).
\]

That is, by using sufficiently many hyperspaces, we can make \( U_i \) disjoint from any compact subset disjoint from \( \text{Cl}(\tilde{\sigma}^*) \). Now taking the dual \( U_i^* \) of \( U_i \) and by equation (2.26) we obtain

\[
U_i^* \subset U_{i+1}^* \subset \cdots \subset U_i^* \subset U_{i+1}^* \subset \cdots \subset \tilde{\sigma}.
\]

Then \( U_i^* \subset \tilde{\sigma} \) is an increasing sequence eventually containing all compact subset of \( \tilde{\sigma} \) by duality from the above disjointness. This completes the proof for the first three items.
7.2.3 Shrinking of lens and horospherical p-end-neighborhoods.

We now discuss the “shrinking” of p-end-neighborhoods. These repeat some results.

Corollary 7.4 Suppose that $\mathcal{O}$ is a strongly tame properly convex real projective orbifold and let $\mathcal{O}$ be a properly convex domain in $\mathbb{S}^n$ (resp. $\mathbb{R}P^n$) covering $\mathcal{O}$. Then the following statements hold:

(i) If $\tilde{E}$ is a horospherical $R$-p-end, every $p$-end-neighborhood of $\tilde{E}$ contains a horospherical $p$-end-neighborhood.

(ii) Suppose that $\tilde{E}$ is a generalized lens-shaped or lens-shaped $R$-p-end. Let $I(\tilde{E})$ be the convex hull of $\bigcup S(v_{\tilde{E}})$, and let $V$ be a $p$-end-neighborhood of $\tilde{E}$ containing $V$. If $V' \supset I(\tilde{E})$, then $V$ contains a lens-cone $p$-end neighborhood of $\tilde{E}$, and a lens-cone contains $\tilde{E}$ properly.

(iii) If $\tilde{E}$ is a generalized lens-shaped $R$-p-end or satisfies the uniform middle eigenvalue condition, every $p$-end-neighborhood of $\tilde{E}$ contains a concave $p$-end-neighborhood.

(iv) Suppose that $\tilde{E}$ is a lens-shaped $T$-p-end or satisfies the uniform middle eigenvalue condition. Then every $p$-end-neighborhood contains a lens $p$-end-neighborhood $L$ with strictly convex boundary in $\tilde{E}$.

(v) We can choose a collection of mutually disjoint end neighborhoods for all ends that are lens-shaped $T$-end neighborhood, concave $R$-end neighborhood or a horospherical ones.

Proof Suppose that $\tilde{O} \subset \mathbb{S}^n$ first.

(i) Let $v_{\tilde{E}}$ denote the $R$-p-end vertex corresponding to $\tilde{E}$. By Theorem 4.3, we obtain a conjugate $G$ of a subgroup of a parabolic or cusp subgroup of $SO(n, 1)$ as the finite index subgroup of $h(\pi_1(\tilde{E}))$ acting on $U$, a $p$-end-neighborhood of $\tilde{E}$. We can choose a $G$-invariant ellipsoid of $d$-diameter $\leq \epsilon$ for any $\epsilon > 0$ in $U$ containing $v_{\tilde{E}}$.

(ii) This follows from Proposition 6.6 since the convex hull of $\bigcup S(v_{\tilde{E}})$ contains a generalized lens with the right properties.

(iii) This was proved in Proposition 6.9.

(iv) The existence of a lens-shaped $p$-end neighborhood of $\tilde{E}$ follows from Theorem 5.4.

(v) We choose a mutually disjoint end neighborhoods for all ends. Then we choose the desired ones by the above.
7.2.4 The mc-p-end neighborhoods

The mc-p-end neighborhood will be useful in other papers.

Definition 7.4 Let $\tilde{E}$ be a lens-shaped R-end of a strongly tame convex projective orbifold $\mathcal{O}$ with the universal cover $\tilde{\mathcal{O}} \subset \mathbb{S}^n$ (resp. $\mathbb{R}P^n$). Let $\mathcal{CH}(\Lambda(\tilde{E}))$ denote the convex hull of $\Lambda(\tilde{E})$. Let $U'$ be any p-end neighborhood of $\tilde{E}$ containing $\mathcal{CH}(\Lambda(\tilde{E})) \cap \tilde{\mathcal{O}}$. We define a maximal concave p-end neighborhood or mc-p-end-neighborhood $U$ to be one of the components of $U' - \mathcal{CH}(\Lambda(\tilde{E}))$ containing a p-end neighborhood of $\tilde{E}$. The closed maximal concave p-end neighborhood is $\text{Cl}(U) \cap \tilde{\mathcal{O}}$. An $\varepsilon$-$d_{\tilde{\mathcal{O}}}$-neighborhood $U''$ of a maximal concave p-end neighborhood is called an $\varepsilon$-mc-p-end-neighborhood.

In fact, these are independent of choices of $U'$. Note that a maximal concave p-end neighborhood $U$ is uniquely determined since so is $\Lambda(\tilde{E})$.

Each radial segment $s$ in $\tilde{\mathcal{O}}$ from $v_E$ meets $\text{bd}U \cap \tilde{\mathcal{O}}$ at a unique point since the point $s \cap \text{bd}U$ is in an $n - 1$-dimensional ball $D = P \cap U$ for a hyperspace $P$ sharply supporting $\mathcal{CH}(\Lambda(\tilde{E}))$ with $\partial \text{Cl}(D) \subset \bigcup S(v_E)$.

Lemma 7.3 Let $D$ be an $i$-dimensional totally geodesic compact convex domain, $i \geq 1$. Let $\tilde{E}$ be a generalized lens-shaped R-p-end with the p-end vertex $v_E$. Suppose $\partial D \subset \bigcup S(v_E)$. Then $D' \subset V$ for a maximal concave p-end neighborhood $V$, and for sufficiently small $\varepsilon > 0$, an $\varepsilon$-$d_{\tilde{\mathcal{O}}}$-neighborhood of $D'$ is contained in $V'$ for any $\varepsilon$-mc-p-end neighborhood $V'$.

Proof Suppose that $\tilde{\mathcal{O}} \subset \mathbb{S}^n$ first. Assume that $U$ is a generalized lens-cone of $v_E$. Then $\Lambda(\tilde{E})$ is the set of endpoints of segments in $S(v_E)$ which are not $v_E$ by Theorems 6.7 and 6.8. Let $P$ be the subspace spanned by $D \cup \{v_E\}$. Since

$$\partial D, \Lambda(\tilde{E}) \cap P \subset \bigcup S(v_E) \cap P,$$

and $\partial D \cap P$ is closer than $\Lambda(\tilde{E}) \cap P$ from $v_E$, it follows that $P \cap \text{Cl}(U) - D$ has a component $C_1$ containing $v_E$ and $\text{Cl}(P \cap \text{Cl}(U) - C_1)$ contains $\Lambda(\tilde{E}) \cap P$. Hence

$$\text{Cl}(P \cap \text{Cl}(U) - C_1) \supset \mathcal{CH}(\Lambda(\tilde{E})) \cap P$$

by the convexity of $\text{Cl}(P \cap \text{Cl}(U) - C_1)$. Since $\mathcal{CH}(\Lambda(\tilde{E})) \cap P$ is a convex set in $P$, we have only two possibilities:

- $D$ is disjoint from $\mathcal{CH}(\Lambda(\tilde{E}))$ or
- $D$ contains $\mathcal{CH}(\Lambda(\tilde{E})) \cap P$.

Let $V$ be an mc-p-end neighborhood of $U$. Since $\text{Cl}(V)$ includes the closure of the component of $U - \mathcal{CH}(\Lambda(\tilde{E}))$ with a limit point $v_E$, it follows that $\text{Cl}(V)$ includes $D$.

Since $D$ is in $\text{Cl}(V)$, the boundary $\text{bd}V' \cap \tilde{\mathcal{O}}$ of the $\varepsilon$-mc-p-end neighborhood $V'$ do not meet $D$. Hence $D' \subset V'$. [\$\mathbb{S}^n\$T]
The following gives us a characterization of \( \varepsilon \)-mc-p-end neighborhoods of \( \tilde{E} \).

**Corollary 7.5** Let \( \mathcal{O} \) be a properly convex real projective orbifold with generalized lens-shaped or horospherical \( \mathcal{A} \)- or \( \mathcal{T} \)-ends and satisfies (IE). Let \( \tilde{E} \) be a generalized lens-shaped \( \mathcal{R} \)-end. Then the following statements hold:

(i) A concave \( p \)-end neighborhood of \( \tilde{E} \) is always a subset of an mc-p-end-neighborhood of the same \( R \)-p-end.

(ii) The closed mc-p-end-neighborhood of \( \tilde{E} \) is the closure in \( \mathcal{O} \) of a union of all concave end neighborhoods of \( \tilde{E} \).

(iii) The mc-p-end-neighborhood \( V \) of \( \tilde{E} \) is a proper \( p \)-end neighborhood, and covers an end-neighborhood with compact boundary in \( \mathcal{O} \).

(iv) An \( \varepsilon \)-mc-p-end-neighborhood of \( \tilde{E} \) for sufficiently small \( \varepsilon > 0 \) is a proper \( p \)-end neighborhood.

(v) For sufficiently small \( \varepsilon > 0 \), the image end-neighborhoods in \( \mathcal{O} \) of \( \varepsilon \)-mc-p-end neighborhoods of \( R \)-ends are mutually disjoint or identical.

**Proof** Suppose first that \( \partial \tilde{E} \subset S^n \). (i) Since the limit set \( \Lambda (\tilde{E}) \) is in any generalized CA-lens \( L \) by Corollary 7.2, a generalized lens-cone \( p \)-end neighborhood \( U \) of \( \tilde{E} \) contains \( \mathcal{C} \mathcal{H}(\Lambda) \cap \partial \tilde{E} \). Hence, a concave end neighborhood is contained in an mc-p-end-neighborhood.

(ii) Let \( V \) be an mc-p-end neighborhood of \( \tilde{E} \). Then define \( S \) to be the subspace of endpoints in \( \text{Cl}(\partial \tilde{E}) \) of maximal segments in \( V \) from \( \gamma_{\tilde{E}} \) in directions of \( \Sigma_{\tilde{E}} \). Then \( S \) is homeomorphic to \( \Sigma_{\tilde{E}} \) by the map induced by radial segments as shown in the paragraph before. Thus, \( S/\pi_1 (\tilde{E}) \) is a compact set since \( S \) is contractible and \( \Sigma_{\tilde{E}}/\pi_1 (\tilde{E}) \) is a \( K(\pi_1 (\tilde{E})) \)-space up to taking a torsion-free finite-index subgroup by Theorem 2.3 (Selberg’s lemma). We can approximate \( S \) in the \( d_{\tilde{E}} \)-sense by smooth convex boundary component \( S_\varepsilon \) outwards of a generalized CA-lens by Theorem 5.5 since \( \tilde{E} \) satisfies the uniform middle-eigenvalue condition. A component \( U - S_\varepsilon \) is a concave \( p \)-end neighborhood. (ii) follows from this.

(iii) Since a concave \( p \)-end neighborhood is a proper \( p \)-end neighborhood by Theorems 6.7(iv) and 6.8, and \( V \) is a union of concave \( p \)-end neighborhoods, we obtain

\[
g(V) \cap V = \emptyset \text{ or } g(V) = V \text{ for } g \in \pi_1 (\mathcal{O}) \text{ by (ii).}
\]

Suppose that \( g(\text{Cl}(V) \cap \partial \tilde{E}) \cap \text{Cl}(V) \neq \emptyset \). Then \( g(V) = V \) and \( g \in \pi_1 (\mathcal{O}) \): Otherwise, \( g(V) \cap V = \emptyset \), and \( g(\text{Cl}(V) \cap \partial \tilde{E}) \) meets \( \text{Cl}(V) \) in a totally geodesic hypersurface \( S \) equal to \( \mathcal{C} \mathcal{H}(\Lambda)^o \) by the concavity of \( V \). Furthermore, for every \( g \in \pi_1 (\mathcal{O}) \), \( g(S) = S \), since \( S \) is a maximal totally geodesic hypersurface in \( \partial \tilde{E} \). Hence, \( g(V) \cup S \cup V = \partial \tilde{E} \) since these are subsets of a properly convex domain \( \partial \tilde{E} \), the boundary of \( V \) and \( g(V) \) are in \( S \), and \( S \) is now in the interior of \( \partial \tilde{E} \). Then \( \pi_1 (\mathcal{O}) \) acts on \( S \), and \( S/G \) is homotopy equivalent to \( \partial \tilde{E} / G \) for a finite-index torsion-free subgroup \( G \) of \( \pi_1 (\mathcal{O}) \) by Theorem 2.3 (Selberg’s lemma). This contradicts the condition (IE).

Hence, only possibility is that \( \text{Cl}(V) \cap \partial \tilde{E} = V \cup S \) for a hypersurface \( S \) and

\[
g(V \cup S) \cap V = \emptyset \text{ or } g(V \cup S) = V \cup S \text{ for } g \in \pi_1 (\mathcal{O}).
\]
Now suppose that $S \cap \partial \tilde{O} \neq \emptyset$. Let $S'$ be a maximal totally geodesic domain in $\text{Cl}(V)$ containing $S$. Then $S' \subset \partial \tilde{O}$ by convexity and Lemma 2.18, meaning that $S' = S \subset \partial \tilde{O}$. In this case, $\tilde{O}$ is a cone over $S$ and the end vertex $v_\tilde{E}$ of $\tilde{E}$. For each $g \in \pi_1(\tilde{O})$,
\[ g(V) \cap V \neq \emptyset \] implies $g(V) = V$
since $g(v_\tilde{E})$ is on $\text{Cl}(S)$. Thus, $\pi_1(\tilde{O}) = \pi_1(\tilde{E})$. This contradicts the condition (IE) of $\pi_1(\tilde{E})$.

We showed that $\text{Cl}(V) \cap \tilde{O} = V \cup S$ for a hypersurface $S$ and covers a submanifold in $\tilde{O}$ which is a closure of an end-neighborhood covered by $V$. Thus, an mc-p-end-neighborhood $\text{Cl}(V) \cap \tilde{O}$ is a proper end neighborhood of $\tilde{E}$ with compact embedded boundary $S/\pi_1(\tilde{E})$.

(iv) Obviously, we can choose positive $\varepsilon$ so that an $\varepsilon$-mc-p-end-neighborhood is a proper p-end neighborhood also.

(v) For two mc-p-end neighborhoods $U$ and $V$ for different R-p-ends, we have $U \cap V = \emptyset$ by reasoning as in (iii) replacing $g(V)$ with $U$: We showed that $\text{Cl}(V) \cap \tilde{O}$ for an mc-p-end-neighborhood $V$ covers an end neighborhood in $\tilde{O}$.

Suppose that $U$ is another mc-p-end neighborhood different from $V$. We claim that $\text{Cl}(U) \cap \text{Cl}(V) \cap \tilde{O} = \emptyset$: Suppose not. $g(\text{Cl}(V))$ for $g \notin \Gamma_\tilde{E}$ must be a subset of $U$ since otherwise we have a situation of (iii) for $V$ and $g(V)$. Since the preimage of the end neighborhoods are disjoint, $g(V)$ is a p-end neighborhood of the same end as $U$. Since both are $\varepsilon$-mc-p-end-neighborhood which are canonically defined, we obtain $U = g(V)$. This was ruled out in (iii).

7.3 The strong irreducibility of the real projective orbifolds.

The main purpose of this section is to prove Theorem 1.2, the strong irreducibility result. But we will discuss the convex hull of the ends first. We show that the closure of convex hulls of p-end neighborhoods are disjoint in $\partial \tilde{O}$. The infinity of the number of these will show the strong irreducibility.

For the following, we need a stronger condition of lens-shaped ends, and not just the generalized lens-shaped property, to obtain the disjointedness of the closures of p-end neighborhoods.

**Corollary 7.6** Let $\tilde{O}$ be a strongly tame properly convex real projective orbifold with generalized lens-shaped or horospherical $R$- or $T$-ends and satisfy (IE) and (NA). Let $\mathcal{U}$ be the collection of the components of the inverse image in $\tilde{O}$ of the union of disjoint collection of end neighborhoods of $\tilde{O}$. Now replace each R-p-end neighborhood of collection $\mathcal{U}$ by a concave p-end neighborhood by Corollary 7.4 (iii). Then the following statements hold:

(i) Given horospherical, concave, or one-sided lens p-end-neighborhoods $U_1$ and $U_2$ contained in $\bigcup \mathcal{U}$, we have $U_1 \cap U_2 = \emptyset$ or $U_1 = U_2$.

(ii) Let $U_1$ and $U_2$ be in $\mathcal{U}$. Then $\text{Cl}(U_1) \cap \text{Cl}(U_2) \cap \partial \tilde{O} = \emptyset$ or $U_1 = U_2$ holds.
Proof Suppose first that $Ω ⊆ \mathbb{R}^n$. (i) Suppose that $U_1$ and $U_2$ are p-end neighborhoods of R-$p$-ends $\tilde{E}_1$ and $\tilde{E}_2$ respectively. Let $U_1'$ be the interior of the associated generalized lens-cone of $U_1$ in $\text{Cl}(\tilde{Ω})$ and $U_2'$ be that of $U_2$. Let $U_i''$ be the concave p-end-neighborhood of $U_i'$ for $i = 1, 2$ by Corollary 7.4 (iii) that cover respective end neighborhoods in $\tilde{Ω}$.

Since the neighborhoods in $\mathcal{U}$ are mutually disjoint,

- $\text{Cl}(U_1'') \cap \text{Cl}(U_2'') \cap \tilde{Ω} = \emptyset$ or
- $U_1'' = U_2''$.

(ii) Assume that $U_i'' \in \mathcal{U}$, $i = 1, 2$, and $U_i'' \neq U_j''$. Suppose that the closures of $U_i''$ and $U_j''$ intersect in $bd\tilde{Ω}$. Suppose that they are both R-$p$-end neighborhoods. Then the respective closures of convex hulls $I_1$ and $I_2$ as obtained by Proposition 7.1 intersect as well. Take a point $z \in \text{Cl}(U_1'') \cap \text{Cl}(U_2'') \cap bd\tilde{Ω}$. Suppose that we choose $p_i = v_{E_i}, i = 1, 2$, for each p-end $\tilde{E}_i$.

Suppose that $p_{1,2}' \subseteq bd\tilde{Ω}$. Then there exists a segment in $bd\tilde{Ω}$ from $v_{E_1}$ to $v_{E_2}$. By Theorems 6.7 and 6.8, these vertices are in the closures of p-end neighborhoods of one other. Hence $U_1'' \cap U_2'' \neq \emptyset$. This is a contradiction.

Hence, $p_{1,2}' \subseteq \tilde{Ω}$ holds. Then $p_{1,2}' \subseteq S(v_{E_1})$ and $p_{1,2}' \subseteq S(v_{E_2})$ and these segments are maximal since otherwise $U_1'' \cap U_2'' \neq \emptyset$. The segments intersect transversely at $z$ since otherwise we violated the maximality in Theorems 6.7 and 6.8. We obtain a triangle $\triangle(p_1p_2z)$ in $\text{Cl}(\tilde{Ω})$ with vertices $p_1, p_2, z$.

Let $P$ be the 2-dimensional plane containing $p_1, p_2, z$. Consider a disk $P \cap \text{Cl}(\tilde{Ω})$ containing $p_1, p_2, z$ in the boundary. However, the disk has an angle $< \pi$ at $z$ since $\text{Cl}(\tilde{Ω})$ is properly convex. We will denote the disk by $\triangle(p_1p_2z)$ and $p_1, p_2, z$ are considered as vertices. Since $p_{1,2}' \subseteq \tilde{Ω}$, we obtain that $\triangle(p_1p_2z)' \subseteq \tilde{Ω}$ by Lemma 2.18.

Let $I_i$ be the convex hull of $\bigcup S(v_{E_i})$ for each $i$, $i = 1, 2$, and define $\partial S I_i$ as in Section 7.2.1. Then $p_1$ is not in $I_i$ since $p_1 \notin \text{Cl}(U_2'')$: Suppose not. $p_1$ is in the interior of a simplex $\sigma$ in $\text{Cl}(\tilde{Ω})$ with all vertices in $\text{Cl}(U_2'')$. Since $p_1 \in bd\tilde{Ω}$, Lemma 2.18 implies that $\sigma \subseteq bd\tilde{Ω}$. Then for each edge $s$ of $\sigma$, we have $s \subseteq \bigcup S(v_{E_i})$ by Theorems 6.7 and 6.8. Then by induction on dimension, we obtain $\sigma \subseteq \bigcup S(v_{E_i})$ and so $p_1 \notin \bigcup S(v_{E_i})$, a contradiction. Conversely, $p_2$ is not in $I_1$.

Since $p_{1,2}'$ ends in $p_1$ and $p_2$, where $p_1 \notin I_2$ and $p_2 \notin I_1$, it follows that $p_{1,2}'$ and hence $\triangle(p_1p_2z)'$ are both not subsets of $I_1$ and $I_2$. Since $bdI_i \cap \tilde{Ω} \subseteq \partial S I_i, i = 1, 2$, by (7.1), $\triangle(p_1p_2z)'$ must meet both $\partial S I_1$ and $\partial S I_2$.

We define a convex curve $\alpha_i := \triangle(p_1p_2z) \cap \partial S I_i$ with an endpoint $z$ for each $i$, $i = 1, 2$. Let $\tilde{E}_i$ denote the R-$p$-end corresponding to $p_i$. Since $\alpha_i$ maps to a geodesic in $\nu_{p_i}(\tilde{Ω})$, there exists a foliation $\mathcal{F}$ of $\triangle(p_1p_2z)$ by maximal segments from the vertex $p_1$. There is a natural parametrization of the space of leaves by $\mathbb{R}$ using the Hilbert metric of the interval. We parameterize $\alpha_i$ by these parameters as $\alpha_i$ intersected with a leaf is a unique point. They give the geodesic length parameterizations under the Hilbert metric of $\nu_{p_i}(\tilde{Ω})$ for $i = 1, 2$.

We now show that an infinite-order element of $\pi_1(\tilde{E}_1)$ is the same as one in $\pi_1(\tilde{E}_2)$: By convexity of $I_1$ and $\alpha_2$,

- either $\alpha_2$ goes into $I_1$ and not leave again or
7.3 The strong irreducibility of the real projective orbifolds.

Fig. 7.2 The diagram of the quadrilateral bounded by $\beta(t), \beta(t+i), \alpha_1, \alpha_2$.

- $\alpha_2$ is disjoint from $I_1$.

Suppose that $\alpha_2$ goes into $I_1$ and not leave it again. Since $\hat{E}_2$ is an R-p-end of lens-type and not just generalized lens-type, $\partial_S I_2/\pi_1(\hat{E}_2)$ is a compact orbifold in $\mathcal{C}$ by Proposition 7.1(i). There is a sequence $t_i$ so that

- the sequence of projections of points of form $\alpha_2(t_i)$ converges to a point of $\partial_S I_2/\pi_1(\hat{E}_2)$, and
- the sequence of projections of points of form $\alpha_2(t_i)$ to $R_{p_1}(\hat{E}_1)$ converges to a point of $R_{p_1}(\hat{E}_1)/\pi_1(\hat{E}_1)$.

Hence, by taking a short path between $\alpha_2(t_i)$'s, there exists an essential closed curve $c_2$ in $\partial_S I_2/\pi_1(\hat{E}_2)$ homotopic to an element of $\pi_1(\hat{E}_1)$ since $c_2$ is in a lens-cone end neighborhood of the end corresponding to $\hat{E}_1$ under the covering map $\hat{O} \to \mathcal{C}$. This contradicts (NA). (The element can be assumed to be infinite order since we can take a finite cover of $\mathcal{C}$ so that $\pi_1(\mathcal{C})$ is torsion-free by Theorem 2.3 (Selberg's lemma).)

Suppose now that $\alpha_2$ is disjoint from $I_1$. We may assume that $\alpha_2$ is disjoint from $I_2$ without loss of generality by switching 1 and 2. Then $\alpha_2'$ and $\alpha_2''$ now must be both in $T' \subset \hat{C}$. Then $\alpha_1$ and $\alpha_2$ have the same endpoint $z$ and by the convexity of $\alpha_2$. We parameterize $\alpha_2$ so that $\alpha_2(t)$ and $\alpha_2(t)$ are on a line segment containing $\alpha_1(t)\alpha_2(t)$ in the triangle with endpoints in $\mathbb{P}^1$ and $\mathbb{P}^2$.

We obtain $d_{\mathcal{C}}(\alpha_2(t), \alpha_2(t)) \leq C$ for a uniform constant $C$: We define $\beta(t) := \alpha_2(t)\alpha_2(t)$. Let $\gamma(t)$ denote the full extension of $\beta(t)$ in $\Delta(p_1p_2z)$. One can project to the space of lines through $z$, a one-dimensional projective space. Then the image of $\beta(t)$ are so that the image of $\beta(t')$ is contained in that of $\beta(t)$ if $t < t'$. Also, the image of $\gamma(t)$ contains that of $\gamma(t')$ if $t < t'$. Thus, the convexity of the boundary $\partial_S I_1$ and $\partial_S I_2$ shows that that the Hilbert-metric length of the segment $\beta(t)$ is bounded above by the uniform constant.
We have a sequence \( \{t_i\} \rightarrow \infty \) so that
\[
\{ p_\varnothing \circ \alpha_2(t_i) \} \rightarrow x, \{ d_\varnothing (p_\varnothing \circ \alpha_2(t_{i+1}), p_\varnothing \circ \alpha_2(t_i)) \} \rightarrow 0, x \in \varnothing.
\]

So we obtain a closed curve \( c_{2,i} \) in \( \varnothing \) obtained by taking a short path jumping between the two points. By taking a subsequence, the sequence of the images of form \( \{ \beta(t_i) \} \) in \( \varnothing \) geometrically converges to a segment of Hilbert-length \( \leq C \). As \( i \rightarrow \infty \), we have
\[
\{ d_\varnothing (p_\varnothing \circ \alpha_2(t_i), p_\varnothing \circ \alpha_2(t_{i+1})) \} \rightarrow 0
\]
by extracting a subsequence. There exists a closed curve \( c_{1,i} \) in \( \varnothing \) again by taking a short jumping path. We see that \( c_{1,i} \) and \( c_{2,i} \) are homotopic in \( \varnothing \) since we can use the image of the disk in the quadrilateral bounded by
\[
\overline{\alpha_2(t_i) \alpha_2(t_{i+1})}, \overline{\alpha_1(t_i) \alpha_1(t_{i+1})}, \overline{\beta(t_i), \beta(t_{i+1})}
\]
and the connecting thin strips between the images of \( \beta_i \) and \( \beta_{i+1} \) in \( \varnothing \). This again contradicts (NA).

Now, consider when \( U_1 \) is a one-sided lens-neighborhood of a \( \mathbb{T} \)-p-end and let \( U_2 \) be a concave \( \mathbb{R} \)-p-end neighborhood of an \( \mathbb{R} \)-p-end of \( \varnothing \). Let \( z \) be the intersection point in \( \text{Cl}(U_1) \cap \text{Cl}(U_2) \). We can use the same reasoning as above by choosing any \( p_1 \) in \( \tilde{\Sigma}_{E_1} \) so that \( \overrightarrow{p_1z} \) passes the interior of \( \tilde{E}_1 \). Let \( p_2 \) be the \( \mathbb{R} \)-p-end vertex of \( U_2 \). Now we obtain the triangle with vertices \( p_1, p_2, \) and \( z \) as above. Then the arguments are analogous and obtain infinite order elements in \( \pi_1(\tilde{E}_1) \cap \pi_1(\tilde{E}_2) \).

Next, consider when \( U_1 \) and \( U_2 \) are one-sided lens-neighborhoods of \( \mathbb{T} \)-p-ends respectively. Using the intersection point \( z \) of \( \text{Cl}(U_1) \cap \text{Cl}(U_2) \cap \varnothing \) and we choose \( p_i \) in \( \partial \tilde{E}_i \) so that \( \overrightarrow{p_i z} \) passes the interior of \( \tilde{\Sigma}_{E_i} \) for \( i = 1, 2 \). Again, we obtain a triangle with vertex \( p_1, p_2, \) and \( z \), and find a contradiction as above.

We finally consider when \( U \) is a horospherical \( \mathbb{R} \)-p-end. Since \( \text{Cl}(U) \cap \partial \varnothing \) is a unique point, (v) of Theorem 4.2 implies the result.

We modify Theorem 6.8 by replacing some conditions. In particular, we don’t assume that \( h(\pi_1(\varnothing)) \) is strongly irreducible.

**Lemma 7.4** Let \( \varnothing \) be a strongly tame properly convex real projective orbifold and satisfy (IE) and (NA). Let \( \tilde{E} \) be a virtually factorizable \( \mathbb{R} \)-p-end of \( \varnothing \) of generalized lens-shaped. Then

- there exists a totally geodesic hyperspace \( P \) on which \( h(\pi_1(\tilde{E})) \) acts,
- \( D := P \cap \varnothing \) is a properly convex domain,
- \( D' \subset \varnothing \), and
- \( D'/\pi_1(\tilde{E}) \) is a compact orbifold.

**Proof** Assume first that \( \varnothing \subset \mathbb{S}^n \). The proof of Theorem 6.8 shows that

- either \( \text{Cl}(\varnothing) \) is a strict join \( \{ v_{E} \} \ast D' \) for a properly convex domain \( D \) in a hyper-space, or
- the conclusion of Theorem 6.8 holds.
In both cases, \( \pi_1(\tilde{E}) \) acts on a totally geodesic convex compact domain \( D \) of codimension 1. \( D \) is the intersection \( P_\tilde{E} \cap \text{Cl}(\tilde{\Theta}) \) for a \( \pi_1(\tilde{E}) \)-invariant subspace \( P_\tilde{E} \). Suppose that \( D^o \) is not a subset of \( \tilde{\Theta} \). Then by Lemma 2.18, \( D \subset \text{bd} \tilde{\Theta} \).

In the former case, \( \text{Cl}(\tilde{\Theta}) \) is the join \( \nu_\tilde{E} \ast D \). For each \( g \in \pi_1(\tilde{E}) \) satisfying \( g(\nu_\tilde{E}) \neq \nu_\tilde{E} \), we have \( g(D) \neq D \) since \( g(\nu_\tilde{E}) \ast g(D) = \{\nu_\tilde{E}\} \ast D \). The intersection \( g(D) \cap D \) is a proper compact convex subset of \( D \) and \( g(D) \). Moreover,

\[
\text{Cl}(\tilde{\Theta}) = \{\nu_\tilde{E}\} \ast g(\{\nu_\tilde{E}\}) \ast (D \cap g(D)).
\]

We can continue as many times as there is a mutually distinct collection of vertices of form \( g(\nu_\tilde{E}) \). Since this process must stop, we have a contradiction since by Condition (IE), there are infinitely many distinct end vertices of form \( g(\nu_\tilde{E}) \) for \( g \in \pi_1(\tilde{\Theta}) \).

Now, we go to the alternative \( D^o \subset \tilde{\Theta} \) where \( D^o / \Gamma_{\tilde{E}} \) is projectively diffeomorphic to \( \tilde{\Sigma}_E / \Gamma_{\tilde{E}} \).

**Proof (Proof of Theorem 1.2)** It is sufficient to prove for \( \tilde{\Theta} \subset S^o \). Let \( h : \pi_1(\tilde{\Theta}) \to \text{SL}_{+}(n+1, \mathbb{R}) \) be the holonomy homomorphism. Suppose that \( h(\pi_1(\tilde{\Theta})) \) is virtually reducible. Then we can choose a finite cover \( \tilde{\Theta} \) so that \( h(\pi_1(\tilde{\Theta})) \) is reducible since it is sufficient to prove for the finite index groups.

We may assume that \( \pi_1(\tilde{\Theta}) \) is torsion-free by taking a finite cover by Theorem 2.3.

We denote \( \tilde{\Theta} \) by \( \tilde{\Theta} \) for simplicity. Let \( S \) denote a proper subspace where \( \pi_1(\tilde{\Theta}) \) acts on. Suppose that \( S \) meets \( \tilde{\Theta} \). Then \( \pi_1(\tilde{E}) \) acts on a properly convex open domain \( S \cap \tilde{\Theta} \) for each \( p \)-end \( E \). Then \( S \cap \tilde{\Theta} \) for any \( p \)-end neighborhood gives a submanifold of a closed end orbifold homotopy equivalent to it. Thus, \( (S \cap \tilde{\Theta}) / \pi_1(\tilde{E}) \) is a compact orbifold homotopy equivalent to one of the end orbifolds. However, \( S \cap \tilde{\Theta} \) is \( \pi_1(\tilde{E}) \)-invariant and cocompact for any other \( p \)-end \( E \). Hence, each \( p \)-end fundamental group \( \pi_1(\tilde{E}) \) is virtually identical to any other \( p \)-end fundamental group. This contradicts (NA). Therefore,

\[
K := S \cap \text{Cl}(\tilde{\Theta}) \subset \text{bd} \tilde{\Theta},
\]

where \( g(K) = K \) for every \( g \in h(\pi_1(\tilde{\Theta})) \).

We divide into steps:

(A) First, we show \( K \neq \emptyset \).

(B) We show \( K = D_j \) or \( K = \{\nu_\tilde{E}\} \ast D_j \) for some properly convex domain \( D_j \subset \text{bd} \tilde{\Theta} \cap \text{Cl}(U) \) for a \( p \)-end neighborhood \( U \) of \( \tilde{E} \).

(C) Finally \( g(D_j) = D_j \) for \( g \in \Gamma \) and we use Corollary 7.6 to obtain a contradiction.

(A) We show that \( K := \text{Cl}(\tilde{\Theta}) \cap S \neq \emptyset \): Let \( \tilde{E} \) be a \( p \)-end. If \( \tilde{E} \) is horospherical, \( \pi(\tilde{E}) \) acts on a great sphere \( \hat{S} \) tangent to an end vertex. Since \( S \) is \( \pi \)-invariant, \( S \) has to be a subspace in \( \hat{S} \) containing the end vertex by Theorem 4.2(iii). This implies that every horospherical \( p \)-end vertex is in \( S \). Let \( p \) be one. Since there is no nontrivial segment in \( \text{bd} \tilde{\Theta} \) containing \( p \) by Theorem 4.2(v), \( p \) equals \( S \cap \text{Cl}(\tilde{\Theta}) \). Hence, \( p \) is \( \Gamma \)-invariant and \( \Gamma = \Gamma_{\tilde{E}} \). This contradicts the condition (IE).
Suppose that $\hat{E}$ is a generalized lens-shaped R-p-end. Then by the existence of attracting subspaces of some elements of $\Gamma_{\hat{E}}$, we have

- either $S$ passes the end vertex $v_\hat{E}$ or
- there exists a subspace $S'$ containing $S$ and $v_\hat{E}$ that is $\Gamma_{\hat{E}}$-invariant.

In the first case, we have $S \cap Cl(\hat{O}) \neq \emptyset$, and we are done for the step (A).

In the second case, $S'$ corresponds to a subspace in $S_{\hat{E}}^{n-1}$ and $S$ is a hyperspace of dimension $\leq n - 1$ disjoint from $v_\hat{E}$. Thus, $\hat{E}$ is a virtually factorizable R-p-end. By Theorem 2.7, we obtain some attracting fixed points in the limit sets of $\pi_1(\hat{E})$. If $S'$ is a proper subspace, then $\hat{E}$ is factorizable, and $S'$ contains the attracting fixed set of some positive bi-semi-proximal $g$, $g \in \Gamma_{\hat{E}}$. The uniform middle eigenvalue condition shows that positive bi-semi-proximal $g$ has attracting fixed sets in $Cl(L)$. Since $g$ acts on $S$, we obtain $S \cap Cl(L) \neq \emptyset$ by the uniform middle eigenvalue condition.

If $S'$ is not a proper subspace, then $g$ acts on $S$, and $S$ contains the attracting fixed set of $g$ by the uniform middle eigenvalue condition. Thus, $S \cap Cl(L) \neq \emptyset$.

If $\hat{E}$ is a lens-shaped T-p-end, we can apply a similar argument using the attracting fixed sets. Therefore, $S \cap Cl(\hat{O})$ is a subset $K$ of $bd\hat{O}$ of $\dim K \geq 0$ and is not empty. In fact, we showed that the closure of each p-end neighborhood meets $K$.

(B) By taking a dual orbifold if necessary, we assume without loss of generality that there exists a generalized lens-shaped R-p-end $\hat{E}$ with a radial p-end vertex $v_\hat{E}$.

As above in (A), suppose that $v_\hat{E} \in \accentset{\circ}{K}$. There exists $g \in \pi_1(\hat{E})$, so that

$$g(v_\hat{E}) \neq v_\hat{E}, \quad \text{and} \quad g(v_\hat{E}) \in K \subset bd\hat{O}$$

since $g$ acts on $K$. The point $g(v_\hat{E})$ is outside the closure of the concave p-end neighborhood of $\hat{E}$ by Corollary 7.6. Since $K$ is connected, $K$ meets $Cl(L)$ for the CA-lens or generalized CA-lens $L$ of $\hat{E}$.

If $v_\hat{E} \notin K$, then again $K \cap Cl(L) \neq \emptyset$ as in (A) using attracting fixed sets of some elements of $\pi_1(\hat{E})$. Hence, we conclude $K \cap Cl(L) \neq \emptyset$ for a generalized CA-lens $L$ of $\hat{E}$.

Let $\Sigma'_E$ denote $D^\alpha$ from Lemma 7.4. Since $K \subset bd\hat{O}$, it follows that $K$ cannot contain $\Sigma'_E$. Thus, $K \cap Cl(\Sigma'_E)$ is a proper subspace of $Cl(\Sigma'_E)$, and $\hat{E}$ must be a virtually factorizable end.

By Lemma 7.4, there exists a totally geodesic domain $\Sigma'_E$ in the CA-lens. A p-end neighborhood of $v_\hat{E}$ equals $U_{v_\hat{E}} := (\{v_\hat{E}\} \ast \Sigma'_E)^\circ$. Since $\pi_1(\hat{E})$ acts reducibly,

$$Cl(\Sigma'_E) = D_1 \ast \cdots \ast D_m,$$

where $K \cap Cl(U_{v_\hat{E}})$ contains a join $D_J := \ast_{i \in J} D_i$ for a proper subcollection $J$ of $\{1, \ldots, m\}$. Moreover, $K \cap Cl(\Sigma'_E) = D_J$.

Since $g(U_{v_\hat{E}})$ is a p-end neighborhood of $g(v_\hat{E})$, we obtain $g(U_{v_\hat{E}}) = U_{g(v_\hat{E})}$. Since $g(K) = K$ for $g \in \Gamma$, we obtain that

$$K \cap g(Cl(\Sigma'_E)) = g(D_J).$$

Lemma 7.4 implies that
Hence, \( v \) since these are components of \( \tilde{J} \) and the fact that \( \text{bd}U_{\tilde{g}} \cap \tilde{\Theta} \) and \( \text{bd}U_{g}(v) \cap \tilde{\Theta} \) are totally geodesic domains.

Let \( \tilde{\lambda}_J(g) \) denote the \((\dim D_J + 1)\)-th root of the norm of the determinant of the submatrix of \( g \) associated with \( D_J \) for the unit norm matrix of \( g \). There exists a sequence of virtually central diagonalizable elements \( \gamma \in \pi_1(\tilde{\Theta}) \) by Proposition 4.4 of [21] so that

\[
\{ \gamma \} \cap D_J \to \{ \gamma \} \cap D_J' \to I \text{ for the complement } J' := \{1, 2, \ldots, m\} - J,
\]

for the complement \( J' := \{1, 2, \ldots, m\} - J \). Since the lens-shaped ends satisfy the uniform middle eigenvalue condition for \( \Gamma_{\tilde{E}} \) implies that one of the following holds:

\[
K = D_J, \quad K = \{ v_{\tilde{E}} \} * D_J \text{ or } K = \{ v_{\tilde{E}} \} * D_J \cup \{ v_{\tilde{E}} \} * D_J
\]

by the invariance of \( K \) under \( \gamma^{-1} \) and the fact that \( K \cap \text{Cl}(\Sigma_{\tilde{E}}) = D_J \). Since \( K \subset \text{Cl}(\tilde{\Theta}) \), the third case is not possible by the proper convexity of \( \text{Cl}(\tilde{\Theta}) \). We obtain

\[
K = D_J \quad \text{or} \quad K = \{ v_{\tilde{E}} \} * D_J.
\]

(C) We will explore the two cases of (7.5). Assume the second case. Let \( g \) be an arbitrary element of \( \pi_1(\tilde{\Theta}) - \pi_1(\tilde{\Theta}) \). Since \( D_J \subset K \), we obtain \( g(D_J) \subset K \). Recall that \( U_{v_{\tilde{E}}} \cup \text{S}(v_{\tilde{E}})^{\circ} \) is a neighborhood of points of \( \text{S}(v_{\tilde{E}})^{\circ} \) in \( \text{Cl}(\tilde{\Theta}) \). Thus, \( g(U_{v_{\tilde{E}}} \cup \text{S}(v_{\tilde{E}})^{\circ}) \) is a neighborhood of points of \( g(\text{S}(v_{\tilde{E}})^{\circ}) \).

Recall that \( D_J' \) is in the closure of \( U_{v_{\tilde{E}}} \). If \( D_J' \) meets

\[
g(\{ v_{\tilde{E}} \} * D_J - D_J') \subset g(U_{v_{\tilde{E}}} \cup \text{S}(v_{\tilde{E}})^{\circ}) \supset g(\text{S}(v_{\tilde{E}})^{\circ}),
\]

then

\[
U_{\tilde{E}} \cap g(U_{\tilde{E}}) \neq \emptyset, \quad \text{and } \text{S}(v_{\tilde{E}})^{\circ} \cap g(\text{S}(v_{\tilde{E}})^{\circ}) \neq \emptyset
\]

since these are components of \( \tilde{\Theta} \) with some totally geodesic hyperspaces removed. Hence, \( v_{\tilde{E}} = g(v_{\tilde{E}}) \) by Theorems 6.7 and 6.8. Finally, we obtain \( D_J = g(D_J) \) since

\[
K = \{ v_{\tilde{E}} \} * D_J = g(\{ v_{\tilde{E}} \}) * g(D_J).
\]
If $D_J$ is disjoint from $g(\{v_\tilde{E}\} \ast D_J - D_J)$, then $g(D_J) \subset D_J$ since $K = \{v_\tilde{E}\} \ast D_J$ and $g(K) = K$. Since $D_J$ and $g(D_J)$ are intersections of a hyperspace with $\partial \tilde{\mathcal{O}}$, we obtain $g(D_J) = D_J$.

Both cases of (7.5) imply that $g(D_J) = D_J$ for $g \in \pi_1(\mathcal{O})$. This implies $g(D_J) = D_J$ for $g \in \pi_1(\mathcal{O})$. Since $v_\tilde{E}$ and $g(v_\tilde{E})$ are not equal for $g \in \pi_1(\mathcal{O}) - \pi_1(\tilde{E})$, we obtain

$$\text{Cl}(U_1) \cap g(\text{Cl}(U_1)) \neq \emptyset.$$ 

Corollary 7.6 gives us a contradiction. Therefore, we deduced that the $h(\pi_1(\mathcal{O}))$-invariant subspace $S$ does not exist.

Since parabolic subgroups of $\text{PGL}(n+1, \mathbb{R})$ or $\text{SL}_\pm(n+1, \mathbb{R})$ are reducible, we are done.

7.3.1 Equivalence of lens-ends and generalized lens-ends for strict SPC-orbifolds

**Corollary 7.7** Suppose that $\mathcal{O}$ is a strongly tame strictly SPC-orbifold with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and satisfying the conditions (IE) and (NA). Then $\mathcal{O}$ satisfies the triangle condition and every generalized lens-shaped $\mathcal{R}$-ends are lens-shaped $\mathcal{R}$-ends.

**Proof** Assume first $\tilde{\mathcal{O}} \subset \mathbb{S}^n$. Let $\tilde{E}$ be a generalized lens-shaped p-end neighborhood of $\tilde{\mathcal{O}}$. Let $L$ be the generalized CA-lens so that the interior $U$ of $\{v_\tilde{E}\} \ast L$ is a lens p-end neighborhood. Then $U - L$ is a concave p-end neighborhood. Recall the triangle condition of Definition 6.7. Let $T$ be a triangle with

$$\partial T \subset \partial \tilde{\mathcal{O}}, T^0 \subset \tilde{\mathcal{O}} \text{ and } \partial T \cap \text{Cl}(U) \neq \emptyset$$

for an R-p-end neighborhood $U$. By the strict convexity $\tilde{\mathcal{O}}$, each edge of $T$ has to be inside a set of form $\text{Cl}(V) \setminus \partial \tilde{\mathcal{O}}$ for a p-end neighborhood $V$. Corollary 7.6 implies that the edges are all in $\text{Cl}(U) \setminus \partial \tilde{\mathcal{O}}$ for a single R-p-end neighborhood $U$. Hence, the triangle condition is satisfied. By Theorem 6.6, $\tilde{E}$ is a lens-shaped p-end. [S^nT]
Chapter 8
The convex but nonproperly convex and non-complete-affine radial ends

In previous chapters, we classified properly convex or complete radial ends under suitable conditions. In this chapter, we will study radial ends that are convex but not properly convex nor complete affine. The main techniques are the theory of Fried and Goldman on affine manifolds, and a generalization of the work on Riemannian foliations by Molino, Carrière, and so on. We will show that these are quasi-joins of horospheres and totally geodesic radial ends under transverse weak middle eigenvalue conditions. These are suitable deformations of joins of horospheres and totally geodesic radial ends. Since this is the most technical chapter, we will give outlines at some places in addition to the main outline in Section 8.1.3.

8.1 Introduction

8.1.1 General setting

In this chapter, we will work with $\mathbb{S}^n$ and $\mathbb{SL}_+ (n + 1, \mathbb{R})$ with only a few exceptions since the purpose is to classify some objects modulo projective automorphisms. However, the corresponding results for $\mathbb{RP}^n$ can be obtained easily by results in Section 2.1.7 and then projecting back to $\mathbb{RP}^n$. Let $E$ be a $p$-end of a convex real projective orbifold $O$ with end orbifold $\tilde{\Sigma}_E$ and its universal cover $\tilde{\Sigma}_E$ and the $p$-end vertex $v_{\tilde{E}}$. Suppose that $\tilde{\Sigma}_E$ is convex but not properly convex and not complete affine. The closure $\text{Cl}(\tilde{\Sigma}_E)$ contains a great $(i_0 - 1)$-dimensional sphere $S^{i_0 - 1}$, and the convex open domain $\tilde{\Sigma}_E$ is foliated by $i_0$-dimensional hemispheres with this boundary $S^{i_0 - 1}$ by Proposition 2.5. (These follow from Section 1.4 of [40]. See also [76].)

The space of $i_0$-dimensional hemispheres in $\mathbb{S}^{n-1}_{\tilde{E}}$ with boundary $S^{i_0 - 1}$ forms a projective sphere $\mathbb{S}^{n-i_0-1}_{\tilde{E}}$: This follows since a complementary subspace $S$ isomorphic to $\mathbb{S}^{n-i_0-1}$ parameterize the space by the intersection points with $S$. The fibration with fibers open hemispheres of dimension $i_0$ with boundary $S^{i_0 - 1}$.
8 The NPNC-ends

\[ \Pi_E : \mathbb{S}^{n-1}_{v_E} - \mathbb{S}^0_{v_E} \longrightarrow \mathbb{S}^{n-i_0-1} \]  \hspace{1cm} (8.1)

\[ \uparrow \hspace{1cm} \uparrow \]
\[ \Sigma_E \longrightarrow K^\alpha \]

gives us an image of \( \Sigma_E \) that is the interior \( K^\alpha \) of a properly convex compact set \( K \).

Let \( \mathbb{S}^n_0 \) be a great \( i_0 \)-dimensional sphere in \( \mathbb{S}^n \) containing \( v_E \) corresponding to the directions of \( \mathbb{S}^n_0^{-1} \) from \( v_E \). The space of \((i_0 + 1)\)-dimensional hemispheres in \( \mathbb{S}^n \) with boundary \( \mathbb{S}^n_0 \) again has the structure of the projective sphere \( \mathbb{S}^{n-i_0-1} \), identifiable with the above one.

Each \( i_0 \)-dimensional hemisphere \( H^i \) in \( \mathbb{S}^n_0^{-1} \) with \( \text{bd}H^i = \mathbb{S}^n_0 \) corresponds to an \((i_0 + 1)\)-dimensional hemisphere \( H^{i+1} \) in \( \mathbb{S}^n \) with common boundary \( \mathbb{S}^n_0 \) that contains \( v_E \).

There is also a fibration with fibers open hemispheres of dimension \( i_0 + 1 \) and boundary \( \mathbb{S}^n_0 \):

\[ \Pi_K : \mathbb{S}^n - \mathbb{S}^n_0 \longrightarrow \mathbb{S}^{n-i_0-1} \]  \hspace{1cm} (8.2)

\[ \uparrow \hspace{1cm} \uparrow \]
\[ U \longrightarrow K^\alpha \]

since \( \mathbb{S}^n_0 \) corresponds to \( \mathbb{S}^n_0 \) in the projection \( \mathbb{S}^n \setminus \{v_E, v_E^-\} \rightarrow \mathbb{S}^{n-1} \).

Let \( \text{SL}_{\pm}(n + 1, \mathbb{R})_{\mathbb{S}^n_0, v_E} \) denote the subgroup of \( \text{Aut}(\mathbb{S}^n) \) acting on \( \mathbb{S}^n_0 \) and \( v_E \).

The projection \( \Pi_K \) induces a homomorphism

\[ \Pi_K^* : \text{SL}_{\pm}(n + 1, \mathbb{R})_{\mathbb{S}^n_0, v_E} \longrightarrow \text{SL}_{\pm}(n - i_0, \mathbb{R}). \]

Suppose that \( \mathbb{S}^n_0 \) is \( h(\pi_1(\hat{E})) \)-invariant. We let \( N \) be the subgroup of \( h(\pi_1(\hat{E})) \) of elements inducing trivial actions on \( \mathbb{S}^{n-i_0-1} \). The above exact sequence

\[ 1 \rightarrow N \rightarrow h(\pi_1(\hat{E})) \xrightarrow{\Pi_K^*} N_K \rightarrow 1 \]  \hspace{1cm} (8.3)

is so that the kernel normal subgroup \( N \) acts trivially on \( \mathbb{S}^{n-i_0-1} \) but acts on each hemisphere with boundary equal to \( \mathbb{S}^n_0 \) and \( N_K \) acts faithfully by the action induced from \( \Pi_K^* \).

Since \( K \) is a properly convex domain, \( K^\alpha \) admits a Hilbert metric \( d_k \) and \( \text{Aut}(K) \) is a subgroup of isometries of \( K^\alpha \). Here \( N_K \) is a subgroup of the group \( \text{Aut}(K) \) of the group of projective automorphisms of \( K \), and \( N_K \) is called the properly convex quotient of \( h(\pi_1(\hat{E})) \) or \( \Gamma_E \).

We showed:

**Theorem 8.1** Let \( \Sigma_E \) be the end orbifold of an NPNC \( R \)-p-end \( E \) of a strongly tame properly convex \( n \)-orbifold \( \mathcal{O} \) with radial or totally geodesic ends. Let \( \hat{\mathcal{O}} \) be the universal cover in \( \mathbb{S}^n \). We consider the induced action of \( h(\pi_1(\hat{E})) \) on \( \text{Aut}(\mathbb{S}^n/E) \) for the corresponding end vertex \( v_E \). Then the following hold:
• $\Sigma_{\mathcal{E}}$ is foliated by complete affine subspaces of dimension $i_0$, $i_0 > 0$. Let $K$ be the properly convex compact domain of dimension $n - i_0 - 1$ whose interior is the space of complete affine subspaces of dimension $i_0$.

• $h(\pi_1(\mathcal{E}))$ fixes the great sphere $S^{i_0-1}_{\mathcal{E}}$ of dimension $i_0 - 1$ in $S^{n-1}_{\mathcal{E}}$.

• There exists an exact sequence

$$1 \rightarrow N \rightarrow \pi_1(\mathcal{E}) \xrightarrow{\Pi_K^*} N_K \rightarrow 1$$

where $N$ acts trivially on quotient great sphere $S^{n-i_0-1}$ and $N_K$ acts faithfully on a properly convex domain $K'$ in $S^{n-i_0-1}$ isometrically with respect to the Hilbert metric $d_K$.

\[ \square \]

8.1.2 Main results.

We begin with a definition of quasi-joined R-ends.

**Definition 8.1** We have the following:

• Let $\mathring{K}$ be a compact properly convex subset of dimension
Let $v$ be a point in $S_i^\infty$. A group $G$ acts on $\hat{K}$, $S_i^\infty$, and $v$ and on an open set $U$ containing $v$ in the boundary.

- $U/G$ is required to be diffeomorphic to a compact orbifold times an interval.
- There is a fibration $\Pi_K : S^n - S_{i+1}^\infty \rightarrow S^{n-i-1}$ with fibers equal to open $(i+1)$-hemispheres with boundary $S_i^\infty$.

- The set of fibering open $(i+1)$-hemispheres $H$ so that $H \cap U$ is a nonempty open set is projected to a convex open set in $S^{n-i-1}$ onto the interior of $\{x\} \ast \Pi_K(\hat{K})$ for a point $x$ not in $\Pi_K(\hat{K})$.
- For each fibering open hemisphere $H$, $H \cap U$ is an open set bounded by an ellipsoid containing $v$ unless $H \cap U$ is empty.
- $\text{Cl}(H_\ast) \cap \text{Cl}(U) = \{v\}$ for $H_\ast$ the fibering open $(i+1)$-hemisphere over $x$.

If an $R$-end $E$ of a real projective orbifold has an end neighborhood projectively diffeomorphic to $U/G$ with the induced radial foliation corresponding to $v$, then $E$ is called a quasi-joined end (of a totally geodesic $R$-end and a horospherical end with respect to $v$) and a corresponding $R$-p-end is said to be a quasi-joined $R$-p-end also. Also, any $R$-end with an end-neighborhood covered by an end-neighborhood of a quasi-joined $R$-end is called by the same name. In these cases, the end holonomy group is a quasi-joined end group (of a totally geodesic $R$-end and a horospherical end with respect to $v$).

We will see the example later. In this chapter, we will characterize this and other types of ends named NPNC-ends. See Proposition 8.3 and Remark 8.4 for detailed understanding of quasi-joined ends.

Let $\hat{E}$ be an NPNC-end. Recall from Chapter 4 that the universal cover $\hat{\Sigma}_E$ of the end orbifold $\Sigma_E$ is foliated by complete affine $i_0$-dimensional totally geodesic leaves for $i_0 > 1$. The end fundamental group $\pi_1(\hat{E})$ acts on a properly convex domain $K$ that is the space of $i_0$-dimensional totally geodesic hemispheres foliating $\hat{\Sigma}_E$.

The main result of this chapter is the following. We need the proper convexity of $\hat{\Sigma}_E$. The following theorem shows that given that $\hat{\Sigma}_E$ is convex but not properly convex, the transverse weak middle eigenvalue condition implies the end is quasi-joined type. For example, this implies that the holonomy group decomposes into semi-simple part and horospherical part. (see (8.55)). This type is much easier to understand. Section 8.3 gives some detailed discussions.

**Theorem 8.2** Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends. Assume that the holonomy group of $\mathcal{O}$ is strongly irreducible. Let $\hat{E}$ be an NPNC $R$-p-end with the end orbifold $\Sigma_{\hat{E}}$. The universal cover $\hat{\Sigma}_E$ is foliated by $i_0$-dimensional totally geodesic hemispheres. The leaf space naturally identifies with the interior of a compact convex set $K$. Suppose that the following hold:

- the end fundamental group satisfies the property $(NS)$ or $\dim K^o = 0, 1$ for the leaf space $K^o$ of $\hat{E}$.
the p-end holonomy group $h(\pi_1(\tilde{E}))$ virtually satisfies the transverse weak middle-eigenvalue condition with respect to a p-end vertex $v_\tilde{E}$.

Then $\tilde{E}$ is a quasi-joined type R-p-end for $v_\tilde{E}$.

See Definition 8.3 for the transverse weak middle-eigenvalue condition for NPNC-ends. Without this condition, we doubt that we can obtain this type of results. However, it is open to investigations. In this case, $\tilde{E}$ does not satisfy the uniform middle-eigenvalue condition as stated in Chapter 4 for properly convex ends.

We again remark that Cooper and Leitner have classified the ends when the end fundamental group is abelian. (See Leitner [129], [128] and [130].) Also, Ballas [6] and [5] has found some examples of quasi-joined ends when the upper-left parts are diagonal groups. Our quasi-joined ends are also classified by [7] when the holonomy group is nilpotent.

Recall the dual orbifold $O^*$ given a properly convex real projective orbifold $O$. (See Section 6.5.2.) The set of ends of $O$ is in a one-to-one correspondence with the set of ends of $O^*$. We show that a dual of a quasi-joined NPNC R-p-end is a quasi-joined NPNC R-p-end.

**Corollary 8.1** Let $O$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends. Assume that the holonomy group of $O$ is strongly irreducible. Let $\tilde{E}$ be a quasi-joined NPNC R-p-end for an end $E$ of $O$ virtually satisfying the transverse weak middle-eigenvalue condition with respect to the p-end vertex. Suppose that the end fundamental group satisfies the property (NS) or $\dim K^0 = 1$ for the leaf space $K^0$ of $\tilde{E}$.

Let $O^*$ denote the dual real projective orbifold of $O$. Let $\tilde{E}^*$ be a p-end corresponding to a dual end of $E$. Then $\tilde{E}^*$ has a p-end neighborhood of a quasi-joined type R-p-end for the universal cover of $O^*$ for a unique choice of a p-end vertex.

In short, we are saying that $\tilde{E}^*$ can be considered a quasi-joined type R-p-end by choosing its p-end vertex well. However, this does involve artificially introducing a radial foliation structure in an end neighborhood. We mention that the choice of the p-end vertex is uniquely determined for $\tilde{E}^*$ to be quasi-joined.

**8.1.3 Outline**

In Section 8.2, we discuss the R-ends that are NPNC. We introduce the transverse weak middle eigenvalue condition. We will explain the main eigenvalue estimates following from the transverse weak middle eigenvalue condition for NPNC-ends. Then we will explain our plan to prove Theorem 8.2.

In Section 8.3, we introduce the example of the joining of horospherical and totally geodesic R-ends. We will now study a bit more general situation introducing Hypothesis 8.3.1. We will try to obtain the splitting under some hypothesis. We will outline the subsection there. By computations involving the normalization conditions, we show that the above exact sequence is virtually split under the condition
(NS), and we can surprisingly show that the R-p-ends are of strictly joined or quasi-joined types. Then we show using the irreducibility of the holonomy group of $\pi_1(\mathcal{O})$ that they can only be of quasi-joined type. We divide the tasks:

- In Section 8.3.2, we introduce Hypothesis 8.3.1 under which we work. We show that $K$ has to be a cone, and the conjugation action on $\mathcal{N}$ has to be scalar orthogonal type changes. Finally, we show the splitting of the NPNC-ends. We will outline more completely in there.

- In Section 8.3.3, we introduce Hypothesis 8.3.2, requiring more than Hypothesis 8.3.1. We will prove Proposition 8.3 that the ends are quasi-joins under the hypothesis.

As a final part of this section in Section 8.3.4, we discuss the case when $N_K$ is discrete. We prove Theorem 8.2 for this case by showing that the above two hypotheses are satisfied.

In Section 8.4, we discuss when $N_K$ is not discrete. There is a foliation by complete affine subspaces as above. We use some estimates on eigenvalues to show that each leaf is of polynomial growth. The leaf closures are suborbifolds $V_l$ by the theory of Carrièere [38] and Molino [142] on Riemannian foliations. They form the fibration with compact fibers. $\pi_1(V_l)$ is solvable using the work of Carrièere [38]. One can then take the syndetic closure to obtain a bigger group that acts transitively on each leaf following Fried and Goldman [86]. We find a unipotent cusp group acting on each leaf transitively normalized by $\Gamma$. Then we show that the end also splits virtually using the theory of Section 8.3. This proves Theorem 8.2 for this case.

In Section 8.5, we prove Corollary 8.1 showing that the duals of NPNC-ends are NPNS-ends, and in Section 8.5.2, we classify complete ends that are not cusps. This was needed in the proof of Theorem 4.1.

In Section 8.6, we will discuss some miscellaneous results. In Section 8.6.3, we discuss a counterexample to Theorem 8.3 when the condition (NS) is dropped.

### 8.1.3.1 The plan of the proof of Theorem 8.2

We will show that our NPNC-ends are quasi-joined type ones; i.e., we prove Theorem 8.2 by proving for discrete $N_K$ in Section 8.3.4 in Section 8.3 and proving for nondiscrete $N_K$ in Section 8.4.3 in Section 8.4.

- Assume Hypotheses 8.3.1.
  - We show that $\Gamma$ acts as $\mathbb{R}_+$ times an orthogonal group on a Lie group $\mathcal{N}$ as realized as an real unipotent abelian group $\mathbb{R}_0$. See Lemmas 8.3. This is done by computations and coordinate change arguments and the distal group theory of Fried [84].
  - We show that $K$ is a cone in Lemma 8.5.
  - We refine the matrix forms in Lemma 8.6 when $\mu_g = 1$. Here the matrices are in almost desired forms.
8.2 The transverse weak middle eigenvalue conditions for NPNC ends

Proposition 8.2 shows the splitting of the representation of $\Gamma_E$. One uses the transverse weak middle eigenvalue condition to realize the compact $(n - i_0 - 1)$-dimensional totally geodesic domain independent of $S^0_{i_0}$ where $\Gamma_E$ acts on.

Now we can assume Hypothesis 8.3.2 additionally. In Section 8.3.3, we discuss joins and quasi-joins. The idea is to show that the join cannot occur by Propositions 2.17 and 2.18.

This will settle the cases of discrete $N_K$ in Theorem 8.3 in Section 8.3.4.

In Section 8.4, we will settle for the cases of non-discrete $N_K$. See above for the outline of this section.

We remark that we can often take a finite index subgroup of $\Gamma_E$ during our proofs since Definition 8.4 is a definition given up to finite index subgroups.

Remark 8.1 The main result of this chapter Theorem 8.2 and Corollary 8.1 are stated without references to $S^n$ or $\mathbb{RP}^n$. We will work in the space $S^n$ only. Often the result for $S^n$ implies the result for $\mathbb{RP}^n$. In this case, we only prove for $S^n$. In other cases, we can easily modify the $S^n$-version proof to one for the $\mathbb{RP}^n$-version proof.

8.2 The transverse weak middle eigenvalue conditions for NPNC ends

We will now study the ends where the transverse real projective structures are not properly convex but not projectively diffeomorphic to a complete affine subspace. Let $E$ be an R-p-end of $O$, and let $U$ be the corresponding p-end-neighborhood in $O$ with the p-end vertex $v_E$. Let $\tilde{\Sigma}_E$ denote the universal cover of the p-end orbifold $\Sigma_E$ as a domain in $S^n_{v_E}$. In Section 8.1.1, we will discuss the general setting that the NPNC-ends satisfy. In Section 8.1.3.1, we will give a plan to prove Theorem 8.2. We accomplished this proof in Sections 8.3 and 8.4.

We denote by $\tilde{\mathcal{F}}_E$ the foliation on $\tilde{\Sigma}_E$ or the corresponding one in $\Sigma_E$. We denote by $\mathcal{F}_E$ the foliation on $\tilde{\Sigma}_E$.

8.2.0.1 The main eigenvalue estimations

We denote by $\Gamma_E$ the p-end holonomy group acting on $U$ fixing $v_E$. Denote the induced foliations on $\Sigma_E$ and $\tilde{\Sigma}_E$ by $\mathcal{F}_E$. We recall

$$\text{length}_K(g) := \inf \{ d_K(x, g(x)) | x \in K^o \}, g \in \Gamma_E.$$ 

We recall Definition 2.5. Let $V_{i_0}^{i_0 + 1}$ denote the subspace of $\mathbb{R}^{i_0 + 1}$ corresponding to $S^0_{i_0}$. By invariance of $S^0_{i_0}$, if

$$\mathcal{R}_\mu(g) \cap V_{i_0}^{i_0 + 1} \neq \{0\}, \mu > 0,$$
then $\mathcal{B}_\mu(g) \cap V_{\infty}^{i+1}$ always contains a $C$-eigenvector of $g$.

**Definition 8.2** Let $\Sigma_E$ be the end orbifold of a nonproperly convex $R$-p-end $E$ of a strongly tame properly convex $n$-orbifold $\vartheta$. Let $\Gamma_E$ be the $p$-end holonomy group.

- Let $\lambda_{\max}^{Tr}(g)$ denote the largest norm of the eigenvalue of $g \in \Gamma_E$ affiliated with $v \neq 0$, $(v) \in \mathbb{S}^n - S^n_0$, i.e.,
  $$\lambda_{\max}^{Tr}(g) := \max \{ \mu \mid \exists v \in \mathcal{B}_\mu(g) - V^{i+1}_{\infty} \},$$
  which is the maximal norm of transverse eigenvalues.
- Also, let $\lambda_{\min}^{Tr}(g)$ denote the smallest one affiliated with a nonzero vector $v$, $(v) \in \mathbb{S}^n - S^n_0$, i.e.,
  $$\lambda_{\min}^{Tr}(g) := \min \{ \mu \mid \exists v \in \mathcal{B}_\mu(g) - V^{i+1}_{\infty} \},$$
  which is the minimal norm of transverse eigenvalues.
- Let $\lambda_{\max}^{\nu}(g)$ be the largest of the norms of the eigenvalues of $g$ with $C$-eigenvectors of form $v$, $(v) \in S^n_0$ and $\lambda_{\min}^{\nu}(g)$ the smallest such one.

We will assume that the $p$-end holonomy group $h(\pi_1(E))$ satisfies the transverse weak middle eigenvalue condition for NPNC-ends:

**Definition 8.3** Let $\lambda_{\nu_E}(g)$ denote the eigenvalue of $g$ at $v_E$. The *transverse weak middle eigenvalue condition* with respect to $v_E$ or the $R$-p-end structure is that each element $g$ of $h(\pi_1(E))$ satisfies

$$\lambda_{\max}^{Tr}(g) \geq \lambda_{\nu_E}(g). \quad (8.4)$$

Theorem A.1 somewhat justifies our approach. We do believe that the weak middle eigenvalue conditions implies the transverse ones at least for relevant cases.

The following proposition is very important in this chapter and shows that $\lambda_{\max}^{Tr}(g)$ and $\lambda_{\min}^{Tr}(g)$ are true largest and smallest norms of the eigenvalues of $g$.

**Proposition 8.1** Let $\Sigma_E$ be the end orbifold of an NPNC $R$-p-end $E$ of a strongly tame properly convex $n$-orbifold $\vartheta$ with radial or totally geodesic ends. Suppose that $\vartheta$ in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$) covers $\vartheta$ as a universal cover. Let $\Gamma_E$ be the $p$-end holonomy group satisfying the transverse weak middle eigenvalue condition for the $R$-p-end structure. Let $g \in \Gamma_E$. Then the following hold:

$$\lambda_{\max}^{Tr}(g) \geq \lambda_{\max}^{\nu}(g) \geq \lambda_{\nu_E}(g) \geq \lambda_{\min}^{\nu}(g) \geq \lambda_{\min}^{Tr}(g),$$

$$\lambda_1(g) = \lambda_{\max}^{Tr}(g), \text{ and } \lambda_{n+1} = \lambda_{\min}^{Tr}(g). \quad (8.5)$$

**Proof** We may assume that $g$ is of infinite order. Suppose that $\lambda_{\max}^{\nu}(g) > \lambda_{\max}^{Tr}(g)$. We have $\lambda_{\max}^{\nu}(g) > \lambda_{\nu_E}(g)$ by the transverse weak uniform middle eigenvalue condition.

Now, $\lambda_{\max}^{Tr}(g) < \lambda_{\max}^{\nu}(g)$ implies that
8.2 The transverse weak middle eigenvalue conditions for NPNC ends

\[ R_{\lambda_{\text{max}}(g)}^{\pm_0} := \bigoplus_{|\mu|=\lambda_{\text{max}}(g)} R_{\mu}(g) \]

is a subspace of \( V_{\omega_0}^{i_0+1} \) and corresponds to a great sphere \( S^j, S^j \subset S_{\omega_0}^{i_0} \). Hence, a great sphere \( S^j, j \geq 0, \) in \( S_{\omega_0}^{i_0} \) is disjoint from \{v_E, v_{E-}\}. Since \( v_E \in S_{\omega_0}^{i_0} \) is not contained in \( S^j \), we obtain \( j + 1 \leq i_0 \).

A vector space \( V_1 \) corresponds \( \bigoplus_{|\mu|<\lambda_{\text{max}}(g)} R_{\mu}(g) \) where \( g \) has strictly smaller norm eigenvalues and is complementary to \( R_{\lambda_{\text{max}}(g)}^{\pm_0} \). Let \( C_1 = S(V_1). \) The great sphere \( C_1 \) is disjoint from \( S^j \) but \( C_1 \) contains \( v_E \). Moreover, \( C_1 \) is of complementary dimension to \( S^j \), i.e., \( \dim C_1 = n - j - 1 \).

Since \( C_1 \) is complementary to \( S^j \subset S_{\omega_0}^{i_0}, \) \( C_1 \) contains a complementary subspace \( C'_1 \) to \( S_{\omega_0}^{i_0} \) of dimension \( n - i_0 - 1 \) in \( S^n \). Considering the sphere \( S_{\omega_E}^{n-1} \) at \( v_E \), it follows that \( C'_1 \) goes to an \( n - i_0 - 1 \)-dimensional subspace \( C''_1 \) in \( S_{\omega_E}^{n-1} \) disjoint from \( \partial l \) for any complete affine leaf \( l \). Each complete affine leaf \( l \) of \( \Sigma_{\omega_E} \) has the dimension \( i_0 \) and meets \( C''_1 \) in \( S_{\omega_E}^{n-1} \) by the dimension consideration.

Fig. 8.2 The figure for the proof of Proposition 8.1.

Hence, a small ball \( B' \) in \( U \) meets \( C_1 \) in its interior.

For any \( \langle \nu \rangle \in B', \nu \in \mathbb{R}^{n+1}, \nu = v_1 + v_2 \) where \( \langle v_1 \rangle \in C_1 \) and \( \langle v_2 \rangle \in S^j \).

We obtain \( g^k(\langle \nu \rangle) = \left( g^k(v_1) + g^k(v_2) \right) \), \( (8.6) \)

where we used \( g \) to represent the linear transformation of determinant \( \pm 1 \) as well (See Remark 1.1.) By Proposition 2.8, the action of \( g^k \) as \( k \to \infty \) makes the former vectors very small compared to the latter ones, i.e.,

\[ \left\{ \left\| g^k(v_1) \right\| / \left\| g^k(v_2) \right\| \right\} \to 0 \text{ as } k \to \infty. \]
Hence, \( \{ g^k((v)) \} \) converges to the limit of \( \{ g^k((v_2)) \} \) if it exists.

Now choose \( (w) \) in \( C_1 \cap B' \) and \( v, (v) \in S' \). We let \( w_1 = (w + \varepsilon v) \) and \( w_2 = (w - \varepsilon v) \) in \( B' \) for small \( \varepsilon > 0 \). Choose a subsequence \( \{ k_i \} \) so that \( \{ g^{k_i}(w_1) \} \) converges to a point of \( S'' \). The above estimation shows that \( \{ g^{k_i}(w_1) \} \) and \( \{ g^{k_i}(w_2) \} \) converge to an antipodal pair of points in \( \Cl(U) \) respectively. This contradicts the proper convexity of \( U \) as \( g^k(B') \subset U \) and the geometric limit is in \( \Cl(U) \).

Also the consideration of \( g^{-1} \) completes the inequality, and the second equation follows from the first one.

[\$m'T$]

### 8.3 The general theory

#### 8.3.1 Examples

First, we give some examples.

##### 8.3.1.1 The standard quadric in \( \mathbb{R}^{i_0+1} \) and the group acting on it.

Let us consider an affine subspace \( A^{i_0+1} \) of \( S^{i_0+1} \) with coordinates \( x_0, x_1, \ldots, x_{i_0+1} \) given by \( x_0 > 0 \). The standard quadric in \( A^{i_0+1} \) is given by

\[
x_{i_0+1} = \frac{1}{2}(x_1^2 + \cdots + x_{i_0}^2).
\]

Clearly the group of the orthogonal maps \( O(i_0) \) acting on the planes given by \( x_{i_0+1} = \text{const} \) acts on the quadric also. Also, the group of matrices of the form

\[
\begin{pmatrix}
1 & 0 & 0 \\
v^T & I_{i_0} & 0 \\
\frac{||v||^2}{2} & v & 1
\end{pmatrix}
\]

acts on the quadric.

The group of affine transformations that acts on the quadric is exactly the Lie group generated by the above cusp group and \( O(i_0) \). The action is transitive and each of the stabilizer is a conjugate of \( O(i_0) \) by elements of the above cusp group.

The proof of this fact is simply that such a group of affine transformations is conjugate into a parabolic isometry group in the \( i_0 + 1 \)-dimensional complete hyperbolic space \( H \) where the ideal fixed point is identified with \( (0, \ldots, 0, 1) \in S^{i_0+1} \) and with \( \text{bd}H \) tangent to \( \text{bd}A^{i_0} \).

The group of projective automorphisms of the following forms is a unipotent cusp group.
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\[ \mathcal{N}(v) := \begin{pmatrix} 1 & 0 & 0 \\ v^T & I_{i_0-1} & 0^T \\ \frac{v^2}{2} & v & 1 \end{pmatrix} \text{ for } v \in \mathbb{R}^{i_0}. \quad (8.7) \]

(see [74] for details.) We can make each translation direction of generators of \( \mathcal{N} \) in \( \Sigma'_p \) to be one of the standard vector. Therefore, we can find a coordinate system of \( V^{i_0+2} \) so that the generators are of \((i_0 + 2) \times (i_0 + 2)\)-matrix forms

\[ \mathcal{N}_j(s) := \begin{pmatrix} 1 & 0 & 0 \\ s e_j^T & I_{i_0} & 0 \\ \frac{1}{2} s e_j & 1 \end{pmatrix} \quad (8.8) \]

where \( s \in \mathbb{R} \) and \((e_j)_k = \delta_{jk} \) a row \( i \)-vector for \( j = 1, \ldots, i_0 \). That is,

\[ \mathcal{N}(v) = \mathcal{N}_1(v_1) \cdots \mathcal{N}_{i_0}(v_{i_0}). \]

### 8.3.1 Examples of generalized joined ends

We first begin with examples. In the following, we will explain the generalized joined type end.

**Example 8.1** Let us consider two ends \( E_1 \), a totally geodesic R-end, with the p-end-neighborhood \( U_1 \) in the universal cover of a real projective orbifold \( \Sigma'_1 \) in \( S^{n-i_0-1} \) of dimension \( n - i_0 - 1 \) with the p-end vertex \( v_1 \), and \( E_2 \) the p-end-neighborhood \( U_2 \), a horospherical type one, in the universal cover of a real projective orbifold \( \Sigma'_2 \) of dimension \( i_0 + 1 \) with the p-end vertex \( v_2 \).

- Let \( \Gamma_1 \) denote the projective automorphism group in \( \text{Aut}(S^{n-i_0-1}) \) acting on \( U_1 \) corresponding to \( E_1 \). We assume that \( \Gamma_1 \) acts on a great sphere \( S^{n-i_0-2} \subset S^{n-i_0-1} \) disjoint from \( v_1 \). There exists a properly convex open domain \( K \) in \( S^{n-i_0-2} \) where \( \Gamma_1 \) acts cocompactly but not necessarily freely. We change \( U_1 \) to be the interior of the join of \( K \) and \( v_1 \).
- Let \( \Gamma_2 \) denote the one in \( \text{Aut}(S^{i_0+1}) \) acting on \( U_2 \) as a subgroup of a cusp group.
- We embed \( S^{n-i_0-1} \) and \( S^{i_0+1} \) in \( S^n \) meeting transversely at \( v = v_1 = v_2 \).
- We embed \( U_2 \) in \( S^{i_0+1} \) and \( \Gamma_2 \) in \( \text{Aut}(S^n) \) fixing each point of \( S^{n-i_0-1} \).
- We can embed \( U_1 \) in \( S^{n-i_0-1} \) and \( \Gamma_1 \) in \( \text{Aut}(S^n) \) acting on the embedded \( U_1 \) so that \( \Gamma_1 \) acts on \( S^{i_0-1} \) normalizing \( \Gamma_2 \) and acting on \( U_1 \). One can find some such embeddings by finding an arbitrary homomorphism \( \rho : \Gamma_1 \to N(\Gamma_2) \) for a normalizer \( N(\Gamma_2) \) of \( \Gamma_2 \) in \( \text{Aut}(S^n) \).

We find an element \( \zeta \in \text{Aut}(S^n) \) fixing each point of \( S^{n-i_0-2} \) and acting on \( S^{i_0+1} \) as a unipotent element normalizing \( \Gamma_2 \) so that the corresponding matrix has only two norms of eigenvalues. Then \( \zeta \) centralizes \( \Gamma_1 \) in \( S^{n-i_0-2} \) and normalizes \( \Gamma_2 \). Let \( U \) be the strict join of \( U_1 \) and \( U_2 \), a properly convex domain. \( U/\langle \Gamma_1, \Gamma_2, \zeta \rangle \) gives us an R-end of dimension \( n \) diffeomorphic to \( \Sigma_{E_1} \times \Sigma_{E_2} \times S^1 \times \mathbb{R} \) and the transverse real projective manifold is diffeomorphic to \( \Sigma_{E_1} \times \Sigma_{E_2} \times S^1 \). We call the results
the a generalized joined end and the generalized joined end-neighborhoods. Those ends with end-neighborhoods finitely covered by these are also called a generalized joined end. The generated group \( \langle \Gamma_1, \Gamma_2, \zeta \rangle \) is called a generalized joined group.

Now we generalize this construction slightly: Suppose that \( \Gamma_1 \) and \( \Gamma_2 \) are Lie groups and they have compact stabilizers at points of \( U_1 \) and \( U_2 \) respectively, and we have a parameter of \( \zeta^t \) for \( t \in \mathbb{R} \) centralizing \( \Gamma_1 | S^{n-i_0-2} \) and normalizing \( \Gamma_2 \) and restricting to a unipotent action on \( S^{n-i_0} \) acting on \( U_2 \). The other conditions remain the same. We obtain a generalized joined homogeneous action of the properly convex actions and cusp actions. Let \( U \) be the properly convex open subset obtained as above as a join of \( U_1 \) and \( U_2 \). Let \( G \) denote the constructed Lie group by taking the embeddings of \( \Gamma_1 \) and \( \Gamma_2 \) as above. \( G \) also has a compact stabilizer on \( U \). Given a discrete cocompact subgroup of \( G \), we obtained a p-end-neighborhood of a generalized joined p-end by taking the quotient of \( U \). An end with an end-neighborhood finitely covered by such a one are also called a generalized joined end.

**Remark 8.2** We will deform this construction to a construction of quasi-joined ends (see Definition 8.1). This will be done by adding some translations appropriately.

We continue the above example to a more specific situation.

**Example 8.2** Let \( N \) be as in (8.16). Let \( N \) be a cusp group in conjugate to one in \( \text{SO}(i_0 + 1, 1) \) acting on an \( i_0 \)-dimensional ellipsoid in \( S^{n+1} \).

We find a closed properly convex real projective orbifold \( \Sigma \) of dimension \( n - i_0 - 2 \) and find a homomorphism from \( \pi_1(\Sigma) \) to a subgroup of \( \text{Aut}(S^{n+1}) \) normalizing \( N \). (We will use a trivial one to begin with.) Using this, we obtain a group \( \Gamma \) so that \[ 1 \rightarrow N \rightarrow \Gamma \rightarrow \pi_1(\Sigma) \rightarrow 1. \]

Actually, we assume that this is “split”, i.e., \( \pi_1(\Sigma) \) acts trivially on \( N \).

We now consider an example where \( i_0 = 1 \). Let \( N \) be 1-dimensional and be generated by \( N_1 \) as in (8.9).

\[
N_1 := \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & \frac{1}{2} & 1 & 1
\end{pmatrix}
\] (8.9)

where \( i_0 = 1 \).

We take a discrete faithful proximal semisimple representation

\[ \tilde{h} : \pi_1(\Sigma) \rightarrow \text{GL}(n-i_0, \mathbb{R}) \]

acting on a convex cone \( C \) in \( \mathbb{R}^{n-i_0} \). We define

\[ h : \pi_1(\Sigma) \rightarrow \text{GL}(n+1, \mathbb{R}) \]
by matrices
\[
    h(g) := \begin{pmatrix}
        \tilde{h}(g) & 0 & 0 \\
        d_1(g) & a_1(g) & 0 \\
        d_2(g) & c(g) & \lambda_{v_E}(g)
    \end{pmatrix}
\] (8.10)

where \(d_1(g)\) and \(d_2(g)\) are \(n-i_0\)-vectors and \(g \mapsto \lambda_{v_E}(g)\) is a homomorphism as defined above for the \(p\)-end vertex and \(\det h(g)a_1(g)\lambda_{v_E}(g) = 1\).

Then the conjugation of \(N_1\) by \(h(g)\) gives us
\[
    h(g)^{-1} := \begin{pmatrix}
        \tilde{h}(g)^{-1} & 0 & 0 \\
        -d_1(g) & a_1(g) & -c(g) \\
        d_2(g) & -c(g) & \lambda_{v_E}(g)
    \end{pmatrix} \begin{pmatrix}
        0 & \frac{1}{a_1(g)} & 0 \\
        \frac{1}{a_1(g)} & \lambda_{v_E}(g) & 0 \\
        \frac{1}{a_1(g)} & \lambda_{v_E}(g) & 1
    \end{pmatrix}.
\] (8.11)

Our condition on the form of \(N_1\) shows that \((0,0,\ldots,0,1)\) has to be a common eigenvector by \(\tilde{h}(\pi_1(\bar{E}))\), and we also assume that \(a_1(g) = \lambda_{v_E}(g)\). The last row of \(\tilde{h}(g)\) equals \((0,\lambda_{v_E}(g))\). Thus, the upper \((n-i_0)\times(n-i_0)\)-part of \(h(\pi_1(\bar{E}))\) is reducible.

Some further computations show that we can take any
\[
    h : \pi_1(\bar{E}) \to \text{SL}(n-i_0,\mathbb{R})
\]
with matrices of form
\[
    h(g) := \begin{pmatrix}
        S_{n-i_0-1}(g) & 0 & 0 \\
        0 & \lambda_{v_E}(g) & 0 \\
        0 & 0 & \lambda_{v_E}(g)
    \end{pmatrix}
\] (8.13)

for \(g \in \pi_1(\bar{E}) \rightarrow N\) by a choice of coordinates by the semisimple property of the \((n-i_0)\times(n-i_0)\)-upper left part of \(h(g)\).

Since \(h(\pi_1(\bar{E}))\) has a common eigenvector, Theorem 1.1 of Benoist [21] shows that the open convex domain \(K\) that is the image of \(\Pi_k\) is reducible. We assume that \(N_k = N'_k \times \mathbb{Z}\) for another subgroup \(N'_k\), and the image of the homomorphism \(g \in N'_k \rightarrow S_{n-i_0-1}(g)\) gives a discrete projective automorphism group acting properly discontinuously on a properly convex subset \(K'\) in \(S^{n-i_0-2}\) with a compact quotient.
Let $\mathcal{E}$ be the one-dimensional ellipsoid where lower-right $3 \times 3$-matrices of $N$ acts on. From this, the end is of the join form $K''/N'_K \times S^1 \times \mathcal{E}/\mathbb{Z}$ by taking a double cover if necessary and $\pi_1(\mathcal{E})$ is isomorphic to $N'_K \times \mathbb{Z} \times \mathbb{Z}$ up to taking an index two subgroups.

We can think of this as the join of $K''/N'_K$ with $\mathcal{E}/\mathbb{Z}$ as $K'$ and $\mathcal{E}$ are on disjoint complementary projective spaces of respective dimensions $n - 3$ and 2 to be denoted $\mathbb{S}(K')$ and $\mathbb{S}(\mathcal{E})$ respectively.

8.3.2 Hypotheses to derive the splitting result

These hypotheses will help us to obtain the splitting. Afterward, we will show the NPNC-ends with transverse weak middle eigenvalue conditions will satisfy these.

We will outline this subsection. In Section 8.3.2.1, we will introduce a standard coordinate system to work on, where we introduce the unipotent cusp group $\mathcal{N} \cong \mathbb{R}^{i_0}$. Our group $\Gamma_{\tilde{E}}$ normalizes $\mathcal{N}$ by the hypothesis. Similarity Lemma 8.3 shows that the conjugation in $\mathcal{N}$ by an element of $\Gamma_{\tilde{E}}$ acts as a similarity, a simple consequence of the normalization property. We use this similarity and the Benoist theory [21] to prove $K$-is-a-cone Lemma 8.5 that $K$ decomposes into a cone $\{k\} \ast K''$ where $\mathcal{N}$ has a nice expression for the adopted coordinates. (If an orthogonal group acts cocompactly on an open manifold, then the manifold is zero-dimensional.) In Section 8.3.2.3, splitting Proposition 8.2 shows that the end holonomy group splits. To do that we find a sequence of elements of the virtual center expanding neighborhoods of a copy of $K''$. Here, we explicitly find a part corresponding to $K'' \subset \text{bd} \tilde{\mathcal{E}}$ explicitly and $k$ is realized by an $(i_0 + 1)$-dimensional hemisphere where $\mathcal{N}$ acts on.

8.3.2.1 The matrix form of the end holonomy group.

Let $\Gamma_{\tilde{E}}$ be an R-p-end holonomy group, and let $l \subset \mathbb{S}^{n-1}_{\tilde{E}}$ be a complete $i_0$-dimensional leaf in $\tilde{\Sigma}_{\tilde{E}}$. Then a great sphere $\mathbb{S}^{i_0+1}_{l}$ contains the great segments from $v_{\tilde{E}}$ in the direction of $l$. Let $V_{i_0+1}$ denote the subspace corresponding to $\mathbb{S}^{i_0}_{l}$ containing $v_{\tilde{E}}$, and $V_{i_0+2}$ the subspace corresponding to $\mathbb{S}^{i_0+1}_{l}$. We choose the coordinate system so that $v_{\tilde{E}} = \{0, \ldots, 0, 1\}$

and points of $V_{i_0+1}$ and those of $V_{i_0+2}$ respectively correspond to

$$
\begin{pmatrix}
\vdots \\
\vdots \\
0, \ldots, 0, \ast, \cdots, \ast
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
\vdots \\
\vdots \\
0, \ldots, 0, \ast, \cdots, \ast
\end{pmatrix}
$$

\begin{align*}
&\left(\begin{pmatrix}
\vdots \\
\vdots \\
0, \ldots, 0, \ast, \cdots, \ast
\end{pmatrix}\right), \\
&\left(\begin{pmatrix}
\vdots \\
\vdots \\
0, \ldots, 0, \ast, \cdots, \ast
\end{pmatrix}\right).
\end{align*}
8.3 The general theory

Since $S_{mn}^0$ and $v_g$ are $g$-invariant, $g, g \in \Gamma_E$, is of standard form

$$
\begin{pmatrix}
S(g) & s_1(g) & 0 & 0 \\
S_2(g) & a_1(g) & 0 & 0 \\
C_1(g) & a_4(g) & A_3(g) & 0 \\
c_2(g) & a_7(g) & a_8(g) & a_9(g)
\end{pmatrix}
$$

(8.14)

where $S(g)$ is an $(n-i_0-1) \times (n-i_0-1)$-matrix and $s_1(g)$ is an $(n-i_0-1)$-column vector, $s_2(g)$ and $c_2(g)$ are $(n-i_0-1)$-row vectors, $C_1(g)$ is an $i_0 \times (n-i_0-1)$-matrix, $a_4(g)$ is an $i_0$-column vectors, $A_3(g)$ is an $i_0 \times i_0$-matrix, $a_8(g)$ is an $i_0$-row vector, and $a_1(g), a_7(g)$, and $a_9(g)$ are scalars.

Denote

$$
\hat{\mathcal{S}}(g) = \begin{pmatrix}
S(g) & s_1(g) \\
S_2(g) & a_1(g)
\end{pmatrix},
$$

and is called an upper-left part of $g$.

Let $\mathcal{N}$ be a unipotent group acting on $S_{mn}^0$ and inducing $I$ on $S_{n-i_0-1}$ also restricting to a cusp group for at least one great $(i_0+1)$-dimensional sphere $S_{n-i_0+1}$ containing $S_{mn}^0$.

We can write each element $g \in \mathcal{N}$ as an $(n+1) \times (n+1)$-matrix

$$
\begin{pmatrix}
I_{n-i_0-1} & 0 & 0 \\
0 & 1 & 0 \\
C_g & * & U_g
\end{pmatrix}
$$

(8.15)

where $C_g > 0$ is an $(i_0+1) \times (n-i_0-1)$-matrix, $U_g$ is a unipotent $(i_0+1) \times (i_0+1)$-matrix, $\mathbf{0}$ indicates various zero row or column vectors, $\mathbf{0}$ denotes the zero row-vector of dimension $n-i_0-1$, and $I_{n-i_0-1}$ is the $(n-i_0-1) \times (n-i_0-1)$-identity matrix. This follows since $g$ acts trivially on $\mathbb{R}^{n+1}/V_{i_0+2}$ and $g$ acts as a cusp group element on the subspace $V_{i_0+2}$.

For $v \in \mathbb{R}^{i_0}$, we define

$$
\mathcal{N}(v) := \begin{pmatrix}
I_{n-i_0-1} & 0 & 0 \cdots 0 \\
0 & 1 & 0 \cdots 0 \\
c_1(v) & v_1 & 1 \cdots 0 \\
c_2(v) & v_2 & 0 \cdots 1 \\
\vdots & \vdots & \vdots \ddots \vdots \\
c_{i_0+1}(v) & \frac{1}{2} \|v\|^2 & v_1 & v_2 & \cdots & 1
\end{pmatrix}
$$

(8.16)

where $\|v\|$ is the norm of $v = (v_1, \cdots, v_{i_0}) \in \mathbb{R}^{i_0}$. We assume that

$$
\mathcal{N} := \{ \mathcal{N}(v) | v \in \mathbb{R}^{i_0} \}$$
is a group, which must be nilpotent. The elements of our nilpotent group \( \mathcal{N} \) are of this form since \( \mathcal{N}(v) \) is the product \( \prod_{j=1}^{i_0} \mathcal{N}(e_j)^{v_j} \). By the way we defined this, for each \( k, k = 1, \ldots, i_0 \), \( c_k : \mathbb{R}^{i_0} \rightarrow \mathbb{R}^{n-i_0-1} \) are linear functions of \( v \) defined as

\[
c_k(v) = \sum_{j=1}^{i_0} c_{kj} v_j \quad \text{for} \quad v = (v_1, v_2, \ldots, v_{i_0})
\]

so that we form a group. (We do not need the property of \( c_{i_0+1} \) at the moment.)

From now on, we denote by \( C_1(v) \) the \( (n-i_0-1) \times i_0 \)-matrix given by the matrix with rows \( c_j(v) \) for \( j = 1, \ldots, i_0 \) and by \( c_2(v) \) the row \( (n-i_0-1) \)-vector \( c_{i_0+1}(v) \). The lower-right \( (i_0+2) \times (i_0+2) \)-matrix is form is called the \textit{standard cusp matrix form}.

We denote by \( A \) the matrix

\[
\begin{pmatrix}
I_{n-i_0-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & A & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(8.17)

for \( A \) an \( i_0 \times i_0 \)-matrix. Denote by the group of form

\[
\{ \hat{O}_5|O_5 \in O(i_0) \}
\]

by \( \hat{O}(n+1, i_0) \).

If \( \mathcal{N} \) is in the form (8.16) \( C_1(v) = 0, c_2(v) = 0 \) for all \( v \), we call \( \mathcal{N} \) the \textit{standard cusp group (of type \((n+1, i_0)\))} in the standard form. The \textit{standard parabolic group (of type \((n+1, i_0)\))} is a group conjugate to \( \mathcal{N} \hat{O}(n+1, i_0) \) where \( \mathcal{N} \) is in the standard cusp group in the standard form. Ones conjugate to these are called \textit{standard cusp group} and \textit{standard parabolic group} respectively.

The assumptions for this subsection are as follows: We will assume that the group satisfies the condition virtually only since this will be sufficient for our purposes.

\textbf{Hypothesis 8.3.1}

- Let \( K \) be defined as above for an R-p-end \( \tilde{E} \). Assume that \( K^\infty \big/ N_K \) is a compact set.
- \( N_K \) acts on \( K = K_1 \ast \cdots \ast K_m \) for the maximal decomposition of \( K \) under the join operations.
- \( \Gamma_{\tilde{E}} \) satisfies the transverse weak middle eigenvalue condition for the R-p-end structure. And elements are in the matrix form of (8.14) under a common coordinate system.
- A group \( \mathcal{N} \) of form (8.16) in the same coordinate system acts on each hemisphere with boundary \( S_i^l \), and fixes \( v_{\tilde{E}} \in S_i^l \) with coordinates \( \langle 0, \cdots, 0, 1 \rangle \).
- \( N \subset \mathcal{N} \) in the same coordinate system as above.
- The p-end holonomy group \( \Gamma_{\tilde{E}} \) normalizes \( \mathcal{N} \).
\( N \) acts on a \( p \)-end neighborhood \( U \) of \( \tilde{E} \), and acts on \( U \cap \Sigma_0 \) for each great sphere \( \Sigma_0 \) containing \( \tilde{E} \) whenever \( U \cap \Sigma_0 \neq 0 \).

\( N \) freely, faithfully, and transitively acts on the space of \( i_0 \)-dimensional leaves of \( \tilde{E} \) by an induced action.

Let \( U \) be a \( p \)-end neighborhood of \( \tilde{E} \). Let \( l' \) be an \( i_0 \)-dimensional leaf of \( \tilde{E} \). The consideration of the projection \( \Pi_K \) shows us that the leaf \( l' \) corresponds to a hemisphere \( H_{i_0+1} \) where

\[
U_{l'} := (H_{i_0+1} - \Sigma_0) \cap U \neq 0
\]

holds.

**Lemma 8.1 (Cusp)** Assume Hypothesis 8.3.1. Let \( l' \) be an \( i_0 \)-dimensional leaf of \( \tilde{E} \). Let \( H_{i_0+1} \) denote the \( i_0 + 1 \)-dimensional hemisphere with boundary \( \Sigma_0 \) corresponding to \( l' \). Then \( N \) acts transitively on \( \partial U_{l'} \) for \( U_{l'} := U \cap H_{i_0+1} \) bounded by an ellipsoid in a component of \( H_{i_0+1} - \Sigma_0 \).

**Proof** Since \( l' \) is an \( i_0 + 1 \)-dimensional leaf of \( \tilde{E} \), we obtain \( H_{i_0+1} \cap U \neq 0 \). Let \( J_{l'} := H_{i_0+1} \cap U \neq 0 \) where \( N \) acts on.

Now, \( l' \) corresponds to an interior point of \( K \). We need to change coordinates of \( S^{n-i_0-1} \) so that \( l' \) goes to

\[
\begin{bmatrix} 0, \ldots, 0, 1 \end{bmatrix}
\]

under \( \Pi_K \).

This involves the coordinate changes of the first \( n - i_0 \) coordinates. Now, we can restrict \( g \) to \( H_{i_0+1} \) so that the matrix form is with respect to \( U_{l'} \). Now give a coordinate system on the open hemisphere \( H_{i_0+1,0} \) given by

\[
\begin{cases}
(1, x, x_{i_0+1}) & \text{for } x \in \mathbb{R}^{i_0+2} \\
\end{cases}
\]

and \( (x, x_{i_0+1}) \) gives the affine coordinate system on \( H_{i_0+1,0} \).

Using (8.16) restricted to \( S^6 \), the lowest row of the lower-right \((i_0 + 1) \times (i_0 + 1)\) restriction matrix has to be of form \((*, v, 1)\). We obtain that each \( g \in N \) then has the form in \( H_{i_0+1} \) as

\[
\begin{pmatrix}
1 & 0 & 0 \\
L(v^T) & I_{i_0} & 0 \\
\kappa(v) & v & 1
\end{pmatrix}
\]

where \( L : \mathbb{R}^{i_0} \to \mathbb{R}^{i_0} \) is a linear map. The linearity of \( L \) is the consequence of the group property. \( \kappa : \mathbb{R}^{i_0} \to \mathbb{R} \) is some function. We consider \( L \) as an \( i_0 \times i_0 \)-matrix.
Suppose that there exists a kernel $K_1$ of $L$. We use $tv \in K_1 - \{0\}$. As $t \to \infty$, consider each orbit of the subgroup $\mathcal{N}'(\mathbb{R}v) \subset \mathcal{N}$ given by

$$(1, x, 0) \mapsto (1, x, \kappa(tv) + tv \cdot x + x_{i_0} + 1).$$

This action fixes coordinates from the 2-nd to $i_0 + 1$-st ones to be $x$. Hence, each orbit lies on an affine line from $(0,0,\ldots,1)$. Since the eigenvalues of every elements of $\mathcal{N}$ all equal 1, the action is unipotent. Since the action is unipotent, either the action is trivial or the orbit is the entire complete affine line on $H_{i_0}^p$. Since the action on each leaf $l'$ is free, the action cannot be trivial. Thus, the orbit is the complete affine line, and this contradicts the proper convexity of $\tilde{O}$.

Also, since $\mathcal{N}$ is abelian, the computations of

$$\mathcal{N}'(v), \mathcal{N}'(w) = \mathcal{N}'(w), \mathcal{N}'(v)$$

shows that $vLw^T = wLv^T$ for every pair of vectors $v$ and $w$ in $\mathbb{R}^i_0$. Thus, $L$ is a symmetric matrix.

We may obtain new coordinates $x_{n-i_0+1}, \ldots, x_n$ by taking linear combinations of these. Since $L$ hence is nonsingular, we can find new coordinates $x_{n-i_0+1}, \ldots, x_n$ so that $\mathcal{N}$ is now of standard form: We conjugate $\mathcal{N}$ by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for nonsingular $A$. We obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ ALv^T & I_{i_0} & 0 \\ \kappa(v) & vA^{-1} & 1 \end{pmatrix}.$$ 

We thus need to solve for $A^{-1}A^{-1T} = L$, which can be done.

Now, we conjugate as we wished to. We can factorize each element of $\mathcal{N}$ into forms

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{i_0} & 0 \\ \kappa(v) - \frac{||v||^2}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ v^T & I_{i_0} & 0 \\ \frac{||v||^2}{2} & v & 1 \end{pmatrix}.$$ 

Again, by the group property, $\alpha_7(v) := \kappa(v) - \frac{||v||^2}{2}$ gives us a linear function $\alpha_7 : \mathbb{R}^i_0 \to \mathbb{R}$. Hence $\alpha_7(v) = \kappa_\alpha \cdot v$ for $\kappa_\alpha \in \mathbb{R}^i_0$. Now, we conjugate $\mathcal{N}$ by the matrix
and this will put $\mathcal{N}$ into the standard form.

Now it is clear that the orbit of $\mathcal{N}(x_0)$ for a point $x_0$ of $\mathcal{Y}$ is an ellipsoid with a point removed since $\mathcal{N}$ acts so in the standard form since the standard form can be recognized as that of the parabolic group in the hyperbolic space in the Klein model in some appropriate coordinates. \hfill \Box

Recall standard cusp group from the end of Section 8.3.2.1. For later purposes, we need:

**Lemma 8.2** Let $C$ be standard cusp group acting on a hemisphere $H$ of dimension $i_0 + 1$ with boundary $S^{i_0}$ fixing a point $p$ in $S^{i_0}$. Then the following hold:

- There exists an affine space $A_C^n$ with $S^{i_0} \subset \text{bd} A_C^n$ and $H^o \subset A_C^n$ where orbits of points have three types:
  - The orbit of each point in $A_C^n$ is an ellipsoid in an affine subspace of dimension $i_0$ parallel to the affine subspace $H^o$.
  - The orbit of each point of a great sphere $S^{n-i_0}$ of dimension $n-i_0$ containing $p$ transverse to $S^{i_0}$ where the orbits are singletons.
  - The orbit of every point of $\text{bd} A_C^n - S^{n-i_0}$ is not contained in a properly convex domain.

- The affine space $A_C^n$ with these orbit properties is uniquely determined.

**Proof** We choose the affine space $A_C^n$ is given by $x_{n-i_0} > 0$ for the coordinate system where $C$ is written as in (8.16) with $c_i, i = 1, \ldots, i_0 + 1$, are zero. Since $C$ is standard, there exists a sphere of dimension $S^{n-i_0-1}$ complementary to $S^{i_0}$ in $\text{bd} A_C^n$ where $C$ acts trivially. Let $S^{n-i_0}$ be the join $\{p, p-\} \ast S^{n-i_0-1}$. On $S^{n-i_0}$ the orbits are singletons. The orbits of points of $\text{bd} A_C^n - S^{n-i_0}$ can be understood by the matrix form. The orbits will always contain a pair of antipodal points in the closures by considering $\mathcal{N}$ with the $n-i_0$-th rows and the $n-i_0$-th columns removed. The affine translations commute with each element of $C$. This shows that each orbit in $A_C^n$ is as claimed.

Also, since the orbit types are characterized, $A_C^n$ is uniquely determined. \hfill \Box

Let $a_5(g)$ denote $|\det(A^5_g)|^{\frac{1}{5}}$. Define $\mu_g := \frac{a_5(g)}{a_1(g)} = \frac{a_9(g)}{a_5(g)}$ for $g \in \Gamma_E$ from Lemma 8.3.

**Lemma 8.3 (Similarity)** Assume Hypothesis 8.3.1. Then any element $g \in \Gamma_E$ induces an $(i_0 \times i_0)$-matrix $M_g$ given by

$$g.\mathcal{N}(v)g^{-1} = \mathcal{N}(vM_g)$$ where
where

\[ C = \begin{pmatrix} a_1(g) & a_2(g) & \ldots & a_n(g) \\ a_1(g) & a_2(g) & \ldots & a_n(g) \\ \vdots & \vdots & \ddots & \vdots \\ a_1(g) & a_2(g) & \ldots & a_n(g) \end{pmatrix} \]

Since we have \( g(C) = Cg \), \( C \) commutes with the result of (8.19). We obtain the relation

\[ av = vA5(g) \]

for every \( v \). The correspondence between the set of \( v \) and \( v' \) is one-to-one, we obtain

\[ v' = a_g v(A5(g))^{-1} \]

for the \( i_0 \times i_0 \)-matrix \( A5(g) \) and we also infer \( a_0(g) \neq 0 \) and \( \det(A5(g)) \neq 0 \). The (3,2)-block of the result of (8.19) equals

\[ a_k(g) + A5(g)v^T. \]

The (3,2)-block of the result of (8.20) equals

\[ C1(v')s_1(v) + a_1(v)v^T + a_4(g). \]

Thus,

\[ A5(g)v^T = C1(v')s_1(v) + a_1(v)v^T. \]
For each \( g \), we can choose a coordinate system so that \( s_1(g) = 0 \) since by the Brouwer fixed point theorem, there is a fixed point in the compact convex set \( K \subset S^{n-i_0-1} \). This involves the coordinate changes of the first \( n-i_0 \) coordinate functions only.

Let \( l' \) denote the fixed point of \( g \). Since \( \mathcal{N} \) acts on \( S^{i_0+1}_p \) for \( l' \) as a cusp group by Lemma 8.1, there exists a coordinate change involving the last \( (i_0+1) \)-coordinates

\[ x_{n-i_0+1}, \ldots, x_n, x_{n+1} \]

so that the matrix form of the lower-right \( (i_0+2) \times (i_0+2) \)-matrix of each element \( \mathcal{N} \) is of the standard cusp form. This will not affect \( s_1(g) = 0 \) as we can check from conjugating matrices used in the proof of Lemma 8.1 as the change involves the above coordinates only. Denote this coordinate system by \( \Phi_{g,l'} \).

We may assume that the transition to this coordinate system from the original one is uniformly bounded: First, we change for \( S^{n-i_0-1} \) with a bounded elliptic coordinate change since we are only picking out a single point to be a coordinate axis. This makes \( L \) and \( \kappa \) in the proof of Lemma 8.1 to be uniformly bounded functions. Hence, \( A \) and \( \kappa_{\alpha} \) in the proof are also uniformly bounded.

Let us use \( \Phi_{g,l'} \) for a while using primes for new set of coordinates functions. Now \( A_{s_1}(g) \) is conjugate to \( A_5(g) \) as we can check in the proof of Lemma 8.1. Under this coordinate system for given \( g \), we obtain \( a_1'(g) \neq 0 \) and we can recompute to show that \( a_9'(g)v = v'A_5'(g) \) for every \( v \) as in (8.21). By (8.23) recomputed for this case, we obtain

\[ v' = \frac{1}{a_1'(g)}v(A_5'(g))^T \quad (8.24) \]

as \( s_1'(g) = 0 \) here since we are using the coordinate system \( \Phi_{g,l'} \). Since this holds for every \( v \in \mathbb{R}^{i_0} \), we obtain

\[ a_9'(g)(A_5'(g))^{-1} = \frac{1}{a_1'(g)}(A_5'(g))^T. \]

Hence

\[ \frac{1}{\det(A_5'(g))}A_5'(g) \in O(i_0). \]

Also,

\[ \frac{a_9(g)}{a_9'(g)} = \frac{a_9'(g)}{a_9'(g)}. \]

Here, \( A_5'(g) \) is a conjugate of the original matrix \( A_5(g) \) by linear coordinate changes as we can see from the above processes to obtain the new coordinate system.

This implies that the original matrix \( A_5(g) \) is conjugate to an orthogonal matrix multiplied by a positive scalar for every \( g \). The set of matrices \( \{A_5(g) | g \in \Gamma_{\mathbb{K}}\} \) forms a group since every \( g \) is of a standard matrix form (see (8.14)). Given such a group of matrices normalized to have determinant \( \pm 1 \), we obtain a compact group
\[ G_E := \left\{ \frac{1}{|\det A_5(g)|^{\frac{1}{6}}} A_5(g) \mid g \in I_E \right\} \]

by Lemma 8.4. This group has a coordinate system where every element is orthogonal by a coordinate change of coordinates \( x_{n-i_0+1}, \ldots, x_n \).

Also, \( a_1(g), a_3(g), a_5(g) \) are conjugation invariant. Hence, we proved

\[
\frac{a_5(g)}{a_3(g)} = \frac{a_3(g)}{a_1(g)}.
\]

We have \( a_5(g) = \lambda_5(g) > 0 \). Since \( a_3(g)^2 = a_1(g)a_5(g) \), we obtain \( a_1(g) > 0 \). Finally, \( a_5(g) > 0 \) by definition.

**Lemma 8.4** Suppose that \( G \) is a subgroup of a linear group \( GL(i_0, \mathbb{R}) \) where each element is conjugate to an orthogonal element by a uniformly bounded conjugating matrices. Then \( G \) is in a compact Lie group.

**Proof** Clearly, the norms of eigenvalues of \( g \in G \) are all 1. Hence, \( G \) is virtually an orthopotent group by Theorem 2.6. (See [143] and [68].) Hence, \( \mathbb{R}^{i_0} \) has subspaces \( \{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{R}^{i_0} \) where \( G \) acts as orthogonal on \( V_{i+1}/V_i \) up to a choice of coordinates. Hence, the Zariski closure \( \mathcal{Z}(G) \) of \( G \) is also orthopotent.

If \( G \) is discrete, Theorem 2.6 shows that \( G \) is virtually unipotent. The unipotent subgroup of \( G \) is trivial since the elements must be conjugate to orthogonal elements. Thus, \( G \) is a finite group, and we finished the proof.

Suppose now that the closure \( \tilde{G} \) of \( G \) is a Lie group of dimension \( \geq 1 \). Each element of the identity component \( \tilde{G} \) is again elliptic or is the identity since each accumulation point is the limit of a sequence of elliptic elements conjugated by a uniformly bounded collection of elements while the limit must preserve a positive definite inner product.

Let \( O(\oplus_{i=1}^m V_i/V_{i-1}) \) denote the group of linear transformations acting on each \( V_i/V_{i-1} \) orthogonally for each \( i = 1, \ldots, m \). By Theorem 2.6, there is a homomorphism \( \mathcal{Z}(G) \to O(\oplus_{i=1}^m V_i/V_{i-1}) \) whose kernel \( U_{i_0} \) is the a group of unipotent matrices. Let \( \hat{G} \) denote the image of \( G \) in the second group. Then \( \hat{G} \cap U_{i_0} \) trivial since every element of \( \hat{G} \) is elliptic or is the identity. Thus, \( \hat{G} \) is isomorphic to a compact group \( \hat{G} \).

From now on, we denote by \( (C_1(v), v^T) \) the matrix obtained from \( C_1(v) \) by adding a column vector \( v^T \) and denote \( O_3(g) := \frac{1}{|\det A_5(g)|^{\frac{1}{6}}} A_5(g) \). We also let

\[
\hat{S}(g) := \left( \begin{array}{c} S(g), \ s_1(g) \\ s_2(g), \ a_1(g) \end{array} \right).
\]

**Lemma 8.5 (K is a cone)** Assume Hypothesis 8.3.1. Suppose that \( \Gamma_E \) acts semisimply on \( K'' \). Then the following hold:

- \( K \) is a cone over a totally geodesic \( (n-i_0-2) \)-dimensional domain \( K'' \).
The rows of \((C_1(v), v^T)\) are proportional to a single vector, and we can find a coordinate system where \(C_1(v) = 0\) not changing any entries of the lower-right \((i_0 + 2) \times (i_0 + 2)\)-submatrices for all \(v \in \mathbb{R}^{i_0}\).

We can find a common coordinate system where

\[
O_5(g)^{-1} = O_5(g)^T, O_5(g) \in O(i_0),
\]

\[
s_1(g) = s_2(g) = 0 \text{ for all } g \in \Gamma_E
\]

(8.25)

where \(O_5(g) = \left|\det(A_5^{-1})\right|^{1/2} A_5(g)\).

In this coordinate system, we have

\[
s_1(g) = 0, s_2(g) = 0, a_9(g) c_2(v) = c_2(\mu_g v O_5(g)^{-1}) S(g) + \mu_g v O_5(g)^{-1} C_1(g).
\]

(8.26)

**Proof** The assumption implies that \(M_g = \mu_g O_5(g)^{-1}\) by Lemma 8.3. We consider the equation

\[
g \cdot \mathcal{N}(v) g^{-1} = \mathcal{N}(\mu_g v O_5(g)^{-1}).
\]

(8.27)

We change to

\[
g \cdot \mathcal{N}(v) = \mathcal{N}(\mu_g v O_5(g)^{-1}) g.
\]

(8.28)

Considering the lower left \((n - i_0) \times (i_0 + 1)\)-matrix of the left side of (8.28), we obtain

\[
\begin{pmatrix}
C_1(g), a_4(g) \\
c_2(g), a_7(g)
\end{pmatrix}
+ \begin{pmatrix}
as_5(g) O_5(g) C_1(v), & as_5(g) O_5(g) v^T \\
as_8(g) C_1(v) + a_0 c_2(v), & a_8(g) v^T + a_9(g) v \cdot v / 2
\end{pmatrix} = \begin{pmatrix}
0, & 0 \\
0, & 0
\end{pmatrix}
\]

(8.29)

where the entry sizes are clear. From the right side of (8.28), we obtain

\[
\begin{pmatrix}
C_1(g), & a_4(g) \\
c_2(\mu_g v O_5(g)^{-1}), & v \cdot v / 2
\end{pmatrix} \hat{S}(g) + \begin{pmatrix}
0, & 0 \\
0, & 0
\end{pmatrix}
\]

(8.30)

From the top rows of (8.29) and (8.30), we obtain that

\[
\begin{pmatrix}
as_5(g) O_5(g) C_1(v), & as_5(g) O_5(g) v^T \\
as_8(g) C_1(v) + a_0 c_2(v), & a_8(g) v^T + a_9(g) v \cdot v / 2
\end{pmatrix} \hat{S}(g).
\]

(8.31)

We multiplied the both sides by \(O_5(g)^{-1}\) from the right and by \(\hat{S}(g)^{-1}\) from the left to obtain
\[
\begin{align*}
(a_5(g)C_1(v), a_5(g)v^T) \hat{S}(g^{-1}) = \\
\left(\mu_gO_5(g)^{-1}C_1(vO_5(g)^{-1}), \mu_gO_5(g)^{-1}O_5(g)^{-1}\mathbf{1}v^T\right).
\end{align*}
\]

Let us form the subspace \(V_C\) in the dual sphere \(\mathbb{R}^{n-i_0}\) spanned by row vectors of \((C_1(v), v^T)\). Let \(S_C^*\) denote the corresponding subspace in \(\mathbb{S}^{n-i_0-1}\). Then

\[
\left\{ \frac{1}{\det \hat{S}(g)^{-\frac{n-i_0-1}{2}}} \hat{S}(g) | g \in \Gamma_E \right\}
\]

acts on \(V_C\) as a group of bounded linear automorphisms since \(O_5(g) \in G\) for a compact group \(G\). Therefore, \(\{\hat{S}(g) | g \in \Gamma_E\}\) on \(S_C^*\) is in a compact group of projective automorphisms by (8.32).

We recall that the dual group \(N_K^*\) of \(N_K\) acts on the properly convex dual domain \(K^*\) of \(K\) cocompactly by Proposition 2.21. Then \(N_K^*\) acts as a compact group on \(S_C^*\). Thus, \(N_K^*\) is reducible.

Now, we apply the theory of Vey [161] and Benoist [21]: Since \(N_K^*\) is semisimple by above premises, \(N_K^*\) acts on a complementary subspace of \(S_N^*\). By Proposition 2.20, \(K^*\) has an invariant subspace \(K_1^*\) and \(K_2^*\) so that we have strict join

\[K^* = K_1^* \ast K_2^*\]

where \(\dim K_1^* = \dim S_C^*, \dim K_2^* = \dim S_N^*\)

Also, \(N_K^*\) is isomorphic to a cocompact subgroup of

\[N_{K,1}^* \times N_{K,2}^* \times A\]

where

- \(A\) is a diagonalizable subgroup with positive eigenvalues only isomorphic to a subgroup of \(\mathbb{R}_+\),
- \(N_{K,i}^*\) is the restriction image of \(N_K^*\) to \(K_i^*\) for \(i = 1, 2\), and
- \(N_{K,i}^*\) acts on the interior of \(K_i^*\) properly and cocompactly.

But since \(N_{K,1}^*\) acts orthogonally on \(S_C^*\), the only possibility is that \(\dim S_C^* = 0\): Otherwise \(K_1^*/N_K^*\) is not compact contradicting Proposition 2.20. Hence, \(\dim S_C^* = 0\) and \(K_1^*\) is a singleton and \(K_2^*\) is \(n-i_0-2\)-dimensional properly convex domain.

Rows of \((C_1(v), v^T)\) are elements of the 1-dimensional subspace in \(\mathbb{R}^{n-i_0-1}\) corresponding to \(S_C^*\). Therefore this shows that the rows of \((C_1(v), v^T)\) are proportional to a single row vector.

Since \((C_1(e_j), e_j^T)\) has 0 as the last column element except for the \(j\)th one, only the \(j\)th row of \(C_1(e_j)\) is nonzero. Let \(C_1(1, e_1)\) be the first row of \(C_1(e_1)\). Thus, each row of \((C_1(e_j), e_j^T)\) equals to a scalar multiple of \((C_1(1, e_1), 1)\) for every \(j\). Now we can choose coordinates of \(\mathbb{R}^{n-i_0}\) so that this \((n-i_0)\)-row-vector now has
coordinates \((0, \ldots, 0, 1)\). We can also choose the coordinates so that \(K_2^*\) is in the zero set of the last coordinate. With this change, we need to do conjugation by matrices with the top left \((n - i_0 - 1) \times (n - i_0 - 1)\)-submatrix being different from \(I\) and the rest of the entries staying the same. This will not affect the expressions of matrices of lower right \((i_0 + 2) \times (i_0 + 2)\)-matrices involved here. Thus, \(C_1(v) = 0\) in this coordinate for all \(v \in \mathbb{R}^{n-i_0}\). Also, \(\langle 0, \ldots, 0, 1 \rangle\) is an eigenvector of every element of \(N^*_K\).

The hyperspace containing \(K_2^*\) is also \(N^*_K\)-invariant. Thus, the \((n - i_0)\)-vector \(\langle 0, \ldots, 0, 1 \rangle\) corresponds to an eigenvector of every element of \(N_K\). In this coordinate system, \(K\) is a strict join of a point for an \((n - i_0)\)-vector \(k = \langle 0, \ldots, 0, 1 \rangle\) and a domain \(K''\) given by setting \(x_{n-i_0} = 0\) in a totally geodesic sphere of dimension \(n - i_0 - 2\) by duality. We also obtain \(s_1(g) = 0, s_2(g) = 0\).

For the final item we have under our coordinate system.

\[
g = \begin{pmatrix}
S(g) & 0 & 0 & 0 \\
0 & a_1(g) & 0 & 0 \\
C_1(g) & a_4(g) & a_5(g)O_5(g) & 0 \\
c_2(g) & a_7(g) & a_8(g) & a_9(g)
\end{pmatrix},
\]

\((8.33)\)

\[
\mathcal{N}(v) = \begin{pmatrix}
I_{n-i_0-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & v^T & 1 & 0 \\
c_2(v) & \frac{1}{2} \|v\|^2 & v & 1
\end{pmatrix}.
\]

\((8.34)\)

The normalization of \(\mathcal{N}\) shows as in the proof of Lemma 8.3 that \(O_5(g)\) is orthogonal now. (See (8.21) and (8.23).) By (8.27), we have

\[g\mathcal{N}(v) = \mathcal{N}(v')g, v' = \mu_4vO_5(g)^{-1}.\]

We consider the lower-right \((i_0 + 1) \times (n - i_0)\)-submatrices of \(g\mathcal{N}(v)\) and \(\mathcal{N}(v')g\). For the first one, we obtain

\[
\begin{pmatrix}
C_1(g), a_4(g) \\
c_2(g), a_7(g)
\end{pmatrix} + \begin{pmatrix}
0 \\
a_5(g)O_5(g)
\end{pmatrix} \begin{pmatrix}
0, v^T \\
c_2(v), \frac{1}{2} \|v\|^2
\end{pmatrix}
\]
For $\mathcal{N}(v')g$, we obtain
\[
\begin{pmatrix}
0, & v'^T \\
c_2(v'), & \frac{1}{2}||v'||^2
\end{pmatrix}
\begin{pmatrix}
S(g), & 0 \\
0, & a_1(g)
\end{pmatrix}
+ 
\begin{pmatrix}
1, & 0 \\
v', & 1
\end{pmatrix}
\begin{pmatrix}
C_1(g), & a_4(g) \\
c_2(g), & a_9(g)
\end{pmatrix}.
\]

Considering (2,1)-blocks, we obtain
\[
c_2(g) + a_9(g)c_2(v) = c_2(v')S(g) + v'C_1(g) + c_2(g).
\]

From now on, we denote $O_5(g) := \det(A_5^g)^{\frac{1}{2}} A_5(g)$.

**Lemma 8.6** Assume Hypothesis 8.3.1 and $N_K$ acts semi-simply. Then we can find coordinates so that the following holds for all $g$:

\[
\frac{a_9(g)}{a_5(g)} O_5(g)^{-1} a_4(g) = a_8(g)^T \text{ or } \frac{a_9(g)}{a_5(g)} a_4(g)^T O_5(g) = a_8(g). \tag{8.35}
\]

If $\mu = 1$, then $a_1(g) = a_9(g) = \lambda_\nu(g)$ and $A_5(g) = \lambda_\nu(g)O_5(g). \tag{8.36}$

**Proof** Again, we use (8.19) and (8.20). We only need to consider lower right $(i_0 + 2) \times (i_0 + 2)$-blocks.

\[
\begin{pmatrix}
a_1(g) & 0 & 0 \\
a_4(g) & a_5(g)O_5(g) & 0 \\
a_7(g) & a_8(g) & a_9(g)
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
v'^T & 1 & 0 \\
\frac{1}{2}||v'||^2 & v' & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a_1(g) & 0 & 0 \\
a_4(g) + a_5(g)O_5(g)v'^T & a_5(g)O_5(g) & 0 \\
a_7(g) + a_8(g)v'^T + \frac{a_9(g)}{2} ||v'||^2 & a_8(g) + a_9(g)v & a_9(g)
\end{pmatrix}. \tag{8.38}
\]

This equals
\[
\begin{pmatrix}
1 & 0 & 0 \\
v'^T & 1 & 0 \\
\frac{1}{2}||v'||^2 & v' & 1
\end{pmatrix}
\begin{pmatrix}
a_1(g) & 0 & 0 \\
a_4(g) & a_5(g)O_5^g & 0 \\
a_7(g) & a_8(g) & a_9(g)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a_1(g) & 0 & 0 \\
a_1(g)v'^T + a_4(g) & a_5(g)O_5(g) & 0 \\
\frac{a_9(g)}{2} ||v'||^2 + v'a_4(g) + a_7(g) & a_5(g)v'O_5(g) + a_8(g) & a_9(g)
\end{pmatrix}. \tag{8.40}
\]

Then by comparing the (3,2)-blocks, we obtain
\[ a_8(g) + a_9(g)v = a_8(g) + a_5(g)v'O_5(g). \]

Thus, \( v = \frac{a_5(g)}{a_9(g)}v'O_5(g). \)

From the \((3,1)\)-blocks, we obtain
\[
\begin{align*}
a_1(g)v'v' / 2 + v'a_4(g) &= a_8(g)v'T + a_9(g)v'v / 2. \\
\end{align*}
\]

Since the quadratic forms have to equal each other, we obtain
\[
\begin{align*}
\frac{a_9(g)}{a_5(g)}(O_5(g)^T a_4(g)) = a_8(g). \\
\end{align*}
\]

Thus, \( \mu_g = 1 \), we obtain
\[
\begin{align*}
a_1(g) &= a_9(g) = a_5(g) = \lambda_{\gamma(E)}(g) \text{ and } A_5(g) = \lambda_{\gamma(E)}(g)O_5(g) \\
\end{align*}
\]

by Lemma 8.3. Also, \( a_1(g) = a_9(g) = a_5(g) = \lambda_{\gamma(E)}(g) \).

Under Hypothesis 8.3.1 and assume that \( N_\gamma \) acts semisimply, we conclude by Lemma 8.6 that each \( g \in \Gamma_E \) has the form
\[
\begin{pmatrix}
S(g) & 0 & 0 & 0 \\
0 & a_1(g) & 0 & 0 \\
C_1(g) & a_1(g)v'_g & a_5(g)O_5(g) & 0 \\
c_2(g) & a_1(g) & a_5(g)v'_gO_5(g) & a_9(g) \\
\end{pmatrix}
\]

(8.41)

**Remark 8.3** Since the matrices are of form (8.41), \( g \mapsto \mu_g \) is a homomorphism.

**Corollary 8.2** If \( g \) of form (8.41) centralizes a Zariski dense subset \( A' \) of \( \mathcal{N} \), then \( \mu_g = 1 \) and \( O_5(g) = I_{i_0} \).

**Proof** \( \mathcal{N} \) is isomorphic to \( \mathbb{R}^{i_0} \). The subset \( A'' \) of \( \mathbb{R}^{i_0} \) corresponding to \( A' \) is also Zariski dense in \( \mathbb{R}^{i_0} \). \( g.\mathcal{N}(v) = \mathcal{N}(v)'g \) shows that \( v = vO_5(g) \) for all \( v \in A'' \). Hence \( O_5(g) = I_{i_0} \).

\[ \square \]

### 8.3.2.2 Invariant \( \alpha_7 \)

We assume \( \mu_g = 1, g \in \Gamma_E \), identically in this subsubsection. When \( \mu_g = 1 \) for all \( g \in \Gamma_E \), by taking a finite index subgroup of \( \Gamma_E \), we conclude that each \( g \in \Gamma_E \) has the form by Lemma 8.6
\[ M(g) := \begin{pmatrix} S(g) & 0 & 0 & 0 \\ 0 & \lambda_{v_E}(g) & 0 & 0 \\ C_1(g) & \lambda_{v_E}(g)v_g^T & \lambda_{v_E}(g)O_5(g) & 0 \\ c_2(g) & \alpha_f(g) & \lambda_{v_E}(g)v_gO_5(g) & \lambda_{v_E}(g) \end{pmatrix}. \] (8.42)

We define an invariant:

\[ \alpha_f(g) := \frac{\alpha_f(g)}{\lambda_{v_E}(g)} - \frac{||v_g||^2}{2}. \]

We denote by \( \hat{M}(g) \) the lower right \((i_0 + 1) \times (i_0 + 1)\)-submatrix of \( M(g) \). An easy computation shows that \( \hat{M}(g)\hat{M}(h) = \hat{M}(gh) \) where \( v_{gh} = v + O_5(g)v_h \) holds. Then it is easy to show that

\[ \alpha_f(g^n) = n\alpha_f(g) \quad \text{and} \quad \alpha_f(gh) = \alpha_f(g) + \alpha_f(h), \]

whenever \( g, h, gh \in G \).

We obtain a homomorphism to the additive group \( \mathbb{R} \)

\[ \alpha_f : \Gamma_E \rightarrow \mathbb{R}. \]

(See (8.43).)

Here \( \alpha_f(g) \) is also determined by factoring the matrix of \( g \) into commuting matrices of form

\[ \begin{pmatrix} I_{n-i_0-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_{i_0} & 0 \\ 0 & \alpha_f(g) & 0 & 1 \end{pmatrix} \times \begin{pmatrix} S_g & 0 & 0 & 0 \\ 0 & \lambda_{v_E}(g) & 0 & 0 \\ C_1(g) & \lambda_{v_E}(g)v_g & \lambda_{v_E}(g)O_5(g) & 0 \\ c_2(g) & \lambda_{v_E}(g)||v_g||^2 & \lambda_{v_E}(g)v_gO_5(g) & \lambda_{v_E}(g) \end{pmatrix}. \] (8.43)

Remark 8.4 We give a bit more explanations. Recall that the space of segments in a hemisphere \( H^{i_0+1} \) with the vertices \( v_E, v_{\bar{E}} \) forms an affine subspace \( \mathbb{A} \) one-dimension lower, and the group \( \text{Aut}(H^{i_0+1})_v \) of projective automorphisms of the hemisphere fixing \( v_E \) maps to \( \text{Aff}(\mathbb{A}) \) with kernel \( K \) equal to transformations of an \((i_0 + 2) \times (i_0 + 2)\)-matrix form.
where $v_\tilde{E}$ is given coordinates $(0, 0, \ldots, 0)$, $0$ denote the $0$-vector in $\mathbb{R}^{i_0}$ and a center point of $H_{i_0+1}$ the coordinates $(1, 0, \ldots, 0)$. In other words, the transformations are of form

$$
\begin{pmatrix}
1 & 0^T & 0 \\
O & I_{i_0} & 0 \\
b & 0^T & 1
\end{pmatrix}
$$

(8.44)

and hence $b$ determines the kernel element. Hence $\alpha_7(g)$ indicates the translation towards $v_\tilde{E} = (0, \ldots, 1)$. (Actually the vertex corresponds to $(1, 0, \ldots, +\infty)$-point in this view.)

We denote by $T(n + 1, n - i_0)$ the group of matrices restricting to (8.44) in the lower-right $(i_0 + 2) \times (i_0 + 2)$-submatrices and equal to $I$ on upper-left $(n - i_0 - 1) \times (n - i_0 - 1)$-submatrices and zero elsewhere.

### 8.3.2.3 Splitting the NPNC end

**Proposition 8.2 (Splitting)** Assume Hypothesis 8.3.1 for $\Gamma_{\tilde{E}}$. Suppose additionally the following:

- Suppose that $N_K$ acts on $K$ in a semi-simple manner.
- $K = \{k\} \ast K''$ a strict join, and $K'' / N_K$ is compact with $k$ a common fixed point of $N_K$.
- Let $H$ be a commutant of a finite index subgroup of $N_K$ that is positive diagonalizable. Assume that $N_K \cap H$ contains a free abelian group of rank $l_0$ provided $K''$ is a strict join of compact convex subsets $K_1, \ldots, K_{l_0}$ where $H$ acts trivially on each $K_j$, $j = 1, \ldots, l_0$.

Then the following hold:

- there exists an exact sequence

$$1 \rightarrow \mathcal{N} \rightarrow \langle \Gamma_{\tilde{E}}, \mathcal{N} \rangle \xrightarrow{\Pi_{\tilde{E}}} N_K \rightarrow 1.$$  

- $K''$ embeds projectively in the closure of $\text{bd} \tilde{\mathcal{O}}$ whose image is $\Gamma_{\tilde{E}}$-invariant, and
• one can find a coordinate system so that every $\mathcal{N}(v)$ for $v \in \mathbb{R}^{b_0}$ is in the standard form and each element $g$ of $\Gamma_{\tilde{E}}$ is written so that
  - $C_1(v) = 0, c_2(v) = 0$, and
  - $C_1(g) = 0$ and $c_2(g) = 0$.

Proof (A) Let $Z$ denote $\langle \Gamma_{\tilde{E}}, \mathcal{N} \rangle$. Since $\mathcal{N} \subset \mathcal{N}$, we have homomorphism

$$Z \xrightarrow{\Pi_k} \mathbb{N}_K \rightarrow 1$$

extending $\Pi_k$ of (8.3). We now determine the kernel.

The function $\lambda_{\tilde{E}} : \Gamma_{\tilde{E}} \rightarrow \mathbb{R}_+$ extends to $\lambda_{\tilde{E}} : Z \rightarrow \mathbb{R}_+$. By (8.41), we deduce that every element $g$ of $Z$ is of form:

$$g = \begin{pmatrix}
  I_{n-i_0-1} & 0 & 0 \\
  0 & 1 & 0 \\
  C_1(v_g) & v_g^T & O_5(g) \\
  c_2(v_g) & \|v_g\|^2 / 2 & v_g O_5(g)
\end{pmatrix}$$

(8.47)

Since $\alpha_7(g) = 0$, and $S(g) = \lambda_{\tilde{E}} \mathbb{I}_{n-i_0-1}$ and the $(n-i_0, n-i_0)$-term must be $\lambda_{\tilde{E}}$ for some $\lambda_{\tilde{E}} > 0$ so that it goes to 1 in $K$.

Theorem 2.6 shows that $\mathcal{N}$ is a subgroup of $\mathcal{N}$ by taking a finite index subgroup of $\Gamma_{\tilde{E}}$. Since the kernel of $\Pi_k|Z$ is generated by $\mathcal{N}$ and $\mathcal{N}$, we proved the first item.

(B) Lemma 8.5 shows that $C_1(v) = 0$ for all $v \in \mathbb{R}^{b_0}$ for a coordinate system where $k$ has the form

$$g = \begin{pmatrix}
  I_{n-i_0-1} & 0 & 0 \\
  0 & 1 & 0 \\
  C_1(v_g) & v_g^T & O_5(g) \\
  c_2(v_g) & \|v_g\|^2 / 2 & v_g O_5(g)
\end{pmatrix}$$

since $\alpha_7(g) = 0$, and $S(g) = \lambda_{\tilde{E}} \mathbb{I}_{n-i_0-1}$ and the $(n-i_0, n-i_0)$-term must be $\lambda_{\tilde{E}}$ for some $\lambda_{\tilde{E}} > 0$ so that it goes to 1 in $K$.
shows that $\tilde{\lambda}_5(g)$ is the largest of norms of every eigenvalue by Proposition 8.1, and $a_9(g) = \lambda_{E^*}$, we have

$$a_1(g) \leq a_5(g) \leq a_9(g) \leq \tilde{\lambda}_5(g)$$ and $\mu_g \geq 1$ for $g \in \Gamma_{E^*}$.

(C) Applying Lemma 8.5, we modify the coordinates so that the elements of $\mathcal{N}$ are of form:

$$k = \begin{pmatrix}
I_{n-i_0-1} & 0 & 0 \\
0 & 1 & 0 \\
C_1(v_k) & v_k^T & I_0 \\
c_2(v_k) & \frac{\|v_k\|^2}{2} & 1
\end{pmatrix}$$

where $C_1(v_k) = 0$. (8.49)

By group property, $\mathbf{v} \mapsto c_2(\mathbf{v})$ is a linear map.

We have coordinates so that $K'' \subset \mathbb{S}^{n-i_0-2}$. There exists a sequence of elements $z_i$ of $N_K \cap H$ in the virtual center $H$ so that a largest norm eigenvalue has a direction in $K''$ and $z_i|K'' \to I_{K''}$.

Since $\text{Cl}(U)$ is in properly convex $\text{Cl}(\tilde{\theta})$, it is in an affine patch where $v_\tilde{\theta}$ is the origin. Let $g_i \in \Gamma_{E^*}$ be the element going to $z_i$ under $\Pi_{E^*}$. Then $\{g_i(x)\}$ for a point $x$ of $U$ converges to $\langle \lambda \mathbf{a}, 0, \mathbf{w}, 1 \rangle$ for $\langle \mathbf{a} \rangle \in K''$ and some $\mathbf{w} \in \mathbb{R}^{i_0}$. Hence by (8.48) and

$$\langle \lambda \mathbf{a}, 0, \mathbf{w}, 1 \rangle \in \text{Cl}(U)$$

(8.50)

Since $z_i|K'' \to I_{K''}$, we may assume that an open subset of $K''$ can be realized as $\langle \mathbf{a} \rangle$ in the above.

By (8.49)

$$\mathcal{N}(v_1)^k \langle \lambda \mathbf{a}, 0, \mathbf{w}, 1 \rangle = \langle \lambda \mathbf{a}, 0, \mathbf{w}, k\lambda \mathbf{a} \cdot c_2(v_1) + k\mathbf{v}_1 \cdot \mathbf{w} + 1 \rangle$$

(8.51)

as $C_1(v_1) = 0$ for every $v_1 \in \mathbb{R}^{i_0}$. Suppose that $\lambda \mathbf{a} \cdot c_2(v_1) + \mathbf{v}_1 \cdot \mathbf{w} \neq 0$. Then as $k \to \infty$, $\{\mathcal{N}(k\mathbf{v}_1)(\langle \mathbf{a} \rangle)\}$ converges to a point and as $k \to -\infty$, it converges to its antipode. The limits form an antipodal pair of points in $\text{Cl}(U)$. This contradicts the proper convexity of $\tilde{\theta}$.

Hence, $\lambda \mathbf{a} \cdot c_2(v_1) + \mathbf{w} \cdot \mathbf{v}_1 = 0$ holds for every $v_1 \in \mathbb{R}^{i_0}$. We write $\tilde{c}_2$ as $(n-i_0) \times i_0$-matrix. Then $\mathbf{w}^T := -\lambda \tilde{c}_2^T \mathbf{a}$. Let $\tilde{K}$ denote the image of

$$\langle \mathbf{a} \rangle \mapsto \langle \lambda \mathbf{a}, 0, \mathbf{w}, 0 \rangle, \mathbf{w}^T := -\lambda \tilde{c}_2^T \mathbf{a}, \langle \mathbf{a} \rangle \in K''.$$

Under $\Pi_{K}$, the compact convex set $\tilde{K}$ embeds onto $K''$.

Since every point of $K''$ is a limit point of the orbit of a point of $K''$ under $z_i$, under $g_i \Gamma_{E^*} \cap \Pi_{K}^{i_0-1}(N_K \cap H)$, every point of $K''$ is a limit point of a point of $K''$. Hence, we obtain $\tilde{K} \subset \text{bd} \tilde{\theta}$ by convexity by (8.50).

Also, $\mathcal{N}$ acts on $\tilde{K}$ by our discussion and hence on $\tilde{K}$. We choose the coordinates so that $\tilde{K}$ corresponds to $x_1 = x_2 = \cdots = x_{n-i_0-1} = 0$. Under this coordinate system,

$$C_1(\mathbf{v}) = 0, c_2(\mathbf{v}) = 0 \text{ for every } \mathbf{v} \in \mathbb{R}^{i_0}.$$ (8.52)
(D) Consider a sequence \( \{g_t\} \) of elements \( g_t \in \Gamma_E \) with \( \{\Pi_E^t(g_t)(y)\} \) converging to \( x \in K'' \). We claim that every limit point \( x' \) of \( g_t(u) \) for \( u \in U \) is in \( K''' \): In our coordinates as above (8.52), we have \( x' = ((\lambda, a, 0, w, 1)) \). (8.51) still holds, and what follows tells us that \( v_1 \cdot w = 0 \) for every \( v_1 \in \mathbb{R}^{b_0} \). Thus, \( w = 0 \), and \( x' \in K''' \).

Since the set of such sequences are invariant under the conjugation by \( \Gamma_E \), it follows that the set of accumulations points of such sequence of elements in \( K''' \) is \( \Gamma_E \)-invariant. Since the set of accumulation points of \( g_t(a) \) for every attracting element \( (a) \) of a virtual center of \( N_K \) span \( K''' \), it follows that \( K''' \) is \( \Gamma_E \)-invariant. This implies that \( \hat{K} \) is \( (\Gamma_E, \mathcal{N}) \)-invariant.

We may assume in our chosen coordinates that

\[
C_1(g) = c_2(g) = C_1(v) = c_2(v) = 0 \quad \text{for every } g \in \Gamma_E, v \in \mathbb{R}^{b_0}.
\]

(8.53)

**Remark 8.5** We can also prove this result with an extra condition more easily: A virtual center of \( \Gamma_E \) maps to \( N_K \) going to a Zariski dense group of the virtual center of \( \text{Aut}(K) \). This was the earlier approach.

### 8.3.3 Strictly joined and quasi-joined ends for \( \mu \equiv 1 \)

We will now discuss joins and their generalizations in-depth in this subsection. That is, we will only consider when \( \mu = 1 \) for all \( g \in \Gamma_E \). We will use a hypothesis and later show that the hypothesis is true in our cases to prove the main results. Again, we assume the hypothesis virtually since it will be sufficient.

**Hypothesis 8.3.2** (\( \mu \equiv 1 \)) Let \( \Gamma_E \) be a p-end holonomy group. We continue to assume Hypothesis 8.3.1 for \( \Gamma_E \).

- Every \( g \in \Gamma_E \to M_\gamma \) is so that \( M_\gamma \) is in a fixed orthogonal group \( O(b_0) \). Thus, \( \mu_\gamma = 1 \) identically.
- \( \Gamma_E \) acts on a subspace \( \mathbb{S}^{b_0} \) containing \( v_E \) and the properly convex domain \( K''' \) in the subspace \( \mathbb{S}^{n-b_0-2} \) forming an independent pair with \( \mathbb{S}^{b_0} \) mapping homeomorphic to the factor \( K'' \) of \( K = \{k\} * K'' \) under \( \Pi_K \).
- \( \mathcal{N} \) acts on these two subspaces fixing every point of \( \mathbb{S}^{n-b_0-2} \).

Let \( H \) be the closed \( n \)-hemisphere defined by \( x_{n-b_0} \geq 0 \). Then by the convexity of \( \bar{\Sigma}_E \), we can choose \( H \) so that \( U \subset H'' \), \( K''' \subset H \) and \( \mathbb{S}^{b_0} \subset \text{bd}H \). We identify \( H'' \) with an affine space \( \mathbb{A}^n \). (See Section 2.1.5.)

By Hypothesis 8.3.2, elements of \( \mathcal{N} \) have the form of (8.16) with

\[
C_1(v) = 0, c_2(v) = 0 \quad \text{for all } v \in \mathbb{R}^{b_0},
\]

and the elements of \( \Gamma_E \) has the form of (8.42) with

\[
s_1(g) = 0, s_2(g) = 0, C_1(g) = 0, \quad \text{and } c_2(g) = 0.
\]
Again we recall the projection $\Pi_K : \mathbb{S}^n - \mathbb{S}_0^0 \to \mathbb{S}^{n-i_0-1}$. $\Gamma_E$ has an induced action on $\mathbb{S}^{n-i_0-1}$ and acts on a properly convex set $K''$ in $\mathbb{S}^{n-i_0-1}$ so that $K$ equals a strict join $k \ast K''$ for $k$ corresponding to $\mathbb{S}^{i_0+1}$. (Recall the projection $\mathbb{S}^n - \mathbb{S}_0^0$ to $\mathbb{S}^{n-i_0-1}$.)

We recall the invariants from the form of (8.43).

$$\alpha_I(g) := \frac{\alpha_I(g)}{\lambda_{v_\tilde{E}}(g)} - \frac{||v_\tilde{E}||^2}{2}$$

for every $g \in \Gamma_E$. It is easy to show $\alpha_I(g^n) = n\alpha_I(g)$ and $\alpha_I(gh) = \alpha_I(g) + \alpha_I(h)$, whenever $g, h, gh \in \Gamma_E$. (8.54)

Under Hypothesis 8.3.2, Lemma 8.6 shows that every $g \in \Gamma_E$ is of form:

$$\begin{pmatrix} S_g & 0 & 0 & 0 \\ 0 & \lambda_g & 0 & 0 \\ 0 & \lambda_g v_\tilde{E}^T & \lambda_g v_\tilde{E} S_5(g) \\ 0 & \lambda_g (\alpha_I(g) + \frac{||v_\tilde{E}||^2}{2}) & \lambda_g S_5(g) \\ \lambda_g \\ \end{pmatrix},$$

(8.55)

and every element of $\mathcal{N}$ is of form

$$\mathcal{N}(v) = \begin{pmatrix} I_{n-i_0-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & v^T & I_{i_0} & 0 \\ 0 & \frac{||v||^2}{2} & v & 1 \end{pmatrix}.$$ (8.56)

We assumed $\mu \equiv 1$. We define

$$\lambda_k(g) := \lambda_{v_\tilde{E}}(g)$$

for $g \in \Gamma_E$. (8.57)

We define $\lambda_{K''}(g)$ to be the maximal norm of the eigenvalue occurring for $S(g)$. We define $\Gamma_{E,+}$ to be a subset of $\Gamma_E$ consisting of elements $g$ so that the largest norm $\lambda_{\max}(g)$ of the eigenvalues occurs at the vertex $k$, i.e., $\lambda_k(g) = \lambda_k(g)$. Then since $\mu_k = 1$, we necessarily have $\lambda_{\max}(g) = \lambda_k(g)$. We also require that all other norms of the eigenvalues occurring at $K''$ is strictly less than $\lambda_k(g)$. The second largest norm $\lambda_2(g)$ equals $\lambda_{K''}(g)$. Thus, $\Gamma_{E,+}$ is a semigroup. The condition that $\alpha_I(g) \geq 0$ for $g \in \Gamma_{E,+}$ is said to be the **nonnegative translation condition**.

Again, we define

$$\mu_I(g) := \frac{\alpha_I(g)}{\lambda_k(g)}$$

for $g \in \Gamma_{E,+}$. 

The condition
\[ \mu_\gamma(g) > C_0, \quad g \in \Gamma_{E,+} \]
for a uniform constant \( C_0 > 0 \) \hspace{1cm} (8.58)
is called the uniform positive translation condition. (Heuristically, the condition means that we don’t translate in the negative direction by too much for bounded \( \lambda_{\gamma}(g) \).

**Lemma 8.7** The condition \( \alpha_\gamma(g) \geq 0 \) for \( g \in \Gamma_{E,+} \) is a necessary condition so that \( \Gamma_{E} \) acts on a properly convex domain.

**Proof** Suppose that \( \alpha_\gamma(g) < 0 \) for some \( g \in \Gamma_{E,+} \). Let \( k' \in K^\circ \). Now, we use (8.18) and see that \( \{g^n(U_{k'})\} \) converges geometrically to an \((i_0 + 1)-\)dimensional hemisphere since \( \{\alpha_\gamma(g^n)\} \to -\infty \) as \( n \to \infty \) implies that \( g \) translates the affine subspace \( H_{\gamma}^n \) a component to \( H_{\gamma}^n(k') \) toward \((-1,0,\ldots,0)\) in the above coordinate system by (8.55). Thus, \( \Gamma_{E} \) cannot act on a properly convex domain. (See Remark 8.4 also.) \( \square \)

From the matrix equation (8.55), we define \( v \) for every \( g \in (\Gamma_{E,+},\mathcal{N}) \). (We just need to do this under a single coordinate system.)

**Lemma 8.8** Given \( \Gamma_{E} \) satisfying Hypotheses 8.3.1 and 8.3.2, let \( \gamma \) be any sequence of elements of \( \Gamma_{E,+} \) so that \( \{\lambda_k(\gamma)/\lambda_{K^n}(\gamma)\} \to \infty \). Then we can replace it by another sequence \( \{g^{-1}_m\gamma_m\} \) for \( g_m \in \Gamma_{E} \) so that
\[ \|v_{g^{-1}_m\gamma_m}\| \text{ and } \Pi^\circ_K(g_m) \in \mathbf{Aut}(K) \]
are uniformly bounded, and
\[ \left\{ \frac{\lambda_k(g^{-1}_m\gamma_m)}{\lambda_{K^n}(g^{-1}_m\gamma_m)} \right\} \to \infty. \]

**Proof** Denote \( v_m := v_{\gamma_m} \). Suppose that \( N_K \) is discrete. Then the subgroup \( \Gamma_{E,+} \) of \( \Gamma_{E} \) acting on a complete affine leaf \( l \) acts cocompactly on \( l \). Since the stabilizer of \( N_K \) on each point of \( K^\circ \) is finite, \( \Gamma_{E} \cap \ker \Pi^\circ_K \) acts on \( l \) cocompactly. The action of \( \mathcal{N} \hat{O}(i_0) \) is proper on each leaf. Hence, \( \Gamma_{E} \cap \mathcal{N} \hat{O}(i_0) \) is a lattice in \( \mathcal{N} \hat{O}(i_0) \). By cocompactness of \( \Gamma_{E} \cap \mathcal{N} \hat{O}(i_0) \) in \( \mathcal{N} \hat{O}(i_0) \), we can multiply \( \gamma_m \) by \( g^{-1}_m \) for an element \( g_m \) of \( \Gamma_{E} \cap \mathcal{N} \hat{O}(i_0) \) nearest to \( \mathcal{N}(v_m) \). The result follows since the action on \( \mathcal{N}^{-1} \) is given by only \( v_x \) and \( S_x \) for \( g \in \Gamma_{E} \) as we can see from (8.55). The last convergence follows since \( S(g_m) = \mathbb{I}_{(i_0-2)} \) and the matrix multiplication form of \( g^{-1}_m \gamma_m \) considering the top left \((n-i_0) \times (n-i_0)\)-submatrix and the bottom right \((i_0+2) \times (i_0+2)\)-submatrix.

We now assume that \( N_K \) is non-discrete. \( \Sigma_{E} \) has a compact fundamental domain \( F \). Thus, given \( v_m \), for \( x \in F \),
\[ \mathcal{N}(v_m)(x) \in g_m(F) \text{ for some } g_m \in \Gamma_{E}. \]

Then \( g^{-1}_m \mathcal{N}(v_m)(x) \in F \). Since
\[ g_m(y) = \mathcal{N}(v_m)(x) \in g_m(F) \quad \text{for} \ y \in F \quad \text{and} \ x \in F, \]

it follows that

\[ d_K\left( \Pi_K(y), \Pi_K^*(g_m)(\Pi_K(y)) = \Pi_K(x) \right) < C_F \quad (8.59) \]

for a constant \( C_F \) depending on \( F \).

(i) \( g_m \) is of form of matrix \( (8.55) \)

(ii) \( S_{g_m} \) is in a bounded neighborhood of \( I \) by above \( (8.59) \) since \( \hat{S}_{g_m} \) moves a point of a compact set \( F \) to a uniformly bounded set. (This follows by considering the Hilbert metric.)

From the linear block form of \( g_{m}^{-1}N(v_m) \) and the fact that \( g_{m}^{-1}N(v_m)(x) \in F \), we obtain that the corresponding \( v_{g_{m}^{-1}N(v_m)} \) can be made uniformly bounded independent of \( v_m \).

For element \( \gamma_m \) above, we take its vector \( v_{\gamma_m} \) and find our \( g_m \) for \( N(v_{\gamma_m}) \). We obtain \( \gamma_m' = g_{m}^{-1}\gamma_m \). Then the corresponding \( v_{g_{m}^{-1}N(v_{\gamma_m})} \) is uniformly bounded as we can see from the block multiplications and the action on \( \tilde{\Sigma}_{E} \) in \( S_{E}^{-1} \). The final part follows from (ii) and the fact that \( \{\lambda_k(\gamma_m)/\lambda_k'(\gamma_m)\} \to \infty \) and the matrix multiplication form of \( g_{m}^{-1}\gamma_m \). Considering the top left \((n-i_0) \times (n-i_0)\)-submatrix and the bottom right \((i_0 + 2) \times (i_0 + 2)\)-submatrix.

**Lemma 8.9** Suppose that the holonomy group of \( \mathcal{O} \) is strongly irreducible. Given \( \Gamma \mathcal{E} \) satisfying Hypotheses 8.3.1 and 8.3.2, let \( U \) be the properly convex \( p \)-end neighborhood of \( \mathcal{E} \). Let \( H_k \) be the \( i_0 + 1 \)-hemisphere mapping to the vertex \( k \) of Hypothesis 8.3.2 under \( \Pi_K \). Then \( \text{Cl}(U) \cap H_k \) cannot contain an open domain \( B \) with \( \text{bd}B \ni \mathcal{E} \).

**Proof** Since \( \mathcal{N} \) acts on \( H_k \), it acts on \( B \) also. The matrix form of \( \mathcal{N} \) is given by the coordinates where \( H_k \) is the projectivization of the span of \( e_{n-k}, \ldots, e_{n+1} \) as we can see from \( (8.33) \) in the proof of Lemma 8.5. Hence, \( B \) is an ellipsoid as we can see from the form of \( \mathcal{N} \) in \( (8.56) \). First of all,

\[ \alpha_\gamma(h) = 0 \quad \text{for all} \ h \in \Gamma \mathcal{E} \quad (8.60) \]

by Lemma 8.7 since otherwise by \( (8.55) \)

\[ \{\gamma'(B)\} \to H_k \quad \text{as} \ i \to \infty \ \text{or} \ -\infty \ \text{for} \ \gamma \ \text{with} \ \alpha_\gamma(\gamma) \neq 0. \]

Since \( \tilde{\Sigma}_{E}/\Gamma \mathcal{E} \) is compact, we have a sequence \( \gamma_i \in \Gamma \mathcal{E}_{+} \) where

\[ \left\{ \frac{\lambda_k(\gamma_i)}{\lambda_k'(\gamma_i)} \right\} \to \infty, \ \alpha_\gamma(\gamma_i) = 0, \ \text{and} \ \{\gamma_i|\mathcal{K}'\} \ \text{are uniformly bounded}. \]

Now modify \( \gamma_i \) to \( g_i^{-1}\gamma_i \) by Lemma 8.8 for \( g_i \) obtained there. Hence, rewriting \( \gamma_i \) as the modified one, we have.
The uniform positive translation condition is equivalent to the existence of a spanned by $K$ of dimension $n \cdot h$ and any such matrix we can find an element of $\Gamma$ acting on $H$. By Proposition 2.18 for $l = 2$ case, $\text{Cl}(\tilde{\partial})$ equals the join of $H_{\text{max}}$ and $K'_{\text{max}}$. This implies that $\Gamma$ is virtually reducible by Proposition 2.17, contradicting the premise of the lemma.

For this proposition, we do not assume $N_k$ is discrete. The assumptions below are just Hypotheses 8.3.1 and 8.3.2. Also, we don’t need the assumption of the proper convexity of $\partial'$. 

**Proposition 8.3 (Quasi-joins)** Let $\Sigma_\xi$ be the end orbifold of an NPNC R-end $\tilde{E}$ of a strongly tame convex $n$-orbifold $\partial'$. Let $\Gamma_{\tilde{E}}$ be the p-end holonomy group. Let $\tilde{E}$ be an NPNC R-p-end and $\Gamma_{\tilde{E}}$ and $\mathcal{N}$ acts on a p-end-neighborhood $U$ fixing $v_{\tilde{E}}$. Let $K, K'', K''', S_k^{0}, S_k^{0+1}$ be as in Hypotheses (8.3.1) and (8.3.2). We assume that $K''/\Gamma_{\tilde{E}}$ is compact, $K = K'' \ast \{k\}$ in $S_k^{0} - S_k^{0+1}$ with a point $k$ corresponding to $S_k^{0+1}$ under the projection $\Pi_k$. Assume that

- $\Gamma_{\tilde{E}}$ satisfies the transverse weak middle-eigenvalue condition with respect to $v_{\tilde{E}}$.
- $\Gamma_{\tilde{E}}$ acts on $K''$ and $k$.
- $\mu_{\xi} = 1$ for all $g \in \Gamma_{\tilde{E}}$.
- Elements of $\Gamma_{\tilde{E}}$ and $\mathcal{N}$ are of form (8.33) and (8.34), with

$$C_1(v) = 0, c_2(v) = 0, C_1(g) = 0, c_2(g) = 0$$

for every $v \in S_k^{0}$ and $g \in \Gamma_{\tilde{E}}$.
- $\Gamma_{\tilde{E}}$ normalizes $\mathcal{N}$, and $\mathcal{N}$ acts on $U$ and each leaf of $\mathcal{F}_{\tilde{E}}$ of $\tilde{E}$.

Then the following hold:

(i) The uniform positive translation condition is equivalent to the existence of a properly convex p-end-neighborhood $U'$ whose closure meets $S_k^{0+1}$ at $v_{\tilde{E}}$ only. This condition furthermore is equivalent to $\tilde{E}$ being quasi-joined p-end.

(ii) $\alpha_1$ is identically zero if and only if $C_\mathcal{H}(U)$ is the interior of the join $K'' \ast B$ for an open ball $B$ in $S_k^{0+1}$, and $C_\mathcal{H}(U)$ is properly convex.
(iii) Suppose that $\tilde{E}$ is a quasi-joined $p$-end. Then a properly convex $p$-end neighborhood $U'$ is radially foliated by line segments from $v_{\tilde{E}}$, and $\bd U' \cap \tilde{\partial}$ is strictly convex with limit points only in $K'' \ast \{v_{\tilde{E}}\}$.

(iv) $\mathcal{C}(U) \cap A^n = K'' \ast \{v_{\tilde{E}}\}$.

**Proof** (A) We will first find some coordinate system and describe the action of $\Gamma_{\tilde{E}}$ on it.

Let $H$ be the unique $n$-hemisphere containing segments in directions of $\tilde{\Sigma}_{\tilde{E}}$ from $v_{\tilde{E}}$ where $\partial H$ contains $S^n_0$ and $K''$ in general position by Hypothesis 8.3.2. Then $H^o$ is an affine subspace to be also denoted by $A^n$ containing $U$. Since $\Gamma_{\tilde{E}}$ and $\mathcal{N}$ act on $K''$ and $S^n_0$, $\Gamma_{\tilde{E}}$ and $\mathcal{N}$ act as an affine transformation group on $A^n$.

Let $H_l$ denote the hemisphere with boundary $S^n_0$ and corresponding to a leaf $l$ of the foliation on $\tilde{\Sigma}_{\tilde{E}}$. Recall that $K''$ denote the leaf space. Let $A^n$ be the affine space whose boundary contains $S^n_0$ and a leaf $l$ is the $i_0 + 1$-dimensional affine subspace with transverse affine space of dimension $n - (i_0 + 1)$ meeting it at a point, considered as the origin. We may further require from $v_{\tilde{E}}$ the space of directions to $A^n$ contains those to $U$. Furthermore, the projection to the affine space of dimension $n - i_0 - 1$ with kernel vector space parallel to $l$, we obtain an affine space $A^{n - i_0 - 1}$ with an origin. The image of the projection of leaves is projectively diffeomorphic to $K''$. We can consider $K''$ as a cone from the origin in $A^{n - i_0 - 1}$. The directions of the cone is isomorphic to $K''$. We may regard $K''$ as a subset of the ideal boundary of $A^{n - i_0 - 1}$. Hence, we projectively identify

$$
\bigcup_{l \in K'} H_l^o = K'' \times \mathbb{R}^{i_0 + 1} \subset A^n.
$$

(Thi might be helpful to see Figure 8.1 where we convert the coordinates so that $(0,0,0)$, $S^n_1$, and $K$ are now in $\bd A^n$.)

There exists a family of affine subspaces in $A^n$ parallel to $A^{n - i_0 - 1}$. Also, there is a transverse family of affine spaces forming a foliation $\mathcal{V}^{i_0 + 1}$ where each leaf is the complete affine space parallel to $A^{i_0 + 1}$. We denote it by $\{(x_1, 1)\} \times \mathbb{R}^{i_0 + 1}$ for $x_1 \in \mathbb{R}^{n - i_0 - 1}$. The affine coordinates are given by $(x_1, 1, x_2, x_{n+1})$ where $1$ is at the $n - i_0$th position, $x_1$ is an $n - i_0 - 1$-vector and $x_2$ is an $i_0$-vector.

Now we describe $H_l^o \cap U$ for each $l \in K''$. We use the affine coordinate system on $A^n$ so that $H_l^o$ are parallel affine $i_0 + 1$-dimensional spaces with origins in $A^{n - i_0 - 1}$. We use the parallel affine coordinates. According to the matrix form (8.56), $\mathcal{N}$ acts on each $x \times \mathbb{R}^{i_0 + 1}$, $x \in K''$.

We denote each point in $H_l^o$ by $(\tilde{x}, x_{n+1})$ where $\tilde{x}$ is a point of $A^{i_0}$. Each of $E_l := H_l \cap U$ is given by

$$
x_{n+1} > \|x_2\|^2 / 2 + C_l, C_l \in \mathbb{R}
$$

(8.61)

since $\mathcal{N}$ acts on each where $C_l$ is a constant depending on $l$ and $U$ by (8.43). (The vertex $v_{\tilde{E}}$ corresponds to the ideal point in the positive infinity in terms of the $x_{n+1}$ coordinate. See Section 8.3.1.1.)
There is a family of quadrics of form $Q_{l,c}$ defined by $x_{n+1} = \|x_2\|^2/2 + C$ for each $C \in \mathbb{R}$ on each leaf $l$ of $\mathcal{V}^{b+1}$ using the affine coordinate system. The family form a foliation $\mathscr{D}_l$ for each $l$.

Now we describe $\Gamma_E$-action. Since $\mu_\eta = 1$ for all $g \in \Gamma_E$, it follows that $\lambda_\eta = \lambda_k$ by definition (8.57). Given a point $x = \langle \mathbf{v} \rangle \in U' \subset \mathbb{S}^n$ where $\mathbf{v} = \mathbf{v}_s + \mathbf{v}_h$ where $\mathbf{v}_s$ is in the direction of $\mathbb{S}(K''')$ and $\mathbf{v}_h$ is in one of $\mathbb{S}_{b+1}$. If $g \in \Gamma_{E,+}$, then we obtain

$$g \langle \mathbf{v} \rangle = \langle g\mathbf{v}_s + g\mathbf{v}_h \rangle \text{ where } \langle g\mathbf{v}_s \rangle \in K''' \text{ and } \langle g\mathbf{v}_h \rangle \in H_k. \quad (8.62)$$

Let $\Pi_{l_0} : U \to \mathbb{R}^{b+1}$ be the projection to the last $i_0 + 1$ coordinates $x_{n-i_0}, \ldots, x_n$.

We obtain a commutative diagram and an affine map $L_g$ induced from $g$

$$
\begin{array}{ccc}
H'_0 & \xrightarrow{g} & g(H'_0) \\
\Pi_{l_0} \downarrow & & \downarrow \Pi_{l_0} \\
\mathbb{R}^{b+1} & \xrightarrow{L_g} & \mathbb{R}^{b+1}.
\end{array}
$$

By (8.43), $L_g$ preserves the family of quadrics $\mathscr{D}_l$ to $\mathscr{D}_{g(l)}$ since $\mathscr{N}$ acts on the quadrics $U \cap H'_0$ for each $l$ and $g$ normalizes $\mathscr{N}$ by Hypothesis 8.3.1. Also, $L_g$ is an affine map since $L_g$ is a projective map sending a complete affine subspace $H'_0$ to a complete affine subspace $g(H'_0)$. Finally, by (8.43), $g$ sends the family of quadrics shifted in the $x_{n+1}$-direction by $\alpha_l(g)$ from $l$ to $g(l)$ using the coordinates $(x, x_{n+1})$ for $x \in \mathbb{R}^b$. That is,

$$g : Q_{l,c} \mapsto Q_{g(l),c+\alpha_l(g)}. \quad (8.64)$$

(B) Now we give proofs. By assumption, $\Gamma_E$ acts on $K = K''' \setminus \{k\}$. Choose an element $\eta \in \Gamma_{E,+}$ by Proposition 2.15 so that $\lambda_{\eta_{\max}}(\eta) > \lambda_2(\eta)$ where $\lambda_1(\eta)$ corresponds to a vertex $k$ and $\lambda_2(\eta)$ is associated with $K'''$, and let $F$ be the fundamental domain in the convex open cone $K''$ with respect to $\langle \eta \rangle$, which is a bounded domain in $\mathbb{A}^{n-i_0-1}$. This corresponds to a radial subset from $v_E$ bounded away at a distance from $K'''$ in $U$.

(i) This long proof will be divided as follows.

(i-a) Forward part: We show $U$ has a convex hull that is properly convex.

(i-a-1) We show that the forward orbit $\langle \eta \rangle$ of the fundamental domain of $U$ under $\langle \eta \rangle$ is contained in a simplex.

(i-a-2) Next, we will try to show that the backward orbit of $\langle \eta \rangle$ is in a neighborhood of $K''' \setminus \{\nu_E\}$.

(i-a-3) Finally, we will show that the convex hull of $U$ is in a properly convex domain.

(i-b) Converse part: We prove the uniform positive translation condition under the assumption.

(i-b) (i-a-1) Let $\lambda_{K'''}(g)$ denote the maximal eigenvalue associated with $K'''$ for $g \in \Gamma_E$. Choose $x_0 \in F$. Let $\Gamma_{E,F} := \{g \in \Gamma_E | g(x_0) \in F\}$. For $g \in \Gamma_{E,F}$,
\[ -C_F < \log \frac{\lambda_{KF}(g)}{\lambda_{K''}(g)} < C_F \]

(8.65)

for a uniform \( C_F > 0 \) a number depending of \( F \) only: Otherwise, we can find

- a sequence \( g_i \) with \( g_i(x_0) \in F \) such that \( \{\lambda_k(g_i)/\lambda_{K''}(g_i)\} \to 0 \) or
- another sequence \( g'_i \) with \( g'_i(x_0) \in F \) such that \( \{\lambda_k(g'_i)/\lambda_{K''}(g'_i)\} \to \infty \).

However, in the first case, let \( \tilde{x}_0 \in U \) be a point mapping to \( x_0 \) under \( \Pi_K \). Then \( \{g_i(\tilde{x}_0)\} \) accumulates only to points of \( K'' \) by Proposition 2.8, which is absurd. The second case is also absurd by taking \( \{g^{-1}_i(x_0)\} \) instead.

Therefore, given \( g \in \Gamma_{E,F} \), we can find a number \( i_0 \in \mathbb{Z} \) dependent only on \( F \) and \( g \) such that \( \eta_{i_0}g \in \Gamma_{E,+} \) since \( \log \frac{\lambda_k(\eta)}{\lambda_{K''}(\eta)} \) is a constant bigger than 1. Now, \( \alpha(\eta_{i_0}g) \) is bounded below by some negative number by the uniform positive eigenvalue condition (8.58) and the fact that \( \log \frac{\lambda_k(\eta_{i_0}g)}{\lambda_{K''}(\eta_{i_0}g)} \) is also uniformly bounded. Since \( \alpha(\eta_{i_0}g) = i_0\alpha(\eta) + \alpha(g) \), we obtain

\[ \{\alpha(g)|g \in \Gamma_{E,F}\} > C \]

(8.66)

for a constant \( C \) by (8.58). \( F \) is covered by \( \bigcup_{g \in \Gamma_{E,F}} g(J) \) for a compact fundamental domain \( J \) of \( K'' \) by \( N_K \).

We will be using a fixed affine coordinate system on \( H_{k'}^o \) parallel under the translations preserving \( \mathbb{A}^{n-\tilde{i}_0-1} \). In the above affine coordinates for \( k' \in F \) of (8.63), the matrix form of (8.55) shows that \( g \in \Gamma_{E,F} \) send paraboloids in affine subspaces \( H_{k''}^o \) for \( k'' \in K'' \) to paraboloids in \( H_{g(k')}^o \) for \( g \in \Gamma_{E} \). (See Section 8.3.1.1.) Now,

\[ x_n(H_{k'}^o \cap U) > C \]

(8.67)

for a uniform constant \( C \in \mathbb{R} \) by (8.66) and the fact that \( H_{k''}^o \) is the image of \( H_{k'}^o \) for \( k'' \in J \) by an element \( g \in \Gamma_{E} \). (See Remark 8.4.)

Since by (8.67),

\[ \bigcup_{k' \in J} \bigcup_{g' \in \Gamma_{E,F}} g(H_{k'}^o \cap U) \]

is a lower-\( x_n \)-bounded set, its convex hull \( D_F \) as a lower-\( x_n \)-bounded subset of \( K'' \times \mathbb{A}^{n-\tilde{i}_0-1} \subset \mathbb{A}^n \). Each region \( D_F \cap H_F \) is contained an \((\tilde{i}_0 + 1)\)-dimensional simplex \( \sigma_0 \) with a face in the boundary of \( H_F \). Since there is a lower \( x_n \)-bound, we may use one \( \sigma_0 \) and translations to contain every \( U \cap H_F \) in \( D_F \).

Therefore, the convex hull \( D_F \) in \( \text{Cl}(\tilde{O}) \) is a properly convex set contained in a proper convex set \( F \times \sigma_0 \).

On \( K'' \), the sequence of norms of eigenvalues of \( \eta^i \) converges to 0 as \( i \to \infty \) and the eigenvalue \( \lambda_{k''} \) at \( \tilde{g}^{\tilde{i}_0+1}_k \) goes to \( +\infty \). Since

\[ \alpha(\eta^i) = i\alpha(\eta) \to +\infty \text{ as } i \to \infty \]

(8.68)

we obtain
Suppose that we have \(\eta'(D_F)\) \(\rightarrow \{v_F\}\) for \(i \rightarrow \infty\) geometrically, i.e., under the Hausdorff metric \(d_H\) by (8.62). Again, for a sufficiently large integer \(I\), because of the lower bound on the \(x_n\)-coordinates, we obtain

\[
\bigcup_{i \geq I} \eta^i(D_F) \subset K^0 \times c_0,
\]

which is our first main result of this proof of the forward part of (i).

(i-a-2) For each \(k' = (x_1, 1) \in K^0\), we can find a point in \(\Pi_k^{-1}(k')\) of form \((x_1, 1, 0, C_{k'})\) on \(\partial U \cap \mathbb{A}^n\) for \(0\) a zero vector in \(\mathbb{R}^6\) and \(C_{k'} \in \mathbb{R}\). Using the \(\mathcal{N}\)-action, we can parameterize \(\partial U \cap \mathbb{A}^n\) starting from the point \((x_1, 1, 0, C_{k'})\). The \(\mathcal{N}\)-orbit of this point is given by

\[
\left(x_1, 1, v, \|v\|^2 / 2 + C_{k'} \right), \ v \in \mathbb{R}^6.
\] (8.70)

Let

\[p_i := \left(x_1(p_i), 1, v(p_i), \|v(p_i)\|^2 / 2 + C_{k'}(p_i) \right).\]

We form a sequence \(\{p_i\}\) of points on \(\partial U\) for \(k'(p_i) = (x_1(p_i), 1)\). Consider \(\eta^i\) in the form (8.55). Since \(\bigcup_{i \in \mathbb{Z}} \eta^i(F)\) covers \(K^0\), for each \(p_i\) there is an integer \(j_i\) for which \(\eta^j_i(F)\) containing \((x_1(p_i), 1)\). Suppose that \(j_i \rightarrow \infty\) as \(i \rightarrow \infty\). From considerations of \(\mathcal{N}(−j_i v_\eta)\eta^{j_i}(F)\), we deduce that

\[
\left\{C_{(x_1(p_i), 1)}\right\} \rightarrow +\infty \text{ as } \|x_1(p_i)\| \rightarrow 0
\]

by (8.68).

Similarly, suppose that \(j_i \rightarrow -\infty\) as \(i \rightarrow \infty\). We deduce that

\[
\left\{C_{(x_1(p_i), 1)} \left\|v(p_i)\right\| \right\} \rightarrow 0 \text{ as } \|x_1(p_i)\| \rightarrow \infty
\]

(8.72)
since \(\alpha_j(\eta^{j_i}) = j_i \alpha_j(\eta) \rightarrow -\infty\) in a linear manner, and every sequence of \(\|x_1(p_i)\|\) coordinates of points of \(\eta^{j_i}(F)\) grows uniformly exponentially as \(j_i \rightarrow -\infty\).

We will now try to find all limit points of \(\{p_i\}\):

- Suppose first that \(\|x_1(p_i)\| \rightarrow \infty\) and \(\frac{\|x_1(p_i)\|}{\|v(p_i)\|^2} \rightarrow \infty\). We obtain that

\[
\left\{\left(x_1(p_i), 1, v(p_i), \|v(p_i)\|^2 / 2 + C_{k'}(p_i)\right) : v(p_i) \in \mathbb{R}^6\right\}
\]

has only limit points of form \(\langle u, 0, 0, 0 \rangle\) for a unit vector \(u\) in the direction of \(K''\) by (8.72). Also, every direction of a point of \(K''\) occurs as a direction of \(u\) for a limit point by taking a sequence \(\{p_i\}\) so that \(\{x_1(p_i)\}\) converges to \(u\) in directions. Hence, the limit is in \(K''\).

- Suppose that we have \(\|x_1(p_i)\| \rightarrow +\infty\) with \(\|v(p_i)\|^2 \rightarrow +\infty\) with their ratios bounded between two real numbers. Then a limit point is of form \(\langle u, 0, 0, C \rangle\) for some \(C > 0\) and a vector \(u\) in the direction a point of \(K''\) by (8.72). Also,
every direction of $K''$ occurs as a direction of $u$ for a limit point by taking a sequence $\{p_i\}$ so that $\{x_i(p_i)\}$ converges to $u$ in directions. Hence, the limits are in $K'' \ast \{v_E\}$.

- Suppose that $\|x_i(p_i)\| \to +\infty$ with $\|v(p_i)\| \to +\infty$ and $\frac{\|x_i(p_i)\|}{\|v(p_i)\|} \to 0$. Then the only limit point is $(0,0,0,1)$ since the last term dominates others.
- Suppose that $1/C' \leq \|x_i(p_i)\| \leq C'$ for a constant $C'$. If $\|v(p_i)\|$ is uniformly bounded, then (8.67) shows that limit points in $\partial U$. If $\|v(p_i)\| \to +\infty$, the only limit point is $(0,0,0,1)$.
- Suppose that $\|x_i(p_i)\| \to 0$. Then the limit is $(0,0,0,1)$ by (8.71).

These gives all the limit points of $U$ in $\partial \mathbb{A}^n$ as we can easily deduce. Therefore, $\partial U \cap \partial \mathbb{A}^n$ is $K'' \ast \{v_E\}$. Since $\text{Cl}(U) \cap \partial \mathbb{A}^n = \partial U \cap \partial \mathbb{A}^n$, we obtain

$$\text{Cl}(U) \cap \partial \mathbb{A}^n = K'' \ast \{v_E\}.$$  

(8.73)

This also shows $\eta^i(D_F)$ geometrically converges to $K'' \ast \{v_E\}$ as $i \to -\infty$. First, the above three items show that for every $\varepsilon > 0$, there exists $i_0$ so that $\eta^i(D_F) \subset N_\varepsilon(K'' \ast \{v_E\})$ for $i > i_0$. (It is equivalent to the statement that all accumulations points are in $K'' \ast \{v_E\}$.) Finally, since $\Pi_k(\eta^i(D_F))$ geometrically converges to $K''$, and we can find a sequence as in the first item converging to any point of $K''$, the geometric limit is $K'' \ast \{v_E\}$.

For every $\varepsilon > 0$, there exists an integer $I$ so that

$$\bigcup_{i < I} \eta^i(D_F) \subset N_\varepsilon(K'' \ast v_E) \cap \text{Cl}(\mathbb{A}^n).$$

(i-a-3) Thus, except for finitely many $i$,

$$\eta^i(D_F) \subset (N_\varepsilon(K'' \ast v_E) \cap \text{Cl}(\mathbb{A}^n)) \cup K \times \sigma_0 \subset \text{Cl}(\mathbb{A}^n).$$

We use the Fubini-Study metric $d$ on $S^n$ where the subspaces spanned and $K$ and $\{k\} \times \mathbb{R}^{i_0}$ are all orthogonal to one another and $k \times O$ for the origin $O$ of $\mathbb{R}^{i_0}$ is in the orthogonal direction to $\partial \mathbb{A}^n$.

Assume $\varepsilon < \pi/8$. Let $p$ denote the vertex of the simplex $\{k\} \times \sigma_0$. We may assume without loss of generality that $p = k \times O$. Then we may assume that $N_\varepsilon(K'' \ast v_E) \cap \text{Cl}(\mathbb{A}^n)$ is in $\hat{K} \times \{k\}$ where $\hat{K}$ is a properly compact $(n-1)$-ball containing $K'' \ast \{v_E\}$ and is contained its $2\varepsilon \cdot d$-neighborhood. $\sigma_\infty := \sigma_0 \cap \partial \mathbb{A}^n$ is an $i_0$-simplex containing $v_E$. The join $\sigma_\infty \ast K''$ is properly convex since $\sigma_\infty$ and $K''$ are properly convex sets in independent subspaces. The join $\sigma_\infty \ast \hat{K}$ is contained in a $2\varepsilon \cdot d$-neighborhood of $\sigma_\infty \ast K'' = \sigma_\infty \ast \sigma_\infty' \ast \{v_E\}$. Hence, for a choice of $\varepsilon$, $\sigma_\infty \ast \hat{K}$ is properly convex. Since $\{k\} \times \sigma_0 = \sigma_\infty \ast \{p\}$, we obtain that $\sigma_0 \ast K''$ is also properly convex. Hence, $\sigma_0 \times \hat{K}$ is also properly convex for a sufficiently small $\varepsilon$. Thus, for a finite set $L$, the convex hull $U_1$ of $\bigcup_{i \in \mathbb{Z}} \eta^i(D_F)$ in $\mathbb{A}^n$ is properly convex.

The convex hull of $U_1$ and $U_L := \bigcup_{i \in L} \eta^i(D_F)$ is still properly convex: Suppose not. Then there exists an antipodal pair in

$$\text{Cl}((C \cdot \mathbb{H}(U_1 \cup U_L)) = C \cdot \mathbb{H}(\text{Cl}(U_1) \cup \text{Cl}(U_L)).$$
These points are in the interior of two respective simplices $\sigma_1$ and $\sigma_2$ with vertices in $\text{Cl}(U_1) \cup \text{Cl}(U_L)$ as follows by the fact that the union of such simplices is the convex hull. (See Section 2.1.7.1.) Since $\sigma_1$ and $\sigma_2$ are in the hemisphere $\text{Cl}(A^n)$, they must be in the boundary $\text{bd}A^n$. The vertices of $\sigma_1$ and $\sigma_2$ are in $\text{bd}A^n$. However, since $\text{Cl}(U_1) \cap \text{bd}A^n$ is a properly convex set $K'' \star \{v_E\}$ and $\text{Cl}(U_L) \cap \text{bd}A^n$ is $\{v_E\}$. This is a contradiction.

Let $U'$ denote the convex hull of $U_1 \cup U_L$ in $A^n$. Hence, we showed that $U'$ is properly convex.

(i-b) Now we prove the converse part of (i). Suppose that $\Gamma_E$ acts on a properly convex p-end-neighborhood $U'$.

By Lemma 8.7, we have $\alpha_l(g) \geq 0$ for $g \in \Gamma_{E,+}$. Suppose that $\alpha_l(g) = 0$ for some $g \in \Gamma_{E,+}$. Then

$$\{g^i(\text{Cl}(U) \cap H_i)\} \to B$$

for a leaf $l$ and a compact domain $B$ at $H_k$ bounded by an ellipsoid. This contradicts the premise of (i). Therefore,

$$\mu_l(h) > 0 \text{ for every } h \in \Gamma_{E,+}. \quad (8.74)$$

Suppose that $\{\mu_l(g_i)\} \to 0$ for a sequence $g_i \in \Gamma_{E,+}$. We can assume that

$$\lambda_{\text{max}}^T(g_i)/\lambda_2(g_i) > 1 + \varepsilon \text{ for a positive constant } \varepsilon > 0$$

since we can take powers of $g_i$ not changing $\mu_l$.

Since $\{\mu_l(g_i)\} \to 0$, we obtain a nondecreasing sequence $\{n_i\}$, $n_i > 0$, so that

$$\{\alpha_l(g_i^{n_i}) = n_i\alpha_l(g_i)\} \to 0 \text{ and } \{\lambda_{\text{max}}^T(g_i^{n_i})/\lambda_2(g_i^{n_i})\} \to \infty.$$ 

However, from such a sequence, we use (8.43) to shows that

$$\{g^i(\text{Cl}(U) \cap H_i)\} \to B$$

to a ball $B$ with a nonempty interior in $H_k$. Again the premise contradicts this. Hence $\mu_l(g) > C$ for all $g \in \Gamma_{E,+}$ and a uniform constant $C > 0$. This proves the converse part of (i).

(ii) Suppose that $\alpha_l$ is identically zero for $\Gamma_{E,+}$. Then by (8.74) in the proof of (i),

$$\{g^i(\text{Cl}(U) \cap H_i)\} \to B$$

for a leaf $l$ and a compact domain $B$ at $H_k$ bounded by an ellipsoid. Choose $\hat{B}$ the maximal domain of form $B$ as arising from the situation. Then we may show that $\mathcal{C}(\mathcal{H}(U) = (K'' \star \hat{B})^\alpha$ by Proposition 2.18 since we can find a sequence $g_i$ so that $g_i(K'' \star \hat{B})$ is bounded and $\{\lambda_{K''}(g_i)/\lambda_{\hat{B}}(g_i)\} \to \infty$ since $(K'' \star k)^\alpha/\Gamma_E$ is compact. Also, $(K'' \star \hat{B})^\alpha$ is properly convex.

Conversely, we have $\alpha_l \geq 0$ by Lemma 8.7. By premise $\mathcal{C}(\mathcal{H}(U) = (K'' \star B)^\alpha$, where $B$ is a convex open ball in a hemisphere $H_k$ in $S^d_k$. Since $\mathcal{N}$ acts on $\mathcal{C}(\mathcal{H}(U)$,
8.3. The general theory

B bounded by an ellipsoid. If $\alpha\{g\} > 0$ for some g, then g acts on B so that g(B) is a translated image of the region B bounded by a paraboloid in the affine subspace $H_k^\alpha$. We obtain $\bigcup_{k=1}^\infty g^{-k}(B) = H_k^\alpha$. This contradicts the proper convexity, and hence $\alpha\{g\}$ is identically 0.

(iii) Since $\tilde{O}$ is convex, we can find a radial p-end neighborhood $U$ of $\tilde{E}$. Let p be a point of U. Then the orbit of p has limit points only in $K''\ast v_{\tilde{E}}$ in (i-b). Then we take the convex hull $U'$ of $\bigcup \Gamma\tilde{E}(p_{\tilde{E}}v_{\tilde{E}})$. Since this is a convex set, it is a radial p-end neighborhood of $\tilde{E}$. bd$U'$ is the boundary of a properly convex domain Cl$(U')$, and hence the union of $(n-1)$-dimensional compact convex domains.

The boundary bd$U'$ of $\tilde{E}$ is a union of compact simplices since it contains no straight segment ending at a point of $K''\ast v_{\tilde{E}}$. Hence, bd$U'$ is a polyhedral hypersurface.

Since $R_p(\tilde{O}) = \tilde{\Sigma}_{\tilde{E}}$ is identical with $R_p(U')$ by Lemma 1.1.

Again, sharply supporting hyperspaces of bd$U' \cap \Lambda^n$ at a fundamental domain under $\Gamma_{\tilde{E}}$ is a compact set. Since bd$U' \cap \Lambda^n$ does not contain segment ending in bd$\Lambda^n$, there are no sequence of supporting hyperplanes at a point $q_i$ where $q_i$ forms an unbounded sequence and meeting a fixed neighborhood of a point p in bd$U' \cap \Lambda^n$. Hence, Lemma 5.16 implies that we can choose a properly convex open p-end neighborhood $U'$ of $\tilde{E}$ so that bd$U' \cap \Lambda^n$ is strictly convex.

(iv) This is proved in (8.73). □

Definition 8.4

- Generalizing Example 8.1, an R-p-end $\tilde{E}$ satisfying the case (ii) of Proposition 8.3 is a strictly joined R-p-end (of a totally geodesic R-end and a horospherical end) and $\Gamma_{\tilde{E}}$ now is called a strictly joined end group. Also, any end finitely covered by a strictly joined R-end is called a strictly joined R-end.

- An R-p-end $\tilde{E}$ satisfying the case (i) of Proposition 8.3, is a quasi-joined R-p-end (of a totally geodesic R-end and a horospherical end) corresponding to Definition 8.1 and $\Gamma_{\tilde{E}}$ now is a quasi-joined end holonomy group.

Also, any p-end $\tilde{E}$ with $\Gamma_{\tilde{E}}$ is a finite-index subgroup of $\Gamma_{\tilde{E}}$ as above is called by the corresponding names.

8.3.3.1 The non-existence of strictly joined cases for $\mu \equiv 1$.

Corollary 8.3 Let $\Sigma_{\tilde{E}}$ be the end orbifold of an NPNC R-p-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $\tilde{O}$ with radial or totally geodesic ends. Assume that the holonomy group of $\tilde{O}$ is strongly irreducible. Let $\Gamma_{\tilde{E}}$ be the p-end holonomy group. Assume Hypotheses 8.3.1 only and $\mu_g = 1$ for all $g \in \Gamma_{\tilde{E}}$. Then $\tilde{E}$ is not a strictly joined end.

Proof Suppose that $\tilde{E}$ is a strictly joined end. By premise, $\mu_g = 1$ for all $g \in \Gamma_{\tilde{E}}$. By Lemma 8.5 and Proposition 8.2, every $g \in \Gamma_{\tilde{E}}$ is of form:
As in the proof of
By Proposition 8.2, we obtain a sequence $\gamma_m$ of form from the step (D) of the proof:

\[
\begin{pmatrix}
\delta_m S_m & 0 & 0 & 0 \\
0 & \lambda_m & 0 & 0 \\
0 & \lambda_m v_m^T & \lambda_m O_5(\gamma_m) & 0 \\
0 & \lambda_m \left( \alpha_7(\gamma_m) + \frac{\|v_m\|^2}{2} \right) & \lambda_m v_m O_5(\gamma_m) & \lambda_m
\end{pmatrix}
\quad (8.76)
\]

as $C_{1,m} = 0$ and $c_{2,m} = 0$ where

- $\{\lambda_m\} \to \infty$,
- $\{\delta_m\} \to 0$,
- $\{S_m\}$ is in a sequence of bounded matrices in $\text{SL}_+ (n-i_0-1)$, and
- $\alpha_7(\gamma_m) = 0$ by Proposition 8.3 (ii).

Moreover, Hypothesis 8.3.2 now holds. By Lemma 8.9, we obtain a contradiction. □

### 8.3.4 The proof for discrete $N_K$.

Now, we go to proving Theorem 8.2 when $N_K$ is discrete. By taking a finite-index torsion-free subgroup if necessary by Theorem 2.3, we may assume that $N_K$ acts freely on $K^\circ$. We have a corresponding fibration

\[
l/N \to \tilde{\Sigma}_E / \Gamma_E
\]

\[
\downarrow
\]

\[
K^\circ / N_K
\]

(8.77)

where the fiber and the quotients are compact orbifolds since $\Sigma_E$ is compact. Here the fiber equals $l/N$ for generic $l$. The action of $N_K$ on $K$ is semisimple by Theorem 3 of Vey [161].

Since $N$ acts on each leaf $l$ of $\mathcal{F}_E$ in $\tilde{\Sigma}_E$, it also acts on a properly convex domain $\partial \tilde{\Sigma}_E$ and $v_E$ in a subspace $\mathbb{H}^{i_0+1}$ in $\mathbb{H}^n$ corresponding to $l$. $l/N \times \mathbb{R}$ is an open real projective orbifold diffeomorphic to $(\mathbb{H}^{i_0+1} \cap \partial \tilde{\Sigma}_E) / N$ for an open hemisphere $\mathbb{H}_l^{i_0+1}$ corresponding to $l$. Since elements of $N$ restricts to $I$ on $K$, we obtain
Otherwise, we see easily \( g \) acts not trivially on \( S^{n-i_0-1} \). By Proposition 8.1, all the norms of eigenvalues are 1.

- Since \( I \) is a complete affine subspace, Lemma 4.1 shows that \( I \) covers a horospherical end of \((S^0_i)^{b_{i+1}} \cap \partial )/N\).
- By Theorem 4.2, \( N \) is virtually unipotent, and \( N \) is virtually a cocompact subgroup of a unipotent group and \( N|\_i^{b_{i+1}} \) can be conjugated into a maximal parabolic subgroup of \( SO(i_0 + 1, 1) \) in \( \text{Aut}(S^0_i^{b_{i+1}}) \) and acting on an ellipsoid of dimension \( i_0 \) in \( H^1_{i_0} \).

We verify Hypothesis 8.3.1.

By the nilpotent Lie group theory of Malcev [133], the Zariski closure \( \mathcal{Z}(N) \) of \( N \) is a virtually simply connected nilpotent Lie group with finitely many components and \( \mathcal{Z}(N)/N \) is compact. Let \( \mathcal{N} \) denote the identity component of the Zariski closure of \( N \) so that \( \mathcal{N}/(\mathcal{N} \cap N) \) is compact. \( \mathcal{N} \cap N \) acts on the great sphere \( S^0_i^{b_{i+1}} \) containing \( \mathcal{V}_E \) and corresponding to \( I \). Since \( \mathcal{N}/(\mathcal{N} \cap N) \) is compact, we can modify \( U \) so that \( \mathcal{N} \) acts on \( U \) by Lemma 2.5: i.e., we take the interior of \( \bigcap_{E \in \mathcal{N}} g(U) = \bigcap_{E \in \mathcal{F}} g(U) \) for the fundamental domain \( F \) of \( \mathcal{N} \) by \( N \).

Since \( \mathcal{G}_E \) normalizes \( N \), it also normalizes the identity component \( \mathcal{N} \).

By above, \( \mathcal{N}|\_i^{b_{i+1}} \) is conjugate to a parabolic subgroup of \( SO(i_0 + 1, 1) \) in \( \text{Aut}(S^0_i^{b_{i+1}}) \), and \( \mathcal{N} \) acts on \( U \cap \_i^{b_{i+1}} \), which is a horoball for each leaf \( I \) of \( \mathcal{Z}_E \).

By taking a finite-index cover of \( U \), we can assume that \( N \subset \mathcal{N} \) since \( \mathcal{Z}(N) \) is a finite extension of \( \mathcal{N} \). We denote the finite index group by \( \mathcal{G}_E \) again.

Since \( S^0_i^{b_{i+1}} \) corresponds to a coordinate \( i_0 + 2 \)-subspace, and \( S^0_i^{b_0} \) and \( \{V_E\} \) are \( \mathcal{G}_E \)-invariant, we can choose coordinates so that (8.14) and (8.16) hold. Hence, Hypothesis 8.3.1 holds.

**Theorem 8.3** Let \( \mathcal{Z}_E \) be the end orbifold of an NPNC \( R \)-end \( E \) of a strongly tame properly convex \( n \)-orbifold \( \mathcal{O} \) with radial or totally geodesic ends. Assume that the holonomy group of \( \pi_1(\mathcal{O}) \) is strongly irreducible. Let \( \mathcal{G}_E \) be the \( p \)-end holonomy group satisfying the transverse weak middle-eigenvalue condition with respect to \( R \)-end structure. Assume also that \( N_K \) is discrete, and \( K^1/N_K \) is compact and Hausdorff. Then \( E \) is a quasi-join of a totally geodesic \( R \)-end and a cusp \( R \)-end.

**Proof** By Lemma 8.3, \( h(g) \mathcal{N}(v) h(g)^{-1} = \mathcal{N}(vM_g) \) where \( M_g \) is a scalar multiplied by an element of a copy of an orthogonal group \( O(i_0) \).

The group \( \mathcal{N} \) is isomorphic to \( \mathbb{R}^{b_0} \) as a Lie group. Since \( N \subset \mathcal{N} \) is a discrete cocompact, \( N \) is virtually isomorphic to \( \mathbb{Z}^{b_0} \). Without loss of generality, we assume that \( N \) is a cocompact subgroup of \( \mathcal{N} \). By normality of \( N \) in \( \mathcal{G}_E \), we obtain \( h(g)Nh(g)^{-1} = N \) for \( g \in \mathcal{G}_E \). Since \( N \) corresponds to a lattice \( L \subset \mathbb{R}^{b_0} \) by the map \( \mathcal{N} \), the conjugation by \( h(g) \) corresponds to an isomorphism \( M_g : L \to L \) by Lemma 8.3. When we identify \( L \) with \( \mathbb{Z}^{b_0} \), \( M_g : L \to L \) is represented by an element of \( SL_{\pm}(i_0, \mathbb{Z}) \) since \((M_g)^{-1} = M_{g^{-1}} \). Also, by Lemma 8.3, \( \{M_g g \in \mathcal{G}_E\} \) is a compact group as their determinants equal \( \pm 1 \). Hence, the image of the homomorphism
given by \( g \in h(\pi_1(\tilde{E})) \mapsto M_g \in SL_\mathbb{Z}(i_0, \mathbb{Z}) \) is a finite order group. Moreover, \( \mu_g = 1 \) for every \( g \in \Gamma_{\tilde{E}} \) as we can see from Lemma 8.3. Thus, \( \Gamma_{\tilde{E}} \) has a finite index group \( \Gamma'_{\tilde{E}} \) centralizing \( \mathcal{N} \).

We can now use Proposition 8.2 by letting \( G = N \) since \( N_K \) is discrete and \( \bar{N}_K = N_K \). We take \( \Sigma_{\tilde{E}'} \) to be the corresponding cover of \( \Sigma_{\tilde{E}} \). By Lemma 8.5 and Proposition 8.2, Hypothesis 8.3.2 holds, and we have the result needed to apply Proposition 8.3. Finally, Proposition 8.3(i) and (ii) imply that \( \Gamma_{\tilde{E}} \) virtually is either a join or a quasi-joined group. Corollary 8.3 shows that a strictly joined end cannot occur.

\[ \square \]

### 8.4 The non-discrete case

This is in part a joint work with Y. Carriere. Let \( \Sigma_{\tilde{E}} \) be the end orbifold of an NPNC R-end \( \tilde{E} \) of a strongly tame properly convex \( n \)-orbifold \( \mathcal{O} \) with radial or totally geodesic ends. Let \( \Gamma_{\tilde{E}} \) be the p-end holonomy group. Let \( U \) be a p-end-neighborhood in \( \mathcal{O} \) corresponding to a p-end vertex \( v_{\tilde{E}} \).

Recall the exact sequence

\[ 1 \to N \to \pi_1(\tilde{E}) \xrightarrow{\pi_{\tilde{E}}^*} N_K \to 1 \]

where we assume that \( N_K \subset \text{Aut}(K) \) is not discrete. Since \( \Sigma_{\tilde{E}}/\Gamma_{\tilde{E}} \) is compact, \( K'/N_K \) is compact also. However, \( N_K \) is not yet shown to be semisimple.

An element \( g \in \Gamma_{\tilde{E}} \) is of form:

\[ g = \begin{pmatrix} K(g) & 0 \\ \ast & U(g) \end{pmatrix}. \] (8.78)

Here \( K(g) \) is an \((n - i_0) \times (n - i_0)\)-matrix and \( U(g) \) is an \((i_0 + 1) \times (i_0 + 1)\)-matrix acting on \( \mathbb{S}_{i_0}^n \). We note \( \det K(g) \det U(g) = 1 \).

### 8.4.1 Outline of Section 8.4

In Section 8.4.2, we will take the leaf closure of the complete affine \( i_0 \)-dimensional leaves. The theory of Molino [142] shows that the space of leaf-closures will be an orbifold. In Section 8.4.2.2, we show that a leaf-closure is a compact suborbifold in \( \Sigma_{\tilde{E}} \). In Section 8.4.2.3, we show that the fundamental group of each leaf-closure is virtually solvable. In Section 8.4.2.4, we find a syndetic closure \( S \) according to a theory of Fried-Goldman [86]. From this, we will find a subgroup acting on each complete affine \( i_0 \)-dimensional leaf in Section 8.4.2.5 which we will show to be a cusp group.
8.4.2 Taking the leaf closure

8.4.2.1 Estimations with $KA\Upsilon$.

Let $\Upsilon$ denote a maximal nilpotent subgroup of $SL_+(n+1,\mathbb{R})_{g_0,\gamma_E}$ given by lower triangular matrices with diagonal entries equal to 1.

The foliation on $\tilde{E}$ given by fibers of $\Pi_k$ has leaves that are $i_0$-dimensional complete affine subspaces. Let us denote it by $\mathcal{F}_E$. Then $K^0$ admits a smooth Riemannian metric $\mu_k$ invariant under $N_k$ by Lemma 2.21. We consider the orthogonal frame bundle $\mathbb{F}K^0$ over $K^0$. A metric on each fiber of $\mathbb{F}K^0$ is induces from $\mu_k$. Since the action of $N_k$ is isometric on $\mathbb{F}K^0$ with trivial stabilizers, $N_k$ acts on a smooth orbit submanifold of $\mathbb{F}K^0$ transitively with trivial stabilizers. (See Lemma 3.4.11 in [159].)

There exists a bundle $\mathbb{F}\Sigma_E$ from pulling back $\mathbb{F}K^0$ by the projection map. Here, $\mathbb{F}\Sigma_E$ covers $\Sigma_E$. Since $\Gamma_E$ acts isometrically on $\mathbb{F}K^0$, the quotient space $\mathbb{F}\Sigma_E/\Gamma_E$ is a bundle $\mathbb{F}\Sigma_E$ over $\Sigma_E$ with compact fibers diffeomorphic to the orthogonal group of dimension $n-i_0$. Also, $\mathbb{F}\Sigma_E$ is foliated by $i_0$-dimensional affine spaces pulled-back from the $i_0$-dimensional leaves on the foliation $\tilde{E}$. One can think of these leaves as being the inverse images of points of $\mathbb{F}K^0$.

**Lemma 8.10** Each leaf $l$ of $\mathbb{F}\Sigma_E$ is of polynomial growth. That is, each ball $B_R(x)$ in $l$ of radius $R$ for $x \in l$ has an area less than equal to $f(R)$ for a polynomial $f$ where we are using an arbitrary Riemannian metric on $\mathbb{F}\Sigma_E$ induced from one on $\mathbb{F}\Sigma_E$.

**Proof** Let us choose a fundamental domain $F$ of $\mathbb{F}\Sigma_E$. Then for each leaf $l$ there exists an index set $I_l$ so that $l$ is a union of $g_i(D_i)$ $i \in I_l$ for the intersection $D_i$ of a leaf with $F$ and $g_i \in \Gamma_E$. We have that $D_i \subset D'_i$ where $D'_i$ is an $\epsilon$-neighborhood of $D_i$ in the leaf. Then

$$\{g_i(D'_i) | i \in I_l\}$$

cover $l$ in a locally finite manner. The subset $G(l) := \{g_i \in \Gamma | i \in I_l\}$ is a discrete subset.

Choose an arbitrary point $d_i \in D_i$ for every $i \in I_l$. The set $\{g_i(d_i) | i \in I_l\}$ and $l$ is quasi-isometric: a map from $G(l)$ to $l$ is given by $f_1: g_i \mapsto g_i(d_i)$ and the multivalued map $f_2$ from $l$ to $G(l)$ given by sending each point $y \in l$ to one of finitely many $g_i$ such that $g_i(D'_i) \ni y$. Let $\Gamma_E$ be given the Cayley metric and $\mathbb{F}\Sigma_E$ a metric induced from $\Sigma_E$. Both maps are quasi-isometries since these maps are restrictions of quasi-isometries $\Gamma_E \to \Sigma_E$ and $\tilde{E} \to \Gamma_E$ defined analogously.

The action of $g_i$ in $K$ is bounded since it sends some points of $\Pi_K$ to ones of $\Pi_K(F)$ which is a compact set in $K^0$. Thus, $\Pi^*_K(g_i)$ goes to a bounded subset of
\[ K(g_i) = \det(K(g_i))^{1/(n-i_0)} \tilde{K}(g_i) \text{ where } \tilde{K}(g_i) \in S_L(n-i_0, \mathbb{R}) \]

where \( \tilde{K}(g_i) \) is uniformly bounded. Let \( \tilde{\lambda}_{\max}(g_i) \) and \( \tilde{\lambda}_{\min}(g_i) \) denote the largest norm and the smallest norm of eigenvalues of \( \tilde{K}(g_i) \). Since \( \Pi_K^i(g_i) \) are in a bounded set of \( \text{Aut}(K) \), we obtain

\[
\frac{1}{C} \leq \tilde{\lambda}_{\max}(g_i), \tilde{\lambda}_{\min}(g_i) \leq C
\]

for \( C > 1 \) independent of \( i \). The largest and the smallest eigenvalues of \( g_i \) equal

\[
\lambda^T_{\max}(g_i) = \det(K(g_i))^{1/(n-i_0)} \tilde{\lambda}_{\max}(g_i) \quad \text{and} \quad \lambda^T_{\min}(g_i) = \det(K(g_i))^{1/(n-i_0)} \tilde{\lambda}_{\min}(g_i)
\]

by Proposition 8.1. Denote by \( a_j(g_i), j = 1, \ldots, i_0+1 \), the norms of eigenvalues of \( g_i \) associated with \( S^i_L \) where \( a_1(g_i) \geq \cdots \geq a_{i_0+1}(g_i) > 0 \) with repetitions allowed. Since \( \det g_i = 1 \), we have

\[
\det(K(g_i)) a_1(g_i) \cdots a_{i_0+1}(g_i) = 1.
\]

If \( \{|\det(K(g_i))|\} \to 0 \), then \( \{a_1(g_i)\} \to \infty \) whereas by (8.79)

\[
\{\det(K(g_i))^{1/(n-i_0)} \tilde{\lambda}_{\max}(g_i)\} \to 0
\]

contradicting Proposition 8.1. If \( \{|\det(K(g_i))|\} \to \infty \), then \( \{a_{i_0+1}\} \to 0 \) whereas by (8.79)

\[
\{\det(K(g_i))^{1/(n-i_0)} \tilde{\lambda}_{\min}(g_i)\} \to \infty
\]

contradicting Proposition 8.1. Therefore, we obtain

\[
1/C < |\det(K(g_i))| < C
\]

for a positive constant \( C \). We deduce that the largest norm and the smallest norm of eigenvalues of \( g_i \)

\[
\det(K(g_i))^{1/(n-i_0)} \tilde{\lambda}_{\max}(g_i) \quad \text{and} \quad \det(K(g_i))^{1/(n-i_0)} \tilde{\lambda}_{\min}(g_i)
\]

are bounded above and below by two positive numbers. Hence, \( \lambda^T_{\max}(g_i) \) and \( \lambda^T_{\min}(g_i) \) are all bounded above and below by a fixed set of positive numbers. \( U'(g_i) \) consists of \( a_5(g_i) O_5(g_i) \) for an orthogonal element \( O_5(g_i) \) and \( a_0(g_i) \). The remaining eigenvalues of \( g_i \) have norms \( a_5(g_i) \) and \( a_0(g_i) \). By Proposition 8.1, these are bounded by the same fixed set of positive numbers.

By Corollary 2.1, \( \{g_i\} \) is of bounded distance from \( U' \). Let \( N_c(U') \) denote a \( c \)-neighborhood of \( U' \). Then

\[
G(l) \subset N_c(U') \text{ for some } c > 0.
\]
8.4 The non-discrete case

Let \( d \) denote the left-invariant metric on \( \text{Aut}(\mathbb{S}^n) \). By the discreteness of \( \Gamma \), the set \( G(l) \) is discrete and there exists a lower bound to

\[
\{ d(g_i, g_j) | g_i, g_j \in G(l), i \neq j \}.
\]

Also given any \( g_i \in G(l) \), there exists an element \( g_j \in G(l) \) so that \( d(g_i, g_j) < C \) for a uniform constant \( C \). (We need to choose \( g_j \) so that \( g_j(F) \) is adjacent to \( g_i(F) \).

Let \( B_R(1) \) denote the ball in \( \text{SL}(n+1, \mathbb{R}) \) of radius \( R \) with the center \( 1 \). Then \( B_R(1) \cap N_c(\mathbb{U}) \) is of polynomial growth with respect to \( R \), and so is \( G(l) \cap B_R(1) \). Since the collection \( \{ g_i(D)^{\tau} | g_i \in G(l) \} \) of uniformly bounded balls cover \( l \) in a locally finite manner, \( l \) is of polynomial growth as well. \( \square \)

8.4.2.2 Closures of leaves

The foliation on \( \tilde{\Sigma}_E \) given by fibers of \( \Pi_K \) has leaves that are \( i_0 \)-dimensional complete affine subspaces. Let us denote it by \( \mathcal{F}_E \). Then \( K'' \) admits a smooth Riemannian metric \( \mu_K \) invariant under \( N_K \) by Lemma 2.21. We consider the orthogonal frame bundle \( \mathbb{F}K'' \) over \( K'' \). A metric on each fiber of \( \mathbb{F}K'' \) is induced from \( \mu_K \).

Since the action of \( N_K \) is isometric on \( \mathbb{F}K'' \) with trivial stabilizers, we find that \( N_K \) acts on a smooth orbit submanifold of \( \mathbb{F}K'' \) transitively with trivial stabilizers. (See Lemma 3.4.11 in [159].)

There exists a bundle \( \mathbb{F}\Sigma_E \) from pulling back \( \mathbb{F}K'' \) by the projection map. Here, \( \mathbb{F}\Sigma_E \) covers \( \Sigma_E \). Since \( \Gamma \) acts isometrically on \( \mathbb{F}K'' \), the quotient space \( \mathbb{F}\Sigma_E / \Gamma \) is a bundle \( \mathbb{F}\Sigma_E \) over \( \Sigma_E \) with a subbundle with compact fibers isomorphic to the orthogonal group of dimension \( n - i_0 \). Also, \( \mathbb{F}\Sigma_E \) is foliated by \( i_0 \)-dimensional affine spaces pulled-back from the \( i_0 \)-dimensional leaves on the foliation \( \Sigma_E \). One can think of these leaves as being the inverse images of points of \( \mathbb{F}K'' \).

8.4.2.3 \( \pi_1(V_l) \) is virtually solvable.

Recall the fibration

\[
\Pi_K : \Sigma_E \to K'' \text{ which induces } \tilde{\Pi}_K : \mathbb{F}\Sigma_E \to \mathbb{F}K''.
\]

Since \( N_K \) acts as isometries of a Riemannian metric on \( K'' \), we can obtain a metric on \( \Sigma_E \) so that the foliation is the Riemannian foliation. Let \( p_{\Sigma_E} : \mathbb{F}\Sigma_E \to \Sigma_E \) be the covering map induced from \( \Sigma_E \to \Sigma_E \). The foliation on \( \Sigma_E \) gives us a foliation of \( \mathbb{F}\Sigma_E \).

Let \( A_K \) be the identity component of the closure of \( N_K \) the image of \( \Gamma \) in \( \text{Aut}(K) \), which is a Lie group of dim \( \geq 1 \).

**Proposition 8.4** \( A_K \) is a normal connected nilpotent subgroup of the closure of \( N_K \).

**Proof** Since the closure of \( N_K \) is normalized by \( N_K \), \( A_K \) is a normal subgroup of \( N_K \). Since \( l \) maps to a polynomial growth leaf in \( \mathbb{F}\Sigma_E \) by Lemma 8.10, Carrière [38]
shows that $A_K$ is a connected nilpotent Lie group in the closure of $N_K$ in $\text{Aut}(K)$ acts on $\mathbb{F}K'$ freely. □

Let $l$ be a leaf of $\mathbb{F}\bar{\Sigma}_E$, and $p$ be the image of $l$ in $\mathbb{F}K'$. Moreover, we have

\[
\bar{\Pi}_K^{-1}(A_K(p)) =: \bar{V}_l \hookrightarrow \mathbb{F}\bar{\Sigma}_E
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
V_l \hookrightarrow \mathbb{F}\Sigma_E
\]

(8.80)

for $V_l := \overline{p\Sigma}(l)$ in $\mathbb{F}\Sigma_E$. Since $V_l$ is closed and is a component of the inverse image of $V_l$ which is a union of copies of $\bar{V}_l$, the image $V_l$ is a compact submanifold. Note $V_l$ has a dimension independent of $l$ since $A_K$ acts freely.

Now, $N$ is precisely the subgroup of $\pi_1(V_l)$ fixing a leaf $l$ in $\mathbb{F}K'$. For each closure $V_l$ of a leaf $l$, the manifold $V_l$ is a compact submanifold of $\mathbb{F}\Sigma_E$, and we have an exact sequence

\[
1 \to N \to h(\pi_1(V_l)) \xrightarrow{\bar{\Pi}_K} A_K' \to 1.
\]

(8.81)

Since the leaf $l$ is dense in $V_l$, it follows that $A_K'$ is dense in $A_K$. Each leaf $l'$ of $\bar{\Sigma}_E$ has a realization a subset in $\overline{k}$. We have the norms of eigenvalues $\lambda_i(g) = 1$ for $g \in N$ by Proposition 8.1. By Theorem 2.6, $N = N_l$ is virtually unipotent since the norms of eigenvalues equal 1 identically and $N_l$ is discrete.

We take a finite cover of $\Sigma_E$ so that $N$ is nilpotent. Hence, $h(\pi_1(V_l))$ is solvable being an extension of a nilpotent group by a nilpotent group. We summarize below:

**Proposition 8.5** Let $l$ be a generic fiber of $\mathbb{F}\bar{\Sigma}_E$ and $p$ be the corresponding point of $\mathbb{F}K'$. Then there exists a nilpotent group $A_K$ acting on $\mathbb{F}K'$ so that $\bar{\Pi}_K^{-1}(A_K(p)) = \bar{V}_l$ covers a compact suborbifold $V_l$ in $\mathbb{F}\Sigma_E$, a conjugate of the image of the holonomy group of $V_l$ is a dense subgroup of $A_K$, and the holonomy group of $V_l$ is solvable. Moreover, $\bar{V}_l$ is homeomorphic to a torus times a cell or a cell.

**Proof** We just need to prove the last statement. Since $A_K$ is a connected nilpotent group, $A_K$ is homeomorphic to a torus times a cell or a cell, and so is the free orbit in $\mathbb{F}K'$. Since $\bar{\Pi}_K$ has fibers that are $i_0$-dimensional open hemispheres, this last statement follows. □

We remark that $A_K$ is nilpotent but may not be unipotent.

**Remark 8.6** The leaf holonomy acts on $\mathbb{F}\Sigma_E / \mathcal{F}_E$ as a nilpotent killing field group without any fixed points. Hence, each leaf $l$ is in $\bar{V}_l$ with a constant dimension. Thus, $\mathcal{F}_E$ is a foliation with leaf closures of identical dimensions.

The leaf closures form another foliation $\mathcal{F}_E$ with compact leaves by Lemma 5.2 of Molino [142]. We let $\mathbb{F}\Sigma_E / \mathcal{F}_E$ denote the space of closures of leaves has an orbifold structure where the projection $\mathbb{F}\Sigma_E \to \mathbb{F}\Sigma_E / \mathcal{F}_E$ is an orbifold morphism by Proposition 5.2 of [142].
8.4.2.4 The holonomy group for a leaf closure is normalized by the end holonomy group.

Note that $\Gamma_l$ is the deck transformation group of $\tilde{V}_l$ over $V_l$. Since $\tilde{V}_l$ is the inverse image of $A_K(x)$ for $x \in \mathbb{F}K$, $\Gamma_l$ is the inverse image of $N_K \cap A_K$ under $\Pi_k$. Since $N_K \cap A_K$ is normal in $N_K$, $\Gamma_l$ is a normal subgroup of $\Gamma_\ell$.

Recall that $\Gamma_l$ is virtually solvable, as we showed above. We let $Z(\Gamma_\ell)$ and $Z(\Gamma_l)$ denote the Zariski closures in $\text{Aut}(S_n)$ of $\Gamma_\ell$ and $\Gamma_l$ respectively.

By Theorem 1.6 of Fried-Goldman [86], there exists a closed virtually solvable Lie group $S_l$ containing $\Gamma_l$ with the following four properties:

- $S_l$ has finitely many components.
- $\Gamma_l \setminus S_l$ is compact.
- The Zariski closure $Z(S_l)$ is the same as $Z(\Gamma_l)$.
- Finally, we have solvable ranks
  \[\text{rank}(S_l) \leq \text{rank}(\Gamma_l).\] (8.82)

We will call this the syndetic hull of $\Gamma_l$.

We summarize:

**Lemma 8.11** $h(\pi_1(V_l))$ is virtually solvable and is contained in a virtually solvable Lie group $S_l \subset Z(h(\pi_1(V_l)))$ with finitely many components, and $S_l/h(\pi_1(V_l))$ is compact. $S_l$ acts on $\tilde{V}_l$. Furthermore, one can modify a p-end-neighborhood $U$ so that $S_l$ acts on it. Also the Zariski closure of $h(\pi_1(V_l))$ is the same as that of $S_l$.

**Proof** By above, $Z(S_l) = Z(\Gamma_l)$ acts on $\tilde{V}_l$ and normalizes $\Gamma_l$. We need to prove about the p-end-neighborhood only. Let $F$ be a compact fundamental domain of $S_l$ under the $\Gamma_l$. Then we have

\[\bigcap_{g \in S_l} g(U) = \bigcap_{g \in F} g(U).\]

By Lemma 2.5, the latter set contains a $S_l$-invariant p-end-neighborhood. \[\square\]

From now on, we will let $S_l$ to denote the only the identity component of itself for simplicity as $S_l$ has finitely many components to begin with. We are taking a finite cover of $E$ if necessary. This will be sufficient for our purposes since we only need a cusp group.

Since $S_l$ acts on $U$ and hence on $\Sigma_E$ as shown in Lemma 8.11, we have a homomorphism $S_l \rightarrow \text{Aut}(K)$. We define by $S_{l,0}$ the kernel of this map. Then $S_{l,0}$ acts on each leaf of $\Sigma_E$. We have an exact sequence

\[1 \rightarrow S_{l,0} \rightarrow S_l \rightarrow A_K \rightarrow 1.\] (8.83)
8.4.2.5 The form of $\mathcal{U}S_l$,  

Let $S_l^{i_0+1}$ denote the $i_0 + 1$-dimensional great sphere containing $S_l^{i_0}$ corresponding to each $i_0$-dimensional leaf $l$ of $\mathcal{F}_E$.

**Proposition 8.6** Let $l$ be a generic fiber so that $\Lambda_K$ acts with trivial stabilizers.

(i) $S_l$ acts on $\tilde{V}_l$ cocompactly, acts on $\partial U$ properly, and acts as isometries on these spaces with respect to some Riemannian metrics.

(ii) A closed subgroup $C_{1,0}$ of unipotent elements of acts transitively on each leaf $l$ with trivial stabilizers, and $C_{1,0}$ acts on an $i_0$-dimensional ellipsoid $\partial U \cap S_l^{i_0+1}$ passing $v_E$ with an invariant Euclidean metric. Here, we may need to modify $U$ further.

(iii) $S_{1,0}$ normalizes an $i_0$-dimensional partial cusp group $C_{1,0}$ where $S_{1,0} \cap C_{1,0}$ are cocompact subgroups in both $S_{1,0}$ and $C_{1,0}$.

(iv) $C_{1,0}$ is virtually normalized by $\Gamma_E$ and also by $S_l$. Also, $C_{1,0}$ acts freely and properly on $m$ for each leaf $m$ of $\mathcal{F}_E$.

(v) With setting $\mathcal{N} := C_{1,0}$, Hypothesis 8.3.1 holds virtually by $\Gamma_E$ for a coordinate system.

**Proof** (i) By Lemma 4.2, $S_l$ acts properly on $\tilde{V}_l$. Since $\partial U$ is in one-to-one correspondence with $\tilde{S}_E$, $S_l$ acts on $\partial U$ properly. Hence, these spaces have compact stabilizers with respect to $S_l$. The existence of an invariant metric follows from an argument similar to one in the proof of Lemma 2.21. Hence, the action is proper and the orbit is closed.

Since $\tilde{V}_l/\Gamma_l$ is compact, $\tilde{V}_l/S_l$ is compact also.

(ii) We may assume that $\Gamma_E$ is torsion-free by Theorem 2.3 taking a finite index subgroup.

Proposition 8.1 implies that for $g \in \Gamma_l$

$$\lambda_{\text{max}}^{\text{Tr}}(g) \geq \lambda_{\text{max}}^{\text{co}}(g) \geq \lambda_{\text{min}}^{\text{co}}(g) \geq \lambda_{\text{min}}^{\text{Tr}}(g).$$

Since $S_l = F\Gamma_l$ for a compact set $F$, the inequality

$$C_1\lambda_{\text{max}}^{\text{Tr}}(g) \geq \lambda_{\text{max}}^{\text{co}}(g) \geq C_2\lambda_{\text{min}}^{\text{co}}(g) \geq C_3\lambda_{\text{min}}^{\text{Tr}}(g), g \in S_l,$$

$$C_1\lambda_{\text{max}}^{\text{Tr}}(g) \geq \lambda_{\text{max}}^{\text{co}}(g) \geq \lambda_{n+1}(g) \geq C_2\lambda_{\text{min}}^{\text{Tr}}(g), g \in S_l,$$  

(8.84)

hold for constants $C_1 > 1, 1 > C_2 > C_3 > 0$ by (8.1). Since $S_{1,0}$ acts trivially on $K^o$, we have $\lambda_{\text{max}}^{\text{Tr}}(g) = \lambda_{\text{min}}^{\text{Tr}}(g)$ for $g \in S_{1,0}$. Since the maximal norm $\lambda_{\text{max}}^{\text{co}}(g)$ of the eigenvalues of $g$ equals $\lambda_{\text{min}}^{\text{co}}(g)$ and the minimal norm of the eigenvalues of $g$ equals $\lambda_{\text{min}}^{\text{Tr}}(g)$, all the norms of the eigenvalues of $g \in S_{1,0}$ are bounded above. (8.84) implies that $|\log \lambda_{\text{max}}^{\text{co}}(g)|, |\log \lambda_{\text{min}}^{\text{co}}(g)|, g \in S_{1,0}$ are both uniformly bounded above. Of course we have

$$|\log \lambda_{\text{max}}^{\text{co}}(g^n)| = |n \log \lambda_{\text{max}}^{\text{co}}(g)|, |\log \lambda_{\text{min}}^{\text{co}}(g^n)| = |n \log \lambda_{\text{min}}^{\text{co}}(g)|, g \in S_{1,0}.$$
8.4 The non-discrete case

We conclude that the norms of eigenvalues of \( g \in S_{l,0} \) are all 1.

Theorem 2.6 implies that \( S_{l,0} \) is a closed orthopotent group and hence a solvable Lie group. Lemma 4.3 gives a unipotent group \( C_{l,0} \) acting on \( l \) where \( C_{l,0} \) is the Zariski closure of the unipotent subgroup \( S_{l,0}^{u} \) of \( S_{l,0} \). We have \( C_{l,0} \cap S_{l,0} = S_{l,0}^{u} \). Proposition 4.4 shows that \( C_{l,0} \) is a cusp group. Since \( S_{l} \) normalizes \( S_{l,0} \) and \( C_{l,0} \cap S_{l,0} = S_{l,0}^{u} \) is cocompact in \( S_{l,0} \), it follows that \( S_{l} \) normalizes \( C_{l,0} \). This also proves (iii).

(iv) Proposition 4.4 shows that the action of \( C_{l,0} \) on any leaf \( m \) is a free and proper action. Since \( C_{m,0} \) acts on \( m \), \( B_{m} := H_{m} \cap U \) is again bounded by an ellipsoid. Since \( B_{m} \) has a hyperbolic metric as a Klein model, such a unipotent group is unique and hence it follows that \( gC_{1,0}g^{-1} \) and \( C_{1,0} \) restrict to a same group in \( H_{m} \).

Let \( g \in \Gamma_{E} \). By using these argument for \( g(l) \) instead of \( l \), \( gC_{1,0}g^{-1} \) also acts on an ellipsoid \( E_{m} \) in the subspace corresponding to \( m \) from \( \nu_{E} \) as a unipotent Lie group freely, transitively, and faithfully. Since \( E_{l} \) bounds a \((i_{0} + 1)\)-dimensional ball with a hyperbolic metric of the Klein model, such a unipotent group is unique and hence it follows that \( gC_{1,0}g^{-1} \) and \( C_{1,0} \) restrict to a same group in \( H_{m} \).

Let \( \hat{C} \) denote the group generated by \( C_{1,0} \) and its conjugates. \( \hat{C} \) is obviously unipotent. Also, \( \hat{C} \) acts properly on \( \tilde{E}_{E} \) since \( \Gamma_{E} \) and \( C_{1,0} \) preserve a Riemannian metric.

Let \( g' \in C_{1,0} \) and \( g'' \in gC_{1,0}g^{-1} \) so that \( g'|H_{m} = g''|H_{m} \). Then \( g'^{-1}g'' \) fixes every point in \( m \). Since and the stabilizer of the unipotent group acting properly on \( \tilde{E}_{E} \) is trivial, \( g' = g'' \). Hence, the normality follows.

(v) The first two properties of Hypothesis 8.3.1 follow from Propositions 2.15 and 2.20. \( \Gamma_{E} \) satisfies the third transverse weak middle eigenvalue condition by the premise. Since \( S_{l,1}^{u} \) goes to \( l \) under \( \Pi_{K} \), it is in the standard form where \( l \) corresponds to a great sphere \( S_{i_{0}+1}^{i_{0}} \) containing \( S_{\infty}^{i_{0}} \). This proves the fourth property.

Since \( N \) acts on \( \tilde{V}_{l} \), \( N \) is a subgroup of \( \tilde{I}_{l} \) virtually. Hence, \( N \) is a subgroup of \( S_{l} \) and hence of \( S_{l,0} \) virtually. Theorem 2.6 tells us that \( N \) is unipotent virtually and hence \( N \cap \Gamma_{E}' \) is in \( S_{l,0}^{u} \) for a finite index subgroup \( \Gamma_{E}' \) of \( \Gamma_{E} \). (iv) showed that \( N \) is normalized by \( \Gamma_{E} \). (vi) also shows that \( N \) acts freely and properly on each complete affine leaf of \( \tilde{E}_{E} \).

\[ \Box \]

8.4.3 The proof for non-discrete \( N_{K} \).

Now, we go the splitting argument for this case. We can parametrize \( US_{l,0} \) by \( \mathcal{N}(v) \) for \( v \in \mathbb{R}^{i_{0}} \) by Proposition 8.6. We showed that Hypothesis 8.3.1 holds virtually. For convenience, let us assume that Hypothesis 8.3.1 here.

We outline the proof strategy:

- \( N_{K} \) is semisimple in Proposition 8.7.
- \( \mu_{k} = 1 \) for every \( g \in \Gamma_{E} \).
- Hypothesis 8.3.2 holds. Now we use the results in Section 8.3.3.
Also, \( N \cap US_{l^0} \) is of finite index in \( N \) since both acts on \( l \) and we took many finite index subgroups in the processes above. Again using Proposition 2.6, we can consider the finite covers of \( p \)-end neighborhoods. We assume \( N \subset US_{l^0} \). Hypothesis 8.3.1 holds now as we showed in the above subsections. As above by Lemmas 8.3, we have that the matrices are of form:

\[
\mathcal{N}(v) = \begin{pmatrix}
I_{n-i_0-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
C_1(v) & v^T & I_{i_0} & 0 \\
c_2(v) & \|v\|^2/2 & v & 1
\end{pmatrix},
\]

(8.85)

\[
g = \begin{pmatrix}
S(g) & s_1(g) & 0 & 0 \\
s_2(g) & a_1(g) & 0 & 0 \\
C_1(g) & a_1(g)v_5^T & a_5(g)O_5(g) & 0 \\
c_2(g) & a_5(g)v_5^T O_5(g) & a_9(g)
\end{pmatrix},
\]

(8.86)

where \( g \in \Gamma_E \). (See (8.41).) Recall \( \mu_g = a_5(g)/a_1(g) = a_9(g)/a_5(g) \). Since \( S_l \) is in \( \mathcal{E}(\Gamma_l) \) and the orthogonality of normalized \( A_5(g) \) is an algebraic condition, the above form also holds for \( g \in S_l \).

**Proposition 8.7 (Semisimple \( N_K \))** Assume hypothesis 8.3.1. Suppose that \( \pi_1(E) \) satisfies (NS) or \( \dim K = 0, 1 \). Then the following hold:

- \( N_K \) or any of its finite index group acts semi-simply on \( K^o \).
- There is a finite index subgroup \( N'_K \) of \( N_K \) acting on each \( K_i \) irreducibly and has the diagonalizable commutant \( H \) isomorphic to \( \mathbb{R}_{+}^{\bar{l}-1} \) for some \( \bar{l} \geq 1 \).
- \( K \) is projectively diffeomorphic to \( K_1 \ast \cdots \ast K_{\bar{l}} \), where \( H \) acts trivially on each \( K_j \) for \( j = 1, \ldots, \bar{l} \).
- Let \( A_K \) denote the identity component of the closure \( \bar{N}_K \) of \( N_K \) in \( \text{Aut}(K) \). Let \( A'_K \) denote the image in \( A_K \) of \( \Gamma_E \) in \( N_K \). Then \( A'_K \cap N'_K \) is free abelian and is a diagonalizable group of matrices and is in the virtual center \( N_K \cap H \).
- \( N'_K \) acts on each \( K_i \) strongly irreducibly, and \( N'_K \cap K_i \) is semisimple and discrete and acts on \( K_i^* \) as a divisible action.
- \( N_K \) contains a free abelian group in \( N_K \cap H \) of rank \( \bar{l} - 1 \).

**Proof** It is sufficient to prove for \( N_K \) itself since \( N'_K \) for a finite-index subgroup \( N'_K \) of \( N_K \) acts cocompactly.

If \( N_K \) is discrete, then the conclusion follows from Proposition 2.15.

Suppose \( N_K \) is not discrete. For the case when \( \dim K = 0, 1 \), the conclusions are obvious. We will prove using induction on \( \dim K \).

We now use the notation of Section 8.4.2.2. By Theorem 2.3, we assume that \( \Gamma_E \) is torsion-free. By condition (NS), since \( \Gamma_l \) is virtually normal in \( \Gamma_E \), \( \Gamma_l \cap G \) is central in a finite index subgroup \( G \) of \( \Gamma_E \) and is free-abelian.
Now, we prove by induction on \( \dim K \). We recall the exact sequence

\[
1 \to N \to \Gamma_l \to A'_K \to 1.
\]

Here, \( A'_K \) is dense in a nilpotent Lie group \( A_K \) normalized by \( N_K \). Let \( N'_K \) denote the image of \( G \) in \( N_K \). Let \( \tilde{N}'_K \) denote the closure of \( N'_K \) in \( \text{Aut}(K) \).

We give a bit vague outline of the rest of the proof:

(i) First, we show that there is no unipotent action on \( K^o \) by length arguments.

(ii) We decompose the space into invariant subspaces where \( N_K \) virtually acts on which it virtually acts discretely.

(iii) Now, we prove that \( N_K \) acts semisimply.

(iv) Finally, we show that \( N_K \) contains a free abelian group of certain rank.

(i) We prove some fact: We take the unipotent subgroup \( \Gamma_{l,u}, A'_{K,u}, A_{K,u} \) of solvable groups \( \Gamma_l, A'_K, \) and \( A_K \) respectively. These are normalized by \( N_K \).

Suppose that \( A'_{K,u} \cap N'_K \) is nontrivial. Choose a nontrivial unipotent element \( g_u \) in \( A'_{K,u} \cap N'_K \). By Lemma 2.11, there exists a sequence of elements \( x_i \in K^o \) so that \( d_K(x_i; g_u(x_i)) \to 0 \). Since \( N'_K \) is still a sweeping action, let \( F' \) be a compact set in \( K^o \) so that \( \bigcup_{g \in N'_K^o} g(F) = K^o \). Now, we can choose \( g_i \in G \) so that \( g_i(x_i) \in F' \). Then

\[
\{d_\Omega(g, g_u x_i, g_i(x_i)) = d_\Omega(g, g_u g_i^{-1} (g_i(x_i)), g_i(x_i))\} \to 0 \quad \text{(8.87)}
\]

since \( g_i \) is a \( k \)-isometry. This implies that \( \{g_i g_u g_i^{-1}\} \) converges to an element of a stabilizer of a point \( f, f \in F \) in \( N^*_K \).

Since \( \Gamma_l \cap G \) is central in \( G, g_u g_i^{-1} = g_u \). Since \( g_u \) is unipotent, if \( g_u \) stabilizes a point of \( K^o \), then \( g_u \) is the identity element. This is a contradiction. Therefore, we conclude \( A'_{K,u} \cap N'_K \) is a trivial group.

(ii) Since \( \Gamma_l \) is central in \( G, A_K \cap N'_K \) and its closure \( A_K \cap \tilde{N}'_K \) are abelian groups. Since \( A_K \cap \tilde{N}'_K \) is abelian, we can decompose \( \mathbb{C}^{\alpha-i_0} = V'_1 \oplus \cdots \oplus V'_r \) so that each element \( g \) of \( A_K \cap \tilde{N}'_K \) acts irreducibly with a single eigenvalue \( \lambda_i(g) \) on \( V'_i \) for each \( i \) and its conjugate \( \bar{\lambda}_i(g) \) by the primary decomposition theorem. (See Theorem 12 of Section 6 of [107] and Definition 2.5.) The map

\[
g \in A_K \cap \tilde{N}'_K \mapsto (\lambda_1(g), \ldots, \lambda_r(g)) \in \mathbb{C}^r
\]

gives us an isomorphism to the image set where we choose a representative eigenvalue \( \lambda_i(g) \) for each \( V_i \).

We define \( S_1, \ldots, S_r \) to be the subspace in \( \mathbb{S}^{\alpha-i_0} \) corresponding to a real primary subspace for every \( g \) by the commutativity of elements in \( A_K \cap \tilde{N}'_K \).

Since \( N'_K \) commutes with \( A_K \), \( N'_K \) also acts on or permutes the corresponding subspaces \( S_1, \ldots, S_r \) in \( \mathbb{S}^{\alpha-i_0-i} \). We take the finite-index subgroup \( N''_K \) of \( N'_K \) acting on each \( S_i \). Since \( N''_K \) has a sweeping action on \( K^o \). By Proposition 2.20, \( S_i \cap K \neq \emptyset \) and \( S_i \cap K^o = \emptyset \). Denote by \( K_i := S_i \cap K \).

Suppose that \( \lambda_i(g) \) is not real for some \( i \) and \( g \in A_K \cap \tilde{N}'_K \). Then there is an eigenspace for \( \lambda_i(g) \) and one for \( \bar{\lambda}_i(g) \) for all \( g \in N_K \). Since \( K \cap S \) for a corresponding subspace \( S' \) for the direct sum of two eigenspaces is properly convex, there is a
global fixed point of $g$. The point corresponds to a positive real eigenvalue of $g$. This is a contradiction. The negative case violates the proper convexity. Therefore, every $\lambda_+(g) > 0$ for $g \in A_K \cap \bar{N}_K''$.

Let $\bar{N}_K$ denote the closure of $N_K''$ in $\bar{N}_K$. Suppose that $A_K \cap \bar{N}_K''$ acts on $K_i$ non-trivially. $A_K \cap \bar{N}_K''|K_i$ is a unipotent action since each element of $A_K \cap \bar{N}_K''|K_i$ has a single positive eigenvalue affiliated with $K_i$. Since $K''/A_K \cap \bar{N}_K''$ is compact, we can apply the arguments in the paragraph containing (8.87) and the following one, and we obtain a contradiction. Hence, $A_K \cap \bar{N}_K''|K_i$ is trivial. Hence, $A_K \cap \bar{N}_K$ is a positive diagonalizable group.

Since $\bar{A}_K$ is the identity component of $\bar{N}_K$, and $A_K$ restricts to a trivial group for each $K_i$, $N_{K_i} := N''_K|K_i$ is discrete.

(iii) If $l' = 1$, then this shows $A_K \cap \bar{N}_K'$ is trivial. Then we are in the case of $N_K$ being discrete and the result follows by Proposition 2.15.

Suppose now that $l' \geq 2$. Then since $\dim K_i < \dim K$, we deduce that $N_{K_i} := N''_K|K_i$ still acts cocompactly on $K''$ since otherwise this fails for $N_K''$ action on $K''$. Hence, $N_{K_i}$ is semi-simple and the conclusions of this proposition hold for $N_{K_i}$ and $K_i$ by induction.

By induction, we decompose each $K_j$ into $K_j^{(1)} \ast \cdots \ast K_j^{(l')}$ with a positive diagonalizable commutant $H_j$ for $N_{K_j}$. The finite index subgroup $N''_{K_i}|K_i$ acting on each $K_j^{(i)}$ is a cocompact subgroup of $N_{K_j^{(1)}} \times \cdots \times N_{K_j^{(l')}}(j) \times \Lambda_j$ for a Zariski dense subgroup $\Lambda_j$ in $H_j$ and $N_{K_j^{(i)}} := N''_{K_j}|K_j^{(i)}$ for $i = 1, \ldots, l'(j)$ by Proposition 2.15. Also, $N_{K_j^{(i)}}$ acts strongly irreducibly on each $K_j^{(i)}$ also by Proposition 2.15.

There is a commutant $H_K$ of $N''_K$ that just the positive diagonalizable group acting trivially on each $K_i$. Hence, it is isomorphic to $\mathbb{R}^{l'-1}$. Since $A_K \cap N''_K|K_i$ for each $i$ is trivial, $H_K \cap N''_K$ contains $A_K \cap N''_K$ by the second paragraph above. This proves the third item. We define $H$ to be the product of $H_K \times H_1 \times \cdots \times H_{l'}$.

We list all $K_j^{(i)}$ as a single list $K_1, \ldots, K_{\tilde{l}}$. Define $N''_K$ as a subgroup acting on each $K_i$ for $i = 1, \ldots, \tilde{l}$. Now $N''_K$ is a subgroup of $N_{K_1} \times \cdots \times N_{K_{\tilde{l}}}$ acting on $L$ for a Zariski dense $L$ in $H$. Thus, $N''_K$ is semi-simple. This means that $N''_K$ is semi-simple since $N''_K/A_K''$ acts only as a permutation group of $K_1, \ldots, K_{\tilde{l}}$.

Since $N_K''$ is discrete, and $N''_K$ is isomorphic to a subgroup of $N_{K_1} \times \cdots \times N_{K_{\tilde{l}}} \times \mathbb{R}^{l'-1}$, it follows that the finite extension $N_K''$ is semi-simple. This proves the first to the fourth item.

(iv) Now, we prove the last item in particular. Proposition 2.20 also shows the existence of a free diagonalizable subgroup $\Lambda$ of $\bar{N}_K \cap H$ of rank $\tilde{l} - 1$. Hence, there must be a lattice in $L \subset A$ that is Zariski dense in $\bar{N}_K \cap H$. Choose generators $\eta_1, \ldots, \eta_\tilde{l}$ of the lattice. For each $\eta_j$, there is a sequence $\{\kappa_j^i\}$ in $N_K$ converging to $\eta_j$ in $\text{Aut}(K)$. Each $\kappa_j^i|K_i$ is in a discrete group $N_{K_i}$. Hence, we may assume that $\kappa_j^i|K_i = I_{K_i}$ for every $i$ since $\eta_j|K_i = I_{K_i}$. Hence, $\kappa_j^i \in H \cap N_K$ for every $i$. Since $\kappa_j^i$ are sufficiently close to $\eta_j$ for each $j = 1, \ldots, \tilde{l}$, we can choose a set of generators $\kappa_1^1, \ldots, \kappa_\tilde{l}^1$ of $H \cap N_K$. This completes the proof. \qed
However, we have not shown Hypothesis 8.3.2 yet. We continue to have Hypothesis 8.3.1 for $\Gamma_E$.

**Proposition 8.8** Suppose that $\mathcal{E}$ is properly convex. We assume Hypothesis 8.3.1 and $N_K$ is non-discrete. Suppose that $\pi_1(\mathcal{E})$ satisfies (NS) or $\dim K = 0, 1$. Then we have $\mu_g = 1$ for every $g \in \Gamma_E$.

**Proof** We can take finite-index subgroups for $\Gamma_E$ during the proof and prove for this group since $\mu$ is a homomorphism to the multiplicative group $\mathbb{R}^*$. By Proposition 8.7, $N_K$ is semi-simple and Proposition 8.5 and Lemma 8.6 hold.

Propositions 8.7 and 2.20 show that $K = K_1 \ast \cdots \ast K_l$ for properly convex sets $K_i$ and $N_K$ is virtually isomorphic to a cocompact subgroup of $N_{K_1} \times \cdots \times N_{K_l} \times \Lambda$ where $\Lambda$ is Zariski dense in a diagonalizable group $\mathbb{R}_+^{l-1}$ acting trivially on each $K_i$ and $N_K$ acts semisimply on each $K_i$ for $i = 1, \ldots, l$. We take a finite-index subgroup $N_K'$ so that $N_K'$ acts on $K_i$ for each $i = 1, \ldots, l$. We assume that $N_K'$ is this $N_K'$ without loss of generality.

We apply Lemma 8.5. Then one of $K_i$ is a vertex $k$. Also, Lemma 8.5 shows that $C_1(v) = 0$ for all $v \in \mathbb{R}^0$ for a coordinate system where $k$ has the form $$\langle 0, \ldots, 0, 1 \rangle \in S^{n-\ell_0-1}.$$ By Proposition 8.2, we have a coordinate system where

$$C_1(g) = O, c_2(g) = 0 \text{ for every } g \in \Gamma_E \text{ and }$$

$$C_1(v) = O, c_2(v) = 0 \text{ for every } \mathcal{N}(v), v \in \mathbb{R}^0. \quad (8.88)$$

Let $\lambda_{S_k}$ denote the maximal norm of the eigenvalues of the upper-left part $S_k$ of $g$. We define

$$\Gamma_{E,+} := \{ g | \lambda_{S_k}(g) < a_1(g) \}.$$ 

There is always an element like this. In particular, we take the inverse image of suitable diagonalizable elements of the center $H \cap N_K$ denoted in Proposition 8.7. We take the diagonalizable element in $N_K$ with $k$ having a largest norm eigenvalue. Let $g$ be such an element. Then by transverse weak middle eigenvalue condition shows that $a_1(g)$ is the largest of norms of every eigenvalue by Proposition 8.1, and

$$a_1(g) \geq a_0(g) \text{ or } \mu_k \leq 1 \text{ for } g \in \Gamma_{E,+}. (*)$$

By Proposition 8.7, $N_K \cap H$ contains a free abelian group of rank $\dim H$ which is positive diagonalizable. Hence, there exists $g_c \in \Gamma_{E,+}$ going to a center of $N'_K$ with $\mu_{g_c} < 1$.

(A) We will obtain a nontrivial element of $N'$:

Since $v_E$ has a different eigenvalue $a_0(g_c)$ from that $a_1(g_c)$ as $\mu_{g_c} < 1$, $a_1(g_c)$ is the largest norm of a unique eigenvalue and has the multiplicity one. Hence, we obtain a fixed point $k_{g_c} \neq v_E$, and $k_{g_c} \in \mathbb{S}_{l_0}^{0} + 1$ in the direction of $k$ from $v_E$. The great sphere of dimension $n - 1$ contains $\mathbb{S}_{l_0}$ and the $l_0 + 1$-dimensional hemispheres in
directions of points of $K''$ bounding an affine space $\mathbb{A}$. Since $k_g$, is an attracting fixed point of multiplicity one, and $U$ is in the attracting basin $\mathbb{A}$ of $g$, $k_g$ is in $\text{Cl}(U)$.

Since $S_t$ acts on $U$, it follows that $S_t$ acts on $\text{Cl}(U) \cap \mathcal{S}_k^{(k+1)}$. By the form of the matrices (8.85), $\mathcal{N}$ acts on $\mathcal{S}_k^{(k+1)}$. Hence, $\mathcal{N}$ of form (8.85) acts on $\text{Cl}(U) \cap \mathcal{S}_k^{(k+1)} \ni k_g$. Hence, we have the orbit

$$\mathcal{N}(k_g) \subset \text{Cl}(U) \cap \mathcal{S}_k^{(k+1)}.$$ 

Since $\text{Cl}(U)$ is convex, a convex domain $B = \text{Cl}(U) \cap \mathcal{S}_k^{(k+1)}$ bounded by an ellipsoid is in a hemisphere $H_k$ in $\mathcal{S}_k^{(k+1)}$ bounded by $\mathcal{S}_k^{(k)}$.

There exists a hyperspace $P_{k_g}$ in $\mathcal{S}_k^{(k+1)}$ tangent to $\partial B$ at $k_g$, where $g_c$ acts on. We let $\tilde{P}_{k_g} = P_{k_g} \cap \mathcal{S}_k^{(k)}$. We choose a coordinate system so that

$$k_g = \left\{ (0, \ldots, 0, x_{n-i_0}, 0, \ldots, 0) \right\}, x_{n-i_0} > 0,$n_k^{(k+1)} = \left\{ (0, \ldots, 0, x_{n-i_0}, \ldots, x_{n+1}) \right\} | x_i \in \mathbb{R},$$

$$\tilde{P}_{k_g} = \left\{ (0, \ldots, 0, x_{n-i_0}, x_{n-i_0+1}, \ldots, x_n, 0) \right\} | x_i \in \mathbb{R},$$

$$B = \left\{ (0, \ldots, 0, x_{n-i_0}, \ldots, x_{n+1}) \right\} | x_i \in \mathbb{R}, x_{n+1} \geq \frac{1}{2} \left( x_{n-i_0+1}^2 + \cdots + x_n^2, x_{n-i_0} = 1 \right).$$

Here, $k_g$, may be regarded as the origin of $H_k^c$.

Now, let $g_1$ be any element of $\Gamma_E$. We factorize the lower-right $(i_0+2) \times (i_0+2)$-submatrix of $g_1, g_1 \in \Gamma_E$.

$$\begin{pmatrix}
    a_1(g_1) & 0 & 0 \\
    a_1(g_1)v_{g_1}^T & a_5(g_1)O_5(g_1) & 0 \\
    a_7(g_1) & a_5(g_1)v_{g_1}O_5(g_1) & a_9(g_1)
\end{pmatrix} = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
    1 & 0 & 0 \\
    v_{g_1}^T & 1 & 0 \\
    \frac{\|v_{g_1}\|^2}{2} & v_{g_1} & 1
\end{pmatrix} \begin{pmatrix}
    1 & 0 & 0 \\
    0 & \mu_{g_1}O_5(g_1) & 0 \\
    0 & 0 & \mu_{g_1}^2
\end{pmatrix}.$$}

(8.91)

Since the right two matrices act on $B$, we obtain that

$$a_7(g_1) := \frac{a_5(g_1)}{a_1(g_1)} - \frac{\|v_{g_1}\|^2}{2} = 0 \text{ for any } g_1 \in \Gamma_E.$$

Also, $v_{g_c} = 0$ since $k_g$, is a fixed point of $g_c$. Thus, $g_c$ is a block diagonal matrix and so is $g_c^{-1}$.

Suppose that $v_{g_c} = 0$ for every $\Gamma_E$. The $\Gamma_E$ fixes $k_g$. By Proposition 8.7 and Lemma 8.5, $K = K'' \ast \{k\}$ for a compact convex set $K''$. There is a set $K''$ in $\text{bd} \bar{O}$.
corresponding to \( K'' \). Then the interior \( K'' \ast k_c \) in \( U \) maps to \( K' \) under \( \Pi_K \). By (8.88), \( K'' \ast k_c \) is \( \Gamma_E \)-invariant. Also, under the radial projection to \( R_{\tilde{E}}(\mathcal{O}) = \tilde{\Sigma}_E \), the interior of \( K'' \ast k_c \) goes to \( \Gamma_E \)-invariant subspace in \( \tilde{\Sigma}_E \) meeting each complete affine leaf at a point. This contradicts the cocompactness of the action on \( \tilde{\Sigma}_E \).

Let us take nonidentity \( g \in \Gamma_E \) going to \( N'_K \). With nonzero \( v_g \). Then conjugation \( gcg_c^{-1} \) gives us an element with \( v_g \mu_g O_5(g_c)^{-1} \) by Lemma 8.3. This is not equal to \( v_g \) since \( \mu_g < 1 \). Hence, a block matrix computation shows that \( gcg_c^{-1}g^{-1} \) is not an identity element in \( \Gamma_E \) but maps to 1 in \( N_K \). We obtain a nontrivial element \( n_0 \) of \( N \). By Hypothesis 8.3.1, \( g_1 g g_c^{-1}g^{-1} \in N \). Since \( n_0 := g_1 g g_c^{-1}g^{-1} \neq 1 \) has the form (8.85), \( n_0 \) is a unipotent element. Since \( N \subset N \), we may let \( n_0 = N(v_0) \) for some nonzero vector \( v_0 \).

(B) Now we show \( \mu_g = 1 \) for all \( g \in \Gamma_E \):

Suppose that we have an element \( g \in \Gamma_E \) and \( \mu_g < 1 \). Then we have as above

\[
v_g^k n_0 g^{-k} = v_{n_0} \mu_g O_5(g)\nu.
\]

Also, \( g^k n_0 g^{-k} \) goes to \( 1 \) in \( N_K \) since \( \Pi_E(n_0) = 1 \) in \( N_K \). Hence, \( \{g^k n_0 g^{-k}\} \to 1 \) as \( k \to \infty \) since \( n_0 \) is in the forms (8.85) given by (8.88). This contradicts the discreteness of \( N \). Hence, \( \mu_g = 1 \) for all \( g \in \Gamma_E \).

Since any element of \( g \in \Gamma_E \), we can take \( g', g' \in \Gamma_E \) so that \( gg' \in \Gamma_E \) and so \( \mu_{gg'} = \mu_g \mu_g = 1 \). We obtain \( \mu_g = 1 \) for all \( g \in \Gamma_E \).

\[\Box\]

**Proof (The proof of Theorem 8.2)** Suppose that \( E \) is an NPNC R-end. When \( N_K \) is discrete, Theorem 8.3 gives us the result.

When \( N_K \) is non-discrete, Hypothesis 8.3.1 holds by Propositions 8.6. Also, \( N_K \) is semi-simple by Proposition 8.7.

By Proposition 8.8, \( \mu \equiv 1 \) holds. Lemmas 8.3 and 8.5, (vii) of Proposition 8.6 show that the premise of Proposition 8.2 holds. Proposition 8.2 shows that Hypothesis 8.3.2 holds. Proposition 8.3 shows that we have a strictly joined or quasi-joined end. Corollary 8.3 implies the result.

Note here that we may prove for finite index subgroups of \( \Gamma_E \) by the definition of strictly joined or quasi-joined ends.

\[\Box\]

We give a convenient summary.

**Corollary 8.4** Let \( \mathcal{O} \) be a properly convex strongly tame real projective orbifold. Assume that its holonomy group is strongly irreducible. Let \( \tilde{E} \) be an NPNC p-end of the universal cover \( \tilde{\mathcal{O}} \) or \( \mathcal{O} \) satisfying the transverse weak middle eigenvalue condition for the R-p-end structure of \( \tilde{E} \). Suppose that \( \pi_1(\tilde{E}) \) satisfies (NS) or \( \dim K = 0, 1 \). Then the holonomy group \( h(\Gamma_E) \) is a group whose element under a coordinate system is of form:
\[
g = \begin{pmatrix}
S(g) & 0 & 0 & 0 \\
0 & \lambda(g) & 0 & 0 \\
0 & \lambda(g)v(g)^T & \lambda(g)O_5(g) & 0 \\
0 & \lambda(g)\left(\alpha_7(g) + \frac{||v(g)||^2}{2}\right) & \lambda(g)v(g)O_5(g) & \lambda(g)
\end{pmatrix}
\] (8.92)

where \( \{S(g) | g \in \Gamma_{\tilde{E}}\} \) acts cocompactly on a properly convex domain in \( \text{bd} \tilde{S} \) of dimension \( n - i_0 - 1 \), \( O_5 : \Gamma_{\tilde{E}} \to O(i_0 + 1) \) is a homomorphism, and \( \alpha_7(g) \) satisfies the uniform positive translation condition given by (8.58).

And \( \Gamma_{\tilde{E}} \) virtually normalizes the group

\[
\mathcal{N}(v) = \begin{pmatrix}
I_{n-i_0-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & v^T & I_{i_0} & 0 \\
0 & \|v\|^2 / 2 & v & 1
\end{pmatrix}, \quad v \in \mathbb{R}^{i_0}\}
\] (8.93)

**Proof** The proof is contained in the proof of Theorem 8.2. \(\square\)

### 8.5 Applications of the NPNC-end theory

#### 8.5.1 The proof of Corollary 8.1

**Proof (Proof of Corollary 8.1)** We may always take finite index subgroups for \( \Gamma_{\tilde{E}} \) and consider as the end holonomy group. By Corollary 8.4, we obtain that the dual holonomy group \( g^{-1T} \in \Gamma_{\tilde{E}}^* \) has form under a coordinate system:

\[
g^{-1T} = \begin{pmatrix}
S(g)^{-1T} & 0 & 0 & 0 \\
0 & \lambda(g)^{-1} & -\lambda(g)^{-1}O_5(g)^{-1}v_g & 0 \\
0 & 0 & \lambda(g)^{-1}O_5(g)^{-1} & -\lambda(g)^{-1}v_g^T \\
0 & 0 & 0 & \lambda(g)^{-1}
\end{pmatrix}
\] (8.94)

And \( \Gamma_{\tilde{E}}^* \) virtually normalizes the group \( \mathcal{N}(v)^{-1T} | v \in \mathbb{R}^{i_0} \) where

\[
\mathcal{N}(v)^{-1T} = \begin{pmatrix}
I_{n-i_0-1} & 0 & 0 & 0 \\
0 & 1 & -v & \|v\|^2 / 2 \\
0 & 0 & I_{i_0} & -v^T \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (8.95)
By using coordinate changes by reversing the order of the \(n - i_0 + 1\)-th coordinate to the \(n + 1\)-th coordinate, we can make the lower right matrix of \(\Gamma_E\) and \(N^*\) into a lower triangular form. Hence, \(N^*\) is a partial \(i_0\)-dimensional cusp group.

Suppose that \(\langle S(g), g \in \Gamma_E \rangle\) acts on properly convex set \(K := K'' \ast \{k\} \in S^{n-i_0-1}\), a strict join, for a properly convex set \(K'' \subset S^{n-i_0-2} \subset S^{n-i_0-1}\) and \(k\) from the proof of Proposition 8.3. \(N^*\) acts on \(S_{i_0}^{i_0+1}\) containing \(S_{i_0}^0\) and corresponding to a point \(k\) under the projection \(\Pi_K : S^n - S_{i_0}^0 \to S^{n-i_0-1}\). Let \(K'''\) denote the compact convex set in \(S^n - S_{i_0}^0\) mapping homeomorphic to \(K''\) under \(\Pi_K\) as we showed in Proposition 8.2. There is also a subspace \(S_{K'''}^{n-i_0-2}\) that is the span of \(K''\). Also, \(K_4 := K''' \ast V_E\) is in \(S^{n-i_0-1}\) the great sphere containing \(v_E\) and project to \(S^{n-i_0-2}\) under \(\Pi_K\).

Recall Proposition 2.22 for the following: We have \(\mathbb{R}^{n+1} = V \oplus W\) for subspaces \(V\) and \(W\) corresponding to \(S_{K'''}^{n-i_0-2}\) and \(S_{i_0}^{i_0+1}\) respectively. We may assume that \(N^*\) acts on both spaces and \(\Gamma_E\) acts on \(K'''\) and both spaces. Then \(\mathbb{R}^{n+1} = V^\perp \oplus W^\perp\) for subspaces \(V^\perp\) of \(V^0\) on zero on \(W\) and \(W^\perp\) of \(W^0\) zero on \(V\). Then \(V^\perp\) corresponds to the subspace \(S_{K'''}^{n-i_0-2} - \{v_E\}\) which equals \(S_{i_0}^{i_0+1}\), and \(W^\perp\) corresponds to \(S_{K'''}^{n-i_0-2} - \{v_E\}\) which equals \(S_{K'''}^{n-i_0-2} - \{v_E\}\). We let \(S_{K'''}^{n-i_0-2} - \{v_E\}\) denote the dual subspaces in \(S^{n-1}\).

Hence, \(\Gamma_E\) and \(N^*\) act on both of these two spaces.

Since \(K'''\) is bd \(\Sigma_{E}\) as in Proposition 8.2, we obtain \(K_4 \subset \text{bd} \tilde{\mathcal{O}}\).

Let us choose a properly convex p-end neighborhood \(U\) where \(N^*\) acts on. \(U \cap Q\) for any \(i_0 + 1\)-dimensional subspace containing \(S_{i_0}^0\) is either empty or is an ellipsoid since \(N^*\) acts on \(U\). Any sharply supporting hyperplane \(P'\) at \(v_E\) to \(U \cap Q\) must containing \(S_{i_0}^0\) since \(P' \cap Q\) for any \(i_0 + 1\)-dimensional subspace \(Q\) containing \(S_{i_0}^0\) must be disjoint from \(U \cap Q\) and hence \(P' \cap Q \subset S_{i_0}^0\) and hence \(P' \cap Q \subset S_0^0\).

Let \(P \subset S^n\) be an oriented hyperspace sharply supporting \(\mathcal{O}\) at \(v_E\) and containing \(S_{i_0}^0\) and \(S_{K'''}^{n-i_0-2}\). This is unique such one since the hyperspace is the join of the two. Hence, \(K''' \ast \{v_E\} \subset P\) where

\[
\text{Cl}(U) \cap P = \text{Cl}(\tilde{\mathcal{O}}) \cap P = K_4 := K''' \ast \{v_E\}
\]

by Proposition 8.3.

We have the subspaces \(S^{n-i_0-1}\) and \(S_0^{i_0+1}\) meeting at \(v_E\) and \(S_0^0\) containing \(v_E\) meeting with \(S^{n-i_0-1}\) at the same point.

Let \(P^*\) denote the dual space of \(P\) under its intrinsic duality. Let us denote by \(K_4^* \subset P^*\) the dual of \(K_4\) with respect to \(P\).

Consider a pencil \(P_{t,Q}\) of hyperplanes supporting \(\tilde{\mathcal{O}}\) starting from \(P\) sharing a codimension-two subspace \(Q \subset P\). Then \(P_{t,Q}\) exists for \(t\) in a convex interval \(I_0\) in a projective circle. \(Q\) supports \(P \cap \text{Cl}(\mathcal{O}) = K''' \ast \{v_E\}\). We assume \(P_{0,Q} = P\) for all \(Q\). Thus the space \(S_{P,Q}\) of all hyperspaces in one of \(P_{t,Q}\) is projectively equal to a fibration over \(K_4^*\) in \(P^*\) with fibers a singleton or a compact interval. Since \(S_{P,Q}\) is convex, the set corresponding to the nontrivial interval fibers is the interior of \(K_4^*\).

By Proposition 2.22(iii), \(K_4^* \subset S_{i_0-1}\) for a proper subspace dual \(K_4^*\) in \(S^{n-1}\) to \(K_4\) and a great sphere \(S_{i_0-1}\) in \(P^*\).
Consider $S_{P,\Omega}$ as a subset of $\mathbb{S}^{\ast}$ now. Then $P_{\Omega}$ under duality goes to a ray in $\hat{\mathcal{O}}^\ast$ from $P^\ast$ to a boundary point of $\hat{\mathcal{O}}^\ast$. Hence, the space $R_P(S_{P,\Omega})$ of such open rays are projectively diffeomorphic to the interior of $K_4^\ast$ in its span $S_{P,\Omega}^{\ast}$. Let $K''^\ast$ denote the dual of $K''$ in its span $S_{K''}^{\ast}$. Since $K''^\ast$ is projectively diffeomorphic to $K''^\ast \cup \{v\}$ for a singleton $v$, we have $S_{P,\Omega}$ is projectively diffeomorphic to the interior of $K''^\ast \cup H_{i_0}$ for a hemisphere $H_{i_0}$ of dimension $i_0$.

By the duality argument, this space equals $R_P(\hat{\mathcal{O}}^\ast)$ since such rays correspond to supporting pencils of $\mathcal{O}$ and vice versa.

Now, recall that our matrices of $\Gamma_{\mathcal{E}}^\ast$ in the form of (8.92) and matrices in $N$ in the form (8.93). We can directly show the properness of the action on $R_P(\hat{\mathcal{O}}^\ast)$:

- Let $g_i$ be a sequence of elements of $\Gamma_{\mathcal{E}}^\ast$. Suppose that $S(g_i)^\ast$ is bounded for our matrices in $\Gamma_{\mathcal{E}}^\ast$. Then $v_{g_i}$ blows up: Otherwise the properness of the $\Gamma_{\mathcal{E}}^\ast$ does not hold for $\hat{\Sigma}_{\mathcal{E}}$ and our action splits. Our action is basically that on $K''^\ast \times R_{i_0}$, an affine form of the interior of $K^\ast S_{i_0}^{\ast -1}$, preserving the product structure where $K = K''^\ast \cup \{k\}$.

Hence, it follows by our matrix form (8.94) that $\Gamma_{\mathcal{E}}^\ast$ acts properly on $R_P(\hat{\mathcal{O}}^\ast)$ projectively isomorphic to the interior of $K''^\ast S_{i_0}^{\ast -1}$. By Lemma 1.1, $\hat{\mathcal{O}}^\ast$ can be considered a p-end neighborhood with a radial structure. Hence, we can apply our theory of the classification of NPNC-ends. The transverse weak uniform middle eigenvalue condition is satisfied by the form of the matrices. Also the uniform positive translation condition holds by the matrix forms again. Proposition 8.3 completes the proof.

Finally, we mention the following: Since each $i_0$-dimensional ellipsoid in the fiber in $\hat{\Sigma}_{\mathcal{E}}^\ast$ has a unique fixed point that should be common for all ellipsoid fibers, the choice of the p-end vertex is uniquely determined for $\hat{\mathcal{E}}^\ast$ so that $\hat{E}^\ast$ is to be quasi-joined. \hfill $\square$

### 8.5.2 Complete affine ends again

We now study the case deferred from Theorem 4.3.

**Corollary 8.5 (non-cusp complete-affine p-ends)** Let $\mathcal{O}$ be a strongly tame properly convex $n$-orbifold for $n \geq 3$. Let $\hat{\mathcal{E}}$ be a complete affine $\mathcal{R}$-p-end of its universal cover $\hat{\mathcal{O}}$ in $\mathbb{S}^n$. Let $v_{\mathcal{E}} \in \mathbb{S}^n$ be the p-end vertex with the p-end holonomy group $\Gamma_{\mathcal{E}}^\ast$. Suppose that $\hat{\mathcal{E}}$ is not a cusp p-end. Then we can choose a different point as the p-end vertex for $\hat{\mathcal{E}}$ so that $\hat{\mathcal{E}}$ is a quasi-joined R-p-end with fiber homeomorphic to cells of dimension $n - 2$. Also, the end fundamental group is virtually abelian.

**Proof** We will use the terminology of the proof of Theorem 4.3. Theorem 4.3 shows that $\Gamma_{\mathcal{E}}^\ast$ is virtually nilpotent and with at most two norms of eigenvalues for each element. By taking a finite-index subgroup, we assume that $\Gamma_{\mathcal{E}}^\ast$ is nilpotent. Let $Z$ be the Zariski closure, a nilpotent Lie group. We may assume that $Z$ is connected.
by taking a finite index subgroup of \( \Gamma_{\tilde{E}} \). Theorem 4.3, says \( \Gamma_{\tilde{E}} \) is isomorphic to a virtually unipotent group by restricting to the affine space \( \tilde{\Sigma}_{\tilde{E}} \). Hence, \( Z \) is simply connected and hence contractible. Since \( \Gamma_{\tilde{E}} \cap Z \) is a cocompact lattice in \( Z \), and \( \Gamma_{\tilde{E}} \) has the virtual cohomological dimension \( n - 1 \), it follows that \( Z \) is \((n - 1)\)-dimensional.

By Lemma 4.2, \( Z \) acts transitively on the complete affine subspace \( \tilde{\Sigma}_{\tilde{E}} \) since \( \Gamma_{\tilde{E}} \) acts cocompactly on it.

The orbit map \( Z \to Z(x) \) for \( x \in \tilde{\Sigma}_{\tilde{E}} \) is a fiber bundle over the contractible space with fiber the stabilizer group. Since \( \dim Z = n - 1 \), it follows that it must be trivial. Since \( \tilde{\Sigma}_{\tilde{E}} \) is contractible, the stabilizer is trivial.

Since \( Z \) fixes \( v_{\tilde{E}} \), we have a homomorphism

\[
\lambda_{v_{\tilde{E}}} : Z \ni g \to \lambda_{v_{\tilde{E}}}(g) \in \mathbb{R}. \tag{8.96}
\]

Let \( N \) denote the kernel of the homomorphism. By Theorem 2.6, \( N \) is an orthopotent Lie group since it has only one norm of the eigenvalues equal to 1.

Since \( \tilde{E} \) is a complete affine R-p-end, \( R_{p}(U) \) is a complete affine space equal to \( \tilde{\Sigma}_{\tilde{E}} \). By taking a convex hull of finite number of radial rays from \( v_{\tilde{E}} \), we may choose a properly convex p-end neighborhood \( U \) of \( \tilde{E} \). Also, we may choose so that the closure of \( U \) is in another such p-end neighborhood. Thus, \( \partial U \cap \tilde{\partial} \) is in one-to-one correspondence with \( \tilde{\Sigma}_{\tilde{E}} \). We modify \( U \) to

\[
\bigcap_{g \in Z} g(U) = \bigcap_{g \in F} g(U).
\]

This contains a nonempty properly convex open set by Lemma 2.5. We may assume that \( Z \) acts on a properly convex p-end neighborhood \( U \) of \( \tilde{E} \). Since \( Z \) acts transitively on \( \tilde{\Sigma}_{\tilde{E}} \), it acts so on an embedded convex hypersurface \( \partial U = \partial U \cap \tilde{\partial} \). This is the set of endpoints of maximal segments from \( v_{\tilde{E}} \) in the directions of the complete affine space \( \tilde{\Sigma}_{\tilde{E}} \). Since this characterization is independent of \( \tilde{\partial} \), \( \partial U \) is an orbit of \( Z \). Since \( Z \) is a Lie group, \( \partial U \) is smooth.

The smooth convex hypersurface \( \partial U \) is either strictly convex or has a foliation fibered by totally geodesic submanifolds. Since \( \tilde{\Sigma}_{\tilde{E}} \) is complete affine, these submanifolds must be complete affine subspaces. Since \( Cl(U) \) contains these and \( Cl(U) \) is properly convex, this is a contradiction. Hence, \( \partial U \) is strictly convex.

Finally, since \( \partial U \) is in one-to-one correspondence with \( \tilde{\Sigma}_{\tilde{E}} \), \( \partial U / \Gamma_{\tilde{E}} \) is a compact orbifold of codimension 1.

Let \( A \) be a hyperspace containing \( v_{\tilde{E}} \) in the direction of \( \partial \tilde{E} = S^{n-2} \subset S^{n-1}_{v_{\tilde{E}}} \). Then \( U_{A} := A \cap Cl(U) \) is a properly convex compact set on which \( Z \) acts. By Lemma 8.13, \( U_{A} / Z \) is compact. By Lemma 8.12, \( U_{A} \) is a properly convex segment.

We now need Lemmas 8.12 and 8.13.

**Lemma 8.12** Let a simply connected nilpotent Lie group \( S \) act cocompactly and effectively on a properly convex open domain \( J \). Suppose that each element of \( S \) has at most two norms of eigenvalues and it fixes a point \( p \) in the boundary of \( J \). Then the dimension of the domain is 0 or 1.
Proof Suppose that \( \dim J > 1 \). By Lemma 4.2, \( S \) acts transitively on \( J \). The action is proper since there is a Hilbert metric on \( J \). Since \( S \) is nilpotent and is simply connected, \( S \) is contractible. Since the stabilizer of \( S \) at a point \( x \in J \) is compact, it is trivial in \( S \). Hence \( S \) is diffeomorphic to a \( \dim J \)-dimensional cell.

Let \( \lambda_p(g) \) for \( g \in S \) denote the associated eigenvalue of \( g \) at \( p \) for unit determinant matrix representatives. Let \( N_S \) denote the kernel of the homomorphism \( S \to \mathbb{R}_+ \) given by \( g \mapsto \lambda_p(g) \). Then \( N_S \) is an orthopotent group of dimension \( \dim J - 1 \).

Then \( N_S \) acts on \( R_p(J) \) as an orthopotent Lie group. \( R_p(J)/N_S \) is compact since otherwise \( J/S \) is not compact. A stabilizer of a point of \( R_p(J) \) acts on a segment \( s \) from \( p \) in \( J \). The existence of two fixed directions of eigenvalue 1 implies that each point of \( s \) is a fixed point, and hence the stabilizer is trivial. Therefore, the properness and freeness of the action of \( N_S \) on \( R_p(J) \) follow.

By Proposition 4.4, the orbit \( N_S(x) \) for \( x \in J \) is an ellipsoid of dimension \( \dim J - 1 \). Hence, \( S/N_S \) is a 1-dimensional group. The elements of \( N_S \) are of form

\[
k = \begin{pmatrix} 1 & 0 & 0 \\ \frac{v_k^T}{\| v_k \|^2} & |_{4 \dim J - 1} & 0 \\ v_k \\ \end{pmatrix} \quad \text{for} \quad v_k \in \mathbb{R}^{\dim J - 1}. \tag{8.97}
\]

We may write for \( g \in N \),

\[
g = \begin{pmatrix} a_1(g) & 0 & 0 \\ a_7(g) & A_5(g) & 0 \\ a_9(g) & a_5(g) \\ \end{pmatrix} . \tag{8.98}
\]

By arguing as in the proof of Proposition 8.3, any element \( g \in S \) induces an \((\dim J - 1) \times (\dim J - 1)\)-matrix \( M_g \) given by \( g \mathcal{N}(v)g^{-1} = \mathcal{N}(vM_g) \) where

\[
M_g = \frac{1}{a_1(g)} (A_5(g))^{-1} = \mu_g O_5(g)^{-1}
\]

for \( O_5(g) \) in a compact Lie group \( G_E \) where \( \mu_g = \frac{a_5(g)}{a_1(g)} = \frac{a_9(g)}{a_7(g)} \).

Reasoning as in the proof of Lemma 8.6, we can find coordinates so that for every \( g \in N \),

\[
g = \begin{pmatrix} a_1(g) & 0 & 0 \\ a_1(g)v_g^T & a_5(g) & 0 \\ a_7(g) & a_5(g)v_gO_5(g) & a_9(g) \\ \end{pmatrix} , \quad O_5(g)^{-1} = O_5(g)^T . \tag{8.99}
\]
and the form of $N_S$ is not changed. Also, $a_7(g) = a_1(g)(\alpha_7(g) + \|v\|^2/2)$, as we can show following the beginning of Section 8.3.3. Recall $\alpha_7$ from Section 8.3.2.2. Here, $\alpha_7(g) = 0$ since otherwise $J$ cannot be properly convex since $g$ will translate the orbits in the affine space where $p$ is the infinity as in Remark 8.4.

If there is an element $g$ with $\mu_k \neq 1$, then the group $N$ is solvable and not nilpotent. If $\mu_k = 1$ for all $g \in N$, then from the matrix form we see that $N$ has only 1 as norms of eigenvalues with $a_7(g) = a_5(g) = a_0(g)$ for $g \in N$, and $N$ acts on each ellipsoid orbit of $N_S$. Hence, $J/N = J/N_S$ is not compact. This is a contradiction. Therefore $\dim J = 1$. □

Lemma 8.13 $U^n_\alpha/Z$ is compact.

Proof Suppose that $\dim U^n_\alpha = n - 1$. The orbit map $Z \to Z(x)$ for $x \in U^n_\alpha$ is a fibration over a simply connected domain. The stabilizer must be compact since $U^n_\alpha$ has a Hilbert metric. Since $Z$ being a simply connected nilpotent Lie group is contractible, the stabilizer has to be trivial. Since $\dim Z = n - 1$, $Z$ acts transitively on $U^n_\alpha$ and $U^n_\alpha/Z$ is compact.

Suppose now that $\dim U^n_\alpha = j_0 < n - 1$. Let $L$ be an $(j_0 + 1)$-dimensional subspace containing $U_\alpha$ meeting $A$ transversely. Let $l$ be the $j_0$-dimensional affine subspace of $H$ corresponding to $L$. Since

$$g(U_\alpha) = U_\alpha, U_\alpha \subset L, g(L)$$

it follows that

$$g(L) \cap L = \langle U_\alpha \rangle \text{ or } g(L) = L, \text{ which implies } g(l) = l \text{ or } g(l) \cap l = \emptyset.$$ 

Recall that $Z$ acts transitively and freely on the complete affine space $\mathfrak{g}$ from the beginning of the Section 8.5.2. Since $\dim l = j_0$, it follows that the subgroup $\mathfrak{g}_l := \{g \in Z | g(l) = l\}$ has the dimension $j_0$.

Now $\mathfrak{g}_l$ acts on on $U^n_\alpha$. As proved above, the stabilizer of $\mathfrak{g}_l$ of a point of $U^n_\alpha$ is trivial since $U_\alpha \cap L$ is properly convex and $\mathfrak{g}_l$ is nilpotent without a compact subgroup of dimension $> 0$. Hence, $\mathfrak{g}_l$ acts transitively on $U^n_\alpha$ as in the first part since $\dim \mathfrak{g}_l = \dim U^n_\alpha$, and $U^n_\alpha/\mathfrak{g}_l = U^n_\alpha/Z$ is compact. □

Proof (Proof of Proposition 8.5 continued) If $\dim U_\alpha = 0$, then $U$ is a horospherical $p$-end neighborhood where $\Gamma_\xi$ is unimodular and cuspidal by Theorem 4.2. Hence, $\dim U_\alpha = 1$ by Lemma 8.12.

Let $q$ denote the other endpoint of the segment $U_\alpha$ than $v_\xi$. Since $U$ is convex, $R_q(U)$ is a convex open domain. Since an element of $\Gamma_\xi$ has two eigenvalues, Each radial segment from $q$ maximal in $\partial$ meets the smooth strictly convex hypersurface $\delta U$ and transversely since a radial segment cannot be in $\delta U$.

We have $R_q(U) = R_q(\delta U)$: Since $\Sigma_{\gamma_\xi}$ is complete affine, and $U$ is properly convex, each segment from $v_\xi$ passes $\delta U$ as we lengthen it. Hence, $bdU = \delta U \cup U_\alpha$ since $U_\alpha$ is precisely bd$U \cap A$ for the hyperspace $A$ as defined earlier. Hence, for each segment in $U$ from $q$ must end at a point of $bdU \cap \partial = \delta U$ since $\delta U$ is strictly
convex. By the transversality, a segment from \( q \) ending at \( \delta U \) must have one-side in \( U \). By strict convexity of \( \delta U \), it is clear that each ray from \( q \) meets \( \delta U \) transversely.

Since the \( \Gamma_E \)-action on \( \delta U \) is proper, so is its action on \( R_q(U) \). Hence, \( U \) can be considered a p-end neighborhood with radial lines from \( R_q \) foliating \( U \) by Lemma 1.1.

There is an embedding from \( \delta U \) to \( R_q(\delta U) = R_q(U) \subset R_q(\bar{\delta}) \). Since \( \delta U / \Gamma_E \) is a compact orbifold, so is \( R_q(U) / \Gamma_E \).

By the third item in the second item of Theorem 4.3, \( R_q(U) \) is not complete-affine since the norm \( \lambda_q(g) \) for some \( g \in \Gamma_E \) has a multiplicity \( n, n < n + 1 \), and \( n > 2 \) by assumption.

Suppose that \( R_q(U) \) is properly convex. Elements of \( \Gamma_E \) have at most two distinct norms of eigenvalues. Since \( R_q(U) \) is homeomorphic to \( \partial U \cap \bar{\delta} \) with a compact quotient by \( \Gamma_E \), \( R_q(U) \) has a compact quotient by \( \Gamma_E \), and \( \dim R_q(U) = n - 1 \geq 2 \).

By Lemma 8.12, this is a contradiction. Hence, \( U_q \) is not properly convex.

Thus, \( q \) is the p-end vertex of an NPNC-end for \( U \) foliated by radial segments from \( q \). Since the associated upper-left part has only two norms of eigenvalues by Lemma 8.12, and the properly convex leaf space \( K^q \) is 1-dimensional and has a compact quotient, the fibers have the dimension \( n - 2 = n - 1 - 1 \). Therefore, \( U_q \) is foliated by \( n - 2 \)-dimensional complete affine subspaces. The leaf space \( K^q \) is a properly convex segment.

Suppose that \( R_q(\bar{\delta}) \) is different from \( R_q(U) \). Then \( R_q = K_R * S^{n-1} \) and \( R_q = K_q * S^{n-1} \) for a properly convex domain \( K_q \) and a convex domain \( K_R \) containing \( K_q \).

Then there is some point \( x \) with \( \frac{q}{\Gamma} \) not in \( R_q(U) \). Then by taking elements \( g \) with maximal norm in \( \dot{K} \) from the forth item of Proposition 2.20 and using Proposition 8.3 and \( \lambda_q(g) = 1 \), we obtain that \( g^n(x) \) converges to the point antipodal to a point of \( \dot{K} \). This contradicts the proper convexity of \( \bar{\delta} \). Hence, \( R_q(U) = R_q(\bar{\delta}) \) and by Lemma 1.1 we obtain that \( U \) is a R-p-end neighborhood of a p-end vertex \( q \).

Recall the Lie group \( N \) from (8.96). Since \( Z \) acts on \( U_q \) transitively and \( \lambda_q(g) = \lambda_q, g \in N \), the Lie group \( N \) acts on each complete affine leaf transitively. \( N \) is a nilpotent Lie group since it is a subgroup of \( Z \). Also, \( N \) is orthopotent since elements of \( N \) have only one norm of the eigenvalues by Theorem 2.6. We can apply Proposition 4.4 to the hyperspace \( P \) containing the leaves with a cocompact subgroup of \( N \) acting on it. As \( U \cap P \) is properly convex, \( r_P(N) \) is a cusp group.

Let \( x_1, \ldots, x_{n+1} \) be the coordinates of \( \mathbb{R}^{n+1} \). Now give coordinates so that \( q = (0, 0, \ldots, 1) \) and \( v_E = (1, 0, \ldots, 0) \). Since these are fixed points, we obtain that elements of \( N \) can be put into forms:

\[
N(v) := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & v^T & 1 & 0 \\
0 & \frac{1}{2} \|v\|^2 & v & 1
\end{pmatrix}
\quad \text{for } v \in \mathbb{R}^{n-2}.
\]
Now, \( \Gamma_{\tilde{E}} \) satisfies the transverse weak middle eigenvalue condition with respect to \( q \) since \( \Gamma_{\tilde{E}} \) has just two eigenvalues and \( Z \) is generated by \( N \) and \( g^t \) for a nonunipotent element \( g \) of \( \Gamma_{\tilde{E}} \). \( N \) admits an invariant Euclidean structure being a cusp group. Since \( \dim K = 1 \), Theorem 8.2 shows that \( q \) is an NPNC R-p-end vertex for \( U \) covering the strongly tame convex real projective orbifold \( U / \Gamma_{\tilde{E}} \).

Finally, \( \Gamma_{\tilde{E}} \) is virtually abelian: From (8.92), \( S(g) \) is a \( 1 \times 1 \)-matrix or \( 0 \times 0 \)-one. From the matrix form, the Zariski closure \( Z \) is an extension of an orthopotent Lie group. Since \( \tilde{\Sigma}_{\tilde{E}} \) equals \( A_{i_0}^{b_0} \times I \) for an interval or a singleton \( I, i_0 = n - 2, n - 1 \), \( Z \) acts on it. \( O_S \) extends to a homomorphism \( O_S : Z \to O(i_0) \). Let \( Z_K \) denote the kernel. If \( Z/K \) also acts properly and cocompactly on \( \tilde{\Sigma}_{\tilde{E}} \) since \( Z/Z_K \) is compact. It is easy to see \( Z_K \) is abelian from the matrix form. Also, we can put a \( Z \)-invariant Euclidean metric on the complete affine space \( \tilde{\Sigma}_{\tilde{E}} \) by the product metric form. Then the Bieberbach theorem implies the result. \( \square \)

If we require the weak middle eigenvalue conditions for a given vertex, the completeness of the end implies that the end is cusp.

**Corollary 8.6 (cusp and complete affine)** Let \( \hat{O} \) be a strongly tame properly convex n-orbifold. Suppose that \( \hat{E} \) is a complete affine R-p-end of its universal cover \( \hat{O} \) in \( S^n \) (resp. in \( \mathbb{RP}^n \)). Let \( v_{\tilde{E}} \in S^n \) (resp. \( \in \mathbb{RP}^n \)) be the p-end vertex with the p-end holonomy group \( \Gamma_{\tilde{E}} \). Suppose that \( \Gamma_{\tilde{E}} \) satisfies the weak middle eigenvalue condition with respect to \( v_{\tilde{E}} \). Then \( \tilde{E} \) is a complete affine R-end if and only if \( \tilde{E} \) is a cusp R-end.

**Proof** It is sufficient to prove for the case \( \hat{O} \subset S^n \). Since a horospherical end is a complete affine end by Theorem 4.2, we need to show the forward direction only. In the second possibility of Theorem 4.3, the norm of \( \lambda_{v_{\tilde{E}}} (\gamma) \) has a multiplicity one for a nonunipotent element \( \gamma \) with \( \lambda_{v_{\tilde{E}}} (\gamma) \) or \( \lambda_{v_{\tilde{E}}} (\gamma^{-1}) \) equal to the maximal norm. This violates the weak middle eigenvalue condition, and only the first possibility of Theorem 4.3 holds.

\[ \square \]

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8.6 Some miscellaneous result needed above

8.6.1 J. Porti’s question

The following answers a question that we discussed with J. Porti at the UAB, Barcelona in 2013 whether there is a noncuspidal unipotent group acting as an end holonomy group of an R-end. The following is also proved by Cooper-Long-Tillman [73] with some difference.

The following generalizes the result of D.Fried [84].

**Corollary 8.7** Assume that \( \hat{O} \) is a convex real projective strongly tame orbifold with an end \( E \). Suppose that eigenvalues of elements of \( \Gamma_{\tilde{E}} \) have unit norms only. Then \( \tilde{E} \) is horospherical, i.e., cuspidal.
Proof First, we assume that $\tilde{\mathcal{O}} \subset \mathbb{S}^n$. By Lemma 4.1, we only need to show that $\tilde{E}$ is a complete affine end.

By Theorem 2.6, $\Gamma_{\tilde{E}}$ is orthopotent. Theorem 2.13 shows that $\tilde{\Sigma}_{\tilde{E}}$ is complete affine.\[\mathbb{S}^n\]

8.6.2 Why $\lambda_{v_{\tilde{E}}}(g) \neq 1$?

Corollary 8.8 Let $\mathcal{O}$ be a strongly tame properly convex n-orbifold. Suppose that $\tilde{E}$ is an NPNC R-p-end of its universal cover $\tilde{\mathcal{O}}$ in $\mathbb{S}^n$ or (resp. in $\mathbb{R}P^n$). Let $v_E$ be the p-end vertex with the p-end holonomy group $\Gamma_{\tilde{E}}$. Suppose that $\pi_1(\tilde{E})$ satisfies (NS) or $\dim K^o = 0, 1$ for the leaf space $K^o$. Then for some $g \in \Gamma_{\tilde{E}}, \lambda_{v_{\tilde{E}}}(g) \neq 1$.

Proof It is sufficient to prove for the case $\tilde{\mathcal{O}} \subset \mathbb{S}^n).$ Suppose that $\lambda_{v_{\tilde{E}}}(g) = 1$ for all $g \in \Gamma_{\tilde{E}}$.

We show that the transverse weak middle eigenvalue condition for $\tilde{E}$ holds: Suppose not. A $\Gamma_{\tilde{E}}$-invariant $i_0$-dimensional subspace $\mathbb{S}_{i_0}^n$ contains $v_E$ as we discussed in Section 8.1.

Suppose that every element of $\Gamma_{\tilde{E}}$ is unit-norm-eigenvalued. By Theorem 2.6, $\Gamma_{\tilde{E}}$ is orthopotent. By Fried [84], there exists a nonorthopotent element in $\Gamma_{\tilde{E}}$ since $\tilde{\Sigma}_{\tilde{E}}$ is not complete affine. Hence, there exists an element $g \in \Gamma_{\tilde{E}}$ that is not unit-norm-eigenvalued.

We find an element $g$ of $\Gamma_{\tilde{E}}$ not satisfying the condition of the transverse weak middle eigenvalue condition with $\lambda_1(g) > 1$. For a real number $\mu$ equal to $\lambda_1(g)$, the subspace $\mathcal{R}_\mu(g)$ projects to $\mathbb{S}_{i_0}^n$ since otherwise the transverse weak middle eigenvalue condition holds. There exists a $g$-invariant subspace $\hat{P}_g \subset \mathbb{S}_{i_0}^n$ that is the projection of $\bigoplus_{\mu = \lambda_1(g)} \mathcal{R}_\mu(g)$. (See Definition 2.5.) This is a proper subspace since $v_E$ has associated eigenvalue 1 strictly less than $\lambda_g$. Hence, $\dim \hat{P}_g \leq i_0 - 1$.

We define $P_g$ the projection of $\bigoplus_{\mu < \lambda_1(g)} \mathcal{R}_\mu(g) \subset \mathbb{R}^{n+1}$.

Then $P_g$ is complementary to $\hat{P}_g$. Thus, $\dim P_g \geq n - i_0$, and $P_g \not\supset v_E$.

Under the projection to $\mathbb{S}^{n-1}(v_E), P_g$ goes to a subspace $P'_g$ of $\dim P_g - 1 \geq n - i_0 - 1$. As described in Section 8.2, $\tilde{S}_E$ is foliated by $i_0$-dimensional complete affine subspaces. Since $\dim \tilde{S}_E = n - 1$, these $i_0$-dimensional leaves must meet $P'_g$ of dimension $\geq n - 1 - i_0$.

Thus, any p-end neighborhood $U$ of $\tilde{E}$ meets $P_g$. There must be an antipodal pair $\tilde{P}_g$ in $\tilde{P}_g$ that is the projection of the eigenspace of $g$ with the associated eigenvalue whose norm is $\lambda_1(g) > 1$.\[\mathbb{S}^n\]
is a nonempty open domain meeting $P_g$, and $L - P_g$ has two components. Let $x, y$ be generic points in distinct components in $L - P_g$. Then $\{g^n(\{x, y\})\}$ geometrically converge to an antipodal pair of points in $\hat{P}_g$. Since this set is in $\text{Cl}(\mathcal{O})$, this contradicts the proper convexity of $\mathcal{O}$. Thus, the transverse weak middle eigenvalue condition of $\hat{E}$ holds.

The premises of Theorem 8.2 except for the strong irreducibility of the holonomy group of $\pi_1(\mathcal{O})$ are satisfied, and Theorem 8.2 classifies the ends. Suppose that $h(\pi_1(\mathcal{O}))$ is strongly irreducible. Let $\hat{E}$ be a $p$-end corresponding to one of these, and $\pi_1(\hat{E})$ acts on the leaf space properly convex domain $K$ disjoint from $v_{\hat{E}}$. We showed in the proof of Proposition 8.2 the existence of elements where all the associated norms of eigenvalues of the subspace containing $K$ are $> 1$ and the rest of the norms of the eigenvalues are $< 1$ by (8.46). This is a contradiction to the assumption $\lambda_{v_{\hat{E}}}(g) = 1$. Thus, these types of ends do not occur.

Suppose that $h(\pi_1(\mathcal{O}))$ is virtually reducible. Then (ii) of the conclusion of Proposition 8.3 could hold. Again $\mu_g = 1$ for all $g \in \Gamma_{\hat{E}}$ by Proposition 8.8 and the corresponding part showing $\mu_g = 1$ of the proof of Theorem 8.3 where we do not need strong irreducibility of $h(\pi_1(\mathcal{O}))$. Now, matrices of form (8.46) give us the same contradiction as in the above paragraph. 

8.6.3 A counterexample in a solvable case

We will find some nonsplit NPNC-end where the end holonomy group is solvable. Our construction is related to the construction of Carrière [37] and Epstein [82]. A related work is given by Cooper [69]. These are not a quasi-join nor a join and do not satisfy our conditions (NS) and neither the end orbifolds admit properly convex structure nor the fundamental groups are virtually abelian. Define

$$ N(w, v) = \begin{pmatrix} 1 & w & w^2/2 & 0 & 0 \\ 0 & 1 & w & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & v & 1 & 0 \\ 0 & 0 & v^2/2 & v & 1 \end{pmatrix}, \quad \text{and} \quad (8.101) $$
We compute \( g_\lambda N(w, v)g_\lambda^{-1} = N(\lambda w, v/\lambda) \). We define the group
\[
S := \langle N(w, v), g_\lambda : v, w \in \mathbb{R}, \lambda \in \mathbb{R}_+ \rangle.
\]
Consider the affine space \( \mathbb{A}^4 \) given by \( x_3 > 0 \) with coordinates \( x_1, x_2, x_4, x_5 \) where \( S \) acts on. Then \( \langle N(w, 0), w \in \mathbb{R}, g_\lambda \rangle, \lambda \in \mathbb{R}_+ \), acts on an open disk \( B_{1.2} \) bounded by a quadric \( x_1 > x_2^2/2 \) in the plane \( x_4 = 0, x_5 = 0 \). (See Section 8.3.1.1.) \( \langle N(0, v), v \in \mathbb{R}, g_\lambda, \lambda \in \mathbb{R}_+ \rangle \) acts on an open disk \( B_{3.5} \) bounded by a quadric \( x_5 > x_3^2/2 \) in the plane \( x_2 = 0, x_3 = 0 \).

Hence, an orbit \( S([1, 0, 1, 0, 1]) \) is given by the following set as a subset of \( \mathbb{A}^4 \):
\[
\left\{ (x_1, x_2, x_4, x_5) | x_1 = \frac{x_2^2}{2} + C^2, x_5 = \frac{x_4^2}{2} + \frac{1}{C^2}, C > 0 \right\}.
\]
This is a 3-cell. Moreover,
\[
N(w, v)g_\lambda \left( \frac{x_2^2}{2} + C^2, x_2, x_4, \frac{x_4^2}{2} + \frac{1}{C^2} \right)
= \left( \lambda^2 \left( \frac{x_2^2}{2} + C^2 \right) + \lambda w x_2 + \frac{w^2}{2}, \lambda x_2 + w, x_4 + \frac{x_4^2}{2\lambda} + \frac{1}{C^2\lambda^2} + v^2 \right)
= \left( \frac{(\lambda x_2 + w)^2}{2} + \lambda^2 C^2, \lambda x_2 + w, x_4 + \frac{x_4^2}{\lambda} + \frac{(\frac{x_4}{\lambda} + v)^2}{2} \right)
= g_\lambda N(w/\lambda, v) \left( \frac{x_2^2}{2} + C^2, x_2, x_4, \frac{x_4^2}{2} + \frac{1}{C^2} \right).
\]
Hence, there is an exact sequence
\[
1 \to \{ N(w, v) | w, v \in \mathbb{R} \} \to S \to \{ g_\lambda, \lambda > 0 \} \to 1,
\]
telling us that \( S \) is a solvable Lie group (Thurston’s Sol [159]).

We find a discrete solvable subgroup. We take a lattice \( L \) in \( \mathbb{R}^2 \) and obtain a free abelian group \( N(L) \) of rank two. We can choose \( L \) so that the diagonal matrix with diagonal \( (\lambda, 1/\lambda) \) acts as an automorphism. Then the group \( S_L \) generated by \( \langle N(L), g_\lambda \rangle \) is a discrete cocompact subgroup of \( S \).

We remark that such a group exists by taking a standard lattice in \( \mathbb{R}^2 \) and choosing an integral Anosov linear map \( A \) of determinant 1 with two eigendirections. We
choose a new coordinate system so that the eigendirections are parallel to the \( x \)-axis and the \( y \)-axis. Then now \( L \) can be read from the new coordinate system, and \( \lambda \) is the eigenvalue of \( A \) bigger than 1.

The orbit \( S((1,0,1,0,1)) \) is a subset of \( B_{1,2} \times B_{4,5} \). We may choose our end vertex \( v \) to be \((0,0,0,1,0)\) or \((1,0,0,0,0)\).

The orbit \( S((1,0,1,0,1)) \) is strictly convex: We work with the affine coordinates. We consider this point with affine coordinates \((1,0,0,1)\). The tangent hyperspace at this point is given by \( x_1 + x_5 = 2 \). We can show that locally the orbit meets this hyperplane only at \((1,0,0,1)\) and is otherwise in one-side of the plane.

Since \( Z \) acts transitively, the orbit is strictly convex. Also, it is easy to show that the orbit is properly embedded. Hence the orbit is a boundary of a properly convex open domain. It is now elementary to show that this is an \( \mathbb{R} \)-\( p \)-end neighborhood for a choice of \( p \)-end vertex \((0,0,0,0,1)\) or \((1,0,0,0,0)\).
Part III

The deformation space of convex real projective structures
The third part is devoted to understanding the deformation spaces of convex real projective structures on orbifolds with radial or totally geodesic ends. The end goal is to prove some versions of the Ehresmann-Thurston-Weil principle.

In Chapter 9, we give the precise definition of the deformation spaces. We show that the deformation space of real projective structures on a strongly tame orbifold with some conditions on the ends is mapped locally homeomorphically under the holonomy map to the character space of the fundamental group of the orbifold with corresponding conditions. Here, we are not concerned with convexity. The basic idea is to do an affine suspension and consider the parallel ends that are suspensions of radial ends. Thurston’s idea of deformation, as described in Lok [131] by charts, works well.

In Chapter 10, we will show that a convex real projective orbifold is strictly convex with respect to the ends if and only if the fundamental group is hyperbolic with respect to the end fundamental groups. Basic tools are from Yaman’s work [167] generalizing the Bowditch’s description of hyperbolic groups. That is, we look at triples of points in the boundary of the universal cover and show that the action is properly discontinuous. In addition, we show that the action of the group on the fixed points of end fundamental groups is parabolic in their sense. This generalizes the prior work of Cooper-Long-Tillman [73] and Crampon-Marquis [74] for convex real projective manifolds with cusp ends. The concept of relative hyperbolic ends depends on the types of ends here unfortunately. However, we aim to generalize.

In Chapter 11, we will show that the deformation space of convex real projective structures on a strongly tame orbifold with some conditions on the ends is identifiable with the union of components of the character space of the fundamental group of the orbifold with corresponding conditions.

The openness part here continues that of Chapter 9. Here, the point is to prove the preservation of convexity under small perturbations. The proof consists of showing that we can patch the Hessian functions on the perturbed compact part with the Hessian functions on the end neighborhoods approximating the original Hessian metrics by finding approximating convex domains to the original covering convex domains. Cooper-Long-Tillman [73] uses the intrinsic Hessian metric instead.

The closedness part generalizes the previous work Choi-Goldman [59]. We use the end condition showing us that the sequence of covering convex domains can only degenerate to a point or a hemisphere. Then using Benzecri’s work [25] putting the domains in a fixed ball and containing a fixed smaller ball, we show that the domain has to be actually properly convex.
Chapter 9
The openness of deformations

A real projective structure sometimes admits deformations to parameters of real projective structures. We will prove the local homeomorphism between the deformation space of real projective structures on such an orbifold with radial or totally geodesic ends with various conditions with the $\text{SL}_\pm(n+1,\mathbb{R})$-character space (resp. $\text{PGL}(n+1,\mathbb{R})$-character space) of the fundamental group with corresponding conditions. However, the convexity issue will not be studied in this chapter. Our approach will be to work with radiant affine structures of one dimension higher by affine suspension construction and prove the results. Then the real projective versions will follow easily. In Section 9.1, we will state the main results recall some definitions such as geometric structures, boundary restrictions, and the deformation spaces. In Section 9.2, we prove the main result of the chapter Theorems 9.1 and 9.3, showing the openness of the deformation space in the character space. We first define the end conditions for affine and real projective structures as determined by sections. We describe how to perturb the horospherical ends to lens-shaped ones in the affine setting. Then we state the main results. We give proofs for Theorem 9.2 for affine suspensions following our earlier work [54] or the work of Lok [131] in Section 9.2.5. The affine picture helps us with this to extend the deformations at end neighborhoods. In Section 9.2.6, we prove Theorem 9.3 by pushing Theorem 9.2 to a projective version by unsuspending. In Section 9.3, we will identify the deformation spaces as defined in our earlier papers [55] and [63] as stated in Chapter 3 to the deformation spaces here.

9.1 Introduction

Given a real projective orbifold $\mathcal{O}$, we add the restriction of the end to be a radial or a totally geodesic type. The end will be either assigned $\mathcal{R}$-type or $\mathcal{T}$-type.

- An $\mathcal{R}$-type end is required to be radial.
- A $\mathcal{T}$-type end is required to have totally geodesic properly convex ideal boundary components or be horospherical.
Recall that a strongly tame orbifold will always have such an assignment in this monograph, and finite-covering maps will always respect the types. Let $E_1, \ldots, E_{e_1}$ be the $R$-ends, and $E_{e_1+1}, \ldots, E_{e_1+e_2}$ be the $T$-ends.

Recall that our strongly tame orbifold $O$ comes with a compact orbifold $\overline{O}$ with smooth boundary whose interior equals $O$. Each boundary component of $\overline{O}$ is said to be the ideal boundary component of $O$.

**Definition 9.1** The radial foliation gives us a smooth parameterization of an end neighborhood $U$ of a radial end or horospherical end $E$ of $O$ by $\Sigma_E \times (0, 1)$ where $x \times (0, 1)$ is the radial line for each $x \in \Sigma_E$ since we can choose an embedded hypersurface transverse to the radial rays. We assume that as $t \to 1$, the ray escapes to the end.

Let $E$ be a $T$-end. We are given an end neighborhood $U$ diffeomorphic to $S_E \times (0, 1)$ where $S_E$ is the ideal boundary component. We identify $U$ with $S_E \times (0, 1)$ in $S_E \times [0, 1]$ by identity. Up to isotopies, $S_E \times \{1\}$ identifies with the ideal boundary of $\overline{O}$ corresponding to $E$. This is the compatibility condition of $\overline{O}$ with totally geodesic end structure.

For each $R$-end, we require that a vector field tangent to the leaves extends to a smooth vector field transverse to the corresponding ideal boundary component of $\overline{O}$. For each $T$-end, the identification of $U$ to $S_E \times (0, 1)$ extends to the closure of $U$ in $\overline{O}$ and $S_E \times [0, 1]$. Recall that these are the compatibility condition of $R$-end structures and $T$-end structures with the compactification $\overline{O}$ of $O$ from Section 1.3.

Also, $\overline{O}$ is a very good orbifold since we can identify $\overline{O}$ with $O - U$ for a union $U$ of open end neighborhoods of product forms as above. This is obtained by identifying $\overline{O}$ by $O - Cl(U)$ by an isotopy preserving radial foliations and taking the closures. (See Theorem 2.3.)

There is an obvious isomorphism $\pi_1(O) \to \pi_1(\overline{O})$ since we can perturb any $\mathcal{G}$-path in $\overline{O}$ to one in $O$. For the universal cover $\tilde{O}$ of $\overline{O}$, there is an embedding $\overline{O} \to \tilde{O}$ as an inclusion map to a dense open subset. We will always identify $\overline{O}$ with the dense subset. (See [35].)

### 9.1.1 The local homeomorphism theorems

For technical reasons, we will be assuming $\partial O = \emptyset$. In fact, a proper way to understand the boundary is through understanding the ends as in the hyperbolic manifold theory of Thurston. (As said above, we are not concerned here with the convexity issues.) The following map hol is one induced by sending $(\text{dev}, h)$ to the conjugacy class of $h$ as isotopies preserve $h$:

**Theorem 9.1** Let $O$ be a noncompact strongly tame real projective $n$-orbifold with lens-shaped radial ends or lens-shaped totally-geodesic ends with end structures and given types $\mathcal{R}$ or $\mathcal{T}$. Assume $\partial O = \emptyset$. Then the following map is a local homeomorphism:
hol : Def$^\alpha_{E,u}(\mathcal{O}) \to \text{rep}^\alpha_{E,u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$.

The restrictions of end types are necessary for this theorem to hold. This generalizes results for closed manifolds for hyperbolic structures starting from the classical results of Weil [164]. This is an example of the Ehresmann-Thurston principle. Also, see for an analogous result in Cooper-Long-Tillman [72] where they use different deformation spaces and restrictions on ends. (See Canary-Epstein-Green [36], Goldman [93], Lok [131], Bergeron-Gelander [28], and Choi [54].)

Theorem 9.1 is a corollary of following Theorem 9.3 by using the uniqueness section defined by Lemma 1.5. It is stated here for convenience.

**Theorem 9.3** Let $\mathcal{O}$ be a noncompact strongly tame real projective $n$-orbifold with radial ends or totally geodesic ends with types assigned and a conjugation-invariant open subset $\mathcal{V}$ of a semi-algebraic subset of

$$\text{Hom}^\alpha_{E}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})), $$

and $\mathcal{V}'$ the image in

$$\text{rep}^\alpha_{E}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).$$

Assume $\partial \mathcal{O} = \emptyset$. Let $s_{\mathcal{V}}$ be the fixing section defined on $\mathcal{V}$ with images in $(\mathbb{R}P^n)^{e_1} \times (\mathbb{R}P^{e_2})^{e_2}$. Then the map

$$\text{hol} : \text{Def}^\alpha_{E,s_{\mathcal{V}}}(\mathcal{O}) \to \text{rep}^\alpha_{E}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$

sending the real projective structures with ends compatible with $s_{\mathcal{V}}$ to their conjugacy classes of holonomy homomorphisms is a local homeomorphism to an open subset of $\mathcal{V}'$.

For the definition of fixing sections, see Section 9.2.2.

### 9.1.2 Geometric structures on orbifolds

Let $G$ be a Lie group acting on an $n$-dimensional manifold $X$. For examples, we can let $X = \mathbb{A}^n$ and $G = \text{Aff}(\mathbb{A}^n)$ for the affine group $\text{Aff}(\mathbb{A}^n) = \text{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, i.e., the group of transformations of form $v \mapsto Av + b$ for $A \in \text{GL}(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. Or we can let $X = \mathbb{R}P^n$ and $G = \text{PGL}(n+1, \mathbb{R})$, the group of projective transformations of $\mathbb{R}P^n$.

The complement of $\mathbb{R}P^n$ of a subspace of codimension-one can be identified with an affine subspace. We realize $\text{Aff}(\mathbb{A}^n)$ as a subgroup of transformations of $\text{PGL}(n+1, \mathbb{R})$ fixing a subspace of codimension-one as there is an isomorphism

$$(A, b) \mapsto \begin{bmatrix} A & b^T \\ 0 & 1 \end{bmatrix}, A \in \text{GL}(n, \mathbb{R}), b \in \mathbb{R}^n$$
where $b^T$ is the transpose of $b$.

Recall that an isotopy of an orbifold $\mathcal{O}$ is a map $f : \mathcal{O} \to \mathcal{O}$ with a map $F : \mathcal{O} \times I \to \mathcal{O}$ so that:

- $F_1 : \mathcal{O} \to \mathcal{O}$ for $F_1(x) := F(x, t)$ every fixed $t$ is an orbifold diffeomorphism,
- $F_0$ is the identity, and
- $f = F_1$.

Given an $(X, G)$-structure on another orbifold $\mathcal{O}'$, any orbifold diffeomorphism $f : \mathcal{O} \to \mathcal{O}'$ induces an $(X, G)$-structure pulled back from $\mathcal{O}'$ which is given by using the preimages in $\mathcal{O}$ of the local models of $\mathcal{O}'$.

**Definition 9.2** Let $\iota : \mathcal{O} \to \mathcal{O}$ be an isotopy. We may choose a lift $\tilde{\iota} : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$ of $\iota$ so that for the isotopy $F : \mathcal{O} \times I \to \mathcal{O}$ with $F_0 = I_{\mathcal{O}}$ and $F_1 = \iota$ has a lift $\tilde{F}$ so that $\tilde{F}_0 = I_{\tilde{\mathcal{O}}}$ and $\tilde{F}_1 = \tilde{\iota}$. We call such a map $\tilde{\iota}$ an isotopy-lift.

For now, we restrict to compact orbifolds. Suppose that $\mathcal{O}$ is compact. We define the **isotopy-equivalence space** $\text{Def}_{X, G}(\mathcal{O})$ as the space of development pairs $(\text{dev}, h)$ quotient by the isotopy-lifts of the universal cover $\tilde{\mathcal{O}}$ of $\mathcal{O}$. The **deformation space** $\text{Def}_{X, G}(\mathcal{O})$ is given by the quotient of $\text{Def}_{X, G}(\mathcal{O})$ by the action of $G : g(\text{dev}, h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1})$. (See [54] for details.) We can also interpret as follows: The deformation space $\text{Def}_{X, G}(\mathcal{O})$ of the $(X, G)$-structures is the space of all $(X, G)$-structures on $\mathcal{O}$ quotient by the isotopy pullback actions.

This space can be thought of as the space of pairs $(\text{dev}, h)$ with the compact open $C^r$-topology for $r \geq 1$ and the equivalence relation generated by the isotopy relation:

- $(\text{dev}, h) \sim (\text{dev}', h')$ if $\text{dev}' = \text{dev} \circ \iota$ and $h' = h$ for an isotopy-lift $\iota$ of an isotopy
- $(\text{dev}, h) \sim (\text{dev}', h')$ if $\text{dev}' = k \circ \text{dev}$ and $h(\cdot) = kh(\cdot)k^{-1}$ for $k \in G$.

(See [54] or Chapter 6 of [56].)

### 9.1.3 End structures and end compactifications for topological orbifolds

Let $\mathcal{O}$ be a strongly tame smooth orbifold with ends $E_1, \ldots, E_m, E_{m+1}, \ldots, E_{m+l}$. For this subsection, we do not consider that $\mathcal{O}$ is the interior of a compact orbifold $\mathcal{O}'$, the associated compactification of $\mathcal{O}$, as we remind from Section 1.3.1.

An **ideal boundary structure** of an end neighborhood $U$ of $E_i$ is a pair $(U, f)$ for a smooth embedding $f$ of $U$ into a product space $\Sigma \times (0, 1]$ for a closed $n - 1$-orbifold $\Sigma$ where the image is $\Sigma \times (0, 1]$. An ideal boundary structure $(U_0, f_0)$ with a diffeomorphism $f_0 : U_0 \to \Sigma_0 \times (0, 1)$ for a closed $n - 1$-orbifold $\Sigma_0$ and another one $(U_1, f_1)$ with a diffeomorphism $f_1 : U_1 \to \Sigma_1 \times (0, 1)$ for a closed $n - 1$-orbifold $\Sigma_1$ are **compatible** if there exists another ideal boundary structure $(U_2, f_2)$ so that $U_2 \subset U_0 \cap U_1$ with a diffeomorphism $f_2 : U_2 \to \Sigma'' \times (0, 1)$ so that $f_i \circ f_2^{-1} : \Sigma'' \times (0, 1) \to$
\[ \Sigma_i \times (0, 1) \text{ extends to } \Sigma'' \times (0, 1) \text{ as an embedding restricting to a diffeomorphism } \Sigma'' \times \{1\} \text{ to } \Sigma \times \{0\} \text{ for } i = 0, 1. \]

Given an ideal boundary structure on \( \mathcal{O} \) for an end \( E_i \), we obtain the completion of \( \mathcal{O} \) along \( E_i \). We take \( U \) an end neighborhood of \( E_i \) with an embedding \( f : U \to \Sigma \times (0, 1) \) where the image equals \( \Sigma \times (0, 1) \). We paste \( \mathcal{O} \) with \( \Sigma \times (0, 1) \) by \( f \). The resulting orbifold \( \mathcal{O}'_{(U, f)} \) is said to be the end compactification of \( \mathcal{O} \) along \( E_i \) using \((U, f)\).

Let \( U' \) and \( f' : U' \to \Sigma' \times (0, 1) \) be as above with \((U', f')\) compatible to \((U, f)\), and we obtain an end compactification \( \mathcal{O}'_{(U', f')} \) of \( \mathcal{O} \) along \( E_i \) using \((U', f')\). An isotopy \( \iota \) of \( \mathcal{O} \) with an ideal boundary structure for an end \( E \) is an isotopy of \( \mathcal{O} \) extending to a diffeomorphism \( \iota : \mathcal{O}'_{(U, f)} \to \mathcal{O}'_{(U', f')} \) for at least one compatible pair \((U, f), (U', f')\).

By the following lemma, the definition of an (end-structure extendable) isotopy is independent of the choice of \((U, f)\) and \((U', f')\).

**Lemma 9.1** Let \( U_1 \) and \( U_2 \) be end neighborhoods of an end \( E \). Let \( g \) be an isotopy of \( \mathcal{O} \) extending to an isotopy \( \tilde{g} : \mathcal{O}'_{(U_1, f_1)} \to \mathcal{O}'_{(U_2, f_2)} \) with diffeomorphisms \( f_1 : U_1 \to \Sigma_1 \times (0, 1) \) and \( f_2 : U_2 \to \Sigma_2 \times (0, 1) \) for closed \( n - 1 \)-orbifolds \( \Sigma_1 \) and \( \Sigma_2 \). Then for any pair \((U_1', f_1'), (U_2', f_2')\) for end neighborhoods of end end \( E \) compatible to \((U_i, f_i), i = 1, 2, \) with diffeomorphisms \( f'_1 : U'_1 \to \Sigma'_1 \times (0, 1) \) and \( f'_2 : U'_2 \to \Sigma'_2 \times (0, 1) \) for closed \( n - 1 \)-orbifolds \( \Sigma'_1 \) and \( \Sigma'_2 \), \( g \) extends to an isotopy \( \tilde{g}_1 : \mathcal{O}'_{(U_1, f_1')} \to \mathcal{O}'_{(U_2, f_2')} \).

**Proof** This is straightforward to obtain a diffeomorphism \( \tilde{g}_1 \) since we can take a sufficiently small product neighborhood in each of these end compactifications. To show the isotopy property of \( \tilde{g}_1 \), we simply take \( \mathcal{O} \times I \) and do the same arguments. \( \square \)

A radial structure for \( E_i \) also gives us an end-compactification of \( \mathcal{O} \) along \( E_i \). Let \( U \) be an end neighborhood of \( E_i \) with a foliation by properly embedded arcs. We take a transverse hypersurface \( \Sigma_{E_i} \) transverse to every leaf, which is a closed orbifold. Let \( U' \) denote a component of \( U - \Sigma_{E_i} \) that is an end neighborhood of \( E_i \). We identify each leaf in \( U' \) with a leaf of \( \Sigma_{E_i} \times (0, 1) \) by a function \( f : U' \to \Sigma_{E_i} \times (0, 1) \). We call the identification orbifold \( \mathcal{O}' \) of \( \mathcal{O} \) with \( \Sigma_{E_i} \times (0, 1) \) the end compactification of \( \mathcal{O} \) along \( E_i \). The suborbifold of \( \mathcal{O}' \) corresponding to \( \Sigma_{E_i} \times \{1\} \) is called the ideal boundary component corresponding to \( E_i \). This orbifold is independent of the choices up to isotopoes of the end compactifications extending isotopies of \( \mathcal{O} \) by Lemma 9.2.

**Lemma 9.2**

- Let \( \mathcal{O} \) be a strongly tame orbifold with a radial structure at an end \( E_i \).
- Let \( \mathcal{O}' \) be the end compactification of \( \mathcal{O} \) using \( U \) and \( \Sigma_{E_i} \), a diffeomorphism \( f : U' \to \Sigma_{E_i} \times (0, 1) \).
- Let \( \mathcal{O}'_1 \) be the same orbifold with an isotopic radial structure at \( E_i \).
- Let \( \mathcal{O}'_1 \) be the end compactification of \( \mathcal{O}'_1 \) for the second radial structure using an end neighborhood \( U_1 \) and a hypersurface \( \Sigma_{E_i}' \) and a diffeomorphism \( f' : U'_1 \to \Sigma_{E_i}' \times (0, 1) \) for an end-neighborhood component \( U'_1 \) of \( U_1 - \Sigma_{E_i}' \).
Suppose that an isotopy $t$ of $\mathcal{O}$ sends a radial structure of $U$ for end $E_i$ to that of $U_1$ for $\mathcal{O}_1$ for $E_i$.

Then a diffeomorphism $t' : \mathcal{O} \to \mathcal{O}$ extends to a diffeomorphism $V' : \mathcal{O}' \to \mathcal{O}'_1$ sending the ideal boundary component corresponding to $E_i$ in $\mathcal{O}'_1$ to one in $\mathcal{O}'$.

Proof First, we obtain a diffeomorphism $i$. We may change $t$ so that $t(U') \subset U_1'$ by composing with an isotopy supported in $U$ preserving leaves of the radial foliation.

We consider a diffeomorphism

$$f' \circ t \circ f^{-1} : \Sigma_{E_i} \times (0, 1) \to \Sigma_{E_i} \times (0, 1).$$

We can find an isotopy $t_1 : \Sigma_{E_i} \times (0, 1) \to \Sigma_{E_i} \times (0, 1)$ preserving $\{x\} \times (0, 1)$ for each $x \in \Sigma_{E_i}$ equal to the identity map on $\Sigma_{E_i} \times (0, \varepsilon)$ for small $\varepsilon > 0$ so that $t_1 \circ f' \circ t \circ f^{-1}$ extends to a smooth map at $\Sigma_{E_i} \times \{1\}$. This is fairly simple since every self-embedding of $\Sigma_{E_i} \times (0, 1)$ preserving every fiber of form $\{x\} \times (0, 1)$ for $x \in \Sigma_{E_i}$ are isotopic. Now, we can use $t : \mathcal{O} - U' \to \mathcal{O} - U'_1$ and $f^{-1} \circ t_1 \circ f' \circ t \circ f^{-1}$ on $U'$. Obviously, they extend each other.

The following shows the well-definedness of an orbifold.

Corollary 9.1

- Let $\mathcal{O}$ be a strongly tame orbifold with a $T$-end structure at $E_i$. Then the identity map $1$ of $\mathcal{O}$ is isotopic to a restriction of a diffeomorphism $\mathcal{O}'_1 \to \mathcal{O}'_1'$.

- Let $\mathcal{O}$ be a strongly tame orbifold with an $R$-end structure at $E_i$. Then the identity map $1$ of $\mathcal{O}$ is isotopic to a restriction of a diffeomorphism $\mathcal{O}'_1 \to \mathcal{O}'_2$ for any two end compactifications $\mathcal{O}'_1$ and $\mathcal{O}'_2$ of $\mathcal{O}$.

Proof The first is a corollary of Lemma 9.1. The second one is a corollary of Lemma 9.2.

Finally, we will say about the compactification $\bar{\mathcal{O}}$ associated with $\mathcal{O}$.

If $\bar{\mathcal{O}}$ is the compactification associated with $\mathcal{O}$, the ideal boundary structure is given by $(f, N(\Sigma_{E_i}) \cap \mathcal{O})$ where $f : N(\Sigma_{E_i}) \cap \mathcal{O} \to \Sigma_{E_i} \times (0, 1]$ is an embedding for a tubular neighborhood $N(\Sigma_{E_i})$ of $\Sigma_{E_i}$ in $\bar{\mathcal{O}}$.

Again, if $\bar{\mathcal{O}}$ is the associated compactification of $\mathcal{O}$, and the radial structure at $E_i$ is compatible with $\mathcal{O}$ as in Section 1.3.3, the radial end compactification from $(f, U)$ can be modified to a compatible $(f', N(\Sigma_{E_i}) \cap \mathcal{O})$ for a tubular neighborhood $N(\Sigma_{E_i})$ of $\Sigma_{E_i}$ to $\Sigma_{E_i} \times (0, 1]$ where $f'$ extends to a smooth diffeomorphism $N(\Sigma_{E_i}) \to \Sigma_{E_i} \times (0, 1]$.

Proposition 9.1 We can contract by the above end compactification process a compact orbifold $\bar{\mathcal{O}}$ of which $\mathcal{O}$ is the interior. The end compactification compatible with the given $R$-end and $T$-end structures is always diffeomorphic to $\bar{\mathcal{O}}$ by a diffeomorphism isotopic to the identity in $\mathcal{O}$.

Proof Recall from Sections 1.3.2 and 1.3.3 the definitions of compatibility. Also, it is straightforward to see that the radial foliation is transverse to the added ideal boundary component corresponding to $\Sigma_{E_i} \times \{1\}$. Corollary 9.1 completes the proof.
9.1.4 Definition of the deformation spaces with end structures

We will extend this notion strongly. Two real projective structures $\mu_0$ and $\mu_1$ on $\mathcal{O}$ with R-ends or T-ends with end structures are isotopic if there is an isotopy $i$ on $\mathcal{O}$ so that $i^*(\mu_0) = \mu_1$ where $i^*(\mu_0)$ is the induced structure from $\mu_0$ by $i$

- $i_*(\mu_0)$ has a radial end structure for each R-end or horospherical T-end,
- $i$ sends the radial end foliation for $\mu_0$ from an R-end neighborhood or horospherical T-end to the radial end foliation for real projective structure $\mu_1 = i_*(\mu_0)$ with corresponding R-end neighborhoods or a horospherical T-end, and
- $i$ extends to a diffeomorphism of $\bar{\mathcal{O}}$ using the radial foliations and the totally geodesic ideal boundary components for $\mu_0$ and $\mu_1$ where we use the radial end-compactification for a horospherical T-end. (See Definition 9.1.)

For noncompact orbifolds with end structures, similar definitions hold except that we have to modify the notion of isotopies to preserve the end structures.

**Definition 9.3** We consider the real projective structures on orbifolds with end structures. Let $\mathcal{O}$ be one of this and $\bar{\mathcal{O}}$ be the compactification. Let $\hat{\mathcal{O}}$ denote the universal cover of $\bar{\mathcal{O}}$ containing $\bar{\mathcal{O}}$ as a dense open set.

Let $\text{dev}_\mu$ denote the developing map associated with $\mu$. We now consider only developing maps $\text{dev}_\mu : \hat{\mathcal{O}} \to \mathbb{RP}^n$ extending to a smooth map $\bar{\text{dev}}_\mu : \bar{\mathcal{O}} \to \mathbb{RP}^n$. $\bar{\text{dev}}_\mu$ is also equivariant with respect to $h$ if $\text{dev}_\mu$ was so. We call $(\bar{\text{dev}}_\mu, h)$ an extended developing pair. An extended isotopy is a diffeomorphism $\bar{\mathcal{O}} \to \bar{\mathcal{O}}$ extending an isotopy of $\mathcal{O}$. An extended isotopy-lift is an extension of an isotopy-lift $\tilde{\mathcal{O}} \to \hat{\mathcal{O}}$.

Then the isotopy-equivalence space $\text{Def}_\mathcal{E}(\mathcal{O})$ is defined as the space of extended developing maps $\bar{\text{dev}}_\mu$ of real projective structures on $\mathcal{O}$ with ends with radial structures and lens-shaped totally geodesic ends with end structures under the action of the group the extended isotopy-lifts where an extended isotopy-lift $\tilde{f} : \tilde{\mathcal{O}} \to \hat{\mathcal{O}}$ acting by

$$ (\bar{\text{dev}}_\mu, h) \mapsto (\bar{\text{dev}}_\mu \circ \tilde{f}, h). $$

We explain the topology. Fix a real projective structure $\mu$ with end structure. For any real projective structure $\mu'$ on $\mathcal{O}$ with end structures with an isotopy $t_{\mu, \mu'}$ so that $t_{\mu, \mu'}^*(\mu') = \mu$ the end compactification $\bar{\mathcal{O}}$ has an extended isotopy-lift $\hat{t}_{\mu, \mu'} : \hat{\mathcal{O}} \to \bar{\mathcal{O}}$. The space $\mathcal{D}(\hat{\mathcal{O}})$ of maps of form $\bar{\text{dev}}_{\mu'} \circ \hat{t}_{\mu, \mu'} : \hat{\mathcal{O}} \to \mathbb{RP}^n$ will be given the compact open $C^*$-topology on $\hat{\mathcal{O}}$.

Now, $\bar{\text{dev}}_{\mu'} \circ \hat{t}_{\mu, \mu'}$ is a developing map of $t_{\mu, \mu'}^*(\mu')$ sending the end structures $\hat{\mathcal{O}}$ of $\mu'$ to radial line. Hence, $\bar{\text{dev}}_{\mu'} \circ \hat{t}_{\mu, \mu'}$ is the unique smooth extension. Hence, we can reinterpret $\mathcal{D}(\hat{\mathcal{O}})$ as the space of extensions of developing maps of $\mathcal{O}$ with a fixed end structure for each end.

The quotient space $\mathcal{D}(\hat{\mathcal{O}})/\mathcal{E}(\hat{\mathcal{O}})$ of $\mathcal{D}$ under the group of extended isotopy-lifts $\mathcal{E}(\hat{\mathcal{O}})$ of form $\hat{t}_{\mu, \mu'} : \hat{\mathcal{O}} \to \hat{\mathcal{O}}$ is in one-to-one correspondence with $\text{Def}_\mathcal{E}(\mathcal{O})$. The topology on $\text{Def}_\mathcal{E}(\mathcal{O})$ is given as the quotient topology of this space, which is called a $C^*$-topology.
We define $\text{Def}_E(\mathcal{O}) := \widetilde{\text{Def}}_E(\mathcal{O})/\text{PGL}(n+1, \mathbb{R})$ by the action

$$(\text{dev}, h(\cdot)) \mapsto (\phi \circ \text{dev}, \phi \circ h(\cdot) \circ \phi^{-1}), \phi \in \text{PGL}(n+1, \mathbb{R})$$

as in [54] and [93]. The induced quotient topology is called a $C^r$-topology of $\text{Def}_E(\mathcal{O})$.

We now describe the modification of the developing map by a process that we call the radial-end-projectivization of the developing map with respect to $U$ and $U'$. Let $\widetilde{U}$ and $\widetilde{U}'$ denote the p-end neighborhoods of $\widetilde{E}$ covering end-neighborhoods $U$ and $U'$ respectively. We require $\widetilde{U}$ and $\widetilde{U}'$ to be compatible product neighborhoods corresponding to $\Sigma_{E} \times (0, 1]$. (Recall compatibility from Section 1.3.) Take a maximal radial ray $l_x$ in $\widetilde{U}'$ passing $x \in \text{bd} \widetilde{U} \cap l$. Then there exists a unique projective diffeomorphism $\Pi_x : l_x \rightarrow \mathbb{R}^+_{\times}$ sending

- the endpoint of $l_x$ in $\text{bd} \widetilde{U}'$ to $\infty$,
- the other end to $0$, and
- $\text{bd} \widetilde{U} \cap l = \{x\}$ to $1$.

We define $\Pi_{\widetilde{U}'} : \widetilde{U}' \rightarrow \mathbb{R}^+_{\times}$ by sending $z \in l_x$ to $\Pi_x(z)$. There is also a unique projective diffeomorphism $P_x : \mathbb{R}^+_{\times} \rightarrow \mathbb{RP}^n$ sending

$$(0 \mapsto v_{\widetilde{E}}, 1 \mapsto \text{dev}(l \cap \text{bd} \widetilde{U}) = \text{dev}(x), +\infty \mapsto \text{dev}(l \cap \text{bd} \widetilde{U}')).$$

Define $v_x$ to be the vector at $v_{\widetilde{E}}$ of $(\partial P_x(t)/\partial t)|_{t=0}$. This does depend on $x$ but not on $t$.

Let $\Pi_{\widetilde{E}} : \widetilde{U}' \rightarrow \tilde{\Sigma}_{E}$ denote the map sending a point of a radial ray in $\widetilde{U}'$ to its equivalence class in $\tilde{\Sigma}_{E}$. $\tilde{U}'$ has coordinate functions

$$(\Pi_{\widetilde{U}'}, \Pi_{\widetilde{E}}) : \widetilde{U}' \rightarrow \mathbb{R}^+_{\times} \times \tilde{\Sigma}_{E},$$

which is a diffeomorphism. This commutes with the action of $\Gamma_{\widetilde{E}}$ on $\widetilde{U}'$ and the action on $\mathbb{R}^+_{\times} \times \tilde{\Sigma}_{E}$ acting on the first factor trivial. Also, $\tilde{U}$ goes to $(0, 1] \times \tilde{\Sigma}_{E}$ under the map.

We define a smooth map

$$\text{dev}^N : \tilde{U}' \rightarrow \mathbb{RP}^n$$

given by $\text{dev}^N(y) = P_x \circ \Pi_x(y)$ for $y \in l_x \subset \tilde{U}'$.

Then under the coordinate of $\tilde{U}'$ with affine coordinates on an affine subspace $A^a_{\times}$ containing $\text{dev}(l_x)$ and containing $\text{dev}(v_{\widetilde{E}})$ as the origin, we can write locally

$$\text{dev}^N(x, t) = f_x(t)v_x, \|v_x\| = 1,$$

for $x \in \text{bd} \tilde{U}$ (9.1)

on a neighborhood of $l_{x_0}$ for some $x_0 \in \text{bd} \tilde{U}$ where $v_x$ is a unit vector depending only on $x$ smoothly in the direction of $\text{dev}(v_{\widetilde{E}})\text{dev}(x)$ and
\[ f_x : \mathbb{R}_+ \to \mathbb{R}_+, f_x(0) = 0, f_x(1) = \|\text{dev}(x)\| \]
\[ f_x(\infty) = \|\text{dev}(l_x \cap \text{bd}U')\| \text{ provided } \text{dev}(l_x \cap \text{bd}U') \in \mathbb{R}_+^n \] (9.2)

is a strictly increasing projective function of \( t \). The coefficients of the 1-st order
total visceral function \( f_x \) as a function of \( t \) depend smoothly on \( x \) since \( \partial U' \) and \( \partial U \)are smooth. Here, we can actually change the affine subspaces of form \( \mathbb{A}_x^n \) andthe coordinates so that locally near \( l_x \) we have \( f_x(\infty) < \infty \). It is easy to see that \( \text{dev}^N \)extends to \( \tilde{\Sigma} E \times \{0\} \) as a constant map. The expression of \( f_x \) shows that \( \text{dev}^N \) isasmooth extension also since it has continuous partial derivatives of all orders.

Now we change \( \text{dev}^N \) on \( U' - U \) so that it smoothly extends to \( \tilde{\theta} - U' \). On each
\( l_x \), the lines \( \text{dev}^N l_x \) and \( \text{dev}^N l_x \) have the same image to \( \text{dev}(l_x) \). Hence, \( \text{dev}^{N-1} \circ \text{dev} \)sends \( l_x \) to \( l_x \), which is an increasing map.

We define \( \text{dev}^N \) on \( U' := \bigcup_{g \in \pi_1(\theta)} g(U) \) by defining \( \text{dev}^N(g(x)) = h(g)\text{dev}^N(x) \)for \( x \in U \).

We take a tubular neighborhood \( N \) of \( \text{bd}U' \) in \( \tilde{\theta} \) in \( \tilde{\theta} \) given by \( \Pi_\varepsilon(y) \geq 1/2 \),\( y \in \tilde{\theta} \). We let \( U'' \) be the open p-end neighborhood bounded by \( \Pi_\varepsilon = 1/2 \) for \( x \in \text{bd}U \)containing \( U \). Let \( \mu'_x \) denote the old arcwise metric induced from a leafwise smoothRiemannian metric of \( \theta \). Under \( \mu'_x \), we may assume that there exists \( \varepsilon > 0 \) so that the \( \mu'_x \)-path-length \( l_x \) between
\( \Pi_\varepsilon(y) = 1/2 \) to \( \Pi_\varepsilon(y') = 1/2 + \varepsilon \) equals \( |\varepsilon| \) also for \( y, y' \in l_x \) and \( |\varepsilon| < \delta \):

- We do this by changing \( \mu'_x \) by a leaf-preserving isotopy \( t : l_x \cap N \to l_x \cap U'' \).
- We construct one for each \( l_x \cap N \) and build an isotopy \( t : N \to \tilde{\theta} \).
- We extend \( t : N \to \tilde{\theta} \) to \( t' : \tilde{\theta} \to \tilde{\theta} \) in a leaf preserving manner. (See Palais \[149\]and Cerf \[39\].)
- Leafwise generalization is easy to show by using vector fields arguments radialto leaves supported in a neighborhood of \( N \) in \( U'' \).)
- We change \( \text{dev} \) to \( \text{dev} \circ t' : N \cup (\tilde{\theta} - U') \to \tilde{\theta} \).
- Since the metric \( \mu'_x \) depends continuously on \( x \) we are done.

We take \( \text{dev} \) changed by \( \text{dev} \circ t^{-1} : l_x \cap N \to \mathbb{R}^n \) for each \( x \). Then this map isidentical with \( \text{dev}^N \) on \( N \cap U' \). We patch together \( \text{dev} \) and \( \text{dev}^N \) on \( N \), and denoteit by \( \text{dev}' \). This map is the radial-end-projectivization of \( \text{dev} \) with respect to \( U' \) and\( U \).

We modify \( t' \) in \( U \) so that \( \text{dev}' = \text{dev} \circ t' \) since any map \( l_x \to l_x \) is isotopic byusing just homotopy \( \text{dev}^{-1} \circ \text{dev}' = t_x \) on \( l_x \cap U \) to the identity by taking \((1-t)t_x + t l_x \).

We define a new developing map \( \text{dev}' : \tilde{\theta} \to \mathbb{R}^n \) by using \( \text{dev}^N \) on \( U' \) and lettingit equal to \( \text{dev} \) on the complement \( \tilde{\theta} - U' - N \).

For other components of form \( \gamma(U) \) for \( \gamma \in \pi_1(\theta) \), we do the same constructions.

**Lemma 9.3** Let \( U \) and \( U' \) be a radial end-neighborhoods so that \( \text{Cl}_{\tilde{\theta}}(U) \subset U' \)and compatible to \( \tilde{\theta} \). Let \( \tilde{U} \) and \( \tilde{U}' \) denote the p-end neighborhoods of \( \tilde{\theta} \) covering \( U \) and\( U' \). We assume that \( \text{bd}U \) and \( \text{bd}U' \) are transverse to radial rays by taking \( U \) and \( U' \)smaller if necessary. Then we can modify the developing map in \( \tilde{U} \) so that the newdeveloping map \( \text{dev}' \) agrees with \( \text{dev} \) on \( \tilde{\theta} - p_{\theta}(U) \) so that \( \text{dev}' \) extends smoothly
on the end compactification of $\bar{U}$ and $\text{dev}'$ restricts to each radial line segment is a projective map. Finally, $\text{dev}' = \text{dev} \circ t$ for an isotopy-lift $t$ preserving each radial segment in $\bar{U}'$.

Compare following Proposition 9.2 with Proposition 9.4.

\textbf{Proposition 9.2} The set $\text{Def}_{E}(\partial')$ is identical with the set denoted by the same symbol defined in Section 1.4.

\textbf{Proof} Clearly, the structures representing elements in $\text{Def}_{E}(\partial')$ are structures in Section 1.4.

We show that an arbitrary structure as defined in Section 1.4 can be isotopied so that it has an extended smooth developing map on the universal cover of the end-compactified $\partial'$. Hence, $\text{Def}_{E}(\partial')$ represents the isotopy set of all the elements of the set in Section 1.4.

Let $E$ be a T-end of $\partial$. Here, we can identify an end-neighborhood $U$ of $E$ diffeomorphic to $S_{E} \times (0, 1)$ with $S_{E} \times (0, 1]$. Then the universal cover of $\partial_{E}$ has a developing map extending to a developing map defined on the inverse image of $S_{E} \times (0, 1]$. (See Definition 9.1.)

Let $E$ be a R-end or a horospherical end of $\partial$. Let $\partial_{E}$ denote the orbifold obtained by end-compactifying at $E$. Let $\bar{\partial}_{E}$ denote its universal cover. We can identify an end-neighborhood $U$ of $E$ diffeomorphic to $\Sigma_{E} \times (0, 1)$ with a subset of $\Sigma_{E} \times [0, 1)$ so that each leaf of the radial foliation corresponds to $\{x\} \times (0, 1)$ with $\{x\} \times 1$ is in $\text{bd}U$. We take a union $U'$ of mutually disjoint collection of radially foliated end neighborhoods and another such collection $U$ so that $\text{Cl}_{\bar{E}}(U) \subset U'$. Now, we modify using radial end-projectivization of the developing map.

Let $S_{v}^{-1}$ denote the set of directions of rays at $v$. We consider them as a space of unit vectors. Let $U'$ denote a slightly larger end neighborhood containing the closure of $U$ and has a compatible radial foliation.

Now we go to the second part of the proof. Suppose that two elements $[\mu_{1}]$ and $[\mu_{2}]$ of $\text{Def}_{E}(\partial')$ for real projective structures $\mu_{1}, \mu_{2}$ on $\partial'$ with developing maps $\text{dev}_{1}$ and $\text{dev}_{2}$ extendable to $\bar{\partial}_{E}$ go to isotopic structures on $\bar{\partial}$ in the sense of Section 1.4. There is an isotopy between them. Let $t_{0} : \bar{\partial} \to \bar{\partial}$ denote the isotopy-lift so that $\text{dev}_{2} = \text{dev}_{1} \circ t_{0}$.

For $p$-T-end $E$, since $\text{dev}_{1}(U)$ for a $p$-end neighborhood $U$ extends to $U \cup \bar{S}_{E}$, we see that the isotropy-lift $t_{0}'$ extends to a smooth diffeomorphism $U \cup \bar{S}_{E} \to t_{0}'(U) \cup \bar{S}_{E}$ by the way we defined the isotopy in Section 1.4.

We isotopy $\mu_{i}$ so that the associated $\text{dev}_{i}$ restricted to a $p$-end neighborhood for each R-p-end have been treated with radial end-projectivization with respect to the union of mutually disjoint end-neighborhoods $U'$ with a radial structure and such one $U$ where $\text{Cl}_{\bar{E}}(U) \subset U$. These are for both $\mu_{1}$ and $\mu_{2}$.

We again denote these by $\mu_{1}$ and $\mu_{2}$ respectively. Let $t : \bar{\partial} \to \bar{\partial}$ be the isotopy-lift inducing an isotopy $t_{0}' : \bar{\partial} \to \bar{\partial}$ so that $t_{0}'(\mu_{2}) = \mu_{1}$. Since $\text{dev}_{1}$ and $\text{dev}_{2}$ are defined with respect to $U$ and $U'$, it follows that $t_{0}'$ preserves both spaces.

We may assume that $\text{dev}_{1}(x_{0}) = \text{dev}_{2}(x_{0})$ for a basepoint $x_{0} \in \bar{\partial}$ since we may apply an isotopy-lift of an isotopy supported on a compact subset of $\partial$. Then we
may lift the map \( \text{dev}_2 : \hat{\Omega} \to \mathbb{R}P^n \) with respect to \( \text{dev}_1 \) sending \( x_0 \) to \( x_0 \) and the lift is the isotopy-lift \( \hat{\iota} : \hat{\Omega} \to \hat{\Omega} \).

We give \( \hat{\Omega} \) a Riemannian metric where the end neighborhoods are of the product form. Assume that the isotopy \( \iota'_0 \) is sufficiently close to \( \iota \) on \( \hat{\Omega} \). Then the isotopy extends to \( \hat{\Omega} \to \hat{\Omega} \) smoothly. To show this, we recall the local expression (9.1). We need to compare the coefficients of rational functions \( f_\iota, f_{\iota'} \) and vectors \( v_\iota, v_{\iota'} \) for \( x, z = \iota'_0(x) \in \partial U \). Since \( z = \iota'(x) \) depends smoothly on \( x \) and so do the coefficients of \( f_\iota \) and \( f_{\iota'} \) and vectors \( x \) and \( z \). The isotopy \( \iota'_0 \) extends to \( \text{Cl}_\partial(U) - U \) in \( \hat{\Omega} \) since we can consider \( f_{\iota'}^{-1} \circ f_\iota \).

If \( \iota'_0 \) is not sufficiently close to \( \iota \), we can write it as a composition of such isotopies. Hence, we constructed the needed isotopy. This completes the proof. \( \square \)

We now state the \( S^r \)-version of Definition 9.3:

**Definition 9.4** Let \( \hat{\Omega} \) be a strongly tame orbifold with end structures, and let \( \hat{\Omega} \) denote the universal cover of the end-compactification of \( \hat{\Omega} \). An extended developing map \( \text{dev} : \hat{\Omega} \to S^r \) is a smooth map restricting to a developing map \( \text{dev} : \hat{\Omega} \to S^r \). The isotopy-equivalence space \( \text{Def}_{\mathbb{S}^r, \hat{\Omega}}(\hat{\Omega}) \) is the quotient space of extended developing pairs \( (\text{dev}, h) \) of \( (S^r, \text{SL}_+(n + 1, \mathbb{R})) \)-structures on \( \hat{\Omega} \) under the action of the group of extended isotopy-lifts \( \iota : \hat{\Omega} \to \hat{\Omega} \) by \( (\text{dev}, h) \mapsto (\text{dev} \circ \iota, h). \) The topology is given as the quotient topology induced from the space of developing pairs \( (\text{dev} \circ \iota, h) \) for the extended isotopy-lifts \( \iota \) defined on a fixed \( \hat{\Omega} \) with the compact open \( C^r \)-topology \( r \geq 2. \)

Analogously to Definition 9.3, we define \( \text{Def}_{\mathbb{S}^r, \hat{\Omega}}(\hat{\Omega}) \) as the quotient space of \( \text{Def}_{\hat{\Omega}}(\hat{\Omega}) \) by the action of \( \text{SL}_+(n + 1, \mathbb{R}) \) by \( (\text{dev}, h) \mapsto (\phi \circ \text{dev} \circ \iota, \phi \circ h(\iota) \circ \phi^{-1}) \). These will be called the \( C^r \)-topologies respectively. (See also Definition 9.3.)

**9.1.5 Affine orbifolds**

An *affine orbifold* is an orbifold with a geometric structure modeled on \( (\mathbb{A}^n, \text{Aff}(\mathbb{A}^n)) \). An affine orbifold has a notion of affine geodesics as given by local charts. Recall that a geodesic is *complete* in a direction if the affine geodesic parameter is infinite in the direction. Let \( \hat{\Omega} \) be an affine orbifold.

- Let \( E \) be an end of \( \hat{\Omega} \) with an end neighborhood \( U \).
  - Suppose that we have a smooth complete nonzero vector field \( X_E \) in an end-neighborhood of \( E \) so that its directions are radial with respect to the affine connection in the end-neighborhood. We call \( X_E \) a *parallel end vector field*. There are lines complete in one direction tangent to \( X_E \) in the end neighborhood. Also we require that \( X_E \) is radial in directions under the connection under the flow. The lines are called *parallel lines*.
  - Two pairs of end neighborhoods and radial end vector fields are *compatible* if the vector fields coincide in directions in the intersection of the end neighbor-
The openness of deformations

hoods. The compatibility class of the pairs is called radial end structure. An end with a radial end structure is called a radial end.

- Let $E$ be an end of an affine orbifold $\mathcal{O}$. Suppose that an end neighborhood of the end $E$ diffeomorphic to $E \times [0, 1]$ that compactifies to an orbifold diffeomorphic to $E \times [0, 1]$, and each point of $E \times \{1\}$ has a neighborhood affinely diffeomorphic to a neighborhood of a point $p$ in $\partial H$ for a half-space $H$ of an affine space. Two compactifications of end neighborhoods are compatible if the intersection of the end neighborhoods compactifies so that its inclusion maps extend to embeddings onto a submanifold of the two compactifications. Again the compatibility class of $T$-end structures is called a $T$-end structure of $E$. An end $E$ with a $T$-end structure is called a totally geodesic end ($T$-end) of $\mathcal{O}$.

We will again assume that $\mathcal{O}$ comes with a smooth compact orbifold $\bar{\mathcal{O}}$ with boundary. Again considering the affine orbifold as a real projective one, we assume that the end neighborhoods are compatible with $\bar{\mathcal{O}}$ as in Section 1.3.

**Definition** 9.5 Let $L/\Gamma_L$ be a compact orbifold for a lens $L$. A generalized affine suspension of $L/\Gamma_L$ is called a lens-suspension. It is covered by a properly convex cone $C(L) \subset \mathbb{R}^{n+1}$. $C(L)$ is called suspended CA-lens.

If $L$ is an open or closed horoball, and $L/\Gamma_L$ is a horospherical orbifold homeomorphic to a compact orbifold times $[0, 1)$, then an affine suspension of $L/\Gamma_L$ is called a horoball-suspension. It is covered by a properly convex cone $C(L) \subset \mathbb{R}^{n+1}$.

(See Section 2.2.1 for generalized affine suspension.)

We note that R-ends of a real projective orbifold correspond to radial ends of its affine suspension and vice versa. T-ends of a real projective orbifolds correspond to $T$-ends of its affine suspension and vice versa.

### 9.2 The local homeomorphism theorems

#### 9.2.1 The end condition of affine structures

A suspension of a legion in $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ is the inverse image of the region under the projection $\mathbb{R}^{n+1} - \{O\} \to \mathbb{S}^n$.

Given an affine orbifold $\mathcal{O}$ satisfying our end conditions, and each end is given a radial end type or a totally geodesic lens end type. Each end fundamental group of $\pi_1(\mathcal{O})$ will have a distinguished infinite cyclic group in the center. Each end of our orbifold $\mathcal{O}$ is given an $\mathcal{F}$-type or a $\mathcal{T}$-type.

- An $\mathcal{R}$-type end is allowed to be radial always, and
- a $\mathcal{T}$-type end is allowed to be totally geodesic with a suspended lens neighborhood in some cover of an ambient affine manifold corresponding to the end fundamental group or be radial with a suspended horoball neighborhood. In this case the end is said to satisfy the suspended lens condition.
Here the distinguished cyclic central subgroups are required to go to the groups of scalar dilatations preserving the cones corresponding.

Let us make a choice of conjugacy classes of the fundamental group $\pi_1(E)$ as a subgroup of $\pi_1(\mathcal{O})$ for every radial end $E$ as a subgroup of $\pi_1(\mathcal{O})$.

We define a subspace $\text{Hom}_e(\pi_1(\mathcal{O}), \text{Aff}(\mathbb{A}^{n+1}))$ of

$$\text{Hom}(\pi_1(\mathcal{O}), \text{Aff}(\mathbb{A}^{n+1}))$$

to be the subspace where $h(\pi_1(E_i))$ for each end $E_i$ consists of affine transformations

- with linear parts with at least one common eigenvector if $E_i$ is of $\mathcal{R}$-type or
- acting on an affine hyperspace $P$ and properly discontinuously and cocompactly
  - on a suspension $L$ of a lens meeting $P$ in its interior
  - or on a suspension of a horoball tangent to $P$ if $E_i$ is of $\mathcal{T}$-type. In this case, $h(\pi_1(E_i))$ is said to satisfy the suspended lens-condition. Here we need to fix the generator of the boundary going to a scalar dilatation for each $\mathcal{T}$-ends.

(See Section 1.4.1.)

Let $E_1, \ldots, E_{e_1}$ be the $\mathcal{R}$-ends of $\mathcal{O}$ and let $E_{e_1+1}, \ldots, E_{e_1+e_2}$ be the $\mathcal{T}$-ends of $\mathcal{O}$. Choose a representative $\mathcal{p}$-ends $E_1, \ldots, E_{e_1}, E_{e_1+1}, \ldots, E_{e_1+e_2}$ of $\mathcal{O}$.

Let $\mathcal{U}$ be an open subspace of a semi-algebraic subset of $\text{Hom}_e(\pi_1(\mathcal{O}), \text{Aff}(\mathbb{A}^{n+1}))$

invariant under the conjugation action so that one can choose a continuous section $s^{(1)}_\mathcal{U} : \mathcal{U} \to (\mathbb{R}^{n+1} - \{O\})^{e_1}$ sending a holonomy homomorphism $h$ to a common nonzero eigenvector of $h(\pi_1(E_i))$ for $i = 1, \ldots, e_1$. Here $s^{(1)}_\mathcal{U}$ satisfies

$$s^{(1)}_\mathcal{U}(gh(\cdot)g^{-1}) = gs^{(1)}_\mathcal{U}(h(\cdot)) \text{ for } g \in \text{Aff}(\mathbb{A}^n), h \in \mathcal{U}.$$ 

(The choice of the sections might not be canonical here.) We say that $s^{(1)}_\mathcal{U}$ is the eigenvector-section of $\mathcal{U}$.

Let $AS(\mathbb{R}^{n+1})$ denote the space of oriented affine hyperspaces in $\mathbb{R}^n$. There is a standard action of $\text{Aff}(\mathbb{A}^{n+1})$ on $AS(\mathbb{R}^{n+1})$. One can choose a continuous section also $s^{(2)}_\mathcal{U} : \mathcal{U} \to AS(\mathbb{R}^{n+1})^{e_2}$ sending a holonomy homomorphism $h$ to an invariant hyperspace in $\mathbb{R}^{n+1}$ of $h(\pi_1(E_i))$ for $i = e_1 + 1, \ldots, e_1 + e_2$. $s^{(2)}_\mathcal{U}$ satisfies

$$s^{(2)}_\mathcal{U}(gh(\cdot)g^{-1}) = gs^{(2)}_\mathcal{U}(h(\cdot)) \text{ for } g \in \text{Aff}(\mathbb{A}^{n+1}), h \in \mathcal{U},$$

where the invariant hyperspace is the one satisfying the suspended lens-condition. (The choice of the sections might not be canonical here.) We say that $s^{(2)}_\mathcal{U}$ is the eigen-1-form section of $\mathcal{U}$. We form the eigensection

$$s_\mathcal{U} := s^{(1)}_\mathcal{U} \times s^{(2)}_\mathcal{U} : \mathcal{U} \to (\mathbb{R}^{n+1} - \{O\})^{e_1} \times (AS(\mathbb{R}^{n+1}))^{e_2}.$$
We note that the affine structure with radial and totally geodesic ends also will determine a point of \((\mathbb{R}^{n+1} - \{O\})^{e_1} \times (AS(\mathbb{R}^{n+1}))^{e_2}\). We have a natural inclusion
\[
(\mathbb{R}^{n+1} - \{O\})^{e_1} \times (AS(\mathbb{R}^{n+1}))^{e_2} \to (\mathbb{R}P^n)^{e_1} \times (\mathbb{R}P^n)^{e_2}.
\]

One can identify \(AS(\mathbb{R}^{n+1})\) with an open subspace of \(S^{n+1}\) by sending the affine hyperspace to a hyperspace of \(S^{n+1}\) and hence to a point of \(S^{n+1}\) by duality. In fact, the open subspace is \(S^{n+1} - \{[\alpha], [-\alpha]\} \) where \(\alpha\) is a 1-form determining \(\mathbb{R}^{n+1}\).

### 9.2.2 The end condition for real projective structures

Now, we go over to real projective orbifolds: We are given a real projective orbifold \(\mathcal{O}\) with ends \(E_1, \ldots, E_{e_1}\) of \(\mathcal{I}\)-type and \(E_{e_1+1}, \ldots, E_{e_1+e_2}\) of \(\mathcal{J}\)-type. Let us choose representative p-ends \(\tilde{E}_1, \ldots, \tilde{E}_{e_1}\) and \(\tilde{E}_{e_1+1}, \ldots, \tilde{E}_{e_1+e_2}\).

We define a subspace of \(\text{Hom}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))\) to be as in Section 1.4.1.

Let \(\mathcal{V}\) be an open subset of semi-algebraic subset of
\[
\text{Hom}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]
invariant under the conjugation action so that one can choose a continuous section \(s^{(1)}_{\mathcal{V}} : \mathcal{V} \to (\mathbb{R}P^n)^{e_1}\) sending a holonomy homomorphism to a common fixed point of \(h(\pi_1(\tilde{E}_i))\) for \(i = 1, \ldots, e_1\) and \(s^{(1)}_{\mathcal{V}}\) satisfies
\[
s^{(1)}_{\mathcal{V}}(gh(\cdot)g^{-1}) = g \cdot s^{(1)}_{\mathcal{V}}(h(\cdot)) \text{ for } g \in \text{PGL}(n+1, \mathbb{R}).
\]

There might be more than one choice of a section and the domain of definition, \(s^{(1)}_{\mathcal{V}}\) is said to be a fixed-point section.

Again suppose that one can choose a continuous section \(s^{(2)}_{\mathcal{V}} : \mathcal{V} \to (\mathbb{R}P^n)^{e_2}\) sending a holonomy homomorphism to a common dual fixed point of \(\pi_1(\tilde{E}_i)\) for \(i = e_1 + 1, \ldots, e_2\), and \(s^{(2)}_{\mathcal{V}}\) satisfies
\[
s^{(2)}_{\mathcal{V}}(gh(\cdot)g^{-1}) = (g^*)^{-1} \circ s^{(2)}_{\mathcal{V}}(h(\cdot)) \text{ for } g \in \text{PGL}(n+1, \mathbb{R}).
\]

There might be more than one choice of section in certain cases. \(s^{(2)}_{\mathcal{V}}\) is said to be a dual fixed-point section.

We define \(s_{\mathcal{V}} : \mathcal{V} \to (\mathbb{R}P^n)^{e_1} \times (\mathbb{R}P^n)^{e_2}\) as \(s^{(1)}_{\mathcal{V}} \times s^{(2)}_{\mathcal{V}}\) and call it a fixing section provided the p-end holonomy group of each \(\mathcal{I}\)-type p-end \(\tilde{E}_i\) acts on a horosphere tangent to \(P\) determined by \(s^{(2)}_{\mathcal{V}}\).

Recall from Section 1.5.2. We note that the real projective structure with radial and totally geodesic ends with end structures also will determine a point of \((\mathbb{R}P^n)^{e_1} \times (\mathbb{R}P^n)^{e_2}\). Conversely, if the real projective structure with radial and to-
tally geodesic ends has the end structure determined by a section $s_{\mathcal{V}}$ if the following hold:

- $\tilde{E}_i$ for every $i = 1, \ldots, e_1$ has a p-end neighborhood with a radial foliation with leaves developing into rays ending at the fixed point of the $i$-th factor of $s_{\mathcal{V}}^{(1)}$.
- $\tilde{E}_i$ for every $i = e_1 + 1, \ldots, e_1 + e_2$
  - has a p-end neighborhood with the ideal boundary component in the hyperspace determined by the $i$-th factor of $s_{\mathcal{V}}^{(2)}$ provided $\tilde{E}_i$ is a T-end, or
  - has a p-end neighborhood containing a $\Gamma$-invariant horosphere tangent to the hyperspace determined by the $i$-th factor of $s_{\mathcal{V}}^{(2)}$ provided $\tilde{E}_i$ is a horospherical end.

Example 9.1 If $\mathcal{O}$ is real projective and has some singularity of dimension one in each end-neighborhood of an $R$-type end, then the universal cover of $\mathcal{O}$ has more than two lines corresponding to singular loci. The developing image of the lines must meet at a point in $\mathbb{R}P^n$, which is a common fixed point of the holonomy group of an end. If $\mathcal{O}$ has dimension 3, this is equivalent to requiring that the end orbifold has corner-reflectors or cone-points.

Hence, for an open subspace $\mathcal{V}$ of a semi-algebraic subset of

$$\text{Hom}_{\mathcal{V}}^p(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$$

corresponding to the real projective structures on $\mathcal{O}$, there is a section $s_{\mathcal{V}}^{(1)}$ determined by the common fixed points.

(See Theorems 11.1 and 11.2.)

Remark 9.1 (Cooper) We do caution the readers that these assumptions are not trivial and exclude some important representations. For example, these spaces exclude some incomplete hyperbolic structures arising in Thurston’s Dehn surgery constructions as they have at least two fixed points for the holonomy homomorphism of the fundamental group of a toroidal end as was pointed out by Cooper. Hence, the uniqueness condition fails for this class of examples. However, if we choose a section on a subset, then we can obtain appropriate results. Or if we work with particular types of orbifolds, the uniqueness holds. See Section 3.4.

9.2.3 Perturbing horospherical ends

Theorems 7.1 and 7.2 study the perturbation of lens-shaped R-ends and lens-shaped T-ends.

The following concerns the deformations of $\Gamma_{\mathcal{E}} \to \text{PGL}(n + 1, \mathbb{R})$ near horospherical representations. As long as we restrict to deformed representations satisfying the lens-condition, there exist $n$-dimensional properly convex domains on which the groups act. (This answers a question of Tillmann near 2006. We also benefited from a discussion with J. Porti in 2011.)
Let $P$ be an oriented hyperspace of $\mathbb{S}^n$ with a dual point $P^* \in \mathbb{S}^n$ represented by a 1-form $w_P$ defined on $\mathbb{R}^{n+1}$. Let $P^*$ denote the space of oriented hyperplanes in $P$. Let $S_{P^*}$ be the space of oriented hyperspaces corresponding to hyperspaces in $P$. Then the subspace $P^\ast$ is dual to $S_{P^*}$: each oriented ray in $S_{P^*}$ from $P^*$ defines a hyperspace $S'$ of $P$ as the set of common zeros of the 1-forms in the ray. The orientation of $S'$ is given by the open half-space where the 1-forms near $w_P$ are positive. Conversely, an oriented pencil of oriented hyperspaces determined by an oriented hyperspace of $P$ is a ray in $S_{P^*}$. (We omit the obvious $\mathbb{RP}^n$-version.)

Let

$$\text{Hom}_{\mathcal{E}, \mathfrak{h}, P}(\Gamma', \text{PGL}(n+1, \mathbb{R})) \quad (\text{resp. } \text{Hom}_{\mathcal{E}, \mathfrak{h}, P}(\Gamma', \text{SL}_\pm(n+1, \mathbb{R})))$$

denote the space of representations $h$ fixing a common fixed point $p$ and acts properly and cocompactly on the lens of a lens-cone over vertex $p$ or is horospherical with a horoball with vertex $p$.

Let

$$\text{Hom}_{\mathcal{E}, \mathfrak{h}, P}(\Gamma', \text{PGL}(n+1, \mathbb{R})) \quad (\text{resp. } \text{Hom}_{\mathcal{E}, \mathfrak{h}, P}(\Gamma', \text{SL}_\pm(n+1, \mathbb{R})))$$

denote the space of representations where $h(\Gamma')$ for each element $h$ acts on a hyperspace $P$ satisfying the lens-condition. (See 1.4.1.)

Let a convex cone $B = \partial B \ast \{p\}$ over a point $p$ be diffeomorphic to $\partial B \times (0, 1]$. Then $B$ with a vertex $p$ has a radial foliation. We complete $B$ by identifying with $\partial B \times (0, 1)$ by a diffeomorphism $f$ sending each leaf to $x \times (0, 1)$ and attaching $\partial B \times (0, 1]$ by $f$. We denote the partial completion by $\hat{B}$ diffeomorphic to $\partial B \times [0, 1]$. We call $\hat{B}$ the p-end completion of $B$. An action of a group $\Gamma'$ on $B$ extends to $\hat{B}$ also. $\hat{B}/\Gamma'$ is then the end-compactification of $B/\Gamma$. (See Definition 9.1.)

**Lemma 9.4 (Horospherical-end perturbation)**

(A) \hspace{1cm} Let $B$ be a horoball in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) and $\Gamma$ be a group of projective automorphisms fixing $p, p \in \partial B$ (resp. $p \in \mathbb{S}^n$), so that $B/\Gamma$ is a horospherical-end-type orbifold. Then there exists a sufficiently small neighborhood $K$ of the inclusion homomorphism $h_0$ of $\Gamma$ in $\text{Hom}_{\mathcal{E}, \mathfrak{h}, P}(\Gamma', \text{PGL}(n+1, \mathbb{R}))$

$$\text{(resp. } \text{Hom}_{\mathcal{E}, \mathfrak{h}, P}(\Gamma, \text{SL}_\pm(n+1, \mathbb{R})))$$

where

- for each $h \in K$, $h(\Gamma')$ acts on a properly convex domain $B_h$ so that $B_h/h(\Gamma')$ is diffeomorphic to $B/\Gamma'$ forming a radial end and fixes $p$, and
- there is a diffeomorphism $f_h : B/\Gamma' \to B_h/h(\Gamma')$, $h \in K$, so that the lift $\tilde{f}_h : \hat{B} \to \hat{B}_h$ is a continuous family under the $C^\ast$-topology as a map into $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) where $\tilde{f}_h$ is the identity map.

Let $\hat{B}$ and $\hat{B}_h$ denote the p-end compactifications. Then $f_h$ extends to the end compactifications $\tilde{f}_h : \hat{B}/\Gamma' \to \hat{B}_h/h(\Gamma')$ and $\tilde{f}_{h_0}$ is the identity map. Furthermore, the
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lift of this map \( f_h : B \to \mathbb{R}P^n \) (resp. \( S^n \)) is continuous in the \( C' \)-topology where \( f_{h_0} \) is the identity map.

(B) Let \( P \) be a hyperspace in \( \mathbb{R}P^n \) (resp. in \( S^n \)). Let \( \Gamma' \) denote a projective automorphism group acting on \( P \) and a horoball \( B \) tangent to \( P \) so that \( B/\Gamma' \) is a horospherical-end-type orbifold. Then there exists a sufficiently small neighborhood \( K \) of the inclusion homomorphism \( h_0 \) of \( \Gamma' \) in \( \text{Hom}_{\varepsilon,\text{th},P}(\Gamma', \text{PGL}(n+1, \mathbb{R})) \) (resp. \( \text{Hom}_{\varepsilon,\text{th},P}(\Gamma', \text{SL}(n+1, \mathbb{R})) \)) where

- for each \( h \in K \), \( h(\Gamma') \) acts on a properly convex domain \( B_h \) so that \( B_h/h(\Gamma') \) is homeomorphic to \( B/\Gamma' \) and has a lens-shaped totally geodesic or horospherical-end, and
- there is a diffeomorphism \( f_h : B/\Gamma' \to B_h/h(\Gamma') \), \( h \in K \), so that the lift \( \tilde{f}_h : B \to B_h \) is a continuous family under the \( C' \)-topology where \( f_{h_0} \) is the identity map.

Let \( \hat{B} \) denote the \( p \)-end compactification of \( B \) and \( \hat{B}_h \) denote \( B_h \) union with the \( p \)-end ideal boundary component of \( B_h \) when \( h \) acts properly and cocompactly on a lens. Then \( f_h \) extends to the end compactifications \( \tilde{f}_h : \hat{B}/\Gamma' \to \hat{B}_h/h(\Gamma') \). Furthermore, the lift of this map \( \tilde{f}_h : \hat{B} \to \mathbb{R}P^n \) (resp. \( S^n \)) is a continuous family in the \( C' \)-topology where \( f_{h_0} \) is the identity map.

**Proof** We will prove for the \( S^n \)-version.

(A) Let us choose a larger horoball \( B' \) in \( B \) where \( B'/\Gamma' \) has a boundary component \( S' \) so \( B'/\Gamma' \) is diffeomorphic to \( S' \times [0,1] \). \( S' \) is strictly convex and transverse to the radial foliation. There exists a neighborhood \( O_1 \) in \( \text{Hom}_{\varepsilon,\text{th},p}(\Gamma', \text{SL}(n+1, \mathbb{R})) \) corresponding to the connection on a fixed compact neighborhood \( N \) of \( S' \) changes only by \( \varepsilon \) in the \( C' \)-topology, \( r \geq 2 \). (See the deformation theorem in [93] which generalize to the compact orbifolds with boundary.)

Let \( h \in O_1 \). The universal cover \( \hat{S}' \) is a strictly convex codimension-one manifold, and it deforms to \( \hat{S}'_{E,h} \) that is still strictly convex for sufficiently small \( \varepsilon \). Here, \( \hat{S}'_{E,h} \) may not be embedded in \( S^n \) a priori but is a submanifold of the deformed \( n \)-manifold \( N_h \) from \( N \) by the change of connections. Every ray from \( p \) meets \( \hat{S}'_{E,h} \) transversely also by the \( C' \)-condition.

Let \( \nabla_{x,h} \) be a vector in the direction of \( x \) for \( x \in \hat{S}'_{E,h} \) which we choose equivariant with respect to the action of \( h(\Gamma') \). We may choose so that \( (x,h) \mapsto \nabla_{x,h} \) is continuous. We form a cone

\[
c(\hat{S}'_{E,h}) := \{ t \nabla_{x,h} + (1-t)\nabla_{x,h} | t \in [0,1], x \in \hat{S}'_{E,h} \}.
\]

Let \( \hat{S}'_{E,h} \) denote the space of rays from \( p \) ending at \( \hat{S}'_{E,h} \) in \( c(\hat{S}'_{E,h}) \). Here \( \hat{S}'_{E,h} := \hat{S}'_{E,h}/h(\Gamma') \) is a compact real projective orbifold of \( (n-1) \)-dimension.

Since \( \Gamma' \) is a cusp group, it is virtually abelian. Since \( h \in \text{Hom}_{\varepsilon,\text{th}}(\Gamma', \text{SL}(n+1, \mathbb{R})) \), Lemma A.10 implies that \( D_h : \hat{S}'_{E,h} \to \mathbb{S}_p^{n-1} \) is an embedding to a properly convex domain or a complete affine domain \( \Omega_h \) in \( \mathbb{S}_p^{n-1} \) where \( h(\Gamma') \) acts properly discontinuously and cocompactly when \( h \) is not the inclusion map.
There is a one-to-one correspondence from \( S_\epsilon \) to \( \Sigma \). By convexity of \( S_\epsilon \), the tube domain \( T_\epsilon(\Omega_h) \) with vertices \( p, -p \) is convex. \( S_\epsilon \) meets each great segments in the interior the tube domain with vertices at \( p, -p \) at a unique point transversely since \( \epsilon \) is in \( O_1 \) for sufficiently small \( \epsilon \). The strict convexity of \( S_\epsilon \) implies that \( B_h \) is convex by Lemma 2.17. The proper convexity of \( B_h \) follows since \( S_\epsilon \) is strictly convex and meets each great segment from \( p \) in the interior of the tube domain corresponding to \( S_\epsilon \), and hence \( \text{Cl}(B_h) \) cannot contain a pair of antipodal points.

By Theorem 4.2, the Zariski closure of \( h(\Gamma') \) is a cusp group \( G_h \) extended by a finite group and \( G_h/h(\Gamma') \) is compact. Hence, \( \Gamma' \) is virtually abelian by the Bieberbach theorem. We take the identity component \( \mathcal{N}_h \) of \( G_h \), which is an abelian group with a uniform lattice \( h(\Gamma') \). The set of orbits of \( \mathcal{N}_h \) foliates \( B_h \). Since \( \mathcal{N}_h \) is a normal subgroup of \( G_h \), \( h(\Gamma') \) normalizes \( \mathcal{N}_h \). Hence, the orbits give us a codimension-one foliation on \( B_h/h(\Gamma') \) with compact leaves. The leaves are all diffeomorphic, and hence, we obtain a parameterization \( \partial B/\Gamma' \times [0, 1) \to B_h \).

Now, \( h \) induces isomorphism \( \tilde{h}_h : \mathcal{N}_0 \to \mathcal{N}_h \) where \( \tilde{h}_h \to I \) as \( h \to h_0 := I \).

We choose a proper radial path \( \alpha_h : I \to B_h \) from a point of \( \partial B_h \) and ending at \( p \). We may assume that \( \alpha_h \) is independent of \( h \). We define a parameterization

\[
\tilde{\phi}_h : \mathcal{N} \times [0, 1) \to B_h, (m, t) \mapsto \tilde{h}_h(m)(\alpha_h(t)), t \in [0, 1).
\]

We define \( \tilde{f}_h : B \to B_h := \tilde{\phi}_h \circ \tilde{\phi}_h^{-1} \). This gives us a map \( \tilde{f}_h \). (Here, we might be changing \( S_\epsilon \).) Since \( \tilde{f}_h \) sends radial segments to radial segments, it extends to a smooth map \( \tilde{f}_h : \hat{B} \to \hat{B}_h \). Also, on any compact subset \( J \) of \( \hat{B} \), a compact foliated set \( \hat{J} \) contains it. Let \( J_h \) denote the image of \( \hat{J} \) under \( \tilde{f}_h \). \( J_h \) is coordinatized by \( J \times I \) for a fixed compact set \( J \subset \mathcal{N} \). Under these coordinates of \( J \) and \( J_h \), we can write \( \tilde{f}_h \) as the identity map. Since \( \tilde{h}_h \to h_0 \), we conclude that \( \tilde{f}_h \) uniformly converges to \( I \) as \( h \to h_0 \).

(B) The second item is the dual of the first one. If \( h(\Gamma') \) acts on an open horosphere \( B^* \) tangent to \( P \) with the vertex in \( P \) properly discontinuously, then the dual group \( h(\Gamma')^* \) acts on a horosphere with a vertex the point \( P^* \) dual to \( P \). By duality Proposition 6.11, \( h^* \) is in \( \text{Hom}^* \). \( h^* \) is in \( \text{Hom}(\Gamma', SL_\pm(n + 1, \mathbb{R})) \). We apply the first part, and hence, there exists a properly convex domain \( \Omega_{P,h} \) so that \( \Omega_{P,h}/h^*(\Gamma') \) is an open orbifold for \( h \in K \) for some subset \( K \) of the character space

\[
\text{Hom}^*(\Gamma', SL_\pm(n + 1, \mathbb{R})).
\]

By duality Proposition 6.11, \( \Omega_{P,h} \) is foliated by radial lines from \( P^* \) and \( R_{P^*}(\Omega_{P,h}) \subset S_{P^*} \) is a properly convex domain.

Let \( B_h^* \) be the proper convex domain dual to \( \Omega_{P,h} \) and hence is a properly convex domain, and \( B_h^*/h(\Gamma') \) is a dual orbifold diffeomorphic to \( \Omega_{P,h}/h(\Gamma')^* \) by Theorem 2.16.

Let \( \Gamma \) acts properly on \( \partial B \). Since \( \partial B \) is strictly convex, each point of \( \partial B \) has a unique hyperspace sharply supporting \( B \). By Proposition 2.19, there is an open
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 hypersurface $S, S \subset \partial \Omega$ dual to $\partial B$. Also, $\Gamma^\prime$ acts properly on $S$ so that $S/\Gamma^\prime$ is a closed orbifold. (The closedness again follows since there is a torsion-free group of a finite index and hence a finite regular-covering manifold.) From the first part, there is an open surface $S_h, S \subset \partial \Omega^*_{\rho h}, h \in K$, meeting each radial ray from $P^*$ at a unique point. Also, $S_h/\partial h(\Gamma^*)^\prime$ is diffeomorphic to $\partial B/\Gamma^\prime$.

Again, by Proposition 2.19, we obtain an open surface $S^*_h, S^*_h \subset \partial \Omega^*_{\rho h}$ where $\Gamma_h$ acts properly so that $S^*_h/\Gamma_h$ is a closed orbifold. We define $\tilde{B}_h := \Omega^*_{\rho h} \cup S^*_h$.

Since $\Omega^*_{\rho h}$ is foliated by radial segments from $P^*$ with properly convex

$$R_{P^*}((\Omega^*_{\rho h}) \subset S^*_{P^*}^{-1},$$

$D_h := P \cap \partial \Omega^*_{\rho h}$ and is a properly convex domain in $P$ by Proposition 6.11.

Define $\tilde{B}_h := \Omega^*_{\rho h} \cup S^*_h \cup D_h, h \in K$. For the second and third items, of the second part, we do as above but we choose $\alpha_h : I \to B_h$ to be a single geodesic segment starting from $x_0 \in \partial B_h$ and ending at a point of $D_h$ where $\alpha_h$ converges as a parameter of functions to a geodesic $\alpha_0 : I \to \text{Cl}(B) \subset S^*_{\rho}$ ending at the vertex of the horosphere. We assume that $\alpha_h$ is a $C^\prime$-family of geodesics. Now, the proof is similar to the above using an isomorphism from the identity component $C_h$ of the Zariski closure of $\Gamma^\prime$ to that of $h(\Gamma^\prime)$, which is an abelian group since $\Gamma^\prime$ is virtually abelian. Here, we need the images of $\alpha_h$ under $C_h$ to form a foliation. Since $C_h$ acts on a properly convex set $D_h$, it acts as a diagonalizable group on $P$ by Proposition 2.15. Being a free abelian group satisfying the uniform middle eigenvalue condition, $C_h$ is a diagonalizable group acting on an $n$-simplex. (We dualize the situation and use Theorem 6.8.) We require that the extension of $\alpha_0$ to pass a fixed point of $C_h$ not in $\text{Cl}(D_h)$.

The images of $g \circ \alpha_h, g \in C_h$, form a foliation of $\tilde{B}_h$. Using this we define the map $f_h : B \to B_h$ sending leaves to leaves as given by the function

$$g(\alpha_0(t)) \mapsto g(\alpha_h(t)) \text{ for each } t \in [0, 1], g \in C_h.$$ 

This map extends to $\partial B$ to $\partial B_h$.

Finally notice that our constructions of $f_h$ all are smooth from $\tilde{B}_h$. Hence, these are compatible end neighborhoods.

We remark that we can also reinterpret the parameterization as radial projectivization in Section 9.1.4 by taking a second larger end neighborhood and some modifications of the parameters.

An affine space $\mathbb{A}^n$ with an origin is denoted by $\mathbb{R}^n$. Here, $\text{Aff} (\mathbb{R}^n) = \text{GL}(\mathbb{R}^n)$ really. An affine space $\mathbb{R}^{n+1}$ has a great sphere $S^\prime_{\mathbb{R}^n}$ as a boundary. We set an origin $O$ in this space. We define $S^\prime_{\mathbb{R}^n}$ as the dual sphere on which the dual group $G^\prime$ acts provided an affine group $G$ act on $S^\prime_{\mathbb{R}^n}$. Suppose that $G$ is a countable group with a distinguished central cyclic group $Z$. We assume that the choice of $Z$ is obvious. We denote by $\text{Hom}^\prime(G, \text{GL}(n + 1, \mathbb{R}))$ the space of representations where the central cyclic group $Z$ go to the group of scalar dilatations fixing the origin $O$ of $\mathbb{R}^{n+1}$.

Let $\text{Hom}^\prime_{\alpha, h, p}(G^\prime, \text{GL}(n + 1, \mathbb{R}))$ denote the space of representations $h$ with linear parts with a common eigenvector $v_p$ so that $\{v_p\} = p$ where the restriction group of
along $h \Gamma^\prime$ acts on a lens-cone or a horoball-cone with vertex at $O$ with the end radial along $v_p$ disjoint from the lens-cone or the horoball cone.

Let $\text{Hom}_S^{\Gamma', \Gamma}(F, GL(n+1, \mathbb{R}))$ denote the space of representations $h$ acting on a hyperspace $P$ in $\mathbb{R}^{n+1}$ where $h \Gamma^\prime$ acts on a lens-cone $C(L)$ with vertex at $O$ properly discontinuously and cocompactly with $C(L)^o \cap P = C(L) \cap \partial P \neq \emptyset$ or acts on a horoball-cone tangent to $P$. (The condition here corresponds to the suspended lens-condition. See Section 9.2.1.)

**Lemma 9.5 (Affine horospherical end perturbation)**

(A) Let $B$ be a horoball and $\Gamma^\prime$ be a group of projective automorphisms fixing $p$ so that $B/\Gamma^\prime$ is a horospherical end orbifold. Let $C_B$ an affine cone corresponding to $B$ and $\Gamma^\prime$ denote the affine transformation corresponding to $\Gamma^\prime$ centrally extended by an infinite cyclic scalar dilatation group acting on $C_B$. Let $\hat{C}_B$ denote the end-compactification with respect to $p$. Then there exists a sufficiently small neighborhood $K$ of the inclusion homomorphism $h_0$ of $\Gamma^\prime$ in $\text{Hom}_S^{\Gamma', \Gamma}(F, GL(n+1, \mathbb{R}))$ where

- for each $h \Gamma^\prime$ with $h \in P$ so that $h \Gamma^\prime$ acts on a properly convex cone $C_{B_h}$ so that $C_{B_h}/h \Gamma^\prime$ is homeomorphic to $C_B/\Gamma^\prime$ and has a radial end, and
- there is a diffeomorphism $f_h : C_B/\Gamma^\prime \to C_{B_h}/h \Gamma^\prime$, $h \in K$, so that the lift $\hat{f}_h : \hat{C}_B \to \hat{C}_{B_h}$ is a continuous family under the $C^\prime$-topology.

Let $\hat{C}_B$ and $\hat{C}_{B_h}$ denote the $p$-end compactifications of $C_B$ and $C_{B_h}$ with respect to $p$. Then $f_h$ extends to the end compactifications $\hat{f}_h : \hat{C}_B/\Gamma^\prime \to \hat{C}_{B_h}/h \Gamma^\prime$ and $\hat{f}_{h_0}$ is the identity map. Furthermore, the lift of this map $\hat{f}_h : \hat{C}_B \to \mathbb{RP}^n$ (resp. $S^n$) converges to the inclusion map $\hat{f}_{h_0}$ in the $C^\prime$-topology as $h \to h_0$.

(B) Let $B$ be a horoball and $\Gamma^\prime$ be a group of projective automorphisms acting on a hyperspace $P$ so that $B/\Gamma^\prime$ is a horospherical end orbifold. Let $C_B$ an affine cone corresponding to $B$ and $\Gamma^\prime$ denote the affine transformation corresponding to $\Gamma^\prime$ centrally extended by an infinite cyclic scalar dilatation group acting on $C_B$. Then there exists a sufficiently small neighborhood $K$ of the inclusion homomorphism $f_0$ of $\Gamma^\prime$ in $\text{Hom}_S^{\Gamma', \Gamma}(F, GL(n+1, \mathbb{R}))$ where

- for each $h \Gamma^\prime$ with $h \in K$ so that $h \Gamma^\prime$ acts on a properly convex cone $C_{B_h}$ so that $C_{B_h}/h \Gamma^\prime$ is homeomorphic to $C_B/\Gamma^\prime$ and has a totally geodesic or affine horospherical end, and
- there is a diffeomorphism $f_h : C_B/\Gamma^\prime \to C_{B_h}(h \Gamma^\prime)$, $h \in K$, so that the lift $\tilde{f}_h : C_B \to C_{B_h}$ is a continuous family under the $C^\prime$-topology.

Let $\hat{C}_B$ denote the $p$-end compactifications of $C_B$ with respect to $p$, and let $\hat{C}_{B_h}$ denote $B_h$ union with the $p$-end ideal boundary component of $B_h$. Then $f_h$ extends to the end compactifications $\tilde{f}_h : \hat{C}_B/\Gamma^\prime \to \hat{C}_{B_h}/h \Gamma^\prime$ and $\tilde{f}_{h_0}$ is the identity map. Furthermore, the lift of this map $\tilde{f}_h : \hat{C}_B \to \mathbb{RP}^n$ (resp. $S^n$) converges to the inclusion map in the $C^\prime$-topology as $h \to h_0$. 

\[h(\Gamma^\prime)\]
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**Proof** This follows as in Lemma 9.4. The third item follows similar arguments to those of the proof there using the identity components of the Zariski closures of the holonomy groups and segments. □

9.2.4 Local homeomorphism theorems

Let \( \mathcal{O} \) be a noncompact strongly tame affine \((n + 1)\)-orbifold whose ends are assigned to be of \( \mathcal{R} \)-type or \( \mathcal{I} \)-type as is the convention in this paper.

An affine manifold affinely diffeomorphic to the affine suspension of horospherical end neighborhood is said to be the *affinely suspended horoball neighborhood*. If an end has such a neighborhood, then we call the end *affine horospherical type*. Since the projective automorphism group of a horosphere fixes a point, the fundamental group of the affine horospherical end preserves a direction. Thus, the end of an affine horospherical type is of radial type.

We define the end restricted deformation space for \( \mathcal{O} \) to be the quotient space of affine structures on \( \mathcal{O} \) where

- each end is radial if the end is of \( \mathcal{R} \)-type or
- is totally geodesic satisfying the *suspended lens-condition* if the end is of \( \mathcal{I} \)-type

under the action of group of isotopies preserving the end structures; that is, preserves the radial foliation if the end is radial or horospherical or extends to a smooth diffeomorphism if the end is totally geodesic. (As above, each end has a distinguished infinite cyclic group in the center with holonomies in scalar dilatations in \( \mathbb{R}^{n+1} \).)

An affine orbifold has a real projective structure. The radial end structure becomes a radial foliation end structure. Also, the totally geodesic end structure of an affine structure corresponds to a totally geodesic end structure of the real projective structure. Thus, the end-compactification of \( \mathcal{O} \) of Definition 9.1 makes sense. Let \( \overline{\mathcal{O}} \) be the end-compactification of \( \mathcal{O} \) using the end structures since \( \mathcal{O} \) is an \((n + 1)\)-dimensional real projective orbifold.

Assume that \( \pi_1(\mathcal{O}) \) has a distinguished central infinite cyclic group \( Z \). We also define

\[
\text{Hom}^S_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{GL}(n+1, \mathbb{R}))
\]

as the subspace of

\[
\text{Hom}^S(\pi_1(\mathcal{O}), \text{GL}(n+1, \mathbb{R}))
\]

of elements \( h \) where

- \( h(\pi_1(E)) \) for each end \( E \) has scalar dilatations as images of \( Z \),
- the elements of the representations \( h(\pi_1(E)) \) of the fundamental group of each end \( E \) has a common eigenvector if the end is of \( \mathcal{R} \)-type, or
- \( h(\pi_1(E)) \) acts on a totally geodesic hyperspace \( P \) with \( C_L \cap P = C'_L \cap P \neq \emptyset \) for a lens-cone \( C_L \) or tangent to a horoball-cone \( C_H \) where \( C_L/h(\pi_1(E)) \) is a compact orbifold or \( C_H/h(\pi_1(E)) \) is a horoball suspension if the end is of \( \mathcal{I} \)-type.
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In the second case, \( P \) is said to be the hyperplane satisfying the \textit{suspended lens-condition}.

Let \( AS_0(\mathbb{R}^{n+1}) \) denote the subspace of oriented affine subspaces passing the origin in \( AS(\mathbb{R}^{n+1}) \). Actually, it is identifiable with \( S(\mathbb{R}^{n+1}) \). We define \( \text{Def}_{A,E,s}^S(\mathcal{O}) \) to be the subspace of \( \text{Def}_{A,E}(\mathcal{O}) \) with the corresponding holonomy homomorphism in the open subset \( \mathcal{W} \) of a semi-algebraic subset of

\[
\text{Hom}^S(\pi_1(\mathcal{O}), \text{GL}(n + 1, \mathbb{R}))
\]

invariant under the conjugation action and with affine structures so that the end is determined by the section

\[
s_{\mathcal{W}} : \mathcal{W} \to (\mathbb{R}^{n+1} - \{O\})^{r_1} \times (AS_0(\mathbb{R}^{n+1}))^{r_2}
\]

where \( \mathcal{W} \) is a conjugation-invariant subset of \( \text{Hom}(\pi_1(\mathcal{O}), \text{GL}(n + 1, \mathbb{R})) \) and

\[
s_{\mathcal{W}}(gh(\cdot)g^{-1}) = g \cdot s_{\mathcal{W}}(h(\cdot)) \text{ for } g \in \text{GL}(n + 1, \mathbb{R}).
\]

(See Section 9.2.1.)

We regard \( \mathcal{O} \) as a real projective \( n + 1 \)-orbifold with radial or totally geodesic ends and end-compactify to obtain \( \overline{\mathcal{O}} \). (See Definition 9.1.) Let \( \hat{\mathcal{O}} \) denote the universal cover with a smooth extended developing map \( \text{dev} : \hat{\mathcal{O}} \to \mathbb{RP}^{n+1} \) equivariant with respect to

\[
h : \pi_1(\mathcal{O}) \to \text{GL}(n + 1, \mathbb{R}) \subset \text{PGL}(n + 2, \mathbb{R}).
\]

Here, we require that the structure on \( \mathcal{O} \) so that the extension of \( \text{dev} : \hat{\mathcal{O}} \to \mathbb{RP}^{n+1} \) is smooth. As before, we may assume this without loss of generality by isotopies of any given affine structures.

**Definition 9.6** Let \( \mathcal{O} \) be a strongly tame real projective orbifold with end structures. Let \( \hat{\mathcal{O}} \) be the universal cover of one choice of end compactification of \( \mathcal{O} \). We define the isotopy-equivalence space \( \text{Def}_{\mathcal{A},E,u}^S(\mathcal{O}) \) as the quotient space of all extended developing maps

\[
\text{dev} : \hat{\mathcal{O}} \to \text{Cl}(\mathbb{R}^{n+1}) \subset \mathbb{RP}^{n+1}
\]

for structures in \( \text{Def}_{\mathcal{A},E,s}^S(\mathcal{O}) \) under extended isotopy-lifts defined on \( \mathfrak{l} \). (See Definitions 9.2 and 9.3.) The space has the compact open \( C^0 \)-topology defined for the universal cover \( \hat{\mathcal{O}} \) of one choice of end compactification of \( \mathcal{O} \). Here \( \text{Def}_{\mathcal{A},E,s}^S(\mathcal{O}) \) is the quotient space of \( \text{Def}_{\mathcal{A},E,s}^S(\mathcal{O}) \) under \( \text{GL}(n + 1, \mathbb{R}) \) acting by

\[
g(\text{dev}, h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1}) \text{ for } g \in \text{GL}(n + 1, \mathbb{R}).
\]

(See [54] for details.)

We define \( \text{Def}_{\mathcal{A},E,u}(\mathcal{O}) \) to be the subspace of \( \text{Def}_{\mathcal{A},E}(\mathcal{O}) \) where the end holonomy group of each end \( E \) fixes a unique point if \( E \) is of type \( \mathcal{R} \) and acts on a unique
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hyperplane satisfying the suspended lens condition if $E$ is of type $\mathcal{T}$. Similarly, we define $\tilde{\text{Def}}_{A,E,u}(\mathcal{O})$ as the quotient space of development pairs corresponding to the elements of $\text{Def}_{A,E,u}(\mathcal{O})$ under the isotopies of $\tilde{\mathcal{O}}$ preserving the end structures. We also note that

$$\text{Def}_{A,E,u}(\mathcal{O}) = \tilde{\text{Def}}_{A,E,u}(\mathcal{O})/\text{GL}(n+1, \mathbb{R}).$$

The rest of the proof of the first part of Theorem 9.2 is similar to [54]. We cover $\mathcal{O}$ by open sets covering a codimension-0 compact orbifold $\mathcal{O}'$ and open sets which are end-radial.

**Theorem 9.2** Let $\mathcal{O}$ be a noncompact strongly tame affine $(n+1)$-orbifold with radial ends and totally geodesic ends of suspended lens-type where the types of ends are assigned. Let $\mathcal{U}$ be a conjugation-invariant open subset of a semi-algebraic subset of $\text{Hom}_{E}(\pi_1(\mathcal{O}), \text{GL}(n+1, \mathbb{R}))$ with an eigensection $s_{\mathcal{U}}$. Then the map

$$\text{hol} : \tilde{\text{Def}}_{A,E,u}(\mathcal{O}) \to \text{Hom}_{E}(\pi_1(\mathcal{O}), \text{GL}(n+1, \mathbb{R}))$$

sending affine structures with end structures determined by the section $s_{\mathcal{U}}$ to the conjugacy classes of holonomy homomorphisms is a local homeomorphism on an open subset of $\mathcal{U}$.

Again $\text{Def}_{E,s_{\mathcal{U}}}(\mathcal{O})$ is defined to be the subspace of $\text{Def}_{E}(\mathcal{O})$ with the stable irreducible holonomy homomorphisms in $\mathcal{U}$ and the end determined by $s_{\mathcal{U}}$, i.e.,

- each $\mathcal{R}$-type $p$-end has a $p$-end neighborhood foliated by geodesic leaves that are radial to the vector given by $s_{\mathcal{U}}$ under the developing map, or
- each $\mathcal{F}$-type $p$-end is totally geodesic of suspended lens-type satisfying the lens-condition or horospherical satisfying the suspended lens condition with respect to the hyperspace determined by $s_{\mathcal{U}}$. (See Section 9.2.1.)

**Theorem 9.3** Let $\mathcal{O}$ be a noncompact strongly tame real projective $n$-orbifold with lens-shaped radial ends or lens-shaped totally geodesic ends with types assigned. Let $\mathcal{Y}$ be a conjugation-invariant open subset of the union of semi-algebraic subsets of $\text{Hom}_{E}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$.

Let $s_{\mathcal{Y}}$ be the fixing section defined on $\mathcal{Y}$ with images in $((\mathbb{R}P^n)^e_1 \times (\mathbb{R}P^n)^e_2)$. Then the map

$$\text{hol} : \text{Def}_{E,s_{\mathcal{Y}}}(\mathcal{O}) \to \text{rep}_{E}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$$

sending the real projective structures with ends compatible with $s_{\mathcal{Y}}$ to their conjugacy classes of holonomy homomorphisms is a local homeomorphism to an open subset of $\mathcal{Y}'$.

Define

$$\text{rep}_{E,s_{\mathcal{Y}}}(\pi_1(\mathcal{O}) \times \mathbb{Z}, \text{GL}(n+1, \mathbb{R}))$$

as the subspace of
of holonomy representations whose linear parts are stable and irreducible where each \( \mathcal{R} \)-type end holonomy group has a unique eigenvector and each \( \mathcal{S} \)-type end holonomy group has a unique eigen-1-form. Define \( \text{Def}_{\mathcal{S}, \mathcal{E}, \mu}(\mathcal{O} \times S^1) \) as the subspace of \( \text{Def}_{\mathcal{A}, \mathcal{E}, \mu}(\mathcal{O} \times S^1) \) mapping to the above subspace under \( \text{hol} \).

We will equip \( \bar{\mathcal{O}} \times S^1 \) as the compactification of \( \mathcal{O} \times S^1 \).

Corollary 9.2 Suppose that \( \mathcal{O} \times S^1 \) is a noncompact strongly tame \((n+1)\)-orbifold. Then the map

\[
\text{hol} : \text{Def}_{\mathcal{S}, \mathcal{E}, \mu}(\mathcal{O} \times S^1) \to \text{Hom}_{\mathcal{S}, \mu}(\pi_1(\mathcal{O}) \times \mathbb{Z}, GL(n+1, \mathbb{R}))
\]

sending affine structures to the conjugacy classes of their holonomy homomorphisms is a local homeomorphism. So is the map

\[
\text{hol} : \text{Def}_{\mathcal{S}, \mathcal{E}, \mu}(\mathcal{O} \times S^1) \to \text{rep}_{\mathcal{S}, \mathcal{E}, \mu}(\pi_1(\mathcal{O}) \times \mathbb{Z}, GL(n+1, \mathbb{R})).
\]

\textbf{Proof} For each representation in

\[
\text{Hom}_{\mathcal{S}, \mu}(\pi_1(\mathcal{O}) \times \mathbb{Z}, GL(n+1, \mathbb{R})),
\]

we find a unique set of vectors and 1-forms corresponding to the ends. The continuity follows by considering sequences. We use the above paragraph and Theorems 9.2 and 9.3.

\[\square\]

9.2.5 The proof of Theorem 9.2.

We wish to now prove Theorem 9.2 following the proof of Theorem 1 in Section 5 of [54]. Then we will prove Theorem 9.3 in Section 9.2.6.

Let \( \mathcal{O} \) be an affine orbifold with the universal covering orbifold \( \tilde{\mathcal{O}} \) with the covering map \( p_\mathcal{O} : \tilde{\mathcal{O}} \to \mathcal{O} \) and let the fundamental group \( \pi_1(\mathcal{O}) \) act on it as an automorphism group.

Let \( \mathcal{U} \) and \( s_\mathcal{U} \) be as above. We will now define a map

\[
\text{hol} : \text{Def}_{\mathcal{A}, \mathcal{E}, s_\mathcal{U}}(\mathcal{O}) \to \text{Hom}_{\mathcal{S}, \mu}(\pi_1(\mathcal{O}), GL(n+1, \mathbb{R}))
\]

by sending the affine structure to the pair \( (\text{dev}, h) \) and to the conjugacy class of \( h \) finally.

(A) We show that \( \text{hol} \) is continuous: There is a codimension-0 compact submanifold \( \mathcal{O}' \) of \( \mathcal{O} \) so that \( \pi_1(\mathcal{O}') \to \pi_1(\mathcal{O}) \) is an isomorphism. The holonomy homomorphism is determined on \( \mathcal{O}' \). Since the deformation space has the \( C^r \)-topology, \( r \geq 1 \), induced by \( \text{dev} : \mathcal{O}' \to \mathbb{R}^{n+1} \), it follows that small changes of \( \text{dev} \) on compact domains in \( \mathcal{O}' \) in the \( C^r \)-topology imply sufficiently small changes in \( h(g_i') \) for generators \( g_i' \) of \( \pi_1(\mathcal{O}') \) and hence sufficiently small change of \( h(g_i) \) for generators \( g_i \).
of \( \pi_1(\mathring{\mathcal{O}}) \). Therefore, hol is continuous. (Actually for the continuity, we do not need any condition on ends.)

(B) We will now prove a few local lemmas: For the purpose of this paper, we use \( r \geq 2 \). We will use this fact a few times.

A \( \mathbf{v} \)-parallel set is a subset \( V \) of \( \mathbb{R}^{n+1} \) which satisfy \( T_{av}(V) \subset V \) under the translation \( T_{av} \) along positive multiples \( av, a > 0 \), of a fixed nonzero vector \( \mathbf{v} \). That is, it should be a union of the images under translations along positive multiples of a nonzero vector.

An end-parallel subset of \( \mathring{\mathcal{O}} \) or \( \mathbb{R}^{n+1} \) is a \( \mathbf{v} \)-parallel set where \( \mathbf{v} \) is the eigenvector of the linear parts of the holonomy group of the corresponding p-end.

To show the local homeomorphism property, we take an affine structure \((\mathbf{dev}, h)\) on \( \mathring{\mathcal{O}} \) and the associated holonomy map \( h \). We cover \( \mathring{\mathcal{O}} \) by small precompact open sets as in Section 5 of [54]. We cover \( \mathring{\mathcal{O}} - \mathring{\mathcal{O}}_{\alpha} \) by end-parallel open sets. Consider Lemmas 3, 4, and 5 in [54]. We can generalize these to include the \( \mathbf{v} \)-parallel sets for an invariant direction of \( \mathbf{v} \) of the finite group \( G_B \) where \( \mathbf{v} \) is an eigenvector of eigenvalue 1. We repeat them below. The proofs are very similar and use the commutativity of translation by eigenvectors with the action of \( G_B \).

**Lemma 9.6** Let \( G_B \) be a finite subgroup of \( \text{GL}(n+1, \mathbb{R}) \) acting on a \( \mathbf{v}_0 \)-parallel open subset \( B \) of \( \mathbb{R}^{n+1} \) for an eigenvector \( \mathbf{v}_0 \) of the linear part of \( G_B \). Let \( h_t : G_B \to \text{GL}(n+1, \mathbb{R}), t \in [0, \varepsilon), \varepsilon > 0, \) be an analytic parameter of representations of \( G_B \) so that \( h_0 \) is the inclusion map. Let \( \mathbf{v}_t \) be a nonzero eigenvector of \( h_t(G_B) \) for each \( t \) and we assume that \( t \to \mathbf{v}_t \) is continuous. Then for \( 0 \leq t \leq \varepsilon \), there exists a continuous family of diffeomorphisms \( f_t : B \to B_t \) to a \( \mathbf{v}_t \)-parallel open set \( B_t \) in \( X \) so that \( f_0 \) is the identity map \( B \to B \) and \( f_t \) conjugates the \( h_t(G_B) \)-action to the \( h_t(G_B) \)-action; i.e., \( f_t h_t(g) f_t^{-1} = h_0(g) \) for each \( g \in G_B \) and \( t \in [0, \varepsilon] \).

Here \( \mathbf{v}_t, \mathbf{v}_0, \) and \( \mathbf{v}_0, t \) below are of course determined by \( s_B \).

Since \( \text{Hom}(G_B, \text{GL}(n+1, \mathbb{R})) \) is a semialgebraic set, we obtain that each point has a cone-neighborhood, i.e., a topological neighborhood parameterized by \( I \times S/\sim \) where \( S \) is a semialgebraic set and \( \sim \) is given by \((0, x) \sim (0, y), x, y \in S \). (See Bochnak-Coste-Roy [30].)

**Lemma 9.7** Let \( G_B \) be a finite subgroup of \( \text{GL}(n+1, \mathbb{R}) \) acting on a \( \mathbf{v}_0 \)-parallel open subset \( B \) of \( \mathbb{R}^{n+1} \) for an eigenvector \( \mathbf{v}_0 \) of the linear part of \( G_B \). Suppose that \( h \) is a point of an algebraic set \( V \subset \text{Hom}(G_B, \text{GL}(n+1, \mathbb{R})) \) for a finite group, and let \( C \) be a cone neighborhood of \( h \). Suppose that \( \mathbf{v}_t \) is an eigenvector of the linear part of \( h_t(G_B) \) for each \( h'_t \in C \) and \( h'_t \mapsto \mathbf{v}_t \) forms a continuous function \( C \to \mathbb{R}^{n+1} \). Then for each \( h'_t \in C \), there is a corresponding diffeomorphism

\[
\Phi_{h'_t} : B \to B_{h'_t}, B_{h'_t} = f_{h'_t}(B)
\]

so that \( \Phi_{h'_t} \) conjugates the \( h(G_B) \)-action on \( B \) to the \( h'_t(G_B) \)-action on \( B_{h'_t} \); i.e., \( \Phi_{h'_t} h(g) \Phi_{h'_t}^{-1} = h_t(g) \) for each \( g \in G_B \) where \( B_{h'_t} \) is a \( \mathbf{v}_{h'_t} \)-parallel open set. More-
over, the map \( h' \mapsto f_{h'} \) is continuous from \( C \) to the space \( C'(B, X) \), \( r \geq 2 \), of smooth functions from \( B \) to \( X \).

Continuing to use the notation of Lemma 9.7, we define a parameterization \( l : S \times [0, \varepsilon] \to C \) for a cone-neighborhood which is injective except at \( S \times \{0\} \) mapping to \( h \). (We fix \( l \) although \( C \) may become smaller and smaller.) For \( h' \in S \), we denote by \( l(h') : [0, \varepsilon] \to C \) be a ray in \( C \) so that \( l(h')(0) = h \) and \( l(h')'(\varepsilon) = h' \). Let the finite group \( G_B \) act on a \( v_h \)-parallel relatively compact submanifold \( F \) of a \( v_h \)-parallel open set \( B \) for an eigenvector \( v_h \) of \( h(G_B) \). Let \( v_{h,t} \) be a nonzero eigenvector of \( l(h')(t) \) for \( h' \in S \) and \( t \in [0, \varepsilon] \), and we suppose that \( S \times [0, \varepsilon] \to \mathbb{R}^{n+1} \) given by \( (h', t) \mapsto v_{h', t} \) is continuous.

Let \( F \) be a \( v_h \)-parallel relatively compact submanifold of a \( v_h \)-parallel open set \( B \) for an eigenvector \( v_h \) of \( h(G_B) \). A \( G_B \)-equivariant isometry \( H : F \times [0, \varepsilon] \to \mathbb{R}^{n+1} \) is a map so that \( H_t \) is an embedding for each \( t \in [0, \varepsilon'] \), with \( 0 < \varepsilon' < \varepsilon \), conjugating the \( G_B \)-action on \( F \) to the \( (l(h')(t)) \)-action on \( \mathbb{R}^{n+1} \). Here \( H_0 \) is an inclusion map \( F \to \mathbb{R}^{n+1} \) where the image \( H(F, t) \) is a \( v_{h,t} \)-parallel set for each \( t \). Lemma 9.7 above says that for each \( h' \in C \), there exists a \( G_B \)-equivariant isometry \( H : B \times [0, \varepsilon] \to \mathbb{R}^{n+1} \) so that the image \( H(B, t) \) is a \( v_{h', t} \)-parallel open set for each \( t \). We will denote by \( H_{h,t} : B \to \mathbb{R}^{n+1} \) the map obtained from \( H \) for \( h' \) and \( t = \varepsilon' \). Note also by generalizing Lemmas 3, 4, and 5 of [54], for each \( h' \in S \), there exists a \( G_B \)-equivariant isometry \( \tilde{H} : F \times [0, \varepsilon'] \to \mathbb{R}^{n+1} \).

**Lemma 9.8** Let \( F \) be a \( v_h \)-parallel relatively compact submanifold of a \( v_h \)-parallel open set \( B \) for an eigenvector \( v_h \) of \( h(G_B) \). Let \( H : F \times [0, \varepsilon] \times S \to \mathbb{R}^{n+1} \) be a map so that \( H(h') : F \times [0, \varepsilon'] \times S \to \mathbb{R}^{n+1} \) is a \( G_B \)-equivariant isometry of \( F \) for each \( h' \in S \) where \( 0 < \varepsilon' \leq \varepsilon \) for some \( \varepsilon > 0 \). Then for a neighborhood \( B' \) of \( F \) in \( B \), it follows that \( H \) can be extended to \( \tilde{H} : B' \times [0, \varepsilon''] \times S \to \mathbb{R}^{n+1} \) so that

\[
\tilde{H}(h') : B' \times [0, \varepsilon''] \to \mathbb{R}^{n+1}, 0 < \varepsilon'' \leq \varepsilon
\]

is a \( G_B \)-equivariant isometry of \( B' \) for each \( h' \in S \). The image \( \tilde{H}(h')(t)(B') \) is a \( v_{h', t} \)-parallel open set for each \( h', t \).

**Proof (Proof of Theorem 9.2)** We are aiming to prove the local homeomorphism property of the map

\[
hol : \overline{D}_{A, \varepsilon, s_{\mathcal{G}}}^S(\mathcal{O}) \to \text{Hom}^S_{\mathcal{E}}(\pi_1(\mathcal{O}), \text{GL}(n+1, \mathbb{R}))
\]

sending affine structures determined by the section \( s_{\mathcal{G}} \) to the conjugacy classes of holonomy homomorphisms is a local homeomorphism on an open subset of \( \mathcal{H}' \). The continuity of \( \text{hol} \) was proved at the beginning of this subsection.

Next, we define the local inverse map from a neighborhood in \( \mathcal{H}' \) of the image point. Let \( \mathcal{O}' \) be a compact suborbifold of \( \mathcal{O} \) so that \( \mathcal{O} - \mathcal{O}' \) is a union \( U \) of product end neighborhoods.

The first case is when \( \mathcal{O} \) has only \( \mathcal{R} \)-type ends.

We will show how to change the proof of Theorem 1 of [54]. Let \( h \) be a representation coming from an affine orbifold \( \mathcal{O} \) with parallel or totally geodesic boundary.
The task is to reassemble $\mathcal{O}$ with new holonomy homomorphisms as we vary $h$ as in [54] following approaches of Thurston. Suppose that $h'$ is in a neighborhood of $h$ in $\text{Hom}_S^\delta(\pi_1(\mathcal{O}), \text{GL}(n+1, \mathbb{R}))$.

- As in Lok [131], we consider locally finite collections $\mathcal{V}$ of open domains that cover $\mathcal{O}$. $\mathcal{O}'$ is covered by contractible precompact open sets and each component of $\mathcal{O} - \mathcal{O}'$ is covered by contractible parallel open sets in $U$. We find subcollection $\mathcal{V}'$ of compact or closed domains in the covering open sets which covers $\mathcal{O}'$ again. Here, each precompact element of $\mathcal{V}$ contains a compact domain in $\mathcal{V}'$ forming a cover of $\mathcal{O}'$. The contractible parallel open sets can meet only if they are in the same component of $U$.
- We will give orders to the open sets covering $\mathcal{O}$. Contractible parallel open sets will have orders higher than all precompact sets.
- We regard these as sets in $\mathbb{S}^{n+1}$ by charts.
- We consider the sets that are the intersection

$$U_{i_1} \cap \cdots \cap U_{i_k}, i_1 > \cdots > i_k,$$

where $U_{i_j} \in \mathcal{V}'$ for $j = 1, \ldots, k$.

of the largest cardinality of the compact or closed domains in $\mathcal{V}'$ and find the corresponding sets in $\mathbb{S}^{n+1}$ by charts. We map it by isotopies to the corresponding intersection of deformed collections of domains in $\mathbb{S}^{n+1}$ corresponding to the $h'(\pi_1(\mathcal{O}))$-action by Lemmas 3 and 4 in [54] and Lemmas 9.6 and 9.7. Here, we will follow the ordering as above when we deform as in Lok [131]. That is, we use the isotopy of $U_{i_1}$ restricted to $U_{i_1} \cap \cdots \cap U_{i_k}$ when $U_{i_1}$ has the largest order.

- We extend the isotopies to the sets of intersections of smaller number of sets in $\mathcal{V}'$ by Lemma 5 of [54] and Lemma 9.8. By induction, we extend it to all the images of compact and closed domains in $\mathcal{V}'$.
- We patch these open sets to build an orbifold $\mathcal{O}_{h'}$ with holonomy $h'$ referring back to $\mathcal{O}$ by isotopies.
- $\mathcal{O}_{h'}$ is diffeomorphic to $\mathcal{O}$ by the map constructed by the isotopies.

We go to general cases when $\mathcal{O}$ has both $R$-type and $T$-type ends. We will assume that the two open sets in the covering occurring below are disjoint if they meet different components of $U$.

- For an $R$-type end, this is accomplished as in [54] for precompact open covering sets and for end-parallel open covering sets we use above Lemmas 9.6, 9.7, and 9.8 since we are working with finitely many open sets. Instead of precompact nested compact domains in the paper, for parallel sets, we need to use sets of form $\bigcup_{t \geq 0} T_t(S)$ for a translation $T_t$ by $t\mathbf{v}$ for a nonzero vector $\mathbf{v}$ and a smooth hypersurface $S$ transverse to $\mathbf{v}$. However, during deformations outside a compact set the isotopies will be affine maps restricted on each leaf of the parallel foliation.
- For a $T$-type end that has the totally geodesic ideal boundary, we first complete it with an open subset of a totally geodesic hyperspace. There exists an open subset where the corresponding p-end has a totally geodesic hyperspace invariant under the holonomy group of the pseudo-end and is not horospherical. For a sufficiently small open set in $\mathcal{U}$, we can change each open neighborhood in the
manner described in [54]. The totally geodesic ideal boundary does not present any difficulty here.

• For a $\mathcal{T}$-type end $E$ that is a suspended horospherical end, we take an affinely suspended horospherical end-neighborhood in $U$ projectively isomorphic to $C_B/\Gamma'$ where $\Gamma'$ is the affine suspension group extended by a central infinite cyclic group generated by a scalar dilatation. Lemma 9.5 shows us how to obtain a totally geodesic end under small deformations of holonomy homomorphisms.

Again the methods are similar to the above methods using the ordering of the open sets where the end neighborhoods have higher orders than those of precompact ones. This shows that we can choose a continuous family of developing maps in the $C^r$-topology.

We can show that each constructed orbifold has a developing map extended to the universal cover of end-completions of the orbifold and with the image in $\mathbb{S}^{n+1}$.

We just need to consider the radial ends by Lemma 9.5 since the discussion for the totally geodesic ends is straightforward. Let $\tilde{U}$ be an R-p-end neighborhood of $\tilde{E}$. Here, since the parameterizations are linear in the parallel directions, the affine developing map $\text{dev} : \tilde{O} \to \mathbb{A}^{n+1} \subset \mathbb{S}^{n+1}$ sends each parallel line $l_x$ passing $x \in \text{bd} \tilde{U}$ to $l_x(x) \in \mathbb{A}^{n+1}$ and the forward endpoint of $l_x$ equals $v_E$. Let $v$ be the parallel direction of the radial end $\tilde{E}$. We take a p-end neighborhood $U'$ so that $\text{Cl} \tilde{O}(U') \subset U'$, and we let $\tilde{U}'$ be the inverse image of $U'$ containing $\tilde{U}$. We may assume that $\tilde{U}'$ is obtained from $\tilde{U}$ by flowing by time $-1$ by the parallel vector field of $\tilde{E}$. We parameterize each $l_x$ so that $l_x \cap \text{bd} \tilde{U}' \to 0, x \to 1, v_E \to \infty$.

We reparameterize so that $l_x \cap \text{bd} \tilde{U}' \to \infty, x \to 1, v_E \to 0$.

by post-composing by $t \mapsto 1/t$. Then $\text{dev} : U \to \mathbb{S}^{n+1}$ equals $\text{dev}^\mathcal{Y} : U \to \mathbb{S}^{n+1}$ radial end-projectivization with respect to $U'$ and $U$ constructed in the proof of Lemma 9.3. Hence, $\text{dev}|\tilde{U}$ does extend smoothly on the end-completion.

We now prove the local injectivity of hol. Given two structures $\mu_0$ and $\mu_1$ in a neighborhood of the deformation space, we show that if their holonomy homomorphisms are the same, say $h : \pi_1(\mathcal{O}) \to \text{GL}(n+1, \mathbb{R})$, then we can isotopy one in the neighborhood to the other using vector fields as in [54].

Because of the section $s_{\mathcal{U}}$ defined on $\mathcal{U}$, given a holonomy $h$, we have a direction of the radial end that is unique for the holonomy homomorphism.

First assume that $\mathcal{O}$ has only $\mathcal{R}$-type ends. Recall the compact suborbifold $\mathcal{O}'$ so that $\mathcal{O} - \mathcal{O}'$ is homeomorphic to $E_i \times (0, 1)$ for each end orbifold $E_i$.

We can choose a Riemannian metric on $\mathcal{O}$ so that an end neighborhood has a product metric of form $E_i \times (0, 1]$. Let $\text{dev}_j$ be the developing map of $\mu_j$ for $j = 0, 1$. Then the $C^r$-norm distance of extensions $\text{dev}_0$ and $\text{dev}_1$ to $\mathcal{O}'$ is bounded on each compact set $K \subset \mathcal{O}$. Since we chose $\mu_1$ and $\mu_2$ sufficiently close, $\text{dev}_0$ and $\text{dev}_1$ can be assumed to be sufficiently close in the $C^r$-topology over $K$. The images of $K$
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under each of these maps can be assumed to lie on a neighborhood of the image of a p-end vertex, say v. Since the parallel domains have compact closures, and they map to parallel sets and developing maps are smooth, we can use the argument in the last part of [54] to show that \( \text{dev}_1 \) lifts to an immersion \( \hat{\theta} \to \hat{\theta} \) equivariant with respect to the deck transformation group. Then we use the metrics to equivariantly isotopy it to \( I \) as in the last section of [54]. Hence, \( \mu_0 \) and \( \mu_1 \) represent the same point of \( \text{Def}_{\hat{\theta}}(\hat{\theta}) \). This proves the local injectivity.

Suppose now that \( \mathcal{O} \) has some \( T \)-type ends. Suppose that \( \mu_0 \) and \( \mu_1 \) have a totally geodesic ideal boundary component corresponding to an end of \( \mathcal{O} \). We attach the totally geodesic ideal boundary component for each end, and then we can argue as in [54] proving the local injectivity.

Suppose that \( \mu_0 \) and \( \mu_1 \) have horospherical end neighborhoods corresponding to an end of \( \mathcal{O} \). Then these are radial ends and the same argument as the above one for \( R \)-type ends will apply to show the local injectivity.

Finally, we cannot have the situation that \( \mu_0 \) has the totally geodesic ideal boundary component corresponding to an end while \( \mu_1 \) has a horoball end neighborhood for the same end. This follows since the end holonomy group acts on a properly convex domain in a totally geodesic hyperspace and as such the end holonomy group elements have some norms of eigenvalues > 1. (See Proposition 1.1 of [18] for example.)

To show that the local inverse is a continuous map for the \( C^r \)-topology of \( \hat{\theta} \), we only need to consider compact suborbifolds in \( \mathcal{O} \), and we use the fact that the conjugating maps of above Lemmas 9.6, 9.7, and 9.8 and the second part of Lemma 9.5 depend continuously on \( \mathcal{U} \).

\( \square \)

9.2.6 The proof of Theorem 9.3.

Suppose now that \( \mathcal{O} \) is a real projective orbifold of dimension \( n \). We assume that \( \mathcal{O} \) has ends that are assigned to be \( T \)-type or \( R \)-type ones. Let \( \mathcal{O}' = \mathcal{O} \times S^1 \) be the affine suspension. \( \pi_1(\mathcal{O}') \) is isomorphic to \( \pi_1(\mathcal{O}) \times \mathbb{Z} \). Each end has a distinguished infinite cyclic group in the center given by the factor \( \mathbb{Z} \). \( \mathcal{O}' \) has a radial end with the end direction determined by the lens-shaped radial ends of \( \mathcal{O} \) and lens-shaped totally geodesic ends determined by that of \( \mathcal{O} \). Let \( \mathcal{U} \) be the conjugation invariant subspace of

\[
\text{Hom}^{\pi_1}(\pi_1(\mathcal{O}'), \text{GL}(n+1, \mathbb{R})),
\]

and we are given the fixing section

\[
s_{\mathcal{U}}: \mathcal{U} \to (\mathbb{R}^{n+1} - \{O\})^{c_1} \times (\text{AS}(\mathbb{R}^{n+1}))^{c_2}.
\]

For any element \( \mu \) of \( \text{Def}_{\hat{\theta}}(\mathcal{O}') \), \( \mathcal{O}' \) with \( \mu \) a developing map pulls back a radiant vector field \( \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i} \) on \( \mathbb{R}^{n+1} \). This gives us a radial flow on \( \mathcal{O}' \) with \( \mu \). Each point \( p \) of \( \mathcal{O}' \) has a neighborhood foliated by radial lines. Furthermore, the
radial lines are always closed since a dilation from the central elements acts on each radial line giving us a closed orbit always. By Lemma 2.6, $\mathcal{O}'$ with $\mu$ is an affine suspension from $\mathcal{O}$. Since $\mathcal{O}$ can be embedded transversely to the radial flow, it follows that $\mathcal{O}'$ with $\mu$ gives us an $(S^\mu, SL_{\pm}(n+1, \mathbb{R}))$-structure on $\mathcal{O}$.

We define $\text{rep}_{S^\mu}(\pi_1(\mathcal{O}), GL(n+1, \mathbb{R}))$ and $\text{rep}_{S^\mu}(\pi_1(\mathcal{O}), SL_{\pm}(n+1, \mathbb{R}))$ as the respective subsets where each end holonomy group of $\mathcal{O}$ has a common eigenvector. By sending scalar dilatations to the expansion factors, we obtain that

$$\text{rep}_{S^\mu}^s(\pi_1(\mathcal{O}'), GL(n+1, \mathbb{R}))$$

is identical with

$$\text{rep}_{S^\mu}^s(\pi_1(\mathcal{O}), GL(n+1, \mathbb{R})) \times \mathbb{R}_+$$

which is the subspace of

$$\text{rep}_{S^\mu}^s(\pi_1(\mathcal{O}), GL(n+1, \mathbb{R})) \times \mathbb{R}_+$$

where the holonomy group of each p-end has a common eigendirection or a common eigen-1-form.

$$\text{rep}_{S^\mu}^s(\pi_1(\mathcal{O}), GL(n+1, \mathbb{R})) \times \mathbb{R}_+$$

can be identified with

$$\text{rep}_{S^\mu}^s(\pi_1(\mathcal{O}), SL_{\pm}(n+1, \mathbb{R})) \times H^1(\pi_1(\mathcal{O}), \mathbb{R}) \times \mathbb{R}$$

by using the isomorphism

$$GL(n+1, \mathbb{R}) \rightarrow SL_{\pm}(n+1, \mathbb{R}) \times \mathbb{R}$$

which is given by sending a matrix $L$ to $(L/|\det(L)| \cdot \log(|\det(L)|))$. Let

$$q_S : \text{rep}_{S^\mu}^s(\pi_1(\mathcal{O}'), GL(n+1, \mathbb{R})) \rightarrow \text{rep}_{S^\mu}^s(\pi_1(\mathcal{O}), SL_{\pm}(n+1, \mathbb{R}))$$

denote the obvious projection.

Let $\mathcal{U}$ denote a conjugation invariant open subset of a semi-algebraic subset of

$$\text{Hom}_{S^\mu}^s(\pi_1(\mathcal{O}'), GL(n+1, \mathbb{R}))$$

with a section

$$s_{\mathcal{U}} : \mathcal{U} \rightarrow (\mathbb{R}^{n+1} - \{O\})^{r_1} \times (AS_0(\mathbb{R}^{n+1}))^{r_2}$$

where $AS_0(\mathbb{R}^{n+1})$ is the subspace of $AS(\mathbb{R}^{n+1})$ passing the origin $O$. Theorem 9.2 shows that after taking the Hausdorff quotients,

$$\text{hol} : \text{Def}_{\mathcal{X}, S^\mu}(\mathcal{O}') \rightarrow \text{rep}_{S^\mu}^s(\pi_1(\mathcal{O}'), GL(n+1, \mathbb{R}))$$

(9.3)

is a local homeomorphism to its image since the former space is simply the inverse image of the second space.
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Recall Definition 9.4. Let \( \mathcal{U}' \) denote a conjugation invariant open subset of a semi-algebraic subset of

\[
\mathrm{Hom}_{\mathcal{E}}^s(\pi_1(\mathcal{O}), \mathrm{SL}_\pm(n + 1, \mathbb{R}))
\]

with a section

\[
s'_{\mathcal{U}'} : \mathcal{U}' \to (\mathbb{S}^n)^{e_1} \times (\mathbb{S}^{n*})^{e_2}.
\]

This section is again called a fixing section. We define \( \text{Def}_{\mathcal{E}, E, s'_{\mathcal{U}'}}(\mathcal{O}) \) as the subspace of structures whose holonomy characters are in \( \mathcal{U}' \) and the ends are compatible with \( s'_{\mathcal{U}'} \), i.e., a p-end neighborhood of each R-p-end is foliated by concurrent geodesics ending at the point given by \( s'_{\mathcal{U}'} \) and the ideal geodesic boundary component of each T-p-end is in the totally geodesic hyperspace oriented outwardly as determined by \( s'_{\mathcal{U}'} \).

**Proposition 9.3** Let \( \mathcal{O} \) be a noncompact strongly tame \( n \)-orbifold where the types of ends are assigned. Let \( \mathcal{U}' \) be a conjugation-invariant open subset of

\[
\mathrm{Hom}_{\mathcal{E}}^s(\pi_1(\mathcal{O}), \mathrm{SL}_\pm(n + 1, \mathbb{R}))
\]

with the section

\[
s'_{\mathcal{U}'} : \mathcal{U}' \to (\mathbb{S}^n)^{e_1} \times (\mathbb{S}^{n*})^{e_2}.
\]

The map

\[
\text{hol} : \text{Def}_{\mathcal{E}, E, s'_{\mathcal{U}'}}(\mathcal{O}) \to \text{rep}_{\mathcal{E}}(\pi_1(\mathcal{O}), \mathrm{SL}_\pm(n + 1, \mathbb{R}))
\]

sending \( (\mathbb{S}^n, \mathrm{SL}_\pm(n + 1, \mathbb{R})) \)-structures determined by the fixing section \( s'_{\mathcal{U}'} \) to the conjugacy classes of holonomy homomorphisms is a local homeomorphism to an open subset of \( \mathcal{U}' \).

**Proof** Let \( \mathcal{O}' \) be the product \( \mathcal{O} \times \mathbb{S}^1 \) as above, and this gives each end a distinguished central cyclic group.

Let \( \mathcal{U} \) be the inverse image under \( q \) of \( \mathcal{U}' \) in

\[
\mathrm{Hom}_{\mathcal{E}}^s(\pi_1(\mathcal{O}'), \mathrm{GL}(n + 1, \mathbb{R})).
\]

Let

\[
q : (\mathbb{R}^{n+1} - \{O\})^{e_1} \times (\mathrm{AS}_0(\mathbb{R}^{n+1}))^{e_2} \to (\mathbb{S}^n)^{e_1} \times (\mathbb{S}^{n*})^{e_2}
\]

be the obvious projections. Let the section

\[
s'_{\mathcal{U}} : \mathcal{U} \to (\mathbb{R}^{n+1} - \{O\})^{e_1} \times (\mathrm{AS}_0(\mathbb{R}^{n+1}))^{e_2}
\]

be the one lifting \( s'_{\mathcal{U}'} \). (Here, each hyperspace in \( \mathbb{S}^n \) lifts to a hyperspace in \( \mathbb{R}^{n+1} \) through the fixed points of the holonomy groups of the center.)

By Lemma 2.6, an element of \( \text{Def}_{\mathcal{E}, E, s_{\mathcal{U}}}(\mathcal{O}') \) gives us an element of \( \text{Def}_{\mathcal{E}, E, s'_{\mathcal{U}'}}(\mathcal{O}) \):

We can cover \( \mathcal{O}' \) by radial cones with vertex at the origin and project to \( \mathbb{S}^n \). Each gluing of open radial cones becomes an element of \( \mathrm{SL}_\pm(n + 1, \mathbb{R}) \) acting on \( \mathbb{S}^n \) with
positive scalar factors forgotten. The parallel end structures and totally geodesic ideal boundary components for ends of $\mathcal{O}'$ go to the R-end structures and the totally geodesic ideal boundary components of $\mathcal{O}$. The isotopies in $\mathcal{O}'$ will give rise to isotopies in $\mathcal{O}$ suspending the vector fields on cross-sections preserving the radial vector fields and the totally geodesic ideal boundary components. Also, Proposition 9.1 gives us compatible end-compactification for the induced R-end and T-end structures.

Therefore, the following map $\mathcal{P}$ is defined:

$$
\text{Def}^\mathcal{P}_{\mathcal{A}, \mathcal{E}, \omega_{\mathcal{A}}}(\mathcal{O}') \xrightarrow{\text{hol}} \text{Def}^\mathcal{P}_{\mathcal{A}, \mathcal{E}, \omega_{\mathcal{A}}}(\mathcal{O}) \times H^1(\mathcal{O}, \mathbb{R}) \times (\mathbb{R}^g - \{1\})
$$

$$
\text{hol} \downarrow
$$

$$
\text{rep}_\mathcal{P}(\pi_1(\mathcal{O}'), \text{GL}(n+1, \mathbb{R})) \quad \rightarrow \quad \text{rep}_\mathcal{P}(\pi_1(\mathcal{O}), \text{SL}(n+1, \mathbb{R})) \times H^1(\mathcal{O}, \mathbb{R}) \times (\mathbb{R}^g - \{1\}),
$$

A section to $\mathcal{P}$ is defined by taking an affine suspension by the data in $H^1(\mathcal{O}, \mathbb{R}) \times (\mathbb{R}^g - \{1\})$ and the $(\mathbb{S}^n, \text{SL}(n+1, \mathbb{R}))$-structures on $\mathcal{O}$ using the methods of Section 2.2.1. From this, we deduce that the horizontal maps are local homeomorphisms in the commutative diagram. Since the left downarrow is a local homeomorphism by Theorem 9.2, the result is proved.

The homomorphism $q : \text{SL}(n+1, \mathbb{R}) \rightarrow \text{PGL}(n+1, \mathbb{R})$ induces a continuous map

$$
\hat{q} : \text{rep}_\mathcal{P}(\pi_1(\mathcal{O}), \text{SL}(n+1, \mathbb{R})) \rightarrow \text{rep}_\mathcal{P}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).
$$

Let $\mathcal{U}$ be a conjugation invariant open subset of a semi-algebraic subset of $\text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$. $s_{\mathcal{U}}$ is an arbitrary fixed section defined on $\mathcal{U}$. Let $\mathcal{U}''$ denote the inverse image of $\mathcal{U}$ under $\hat{q}$. We define $s_{\mathcal{U}''} : \mathcal{U}'' \rightarrow (\mathbb{S}^n)^{e_1 + e_2}$ to be a continuous lift of $s_{\mathcal{U}}$. The section is determined up to the action of $\{1, \mathcal{A}\}$ on each of $(e_1 + e_2)$-factors. This gives us a section $\hat{s} : \text{Def}^\mathcal{P}_{\mathcal{A}, \omega_{\mathcal{A}}}(\mathcal{O}) \rightarrow \text{Def}^\mathcal{P}_{\mathcal{A}, \omega_{\mathcal{A}}}(\mathcal{O})$ up to a choice of $s_{\mathcal{U}''}$ by Theorem 2.2. The choice here is determined by the lifting of the development pair $(\text{dev}, h)$. (For the lifting ideas, see p. 143 of Thurston [159].)

The map $\hat{q} : \text{Def}^\mathcal{P}_{\mathcal{A}, \omega_{\mathcal{A}}}(\mathcal{O}') \rightarrow \text{Def}^\mathcal{P}_{\mathcal{A}, \omega_{\mathcal{A}}}(\mathcal{O})$ is induced by the action

$$
(\text{dev}', h') \rightarrow (q \circ \text{dev}', \hat{q} \circ h').
$$

It is easy to see that the section $\hat{s}$ to $\hat{q}$ is well-defined since the lifting $(\text{dev}, h)$ give us development pairs that are equivalent up to $-I \in \text{SL}(n+1, \mathbb{R})$. The map $\hat{s}$ is continuous since for a fixed compact subset of $\mathcal{O}$ the $C'$-closeness of the developing map to $\mathbb{R}P^n$ means the $C'$-closeness of the lifts for $r \geq 1$.

Thus, we showed that

**Theorem 9.4** Assume as in the above paragraphs. We obtain a homeomorphism

$$
\hat{q} : \text{Def}^\mathcal{P}_{\mathcal{A}, \omega_{\mathcal{A}}}(\mathcal{O}') \rightarrow \text{Def}^\mathcal{P}_{\mathcal{A}, \omega_{\mathcal{A}}}(\mathcal{O}).
$$

**Corollary 9.3** $\hat{q} : \text{Def}^\mathcal{P}_{\mathcal{A}, \omega_{\mathcal{A}}}(\mathcal{O}') \rightarrow \text{Def}^\mathcal{P}_{\mathcal{A}, \omega_{\mathcal{A}}}(\mathcal{O})$ is a homeomorphism.
9.3 Relationship to the deformation spaces in our earlier papers

**Proof** In the unique eigenvector or eigen-1-form cases, the existence and the continuity of the sections are clear. □

**Proof (Proof of Theorem 9.3)** We have a commutative diagram:

\[
\begin{array}{ccc}
\text{Def}_{\partial_p, s, \mathcal{W}}(\mathcal{E}) & \xrightarrow{\hat{q}} & \text{Def}_{\partial_p, s, \mathcal{W}}(\mathcal{E}) \\
\downarrow \text{hol} & & \downarrow \text{hol} \\
\text{rep}^s_p(\pi_1(\mathcal{E}), \text{SL}_+(n+1, \mathbb{R})) & \xrightarrow{\hat{q}} & \text{rep}^s_p(\pi_1(\mathcal{E}), \text{PGL}(n+1, \mathbb{R})).
\end{array}
\] (9.5)

First, we remark first that \(\hat{q}\) maps onto the union of components with the associated Stiefel-Whitney number 0 by standard results in bundle theory. (See Section 2 of [92].) Since \(\text{SL}_+(n+1, \mathbb{R}) \rightarrow \text{PGL}(n+1, \mathbb{R})\) is a covering map, so is \(\text{Hom}^s_p(\pi_1(\mathcal{E}), \text{SL}_+(n+1, \mathbb{R})) \rightarrow \text{Hom}^s_p(\pi_1(\mathcal{E}), \text{PGL}(n+1, \mathbb{R}))\) to the union of its image components (i.e., corresponding to ones with the corresponding Stiefel-Whitney classes equal to zero.) Each fiber is in a one to one and onto correspondence with \(\text{Hom}^s_p(\pi_1(\mathcal{E}), \{\pm 1\})\). The induced map \(\hat{q}\) is a local homeomorphism since the conjugation by \(\text{SL}_+(n+1, \mathbb{R})\) on the first space is equivalent to one by \(\text{PGL}(n+1, \mathbb{R})\) since \(\{\pm 1\}\) acts trivially.

Since the left map denoted by hol is locally onto and \(\hat{q}\) is locally onto, so is the right hol by Theorem 9.4.

Given a neighborhood \(V'\) in \(\text{rep}^s_p(\pi_1(\mathcal{E}), \text{PGL}(n+1, \mathbb{R}))\) that is in the image of the left hol, we can find a local section to \(\hat{q}\) as \(\hat{q}\) is a local homeomorphism. Since the left hol is a local homeomorphism, and \(\hat{q}\) is a local homeomorphism, there is a local section to the right hol by Theorem 9.4. □

9.3 Relationship to the deformation spaces in our earlier papers

Recall \(\mathcal{D}(\hat{P})\) for a Coxeter orbifold \(\hat{P}\) that is not necessarily compact in Definition 3.2.

**Proposition 9.4** Let \(P\) be a Coxeter orbifold based on convex polytopes. Then there is a homeomorphism

\[\mathcal{D}(P) = \text{CDef}_p(P).\]

**Proof** The ideal point of \(P\) gives us a radial foliation on each end neighborhood of \(P\). This gives us an end structure on each end of \(P\). Let \(\hat{P}\) denote the universal cover of \(P\). We can form \(\hat{P}\) by using the end-compactification by Definition 9.1. We denote by \(\hat{P}\) the universal cover of \(\hat{P}\) containing \(\hat{P}\) as a dense open set.

We denote by \(D_{\hat{p}}\) the set of maps of form \(\text{dev}\) where \(\text{dev}\) is an extension of a developing map of some Coxeter real projective structure on \(P\) to the universal
cover of the end-compactification of $P$. Recall Definition 9.3. $\text{CDef}_g(P)$ can be considered the quotient space of $D_\hat{P}$ under the extensions of the isotopy-lifts of $\hat{P}$.

Let $D_{\hat{P}}$ denote the space of developing maps $\text{dev}: \hat{P} \to \mathbb{R}P^n$. This has a compact open $C^\epsilon$-topology on $\hat{P}$. We quotient the space by the group of isotopy-lifts and $\text{PGL}(n+1, \mathbb{R})$. Recall that $\mathcal{D}(P)$ is the quotient space of $D_\hat{P}$ by the group of isotopy-lifts and $\text{PGL}(n+1, \mathbb{R})$.

Hence, by restricting the structure on $P$ only, there is a map

$$R'_r: \text{CDef}_g(P) \to \mathcal{D}(P).$$

The map is one-to-one and onto by Proposition 9.2.

Let $J'(f)$ denote the tuples of all jets of $f: \hat{P} \to \mathbb{R}P^n$ of order $\leq r$. The topology on $D_{\hat{P}}$ is given by bases of form

$$B_{K,\epsilon}(\text{dev}) := \{ f \in D_{\hat{P}} | d(J'(\text{dev})(x), J'(f)(x)) < \epsilon, x \in K, f \in C^\epsilon(\hat{P}, \mathbb{R}P^n) \}$$

where $K \subset \hat{P}$ is a compact set, $\epsilon > 0$, and $\text{dev} \in D_{\hat{P}}$. The topology on $D_{\hat{P}}$ is given by bases of form

$$B'_{K,\epsilon}(\text{dev}) := \{ f \in D_{\hat{P}} | d(J'(\text{dev})(x), J'(f)(x)) < \epsilon, x \in K, f \in C^\epsilon(\hat{P}, \mathbb{R}P^n) \}$$

where $K \subset \hat{P}$ is a compact set, $\epsilon > 0$, and $\text{dev} \in D_{\hat{P}}$.

Consider the restriction map

$$R_r: C^r(\hat{P}, \mathbb{R}P^n) \to C^r(\hat{P}, \mathbb{R}P^n)$$

inducing $R'_r$. Since compact subsets of $\hat{P}$ are compact subsets of $\hat{P}$, the inverse image under $R_r$ of a basis element of $C^r(\hat{P}, \mathbb{R}P^n)$ is a basis element in $C^r(\hat{P}, \mathbb{R}P^n)$. Hence, $R'_r: \text{CDef}_g(P) \to \mathcal{D}(P)$ is continuous.

Now we show that $R_r$ is open. Let $\mathcal{G}_P$ denote the group of isotopies of $P$ preserving the radial end structures of $P$, and let $\mathcal{G}_\hat{P}$ denote the group of extended isotopies of $\hat{P}$. The isotopy-lifts $t: \hat{P} \to \hat{P}$ form a group which we denote by $\mathcal{G}^\text{e}$. Also, denote by $\mathcal{G}_{\hat{P}}^\text{e}$ the group of the extensions of isotopy-lifts $t: \hat{P} \to \hat{P}$ of isotopies in $P$ with radial end structures in $\mathcal{G}_P$.

Fix the union $U$ of mutually disjoint $R$-end neighborhoods and radial foliations on each component. We denote by $\mathcal{G}_{P,U}$ the group of isotopies of form $t$ of $P$ fixing each point of $U$ with an isotopy

$$F: P \times [0, 1] \to P \text{ with } F_0 = I_P \text{ and } F_1 = t.$$ 

Also, we choose a union $U'$ of such neighborhoods so that $\text{Cl}_P(U) \subset U'$. Let $\hat{U}$ denote the inverse image of $U$. Let $\mathcal{G}_{\hat{U}}^\text{e}$ denote the isotopy-lifts $t$ of $P$ to $\hat{P}$ fixing each point of $\hat{U}$ which are lifts of elements of $\mathcal{G}_{P,U}$. Each element $\hat{t}$ has a homotopy $F$ so that

$$F(x, 1) = \hat{t}(x), F(x, 0) = I_{\hat{P}}(x) \text{ and } F(x, t) = x \text{ for every } x \in \hat{U}.$$
Let $\mathcal{G}^n_{P,U}$ be the extensions of isotopy-lifts of $P$ to $\hat{P}$ fixing each point of $\text{Cl}_P(\hat{U})$. Clearly there is a natural isomorphism by extension $\mathcal{G}^n_{P,U} \rightarrow \mathcal{G}^n_{\hat{P},\hat{U}}$ with the inverse map given by the restriction to $P$.

Let $D_{P,U,U'}$ denote the space of functions of form $\text{dev}$ in a development pair $(\text{dev},h)$ so that $\text{dev}|\hat{U}$ equals $\text{dev}^N|\hat{U}$ constructed for $U'$ and $U'$ by the radial end-projectivization in Lemma 9.3. By Lemma 9.3, we obtain a natural map $D_{P,U,U'}/\mathcal{G}^n_{P,U} \rightarrow D_P/\mathcal{G}$ that is a one-to-one onto map.

Also, denote by $D_{\hat{P},U,U'}$ denote the space of functions of form $\overline{\text{dev}}$ which are extended developing maps and $\overline{\text{dev}}|\hat{U}$ equals $\text{dev}^N|\hat{U}$ constructed for $U$ and $U'$ by radial end-projectivization in Lemma 9.3.

Recall from the elementary analysis that the $C^r$-topology on a compact manifold is a metric topology. Similarly, $D_{\hat{P},U,U'}$ with the $C^r$-topology is a metric space given by the metric $d_{D_{\hat{P},U,U'}}$ defined by the extended developing maps in the similar way.

We claim that the canonical map $F : D_{P,U,U'}/\mathcal{G}^n_{U} \rightarrow D_P/\mathcal{G}^n_P$ is a homeomorphism: Let us take a compact neighborhood $K_F$ of a fundamental domain $F$ of $\hat{\sigma} - \hat{\nu}$. Then we define a metric on $D_{P,U,U'}$ by $d_{D_{P,U,U'}}$ between two developing maps $f_1, f_2$ is defined as $\sup_{x \in K_F} d_{f_1,f_2}(x)$. Since the developing map on $\hat{U}$ is determined by the developing map restricted on $K_F$ as we can see by a radial end-projectivization with respect to $U$ and $U'$, we obtain a metric $d_{D_{P,U,U'}}$ on $D_{P,U,U'}$ giving us the $C^r$-topology on $D_{P,U,U'}$. The map $F$ is continuous since it is induced by the inclusion map $D_{\hat{P},U,U'} \rightarrow D_P$.

There is an inverse map $G : D_P/\mathcal{G}^n_P \rightarrow D_{P,U,U'}/\mathcal{G}^n_{U}$ given by taking a developing map $f$ and modifying it by radial end-projectivization. There are choices involved, but they are well-defined up to isotopy-lifts.

Any isotopy $t$ induces a homeomorphism $t^*$ in $D_P$. To show the continuity of $G$, we take a ball $B_{d_{D_{P,U,U'}}}(f,\varepsilon)$ for $f \in D_{P,U,U'}$ show that there is a neighborhood of $f \circ t$ in $D_P$ in the $C^r$-topology going into it for $t \in \mathcal{G}^n$. Since we can take $t^*(B)$ as a neighborhood for $f \circ t$, it is sufficient to find a neighborhood $B$ of $f$ in $D_P$ in the $C^r$-topology. Let $K_F$ denote a compact set in $\hat{P}$ disjoint from $\hat{\sigma} - \hat{\nu}$. We take a sufficiently small $\delta$, $\delta > 0$, the neighborhood $B_{K_F,\delta}(f)$ so that each of its element $g$ is still in $B_{d_{D_{P,U,U'}}}(f,\varepsilon)$ after applying the isotopy of radial end-projectivization with respect to $U$ and $U'$. This is because we only need to worry about the compact set $K_F \cap \text{Cl}_P(\hat{U}')$ while $g$ after radial end-projectivization with respect to $U'$ and $U$ is determined in this compact set. If $g$ is sufficiently close to $f$ on $K_F$ in the $C^r$-topology already in the form of Lemma 9.3, then

- the leaves of the radial foliation of $g$ intersected with $K_F$ is $C^r$-close to ones for $f$ and
- the required isotopy $t_g$ for radial end-projectivization of $g$ is also sufficiently $C^r$-close to $I$ on $K_F$ in the uniform $J_*$-topology defined on $U$. 

Hence, by taking sufficiently small $\delta$, $g \circ t_g$ is in $B_{d_{D_{P,U,U'}}}(f,\varepsilon)$. Since we are estimating everything in a compact set $K_F$, and finding $t_g$ depending on $g|K_F$, these methods are possible. (By the uniform topology, we mean the topology using the norm of differences of two functions on $U$ but not just one a compact subset of $U$.)
Also, $D_{P,U,U'}/\mathcal{G}_{P,U}^n \to D_P/\mathcal{G}_P^n$ is a homeomorphism similarly.

It is a triviality that the restriction map $D_{P,U,U'} \to D_{P,U,U'}$ is a homeomorphism. Hence, the induced map $D_{P,U,U'}/\mathcal{G}_{P,U}^n \to D_{P,U,U'}/\mathcal{G}_{P,U}^n$ is a homeomorphism. Since there is a commutative diagram

$$
\begin{array}{ccc}
D_{P,U,U'}/\mathcal{G}_{P,U}^n & \to & D_{P,U,U'}/\mathcal{G}_{P,U}^n \\
\downarrow & & \downarrow \\
D_P/\mathcal{G}_P^n & \to & D_P/\mathcal{G}_P^n.
\end{array}
$$

(9.6)

Since the downarrows maps are homeomorphisms by above, and the upper row map is the identity, the bottom row tells us the homeomorphism we needed. \hfill \Box
Chapter 10
Relative hyperbolicity and strict convexity

We will show the equivalence between the relative hyperbolicity of the fundamental group of the properly convex real projective orbifolds with the lens-shaped radial ends or totally geodesic ends or horospherical ends with the strict convexity of the orbifolds relative to the ends. In Section 10.1, we will show how to add some lenses to the $T$-ends so that the ends become boundary components and how to remove some open sets to make $R$-ends into boundary components. Some constructions preserve the strict convexity and so on. In Section 10.2, we describe the action of the end fundamental group on the boundary of the universal cover of a properly convex orbifold. In Section 10.3, we prove Theorem 10.3 that the strict convexity implies the relative hyperbolicity of the fundamental group using Yaman’s work. We then present the converse of this Theorem 10.5. For this, we use the work of Drutu and Saphir on tree graded spaces and asymptotic cones.

10.1 Some constructions associated with ends

We will discuss some constructions to begin. It will be sufficient to prove for the case $\partial \subset S^n$ in this chapter by Proposition 2.13. So we will not give any $\mathbb{R}^{p^r}$ version here.

The purpose of this chapter is to prove Corollary 1.2 the equivalence of the strict convexity of $\partial$ and the relative hyperbolicity of $\pi_1(\partial)$ with respect to the end fundamental groups. Cooper-Long-Tillman [73] and Crampon-Marquis [74] proved the same result when we only allow horospherical ends. Benoist told us at an IMS meeting at the National University of Singapore in 2016 that he has proof for this theorem for $n = 3$ using trees as he has done in closed 3-dimensional cases in [24] using the Morgan-Shalen’s work on trees [145]. For convex cocompact actions, there are some later work by Islam and Zimmer [110] and [109].

Recall that properly convex strongly tame real projective orbifolds with generalized lens-shaped or horospherical ends satisfying (NA) and (IE) have strongly irreducible holonomy groups by Theorem 1.2. In this chapter, we will fix the union $U$
of all concave end neighborhoods for radial ends and lens end neighborhoods for T-ends and horospherical neighborhoods of ends mutually disjoint from one another. Let $\tilde{U}$ denote the inverse image in $\tilde{O}$.

### 10.1.1 Modifying the T-ends.

For T-ends, by the lens condition, we only consider the ones that have CA-lens neighborhoods in some ambient orbifolds. First, we discuss the extension to bounded orbifolds.

**Theorem 10.1** Suppose that $O$ is a strongly tame properly convex real projective orbifold with generalized lens-shaped or horospherical $R$- or $T$-ends and satisfy (1E). Let $E$ be a lens-shaped T-end, and let $S_E$ be a totally geodesic hypersurface that is the ideal boundary corresponding to $E$. Let $L$ be a lens-shaped end neighborhood of $S_E$ in an ambient real projective orbifold containing $O$. Then

- $L \cup O$ is a properly convex real projective orbifold and has a strictly convex boundary component corresponding to $E$.
- Furthermore, if $O$ is strictly SPC and $\tilde{E}$ is a hyperbolic end, then so is $L \cup O$ which now has one more boundary component and one less T-ends.

**Proof** Let $\tilde{O}$ be the universal cover of $O$, which we can identify with a properly convex bounded domain in an affine subspace. Then $S_E$ corresponds to a T-p-end $\tilde{E}$ and a totally geodesic hypersurface $S = \tilde{S}_E$. And $L$ is covered by a lens $\tilde{L}$ containing $S$. The p-end fundamental group $\pi_1(\tilde{E})$ acts on $\tilde{O}$ and $\tilde{L}_1$ and $\tilde{L}_2$ the two components of $\tilde{L} - \tilde{S}_E$ in $\tilde{O}$ and outside $\tilde{O}$ respectively.

Lemma 10.1 generalizes Theorem 3.7 of [94].

**Lemma 10.1** Suppose that $\tilde{S}_E$ is the totally geodesic ideal boundary of a lens-shaped T-end $\tilde{E}$ of a strongly tame real projective orbifold $O$.

- Given a $\pi_1(\tilde{E})$-invariant properly convex open domain $\Omega_1$ with $\text{bd} \Omega_1 \cap \mathbb{S}^{n-1} = \tilde{S}_E$, for each point $p$ of $\text{bd} \tilde{S}_E$, any sharply supporting hyperspace $H$ of $\tilde{S}_E$ at $p$ in $\mathbb{S}^{n-1}$, there exists an AS-hyperspace to $\tilde{O}$ containing $H$.
- At each point of $\text{bd} \tilde{S}_E$, the hyperspace sharply supporting any $\pi_1(\tilde{E})$-invariant properly convex open set $\Omega$ with $\text{bd} \Omega \cap \mathbb{S}^{n-1} = \tilde{S}_E$ is unique if $\pi_1(\tilde{E})$ is hyperbolic.
- We are given two $\pi_1(\tilde{E})$-invariant properly convex open domains $\Omega_1$ with $\text{bd} \Omega_1 \cap \mathbb{S}^{n-1} = \tilde{S}_E$, and $\Omega_2$ with $\text{bd} \Omega_2 \cap \mathbb{S}^{n-1} = \tilde{S}_E$ from the other side. Then $\text{Cl}(\Omega_1) \cup \text{Cl}(\Omega_2)$ is a convex domain with

$$\text{Cl}(\Omega_1) \cap \text{Cl}(\Omega_2) = \text{Cl}(\tilde{S}_E) \subset \text{bd} \Omega_1 \cap \text{bd} \Omega_2$$

and their AS-hyperspaces at each point of $\text{bd} \tilde{S}_E$ coincide.
10.1 Some constructions associated with ends

Proof Let $\mathbb{A}^n$ denote the affine subspace that is the complement in $\mathbb{R}^n$ of the hyperspace containing $\tilde{S}_E$. Because $\pi_1(\tilde{E})$ acts properly and cocompactly on a lens-shaped domain, By Theorem 6.9, $h(\pi_1(\tilde{E}))$ satisfies the uniform middle eigenvalue condition.

The domain $\Omega$ has an affine half-space $H(x)$ bounded by an AS-hyperspace for each $x \in \partial \tilde{S}_E$ containing $\Omega$. Here, $H(x)$ is uniquely determined by $\pi_1(\tilde{E})$ and $x$ and $H(x) \cap S^{n-1}$ by Theorems 5.1 and 5.2. The respective AS-hyperspaces at each point of $\text{Cl}(\tilde{S}_E) - \tilde{S}_E$ to $\Omega$ and $\Omega$ have to agree by Lemmas 5.9 and 5.3.

The second item follows by the third item and Theorem 1.1 of [22]. □

Proof (Continuation of the proof of Theorem 10.1) By Lemma 10.1, $L_2 \cup \tilde{S}_E \cup \tilde{E}$ is a convex domain. If $L_2 \cup \tilde{E}$ is not properly convex, then it is a union of two cones over $\tilde{S}_E$ over of $[\pm v_i] \in \mathbb{R}^{n+1}$, $[v_i] = x$. This means that $\tilde{E}$ has to be a cone contradicting the strong irreducibility of $h(\pi_1(\tilde{E}))$. Hence, it follows that $L_2 \cup \tilde{E}$ is properly convex.

Suppose that $\tilde{E}$ is strictly SPC and $\pi_1(\tilde{E})$ is hyperbolic. Then every segment in $\partial \tilde{E}$ or a non-$C^1$-point in $\partial \tilde{E}$ is in the closure of one of the p-end neighborhoods. $\text{bd}L_2 - \text{Cl}(\tilde{S}_E)$ does not contain any segment in it or a non-$C^1$-point. $\partial \tilde{E} - \text{Cl}(\tilde{S}_E)$ does not contain any segment or a non-$C^1$-point outside the union of the closures of p-end neighborhoods. $\text{bd}(\tilde{E} \cup L_2 \cup \tilde{S}_E)$ is $C^1$ at each point of $\Lambda(\tilde{E}) := \text{Cl}(\tilde{S}_E) - \tilde{S}_E$ by the uniqueness of the sharply supporting hyperspaces of Lemma 10.1.

Recall that $\tilde{S}_E$ is strictly convex since $\pi_1(\tilde{E})$ is a hyperbolic group. (See Theorem 1.1 of [22].) Thus, $\Lambda$ does not contain a segment, and hence, $\text{bd}(\tilde{E} \cup L_2 \cup \tilde{S}_E)$ does not contain one. Therefore, $L_2 \cup \tilde{E}$ is strictly convex relative to the remaining ends. Now we do this for every copy $g(L_2)$ of $L_2$ for $g \in \pi_1(\tilde{E})$.

Since $L_2 \cup \tilde{E}$ has a Hilbert metric by [120], the action is properly discontinuous. □

Corollary 10.1 Suppose that $\tilde{E}$ is a noncompact strongly tame properly convex real projective orbifold with a p-end $\tilde{E}$, and $\pi_1(\tilde{E})$ is hyperbolic.

(i) Let $\tilde{E}$ be a lens-shaped totally geodesic p-end. Let $L$ be a CA-lens containing a totally geodesic properly convex hypersurface $\tilde{E}$ so that

$$\Lambda := \text{Cl}(\tilde{S}_E) - \tilde{S}_E = \text{bd}L - \partial L.$$  

Then each point of $\Lambda$ has a unique sharply supporting hyperspace of $L$.

(ii) Let $\tilde{E}$ be a lens-shaped radial p-end. Let $L$ be a CA-lens in the p-end neighborhood. Define $\Lambda := \text{bd}L - \partial L$. Then each point of $\Lambda$ has a unique sharply supporting hyperspace of $L$.

Proof (i) is already proved in Lemma 10.1.

(ii) is proved in Proposition 6.4. □

10.1.2 Shaving the R-ends

We call the following construction shaving the ends.
Theorem 10.2 Given a strongly tame SPC-orbifold $\mathcal{O}$ and its universal cover $\tilde{\mathcal{O}}$, there exists a collection of mutually disjoint open concave $p$-end neighborhoods for lens-shaped $p$-ends. We remove a finite union of concave end-neighborhoods of some $R$-ends. Then

- we obtain a convex domain as the universal cover of a strongly tame orbifold $\mathcal{O}_1$ with additional strictly convex smooth boundary components that are closed $(n-1)$-dimensional orbifolds.
- Furthermore, if $\mathcal{O}$ is strictly SPC with respect to all of its ends, and we remove only some of the concave end-neighborhoods of hyperbolic $R$-ends, then $\mathcal{O}_1$ is strictly SPC with respect to the remaining ends.

Proof If $\mathcal{O}_1$ is not convex, then there is a triangle $T$ in $\tilde{\mathcal{O}}_1$ with three segments $s_0,s_1,s_2$ so that $T - s_0' \subset \tilde{\mathcal{O}}_1$ but $s_0' - \mathcal{O}_1 \neq \emptyset$. (See Theorem A.2 of [51] for details.) Since $\tilde{\mathcal{O}}_1$ is an open manifold, $s_0' - \mathcal{O}_1$ is a closed subset of $s_0$. Then a boundary point $x \in s_0' - \tilde{\mathcal{O}}_1$ is in the boundary of one of the removed concave-open neighborhoods or is in $\partial \tilde{\mathcal{O}}$ itself. The second possibility implies that $\mathcal{O}$ is not convex as $\mathcal{O}_1 \subset \mathcal{O}$. The first possibility implies that there exists an open segment meeting $\partial U \cap \tilde{\mathcal{O}}$ at a unique point but disjoint from $U$. This is geometrically not possible since $\partial U \cap \tilde{\mathcal{O}}$ is strictly convex towards the direction of $U$. These are contradictions.

Since $\tilde{\mathcal{O}}$ is properly convex, so is $\tilde{\mathcal{O}}_1$. Since $\partial U \cap \tilde{\mathcal{O}}$ is strictly convex, the new corresponding boundary component of $\tilde{\mathcal{O}}_1$ is strictly convex.

Now we go to the second part. We suppose that $\mathcal{O}$ is strictly SPC. Let $\mathcal{H}$ denote the set of $p$-ends with hyperbolic $p$-end fundamental groups whose concave $p$-end neighborhoods were removed in the equivariant manner. For each $\tilde{E} \in \mathcal{H}$, denote by $U_{\tilde{E}}$ the concave $p$-end neighborhood that we are removing.

Any segment in the boundary of the developing image of $\mathcal{O}$ is in the closure of a $p$-end neighborhood of a $p$-end vertex. For the $p$-end-vertex $v_{\tilde{E}}$ of a $p$-end $\tilde{E}$, the domain $R_{v_{\tilde{E}}}(\mathcal{O}) \subset S^{n-1}_{v_{\tilde{E}}}$ is strictly convex by [22] if $\pi_1(\tilde{E})$ is hyperbolic. Since $\partial R_{v_{\tilde{E}}}(\mathcal{O})$ contains no straight segment, only straight segments in $\text{Cl}(U) \cap \partial \tilde{\mathcal{O}}$ for the concave $p$-end neighborhood $U$ of $\tilde{E}$ are in the segments in $\bigcup S(v_{\tilde{E}})$. Thus, their interiors are disjoint from $\partial \mathcal{O}_1$, and hence $\partial \mathcal{O}_1$ contains no geodesic segment in $\bigcup_{\tilde{E} \in \mathcal{H}} \text{Cl}(U_{\tilde{E}}) \cap \partial \tilde{\mathcal{O}}$.

Since we removed concave end neighborhoods of the lens-shaped ends with the hyperbolic end fundamental groups, any straight segment in $\partial \mathcal{O}_1$ lies in the closure of a $p$-end neighborhood of a remaining $p$-end vertex.

A non-$C^1$-point of $\partial \mathcal{O}_1$ is not on the boundary of the concave $p$-end neighborhood $U$ for a hyperbolic $p$-end $\tilde{E}$ nor in $\partial \mathcal{O} - \bigcup_{\tilde{E} \in \mathcal{H}} \text{Cl}(U_{\tilde{E}})$. We show that points of $\Lambda = L - \partial L$ are $C^1$-points of $\partial \mathcal{O}$. $\text{Cl}(U) \cap \partial \mathcal{O}_1$ contains the limit set $\Lambda = L - \partial L$ for the CA-lens $L$ in a lens-neighborhood. $\mathcal{O}$ has the same set of sharply supporting hyperspaces as $L$ at points of $\Lambda$ since they are both $\pi_1(\tilde{E})$-invariant convex domains by Corollary 10.1. However, the sharply supporting hyperspaces at $\Lambda$ of $L$ are also supporting ones for $\mathcal{O}_1$ by Corollary 10.1 since $L \subset \tilde{\mathcal{O}}$ as we removed the outside component $U$ of $\tilde{\mathcal{O}} - L$. Thus, $\mathcal{O}_1$ is $C^1$ at points of $\Lambda$.

Also, points of $\partial \mathcal{O} - \bigcup_{\tilde{E} \in \mathcal{H}} \text{Cl}(U_{\tilde{E}})$ are $C^1$-points of $\partial \mathcal{O}$ since $\mathcal{O}$ is strictly SPC. Let $x$ be a point of this set. Suppose that $x \in s^o$ for a segment $s$ in $\partial \mathcal{O}_1$. Then
10.2 The strict SPC-structures and relative hyperbolicity

10.2.1 The Hilbert metric on $\mathcal{O}$.

Recall Hilbert metrics from Section 2.1.2. A Hilbert metric on an orbifold with an SPC-structure is defined as a distance metric given by cross ratios. (We do not assume strictness here.)

Given an SPC-structure on $\mathcal{O}$, there is a Hilbert metric which we denote by $d_{\tilde{\mathcal{O}}}$ on $\tilde{\mathcal{O}}$ and hence on $\tilde{\mathcal{O}}$. Actually, we will make $\tilde{\mathcal{O}}$ slightly small by inward perturbations of $\partial \mathcal{O}$ preserving the strict convexity of $\partial \mathcal{O}$ by Lemma 2.16. The Hilbert metric will be defined on original $\tilde{\mathcal{O}}$. (We call this metric the perturbed Hilbert metric.) This induces a metric on $\mathcal{O}$, including the boundary now. We will denote the metric by $d_{\mathcal{O}}$.

Given an open properly convex domain $\Omega$, we note that given any two points $x, y$ in $\Omega$, there is a geodesic arc $\gamma$ with endpoints $x, y$ so that its interior is in $\Omega$. $\Omega$.

**Proposition 10.1** Let $\Omega$ be a properly convex open domain. Let $P$ be a subspace meeting $\Omega$, and let $x$ be a point of $\Omega - P$:

(i) There exists a shortest path $m$ from $x$ to $P \cap \Omega$ that is a line segment.
(ii) The set of shortest paths to $P$ from a point $x$ of $\Omega - P$ have endpoints in a compact convex subset $K$ of $P \cap \Omega$.
(iii) For any line $m'$ containing $m$ and $y \in m'$, the segment in $m'$ from $y$ to the point of $P \cap \Omega$ is one of the shortest segments.
(iv) When $P$ is a complete geodesic in $\Omega$ with $x \in \Omega - P$, outside the compact set $K$, $K \subset P$, of endpoints of shortest segments from $x$ to $P$, the distance function from $P - K$ to $x$ is strictly increasing or strictly decreasing.

**Proof** (i) The distance function $f : P \cap \Omega \to \mathbb{R}$ defined by $f(y) = d_{\Omega}(x, y)$ is a proper function where $f(x) \to \infty$ as $x \to z$ for any boundary point $z$ of $P \cap \Omega$ in $P$. Hence, there exists a shortest segment with an endpoint $x_0$ in $P \cap \Omega$. (iv) is also proved.

(ii) Let $\gamma$ be any geodesic in $P \cap \Omega$ passing $x_0$. We need to consider the 2-dimensional subspace $Q$ containing $\gamma$ and $x$. The set of endpoints of shortest segments of $\Omega$ in $Q$ is a connected compact subset containing $x_0$ by Proposition 1.4.
Hence, by considering all geodesics in $P \cap \Omega$ passing $x_0$, we obtain that the endpoints of the shortest path to $P$ from $x$ is a connected compact set. We take two points $z_1, z_2$ on it. Then the segment connecting $z_1$ and $z_2$ is also in the set of endpoints by Proposition 1.4 of [50]. Hence, the set is convex.

(iii) Suppose that there exists $y \in m'$, so that the shortest geodesic $m''$ to $P \cap \Omega$ is not in $m'$. Consider the 2-dimensional subspace $Q$ containing $m'$ and $m''$. Then this is a contradiction by Corollary 1.5 of [50].

(iv) Again follows by considering a 2-dimensional subspace containing $P$ and $m$. (See Proposition 1.4 of [50] for details.)

An endpoint in $P$ of a shortest segment is called a foot of the perpendicular from $x$ to $\gamma$.

10.2.2 Strict SPC-structures and the group actions

By Corollary 7.7, strict SPC-orbifolds with generalized lens-shaped or horospherical $R$- or $J$-ends have only lens-shaped or horospherical $R$- or $J$-ends.

An elliptic element of $g$ is a nonidentity element of $\pi_1(\tilde{\mathcal{O}})$ fixing an interior point of $\tilde{\mathcal{O}}$. Since $\pi_1(\tilde{\mathcal{O}})$ acts discretely on the space $\tilde{\mathcal{O}}$ with a metric, an elliptic element has to be of finite order.

Lemma 10.2 Let $\mathcal{O}$ be a strongly tame strict SPC-orbifold. Let $\tilde{E}$ be a $p$-end of $\tilde{\mathcal{O}}$.

(i) Suppose that $\tilde{E}$ is a horospherical $p$-end. Let $B$ be a horoball $p$-end neighborhood with a $p$-end vertex $p$ corresponding to $\tilde{E}$. There exists a homeomorphism $\Phi_{\tilde{E}}$:
The strict SPC-structures and relative hyperbolicity

(i) Suppose that $\hat{E}$ is a lens-shaped radial $p$-end. Let $U$ be a lens-shaped radial $p$-end neighborhood with the $p$-end vertex $p$ corresponding to $\hat{E}$. There exists a homeomorphism $\Phi_{\hat{E}} : \text{bd}U \cap \hat{O} \to \text{bd} \hat{O} - \text{Cl}(U)$ given by sending a point $x$ to the other endpoint of the maximal convex segment containing $x$ and $p$ in $\text{Cl}(\hat{O})$.

Moreover, each of the maps denoted by $\Phi_{\hat{E}}$ commutes with elements of $h(\pi_1(E))$.

Proof

(i) By Theorem 4.2(i) $\Phi_{\hat{E}}$ is well-defined. The same proposition implies that $\text{bd} B - \{p\}$ is smooth at $p$ and $\text{bd} \tilde{O}$ has a unique sharply supporting hyperspace. Therefore the map is onto.

(ii) The second item follows from Theorems 6.7 and 6.8 since they imply that the segments in $S(p)$ are maximal ones in $\text{bd} \hat{O}$ from $p$. □

We now study the fixed points in $\text{Cl}(\hat{O})$ of elements of $\pi_1(\hat{E})$. Recall that a great segment is a geodesic arc in $S^n$ with antipodal $p$-end vertices. It is not properly convex.

Note that we can replace a generalized lens to a lens for a strongly tame strictly SPC-orbifold by Corollary 7.7.

Lemma 10.3

Let $\hat{O}$ be a strongly tame strict SPC-orbifold with lens-shaped or horospherical $R$- or $T$-ends. Let $g$ be an infinite order element of a $p$-end fundamental group $\pi_1(E)$. Then every fixed point $x$ of $g$ in $\text{Cl}(\hat{O})$ satisfies one of the following:

- $x$ is in the closure of a $p$-end-neighborhood that is a concave end-neighborhood of an $R$-$p$-end,
- $x$ is in the closure of a $p$-ideal boundary component of a $T$-$p$-end or
- $x$ is the fixed point of a horospherical $R$-$p$-end.

Proof

Suppose that the $p$-end $\hat{E}$ is a lens-shaped $R$-end. The direction of each segment in the interior of the lens cone with an endpoint $v_{\hat{E}}$ is fixed by only the finite-order element of $\pi_1(\hat{E})$ since $\pi_1(\hat{E})$ acts properly discontinuously on $\hat{S}_E$. Thus, the fixed points are on the rays in the direction of the boundary of $\hat{E}$. They are in one of $S(v_{\hat{E}})$ for the $p$-end vertex $v_{\hat{E}}$ corresponding to $\hat{E}$ by Theorems 6.7 and 6.8. Hence, the fixed points of the holonomy homomorphism of $\pi_1(\hat{E})$ is in the closure of the lens-cone with end vertex $v_{\hat{E}}$ and nowhere else in $\text{Cl}(\hat{O})$.

If $\hat{E}$ is horospherical, then the $p$-end vertex $v_{\hat{E}}$ is not contained in any segment $s$ in $\text{bd} \hat{O}$ by Theorem 4.2. Hence $v_{\hat{E}}$ is the only point $S \cap \text{bd} \hat{O}$ of any invariant subset $S$ of $\pi_1(\hat{E})$ by Lemma 10.2. Thus, the only fixed point of $\pi_1(\hat{E})$ in $\text{bd} \hat{O}$ is $v_{\hat{E}}$.

Suppose that $E$ is a lens-shaped $T$-$p$-end. Since $\hat{E}$ is a properly convex real projective orbifold that is closed, we obtain an attracting fixed point $a$ and a repelling fixed point $r$ of $g|\text{Cl}(\hat{S}_E)$ by [17]. Then $a$ and $r$ are attracting and repelling fixed points of $g|\text{Cl}(\hat{O})$ by the existence of the CA-lens neighborhood of $\hat{S}_E$ since Theorem 6.9 implies the uniform middle eigenvalue condition.

Suppose that we have a fixed point $s \in \text{bd} \hat{O}$ distinct from $a$ and $r$. We claim that $\overline{s}$ and $\overline{a}$ are in $\text{bd} \hat{O}$. The norm of the eigenvalue associated with $s$ is strictly between
those of $r$ and $s$ by the uniform middle eigenvalue condition. Let $P$ denote the two-dimensional subspace containing $r, s, a$. Suppose that one of the segment meets $\tilde{\mathcal{O}}$ at a point $x$. We take a convex open-ball-neighborhood $B$ of $x$ in $P \cap \tilde{\mathcal{O}}$. Suppose that $x \in \overline{\pi \mathcal{O}}$. Then using the sequence $\{g^n(B)\}$ we obtain a great segment in $\text{Cl}(\tilde{\mathcal{O}})$ by choosing $n \to \infty$. This is a contradiction. If $x \in \overline{\pi \mathcal{O}}$, we can use $\{g^{-n}(B)\}$ as $n \to \infty$, again giving us a contradiction. Hence, $\overline{\pi \mathcal{O}}, \overline{\mathcal{O}} \subset \text{bd} \tilde{\mathcal{O}}$.

Since $\mathcal{E}$ has a one-sided neighborhood $U$ in a CA-lens neighborhood of $\tilde{\mathcal{S}}_E$ by choosing a smaller such neighborhood $U$ if necessary, we may assume that $\text{Cl}(U) \cap \text{bd} \tilde{\mathcal{O}}$ is in $\text{Cl}(\tilde{\mathcal{S}}_E)$. By the strict convexity of $\mathcal{O}$, we see that the nontrivial segments $\overline{\alpha \mathcal{O}}$ and $\overline{\mathcal{O} \beta}$ have to be in $\text{Cl}(\tilde{\mathcal{S}}_E)$. (See Definition 1.6.)

See Crampon and Marquis [74] and Cooper-Long-Tillmann [73] for similar work to the following. We remind the reader that generalized lens-shaped R-ends are lens-shaped R-ends in the following assumption by Corollary 7.7.

\textbf{Proposition 10.2} Suppose that $\mathcal{O}$ is a strongly tame strict SPC-orbifold with lens-shaped ends or horospherical $\mathcal{B}$- or $\mathcal{T}$-ends satisfying (IE) and (NA). Then each nonidentity and infinite-order element $g$ of $\pi_1(\mathcal{O})$ has two exclusive possibilities:

- $g|\text{Cl}(\mathcal{O})$ has exactly two fixed points in $\text{bd} \tilde{\mathcal{O}}$ none of which is in the closures of the p-end neighborhoods for distinct ends, and $g$ is positive proximal.
- $g$ is in a p-end fundamental group $\pi_1(\mathcal{E})$, and $g|\text{Cl}(\mathcal{O})$ has all fixed points in $\text{bd} \mathcal{O}$ in the closure of a concave p-end neighborhood of a lens-shaped radial p-end $\mathcal{E}$.
- $g$ is in $\text{Cl}(\tilde{\mathcal{S}}_E)$ for the ideal boundary component $\tilde{\mathcal{S}}_E$ of a lens-shaped totally geodesic p-end $\mathcal{E}$, or
- $g$ has a unique fixed point in $\text{bd} \mathcal{O}$ at the horospherical p-end vertex.

\textbf{Proof} Suppose that $g$ has a fixed point at a horospherical p-end vertex $v$ for a p-end $\tilde{\mathcal{E}}$. We can choose the horoball $U$ at $v$ that maps into an end-neighborhood of $\mathcal{O}$. A horoball p-end neighborhood is either sent to a disjoint one or sent to the identical one. Since $g(U) \cap U \neq \emptyset$ by the geometry of a horoball having a smooth boundary at $v$, $g$ must act on the horoball, and hence $g$ is in the p-end fundamental group. The p-end vertex is the unique fixed point of $g$ in $\text{bd} \mathcal{O}$ by Lemma 10.3.

Similarly, suppose that $g \in \pi_1(\mathcal{O})$ fixes a point of the closure $U$ of a concave p-end neighborhood of a p-end vertex $v$ of a lens-shaped end. $g(\text{Cl}(U))$ and $\text{Cl}(U)$ meet at a point. By Corollary 7.6, $g(\text{Cl}(U))$ and $\text{Cl}(U)$ share the p-end vertex and hence $g(U) = U$ as $g$ is a deck transformation. Therefore, $g$ is in the p-end fundamental group of the p-end of $v$. Lemma 10.3 implies the result.

Suppose that $g \in \pi_1(\mathcal{O})$ fixes a point of $\text{Cl}(\tilde{\mathcal{S}}_E)$ for a totally geodesic ideal boundary $\tilde{\mathcal{S}}_E$ corresponding to a p-end $\mathcal{E}$. Again Corollary 7.6 and Lemma 10.3 imply the result for this case.

Suppose that an element $g$ of $\pi_1(\mathcal{O})$ is not in any p-end fundamental subgroup. Then by above, $g$ does not fix any of the above types of points. We show that $g$ has exactly two fixed points in $\text{bd} \tilde{\mathcal{O}}$.

Suppose that $g \in \pi_1(\mathcal{O})$ fixes a unique point $x$ in the closure of $\text{bd} \tilde{\mathcal{O}}$ and $x$ is not in the closure of p-end neighborhoods by the first part of the proof. Then $x$ is a $C^1$-point.
Proposition 10.3 Suppose that $\tilde{\Theta}$ is a noncompact strongly tame strict SPC-orbifold with lens-shaped ends or horospherical $R$- or $T$-ends. Let $\tilde{E}$ be an end. Then for a $p$-end $E$, $(\text{bd} \tilde{\Theta} - K)/\pi_1(E)$ is a compact orbifold where $K = \bigcup S(E)$ for a lens-shaped radial $p$-end $\tilde{E}$, $K = \text{Cl}(\tilde{S}_E)$ for totally geodesic $p$-end $\tilde{E}$, or $K = \{v_E\}$ for horospherical $p$-end $\tilde{E}$.

Proof Suppose that $\tilde{E}$ is a lens-shaped $R$-p-end or horospherical type. By Lemma 10.2, the homeomorphism $\Phi_E : \tilde{S}_E \to \text{bd} \tilde{\Theta} - K$ gives us the result.

Suppose that $\tilde{E}$ is a lens-shaped $T$-p-end. Let $\tilde{\Theta}^*$ denote the dual domain. Then there exists a dual radial $p$-end $\tilde{E}^*$ corresponding to $\tilde{E}$. Hence, $(\text{bd} \tilde{\Theta}^* - K^*)/\pi_1(\tilde{E}^*)$ is compact for $K^*$ equal to the closure of $p$-end neighborhoods of $\tilde{E}^*$ in the radial case or the vertex in the horospherical case.

Recall Section 2.5. Let $\text{bd}^{Ag} \tilde{\Theta}$ be the augmented boundary with the fibration $\Pi_{Ag}$, and let $\text{bd}^{Ag} \tilde{\Theta}^*$ be the augmented boundary with the fibration map $\Pi_{Ag}^*$. Let $K'' := \Pi_{Ag}^{-1}(K)$ and $K'' := \Pi_{Ag}^{-1}(K')$. The discussion on in the proof of Corollary 6.2 shows that there is a duality homeomorphism
\[ \mathcal{D} : \text{bd}^{A_\mathcal{E}} \bar{\mathcal{E}} - K'' \to \text{bd}^{A_\mathcal{E}} \bar{\mathcal{E}}^* - K'''. \]

Now \((\text{bd}^{A_\mathcal{E}} \bar{\mathcal{E}}^* - K''')/\pi_1(\bar{\mathcal{E}}')\) is compact since \(\text{bd}\bar{\mathcal{E}}^* - K'\) has a compact fundamental domain, and the space is the inverse image in \(\text{bd}^{A_\mathcal{E}} \bar{\mathcal{E}}^*\) of \(\text{bd}\bar{\mathcal{E}}^* - K'\). By (iv) of Proposition 2.19, \((\text{bd}^{A_\mathcal{E}} \bar{\mathcal{E}} - K''')/\pi_1(\bar{\mathcal{E}})\) is compact also. Since the image of this set under the map induced by a proper map \(\Pi_{A_\mathcal{E}}\) is \((\text{bd}\bar{\mathcal{E}} - K)/\pi_1(\bar{\mathcal{E}})\). Hence, it is is compact. \(\square\)

### 10.3 Bowditch’s method

#### 10.3.1 The strict convexity implies the relative hyperbolicity

There are results proved by Cooper, Long, and Tillmann [73] and Crampon and Marquis [74] similar to below. However, the ends have to be horospherical in their work. We will use Bowditch’s result [33] to show

**Theorem 10.3** Let \(\mathcal{O}\) be a noncompact strongly tame strict SPC-orbifold with lens-shaped ends or horospherical \(R\)- or \(T\)-ends \(E_1, \ldots, E_k\) and satisfies \((\text{IE})\) and \((\text{NA})\). Assume \(\partial \mathcal{O}\) is smooth and strictly convex. Let \(\mathcal{U}_i\) be the inverse image \(\mathcal{U}_i\) in \(\mathcal{O}\) for a mutually disjoint collection of neighborhoods \(\mathcal{U}_i\) of the ends \(E_i\) for each \(i = 1, \ldots, k\). Then

- \(\pi_1(\mathcal{O})\) is relatively hyperbolic with respect to the end fundamental groups \(\pi_1(E_1), \ldots, \pi_1(E_k)\).

Hence \(\mathcal{O}\) is relatively hyperbolic with respect to \(U := \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k\) as a metric space.

- If \(\pi_1(E_{i+1}), \ldots, \pi_1(E_k)\) are hyperbolic for some \(1 \leq l \leq k\) (possibly some of the hyperbolic ones), then \(\pi_1(\mathcal{O})\) is relatively hyperbolic with respect to the end fundamental group \(\pi_1(E_1), \ldots, \pi_1(E_l)\).

**Proof** We show that \(\pi_1(\mathcal{O})\) is relatively hyperbolic with respect to the end fundamental groups \(\pi_1(E_1), \ldots, \pi_1(E_k)\).

- We now collapse each set of form \(\text{Cl}(\mathcal{U}_i) \cap \text{bd}\bar{\mathcal{E}} = S_E\) for a concave \(p\)-end neighborhood \(\mathcal{U}_i\) to a point and
- collapse \(\text{Cl}(S_E)\) for each lens-shaped totally geodesic end \(E\) to a point.

By Corollary 7.6, these sets are mutually disjoint balls. Let \(\mathcal{C}_B\) denote the collection, and let \(C_B := \bigcup \mathcal{C}_B\).

We claim that for each closed set \(J\) in \(\text{bd}\bar{\mathcal{E}}\), the union of \(C_J\) of elements of \(\mathcal{C}_B\) meeting \(J\) is also closed: Let us choose a sequence \(\{x_i\}\) for \(x_i \in C_i\), \(C_i \cap J \neq \emptyset\), \(C_i \in \mathcal{C}_B\). Suppose that \(x_i \to x\). Let \(y_i \in C_i \cap J\). Let \(v_i\) be the \(p\)-end vertex of \(C_i\) if it is from a \(R\)-\(p\)-end. Then define \(s_i := x_i v_i \cup y_i \cap C_i\) if \(C_i\) is radial or else
Choose a subsequence so that \( \{s_i\} \) geometrically converges to a limit containing \( x \). The limit \( x_\infty \) is a singleton, a segment or a union of two segments. By the strict convexity of \( \hat{\mathcal{B}} \), we obtain that \( s_\infty \) is a subset of an element of \( \mathcal{E}_B \) and \( s_\infty \) meets \( J \). Thus, \( x \in s_\infty \subset C_J \) for \( C_J \cap J \neq \emptyset \). We conclude that \( C_J \) is closed.

We denote this quotient space \( \text{bd} \hat{\mathcal{B}}/\sim \) by \( B \). By Proposition 10.5, \( B \) is a metrizable space.

We show that \( \pi_1(\mathcal{B}) \) acts on the metrizable space \( B \) as a geometrically finite convergence group. By Theorem 0.1 of Yaman [167] following Bowditch [33], this shows that \( \pi_1(\mathcal{B}) \) is relatively hyperbolic with respect to \( \pi_1(E_1), \ldots, \pi_1(E_k) \). The definition of conical limit points and so on are from the article.

(1) We first show that the group acts properly discontinuously on the metric space of ordered mutually distinct triples in \( B = \partial \hat{\mathcal{B}}/\sim \). Suppose not. Then there exists a sequence of nondegenerate triples \( \{ (p_i, q_i, r_i) \} \) of points in \( \text{bd} \hat{\mathcal{B}} \) converging to a mutually distinct triple \( \{ (p, q, r) \} \) so that

\[
\gamma_i(p_i) = \gamma(p_0), \quad \gamma_i(q_i) = \gamma(q_0), \quad \text{and} \quad \gamma_i(r_i) = \gamma(r_0)
\]

where \( \{ \gamma_i \} \) is a sequence of mutually distinct elements of \( \pi_1(\mathcal{B}) \) and the equivalence classes \( [p_0], [q_0], [r_0] \) are mutually distinct and so are \( [p], [q], [r] \). By multiplying by some uniformly bounded element \( R_i \) in \( \text{PGL}(n + 1, \mathbb{R}) \) but not necessarily in \( \hat{h}(\pi_1(\mathcal{B})) \), we obtain that \( R_i \circ \gamma_i \) for each \( i \) fixes \( p_0, q_0, r_0 \) and restricts to a diagonal matrix with entries \( \lambda_i, \delta_i, \mu_i \) on the plane with coordinates so that \( p_0 = e_1, q_0 = e_2, r_0 = e_3 \).

Then we can assume that

\[
\lambda_i \delta_i \mu_i = 1, \quad \lambda_i \geq \delta_i \geq \mu_i > 0
\]

by restricting to the plane and up to choosing subsequences and renaming. Thus \( \{ \lambda_i \} \to \infty \) and \( \{ \mu_i \} \to 0 \); otherwise, both of these two sequences are bounded. Let \( P_i \) denote the 2-dimension subspace spanned by \( p_i, q_i, r_i \). Then \( \gamma_i|P_i \) is a sequence of uniformly bounded automorphisms. Let \( D_i = \text{Cl}(\mathcal{B} \cap P_i) \). Then the sequence of maximal \( d \)-distances from points of \( D_i \) to \( \text{bd} \hat{\mathcal{B}} \) is uniformly bounded below by a positive number: If not, the geometric limit of \( D_i \) is a nontrivial disk in \( \text{bd} \hat{\mathcal{B}} \) containing \( p, q, r \). Since \( p, q, r \) are in mutually distinct equivalence classes, this contradicts the strict convexity. Then for a compact subset \( K \) of \( \mathcal{B} \), \( K \cap D_i \subset \mathcal{B} \) is not empty for sufficiently large \( i \). Choose \( p_i \in K \cap D_i \). Then \( \gamma_i(p_i) \) is in a \( d_{P_i} \)-bounded neighborhood of \( K \) independent of \( i \) since otherwise \( \gamma \) is not uniformly bounded on \( \mathcal{B} \cap P_i \) as indicated by the Hilbert metric. Hence, \( \gamma(K) \) is in a \( d_{P_i} \)-bounded neighborhood of \( K \). This contradicts the proper discontinuity of the action by \( \Gamma \).

Let \( P_0 \) denote the 2-dimensional subspace containing \( p_0, q_0, \) and \( r_0 \). By strictness of convexity, as we collapsed each of the p-end balls, the interiors of the segments \( p_0q_0, q_0r_0, \) and \( r_0p_0 \) are in the interior of \( \mathcal{B} \).

We claim that one of the sequence \( \{ \lambda_i / \delta_i \} \) or the sequence \( \{ \delta_i / \mu_i \} \) are bounded: Suppose not. Then \( \{ \lambda_i / \delta_i \} \to \infty \) and \( \{ \delta_i / \mu_i \} \to \infty \). We choose generic segments \( s_0 \) and \( t_0 \) in \( \mathcal{B} \) with a common endpoint \( q_0 \) and the respective other endpoint \( \delta_0 \) and \( \xi_0 \) in different components of \( P \cap \mathcal{B} - \overline{p_0q_0} \) so that
We choose $s_0$ and $t_0$ so that their directions from $q_0$ differ from those of $p_0q_0$ and $q_0t_0$ at least by a small constant $\delta' > 0$. Then the sequence $\{R_t \circ \gamma(s_0 \cup t_0)\}$ geometrically converges to the segment with endpoint $p_0$ passing $q_0$. The segment is a great segment. Since $R_t$ is bounded, this implies that there exists such a segment in $\text{Cl}(\mathcal{B})$. This is a contradiction to the proper convexity of $\mathcal{B}$.

Suppose now that the sequence $\lambda_i/\delta_i$ is bounded: Now the sequence of segments $\{p_iq_i\}$ converges to $pq$ whose interior is in $\mathcal{B}$. Then we see that $pq$ must be in the boundary of $\mathcal{B}$: Each point in $pq$ must be the limit points of a sequence $\{y_i\}$ for $y_i \in \gamma(s)$ for some compact subsegment $s \subset p\gamma_{\infty}$ by the boundedness of the above ratio and the proper-discontinuity of the action. This contradicts the strict convexity as we assumed that $p, q,$ and $r$ represent distinct points in $B$. If we assume that $\delta_i/\mu_i$ is bounded, then we obtain a contradiction similarly.

This proves the proper discontinuity of the action on the space of distinct triples.

(II) By Propositions 10.2 and 10.3, each group of form $\Gamma_i$ for a point $x$ of $B = \mathcal{B}/\sim$ is a bounded parabolic subgroup in the sense of Bowditch [167].

Now we take lens-cone end neighborhood for each radial end instead. We still choose ones mutually disjoint from themselves and nonradial ones in $U$. We denote by $U'$ the union of the modified end-neighborhoods. Let $U'_1, \ldots, U'_k$ denote its components. Let $U'_k$ denote the inverse image of $U'_k$ in $\mathcal{B}$ for each $k$.

(III) Let $p \in \text{bd}\mathcal{B}$ be a point that is not in a horospherical endpoint or an equivalence class corresponding to a lens-shaped p-end of radial or totally geodesic type of $B$. Hence, $[p] = \{p\}$ in $\text{bd}\mathcal{B}/\sim$. That is, there is no segment containing $p$ in one of the collapsed sets. We show that $[p]$ is a conical limit point. This will complete our proof by Theorem 0.1 of [167].

To show that $[p]$ is a conical limit point, we will find a sequence of holonomy transformations $\gamma$ and distinct points $a, b \in \partial B$ so that $\{\gamma([p])\} \to a$ and $\{\gamma(q)\} \to b$ locally uniformly for $q \in \partial B - \{p\}$: To do this, we draw a line $l$ in $\mathcal{B}$ from a point of the fundamental domain to $p$ where as $t \to \infty$, $l(t) \to p$ in $\text{Cl}(\mathcal{B})$. We may assume that the other endpoint $p'$ of $l$ is in distinct equivalence class from $[p]$. Since $l(t)$ is not eventually in a p-end neighborhood, there is a sequence $\{t_i\}$ going to $\infty$ so that $l(t_i)$ is not in any of the p-end neighborhoods in $U'_1 \cup \cdots \cup U'_k$. Let $p'$ be the other endpoint of the complete extension of $l(t)$ in $\mathcal{B}$. We can assume without generality that $p'$ is not in the closure of any p-end neighborhood by choosing the line $l(t)$ differently if necessary.

Since $(\mathcal{B} - U'_1 \cdots - U'_k)/\Gamma$ is compact, we have a compact fundamental domain $F$ of $\mathcal{B} - U'_1 \cdots - U'_k$ with respect to $\Gamma$. Note that for the minimum distance, we have $d(F, \text{bd}\mathcal{B}) > C_0$ for some constant $C_0 > 0$.

We note: Given any line $m$ passing $F$, the two endpoints must be in distinct equivalence classes because of the convexity of each component of $U'$. We find a sequence of points $z_i \in F$ so that $\gamma(l(t_i)) = z_i$ for a deck transformation $\gamma$. Then $\{\gamma\}$ is an unbounded sequence.

Using Definition 2.6, we may choose a set-convergent subsequence of $\{\{\gamma\}\}$ that is convergent in $S(M_{n+1}(\mathbb{R}))$ to $\{\gamma_\omega\}$ for $\gamma_\omega \in M_{n+1}(\mathbb{R})$. Hence, $A_\star(\{\gamma\}) = d(\tilde{s}_0, q_0), d(\tilde{t}_0, q_0) \geq \delta$ for a uniform $\delta > 0$. 

\[
\text{We choose } s_0 \text{ and } t_0 \text{ so that their directions from } q_0 \text{ differ from those of } p_0q_0 \text{ and } q_0t_0 \text{ at least by a small constant } \delta' > 0. \text{ Then the sequence } \{R_t \circ \gamma(s_0 \cup t_0)\} \text{ geometrically converges to the segment with endpoint } p_0 \text{ passing } q_0. \text{ The segment is a great segment. Since } R_t \text{ is bounded, this implies that there exists such a segment in } \text{Cl}(\mathcal{B}). \text{ This is a contradiction to the proper convexity of } \mathcal{B}. 
\]
10.3 Bowditch’s method

The author obtained a proof of the following theorem from Drutu. See [80] for more details.

**Theorem 10.4 (Drutu)** Let $\mathcal{O}$ be a strongly tame properly orbifold with generalized lens-shaped ends and horospherical $\mathcal{S}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Let $\pi_1(E_1), \ldots, \pi_1(E_m)$ be end fundamental groups where $\pi_1(E_{l-1}), \ldots, \pi_1(E_m)$ for $l \leq m$ are hyperbolic groups. Then $\pi_1(\mathcal{O})$ is a relatively hyperbolic group with respect to $\pi_1(E_1), \ldots, \pi_1(E_m)$ if and only if $\pi_1(\mathcal{O})$ is one with respect to $\pi_1(E_1), \ldots, \pi_1(E_l)$.

**Proof** With the terminology in the paper [80], $\pi_1(\mathcal{O})$ is a relatively hyperbolic group with respect to the end fundamental groups $\pi_1(E_1), \ldots, \pi_1(E_m)$ if and only if $\pi_1(\mathcal{O})$ with a word metric is asymptotically tree graded (ATG) with respect to all the left cosets $g\pi_1(E_i)$ for $g \in \pi_1(\mathcal{O})$ and $i = 1, \ldots, m$.

We claimed that $\pi_1(\mathcal{O})$ with a word metric is asymptotically tree graded (ATG) with respect to all the left cosets $g\pi_1(E_i)$ for $g \in \pi_1(\mathcal{O})$ and $i = 1, \ldots, m$ if and only if $\pi_1(\mathcal{O})$ with a word metric is asymptotically tree graded with respect to all the left cosets $g\pi_1(E_i)$ for $g \in \pi_1(\mathcal{O})$ and $i = 1, \ldots, l$. 

$S(\text{Im } \gamma_p) \cap \text{Cl}(\mathcal{O})$. Also, on $\text{Cl}(\mathcal{O}) - N_\varepsilon(\{\gamma\}) \cap \text{Cl}(\mathcal{O})$, $\{\gamma\}$ is convergent to a subset of $A_\varepsilon(\{\gamma\})$ locally uniformly as we can easily deduce by linear algebra and some estimation.

Since $N_\varepsilon(\{\gamma\})$ is a convex subset of $\text{bd } \mathcal{O}$ and $A_\varepsilon(\{\gamma\})$ is a convex subset of $\text{bd } \mathcal{O}$ by Theorem 2.10, they are in collapsed sets of $\text{bd } \mathcal{O}$ by the strictness of the convexity.

If $p \notin N_\varepsilon(\{\gamma\})$, then $\gamma(l(t_i))$ is also bounded away from $N_\varepsilon(\{\gamma\})$, and hence $\gamma(l(t_i))$ accumulates only to $A_\varepsilon(\{\gamma\})$. This is a contradiction. Thus, $p \in N_\varepsilon(\{\gamma\})$.

Since $N_\varepsilon(\{\gamma\})$ is a convex compact subset of $\text{bd } \mathcal{O}$, we must have

$$N_\varepsilon(\{\gamma\}) \subset [p].$$

Thus, for all $q \in \text{bd } \mathcal{O} - [p]$, we obtain a local uniform convergence under $\gamma$ to $A_\varepsilon(\{\gamma\})$. This shows that $p$ is a conical limit point. We let $b$ be the collapsed set containing $A_\varepsilon(\{\gamma\})$.

Our line $l$ equals the interior of $\overline{pp'}$. We choose a subsequence of $\gamma$ so that the corresponding subsequence $\{\gamma(pp')\}$ geometrically converges to a line passing $F$. Since $p$ and $p'$ are in distinct equivalence classes, $[\gamma(p')]$ converges to $b$, and $\gamma(pp')$ passes $F$, it follows that $\gamma(p)$ converges to a point of the equivalence class $a$ distinct from $b$ by our note above.

Finally, we remove concave end-neighborhoods for $E_{l+1}, \ldots, E_k$ or add lens end neighborhoods by Theorems 10.2 and 10.1. The resulting orbifold is a strict SPC-orbifold again and we can apply the result (i) to this case and obtain (ii). 

10.3.2 The theorem of Drutu
Conditions \((\alpha_1)\) and \((\alpha_2)\) of Theorem 4.9 in [80] are satisfied still when we drop end fundamental groups \(\pi_1(E_{n+1}), \ldots, \pi_1(E_m)\) or add them. (See also Theorem 4.22 in [80].)

For the condition \((\alpha_3)\) of Theorem 4.9 of [80], it is sufficient to consider only hexagons. According to Proposition 4.24 of [80] one can take the fatness constants as large as one wants, in particular \(\theta\) (measuring how fat the hexagon is) much larger than \(\chi\) prescribing how close the fat hexagon is from a left coset.

If \(\theta\) is very large, left cosets containing such hexagons in their neighborhoods can never be cosets of hyperbolic subgroups since hyperbolic groups do not contain fat hexagons. So the condition \((\alpha_3)\) is satisfied too whether one adds \(\pi_1(E_{n+1}), \ldots, \pi_1(E_m)\) or drop them. \(\square\)

### 10.3.3 Converse

We will prove the converse to Theorem 10.3. We will use the theory of tree-graded spaces and asymptotic cones [81] and its appendix.

We expand \(\mathcal{O}\) to \(\mathcal{Oe}\) by adding lens neighborhoods to totally geodesic ideal boundary components by Theorem 10.1, and we shave off every generalized lens-shaped R-ends by Theorem 10.2. We can choose horospherical end-neighbors of horospherical ends so that they are disjoint from the lenses of the \(\mathcal{T}\)-ends and concave end-neighbors of R-ends by Proposition 7.4. We will use the Hilbert metric \(d_{\mathcal{Oe}}\) of \(\mathcal{Oe}\) restricted to every orbifold below. The universal cover of \(\mathcal{Oe}\) is denoted by \(\tilde{\mathcal{Oe}}\) which again can be constructed directed from \(\mathcal{O}\) by adding lenses and removing concave p-end neighborhoods. Now \(\tilde{S_E}\) for every totally geodesic p-end \(E\) is in \(\tilde{\mathcal{O}}\).

**Proposition 10.4** Let \(\mathcal{O}\) be a noncompact strongly tame properly convex real projective orbifold with generalized admissible ends. Suppose that \(\mathcal{Oe}\) is obtained by extensions and shaving all hyperbolic ends. Now we remove the outside of the totally geodesic end neighborhoods and further shave off, and next we remove a collection of the mutually disjoint horospherical end neighborhoods. Let the resulting compact orbifold \(\mathcal{OM}\) have the restricted path-metric \(d_{\mathcal{OM}}\) on \(\tilde{\mathcal{OM}}\) restricted from the infinitesimal Finsler metric associated with \(d_{\mathcal{Oe}}\). Then \(\tilde{\mathcal{OM}}\) is quasi-isometric with \(\pi_1(\mathcal{O})\).

**Proof** Let \(\pi_1(\mathcal{O})\) have the set of generators \(g_1, \ldots, g_q\). Since we removed all the end-neighborhoods of \(\mathcal{Oe}\), our orbifold \(\mathcal{OM}\) is compact. Hence, \(\tilde{\mathcal{OM}}\) has a compact fundamental domain \(F\). We find a function \(\tilde{\mathcal{OM}} \to \pi_1(\mathcal{O})\) by defining \(gF^o\) to go to \(g\) and defining arbitrarily the faces of \(F\) to go to \(gF\), and hence there is a function from it to \(\pi_1(\mathcal{O})\) decreasing distances up to a positive constant.

Conversely, there is a function from \(\pi_1(\mathcal{O})\) to \(\tilde{\mathcal{OM}}\) by sending \(g\) to \(g(x_0)\) for a fixed \(x_0 \in F^o\). This is also distance decreasing up to a positive constant. Hence, this proves the result. \(\square\)
Let $\hat{\Omega}^M$ denote the universal cover of $\Omega^M$. We will denote by $d_{\hat{\Omega}^M}$ the above path metric restricted to $\hat{\Omega}^M$ or $\Omega^M$.

**Theorem 10.5** Let $\Omega$ be a strongly tame properly convex real projective orbifold with generalized lens-shaped or horospherical $R$- or $T$-ends and satisfies (IE) and (NA). Assume $\partial \Omega$ is smooth and strictly convex. Suppose that $\pi_1(\Omega)$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \ldots, \pi_1(E_k)$ where $E_i$ are horospherical for $i = 1, \ldots, m$ and generalized lens-shaped for $i = m + 1, \ldots, k$ for $0 \leq m \leq k$. Then $\Omega$ is strictly SPC with respect to the ends $E_1, \ldots, E_k$.

**Proof** Since an $\epsilon$-mc-p-end-neighborhood is always proper by Corollary 7.5 for sufficiently small $\epsilon$, we choose the end neighborhood $U_i$ of any generalized lens-shaped R-end $E_i$ to be the image of an $\epsilon$-mc-p-end-neighborhood for some $\epsilon > 0$. We can choose all such neighborhoods and horospherical end neighborhoods and lens-shaped end neighborhoods for $T$-ends to be mutually disjoint by Corollaries 7.4 and 7.5. Let $\hat{U}$ denote the union of the inverse images of end neighborhoods $U_1, \ldots, U_k$.

Suppose that $\Omega$ is not strictly convex. We divide into two cases: First, we assume that there exists a segment in $\partial \Omega$ not contained in the closure of a p-end neighborhood. Second, we assume that there exists a non-$C^1$-point in $\partial \Omega$ not contained in the closure of a p-end neighborhood.

1. We assume the first case now. We obtain a triangle with boundary in $\partial \Omega$ and not contained in the convex hull of p-ends: Let $l$ be a nontrivial maximal segment in $\partial \Omega$ not contained in the closure of a p-end neighborhood. First, $l$ does not meet the closure of a horospherical p-end neighborhood by Theorem 4.2. By Theorems 6.7 and 6.8 if $l^\circ$ meets the closure of a lens-shaped R-p-end neighborhood, then $l^\circ$ is in the closure. Also, suppose that $l^\circ$ meets $\hat{S}_E$ for a totally geodesic p-end $E$. Then $l^\circ \cap \partial \Omega(\hat{S}_E) \neq \emptyset$. $l$ is in the hyperspace $\hat{P}$ containing $\hat{S}_E$ since otherwise we have some points of $\hat{S}_E$ in the interior of $\Omega(\partial \hat{\Omega})$. We take a convex hull of $l \cup \hat{S}_E$ which is a domain $D$ containing $\hat{S}_E$ where $\pi_1(E)$ acts on. Then $D$ is still properly convex since so is $\Omega(\partial \hat{\Omega})$. Since $D^\circ$ has a Hilbert metric, $\pi_1(E)$ acts properly on $D^\circ$. By taking a torsion-free subgroup by Theorem 2.3, we obtain that $\pi_1(E)$ acts surjectively. Hence, $D^\circ = \hat{S}_E$. Therefore, $l^\circ \subset \Omega(\partial \hat{\Omega})$. (See Theorem 4.1 of [67] and [20].)

Since $l^\circ$ is disjoint from these sets, we also obtain that $l$ is a maximal segment in $\partial \Omega$ and $\partial \Omega^\circ$ as well. Therefore, $l$ meets the closures of p-end neighborhoods possibly only at its endpoints.

Let $P$ be a 2-dimensional subspace containing $l$ and meeting $\partial \hat{\Omega}$ outside $\hat{U}$. By above, $l^\circ$ is in the boundary of $P \cap \partial \hat{\Omega}$. Draw two segments $s_1$ and $s_2$ in $P \cap \partial \hat{\Omega}$ from the endpoint of $l$ meeting at a vertex $p$ in the interior of $\partial \hat{\Omega}$.

Let $I$ denote the index set of components of $\hat{U}$. Let $U_j$ be a component of $\hat{U}$.

Define $A_i$ to be the set of points $x$ of $l^\circ$ with an open $d$-metric ball-neighborhood in $\Omega(\partial \hat{\Omega}) \cap P$ in the closure of a single component $U_j$. By definition, $A_i$ is open in $l^\circ$. Also, $l^\circ$ is not a subset of single $A_i$ since otherwise $l$ is in the closure of $U_i$, a contradiction. Since $l^\circ$ is connected, $l^\circ - \bigcup_{i \in I} A_i$ is not empty. Choose a point $x$ in it. For any open $d$-metric ball-neighborhood $B$ of $x$ in $\Omega(\partial \hat{\Omega}) \cap P$, we cannot have
Consider any sequence of any maximal straight segment $t_i$ from $x_i$ passing a point $y_i$ of $s_1$ or $s_2$. Let us orient it in the direction of $y_i$ from $x_i$. Then let $\delta_+ t_i$ be the forward endpoint of $t_i$ and $\delta_- t_i$ the backward one. Then the $d$-distance from $y_i$ to $\delta_+ t_i$ goes to zero by the maximality of $l$, which implies the Hilbert metric result by the cross-ratio consideration.

Recall that there is a compact fundamental domain $F$ of $\partial - \bar{U}$ under the action of $\pi_1(E)$. Now, we can take $x_i$ to the fundamental domain $F$ by $g_i$. We choose $g_i$ to be a sequence of mutually distinct elements of $\pi_1(\partial)$. We choose a subsequence so that we assume without loss of generality that $\{g_i(T)\}$ geometrically converges to a convex set, which could be a point or a segment or a nondegenerate triangle. Since $g_i(T) \cap F \neq \emptyset$, and the sequence $\partial g_i(T)$ exits any compact subsets of $\partial$ always while

$$\{d_{\partial\mu}(g_i(x_i), \partial g_i(T))\} \to \infty$$

and $g_i(T)$ passes $F$, we see that a subsequence of $\{g_i(T)\}$ converges to a nondegenerate triangle, say $T_\infty$.

By following Lemma 10.4, $T_\infty$ is so that $\partial T_\infty$ is in $\bigcup S(v_F)$ for a generalized lens-shaped R-end $\bar{E}$.

Now, $T_\infty$ is so that $\partial T_\infty \subset \text{Cl}(U_1)$ for a p-end neighborhood $U_1$ of a generalized lens-shaped end $\bar{E}$. Then for sufficiently small $\epsilon > 0$, the $\epsilon$-$d_{\partial\mu}$-neighborhood of $T_\infty \cap \partial$ is a subset of $U_1$ as $U_1$ was chosen to be an $\epsilon$-mc-p-end-neighborhood (see Lemma 7.3). However as $\{g_i(T)\} \to T_\infty$ geometrically, for any compact subset $K$ of $T_\infty$, $g_i(T) \cap K$ is a subset of $U_1$ for sufficiently large $i$. But $g_i(T) \cap F \neq \emptyset$ for all $i$ and the compact fundamental domain $F$ of $\partial - \bar{U}$, disjoint from $U_1$. This is a contradiction.

(II) Now we suppose that $\text{bd} \partial$ has a non-$C^1$-point $x$ outside the closures of p-end neighborhoods. Then we go to the dual $\partial^*$ and the dual group $\Gamma^*$ where $\partial^*/\Gamma^*$ is a strongly tame properly convex orbifold with horospherical ends, lens-shaped T-ends or generalized lens-shaped R-ends by Corollary 6.3 and Theorem 2.16. Here the type $\bar{T}$ and $\bar{O}$ are switched for the correspondence between the ends of $\partial$ and $\partial^*$ by Corollary 6.3.

Then we have a one-to-one correspondence of the set of p-ends of $\partial$ to the set of p-ends of $\partial^*$, and we obtain that $x$ corresponds to a convex subset of dim $\geq 1$ in $\text{bd} \partial$ containing a segment $l$ not contained in the closure of p-end neighborhoods using the map $\partial$ in Proposition 6.11. Thus, the proof reduces to the case (I).

By Theorem 1.2, we obtain that our orbifold is strictly SPC. □

Recall that the interior of a triangle has a Hilbert metric called the hex metric by de la Harpe [77]. The metric space is isometric with a Euclidean space with norms given by regular hexagons. The unit norm of the metric is a regular hexagon ball for this metric. A regular hexagon of side length $l$ is a hexagon in the interior of a
triangle $T$ with geodesic edges parallel to the sides of the unit norms and with all edge lengths equal to $l$. The regular hexagon is the boundary of a ball of radius $l$. The center of a hexagon is the center of the ball.

**Lemma 10.4** Assume the premise of Theorem 10.5. Let $T$ be a triangle in $\tilde{O}$ with $T^o \cap \tilde{O}_M \neq \emptyset$ and $\partial T \subset \text{bd} \tilde{O}$. Then $\partial T \subset \cup S(\tilde{E})$ for an $R$-p-end $\tilde{E}$.

**Proof** Let $F$ be the fundamental domain of $\tilde{O}_M$.

Using Proposition 10.4, we obtain compact orbifold $O^M$ in $O$ by extending and shaving and removing horospherical end neighborhoods.

Again, we assume that $\pi_1(O)$ is torsion-free by Theorem 2.3 since it is sufficient to prove the result for the finite cover of $O$. Hence, $\pi_1(O)$ acts freely on $\tilde{O}$.

Let $T'$ be a triangle with $T'^o \cap \tilde{O}_M \neq \emptyset$ and $\partial T' \subset \text{bd} \tilde{O}$. Suppose that $T'$ meets infinitely many horoball p-end neighborhoods in $\tilde{O}$ of horospherical p-ends, and the $d_{\tilde{O}_M}$-diameters of $T'$ intersected with these are not bounded. We consider a sequence of such sets $A_i$ with $d_{\tilde{O}_M}$-diameter $A_i$ going to $+\infty$, and we choose a deck transformation $g_i$ so that $g_i(\text{Cl}(A_i))$ intersects the fundamental domain $F$ of $\tilde{O}_M$. We choose a subsequence so that $\{g_i(T')\}$ and $\{g_i(A_i)\}$ geometrically converge to a triangle $T''$ and a compact set $A_{\infty}$, respectively. Here, $T''$ intersects $F$ and the interior of $T''$ is in $\tilde{O}$. $g_i(A_i) = g_i(T'') \cap H_i$ for a horoball $H_i$ whose closure meets $F$. Since only finitely many closures of the horoball p-end neighborhoods in $\tilde{O}$ meet $F$, there are only finitely many such $H_i$, say $H_{i_1}, \ldots, H_{i_m}$. Now, $T''$ meets one such $H_{i_j}$ so that its vertex is in the boundary of $T''$ since the $d_{\tilde{O}_M}$-diameter of $g_i(T'') \cap H_i = g_i(A_i)$ goes to $+\infty$. This contradicts Theorem 4.2.

Thus, the $d_{\tilde{O}_M}$-diameters of horospherical p-end neighborhoods intersected with $L$ are bounded above uniformly. Therefore, by choosing a horospherical end neighborhood sufficiently far inside each horospherical end neighborhood by Corollary 7.4, we may assume that $L$ does not meet any horospherical p-end neighborhoods. That is we choose a horoball $V'$ inside a one $V$ so that

$$d_{\tilde{O}_M}(V', \partial V) > \frac{1}{2} \sup \{d_{\tilde{O}_M}\text{-diam}\{V \cap T'\} | V \in \mathcal{V}, T' \in \mathcal{T}\}$$

where $\mathcal{V}$ is the collection of horoball p-end neighborhoods that we were given in the beginning and $\mathcal{T}$ is the collection of all triangles $T'$ meeting with $\tilde{O}_M$ and with boundary in $\text{bd} \tilde{O}$. 

Recall the discussion on segments in $\text{bd} \tilde{O}$ in the beginning of (I). We obtain that $\partial T'$ is in

$$\text{bd} \tilde{O} \cap \bigcup_{\mathcal{E} \in \mathcal{E}'} (\bigcup_{E \in \mathcal{E}} S(\mathcal{E})) - \bigcup_{E \in \mathcal{T}'} S(\mathcal{E})$$

where $\mathcal{E}'$ denote the set of R-p-ends of $\tilde{O}$ and $\mathcal{E}'$ the set of T-p-ends of $\tilde{O}$ and $\mathcal{E}$ is the set of parabolic fixed points of $\tilde{O}$.

For $i$ in the index set $I$ of p-ends, we define $L_{1,i}$ to be the following subsets of $\tilde{O}_M$:

- $\text{Cl}(U(v_{E})) \cap \tilde{O}_M$ where $U(v_{E})$ is the open shaved-off concave p-end neighborhood of $E$ when $E$ is a generalized lens-shaped R-p-end,
• $\text{Cl}(S_E) \cap \partial^M$ if $E$ is a lens-shaped totally geodesic end, or
• $\text{Cl}(U_E) \cap \partial^M$ for a horoball $U_E$ for a horospherical end $E$.

By Theorem 1.5 of [80], $\pi_1(\mathcal{E})$ is relatively hyperbolic with respect to

$$\pi_1(E_1), \ldots, \pi_1(E_k)$$

if and only if every asymptotic cone $\pi_1(\mathcal{E})$ is asymptotically tree graded with respect to the collection of left cosets of

$$\mathcal{L} = \{ g\pi_1(E_i) | g \in \pi_1(\mathcal{E})/\pi_1(E_i), i = 1, \ldots, k \}.$$ 

By Theorem 5.1 of [81], $\partial^M$ is asymptotically tree graded with respect to $L_{1,i}, i = 1, 2, \ldots$, since $\pi_1(\mathcal{E})$ is quasi-isometric to $\partial^M$ with the cosets of $\pi_1(E_i)$ mapping quasi-isometrically into $L_{1,i}$ by Proposition 10.4.

For any $\theta > 0, \nu \geq 8$, a regular hexagon in $T^{o_1}$ with side length $l > \nu \theta$ is $(\theta, \nu)$-fat according to Definition 5.1 of Drutu [80]. By Theorem 4.22 of [80], there is $\chi > 0$ so that a regular hexagon $H_i$ with side length $l > \nu \theta$ is in $\chi$-neighborhood $V_1$ of $L_{1,i}$ with respect to $\partial^M$ that is either contained in a concave p-end neighborhood of an R-p-end, a CA-lens of a T-p-end or a horoball p-end neighborhood.

Choose a family of regular hexagons

$$\{H_i | H_i \subset T^{o_1}, l > \nu \theta \}$$

with a common center in $T^{o_1}$. Hence, $\bigcup_{\nu \theta} H_i = T^{o_1} - K_i$ for a bounded set $K_i, K_i \subset T^{o_1}$. By the above paragraph, $T^{o_1} - K_i \subset V_1$. Now, $\partial T' \subset \partial^M$ must be in the closure of $L_{1,i}$ by Lemma 10.5.

If $L_{1,i}$ is from a horospherical p-end, $\partial T'$ must be a point. This is absurd. In the case of a T-p-end, the hyperspace containing $T'$ must coincide with one containing $S_E$. This is absurd since $T^{o_1}$ is a subset of $\partial^M$. In the case of an R-p-end $E$, $\partial T^{o_1}$ must lie on a subset that has segments extending those segments in $S(V_1)$. Theorems 6.7 and 6.8, this means $\partial T' \subset \bigcup S(V_1)$. \[\square\]

**Lemma 10.5** Let $V$ be a $\chi$-neighborhood of $L_{1,i}$ in $\partial^M$ under the metric $d_{\partial^M}$. Then $\text{Cl}(V) \cap \partial^M = \text{Cl}(L_{1,i}) \cap \partial^M$.

**Proof** We recall the metric. We first extend $\partial^M$ and shave off to $\partial^\varepsilon$. Then we remove the parts of the lenses outside the ideal end orbifold for T-p-ends and remove horoballs of ends to obtain $\partial^M$. The path-metric induced from the extended $\partial^\varepsilon$ is denoted $d_{\partial^M}$.

Clearly, $\text{Cl}(V) \cap \partial^M = \text{Cl}(L_{1,i}) \cap \partial^M$. Suppose that $L_{1,i}$ is from a horospherical p-end. Then the equality is clear since $V$ is contained in a horospherical p-end neighborhood.

Suppose that $L_{1,i}$ is from a T-p-end of lens type. Then there is a CA-lens $L$ containing $L_{1,i}$. The closure of $V$ in $\partial^M$ has a compact fundamental domain $F_V$. Theorem 6.9 and Lemma 5.14 applied to any sequence of images of $F_V$ imply the equality.
Suppose that $L_{1,i}$ is from an $R$-p-end of lens type. Then the closure of $L_{1,i}$ in $\tilde{\mathcal{O}}^M$ has a compact fundamental domain $F_Y$. Theorem 6.6 and Lemma 6.4 and Proposition 6.3 again show the equality since the limit sets are independent of the choice of neighborhoods.

We recapitulate the results:

**Corollary 10.2** Assume that $\mathcal{O}$ is a strongly tame SPC-orbifold with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{F}$-ends and satisfies (IE) and (NA). Let

$$E_1, \ldots, E_m, E_{m+1}, \ldots, E_k$$

be the ends of $\mathcal{O}$ where $E_{m+1}, \ldots, E_k$ are some or all of the hyperbolic ends. Assume $\partial \mathcal{O} = \emptyset$. Then $\pi_1(\mathcal{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \ldots, \pi_1(E_m)$ if and only if $\mathcal{O}'$ as obtained by Theorem 10.2 is strictly SPC with respect to ends $E_1, \ldots, E_m$.

**Proof** If $\pi_1(\mathcal{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \ldots, \pi_1(E_m)$, then $\pi_1(\mathcal{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \ldots, \pi_1(E_k)$ by Theorem 10.4. By Theorem 10.5, it follows that $\mathcal{O}$ is strictly SPC with respect to the ends $E_1, \ldots, E_k$. Theorem 10.2 shows that $\mathcal{O}'$ is strictly SPC with respect to $E_1, \ldots, E_m$.

For converse, if $\mathcal{O}'$ is strictly SPC with respect to $E_1, \ldots, E_m$, then $\mathcal{O}$ is strictly SPC with respect $E_1, \ldots, E_k$. By Theorem 10.3, $\pi_1(\mathcal{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \ldots, \pi_1(E_k)$. The conclusion follows by Theorem 10.4.

### 10.3.4 A topological result

**Proposition 10.5** Let $X$ be a compact metrizable space. Let $\mathcal{C}_X$ be a countable collection of mutually disjoint compact connected sets. The collection has the property that $C \subset \mathcal{C}_X : \subset C \cap K \neq \emptyset \implies C$ is closed for any closed set $K$. We define the quotient space $X / \sim$ with the equivalence relation $x \sim y$ if and only if $x, y \in C$ for an element $C \in \mathcal{C}_X$. Then $X / \sim$ is metrizable.

**Proof** We show that $X / \sim$ is Hausdorff, 2-nd countable, and regular and use the Urysohn metrization theorem. We define a countable collection $\mathcal{B}$ of open sets of $X$ as follows: We take an open subset $L$ of $X$ that is an $\varepsilon$-neighborhood for $\varepsilon \in \mathbb{Q}, \varepsilon > 0$ of an element of $\mathcal{C}_X$ or a point of a dense countable set $Y$ in $X - \bigcup \mathcal{C}_X$. We form

$$L - \bigcup_{C \cap \text{bd} L \neq \emptyset, C \in \mathcal{C}_X} C$$

for all such $L$ containing an element of $\mathcal{C}_X$ or a point of $Y$. This is an open set by the premise since $\text{bd} L$ is closed. The elements of $\mathcal{B}$ are neighborhoods of elements of $\mathcal{C}_X$ and $Y$. Also, each element of $\mathcal{C}_X$ or a point of $Y$ is contained in an element of
Furthermore, each element of $\mathcal{B}$ is a saturated open set under the quotient map. Hence, $X/\sim$ is Hausdorff and 2-nd countable.

Now, the proof is reduced to showing that $X/\sim$ is regular. For any saturated compact set $K$ in $X$ and a disjoint element $Y$ of $\mathcal{C}_X$ or a point of $X$ not in any of $\mathcal{C}_X$, let $U_K$ and $U_Y$ denote the disjoint neighborhoods of $X$ of $K$ and $Y$ respectively. We form

$$U := U_K - \bigcup_{C \cap \text{bd} U_K \neq \emptyset, C \in \mathcal{C}_X} C,$$

and

$$V := V_K - \bigcup_{C \cap \text{bd} V_K \neq \emptyset, C \in \mathcal{C}_X} C.$$

Then these are disjoint open neighborhoods. \qed
Chapter 11
Openness and closedness

Lastly, we will prove the openness and closedness of the properly (resp. strictly) convex real projective structures on the deformation spaces of a class of orbifolds with generalized lens-shaped or horospherical $R$- or $T$-ends. We need the theory of Crampon and Marquis and Cooper, Long, and Tillmann on the Margulis lemma for convex real projective manifolds. The theory here partly generalizes that of Benoist on closed real projective orbifolds. In Section 11.1, we give some definitions and state the main results of the monograph: various Ehresmann-Thurston-Weil principles holding in certain circumstances. In Section 11.2, we state the openness results that we will prove in this chapter. Mainly, we will use fixing-sections to prove the results here. We will show that the small deformations preserve the convexity. The idea is to use the Hessian functions in the compact part and use the approximation of the original domain by the covering domains of the end neighborhoods. In Section 11.3, we will show the closedness of the convexity under the deformations, first assuming the irreducibility of the holonomy representations. In Section 11.3.3, we show that we actually do not need to assume the irreducibility a priori. Any sequence of properly convex real projective structures will converge to the one whenever the corresponding sequence of representations converges algebraically. In Section 11.4, we prove Theorem 11.3, the most general result of this monograph. Here, we show the natural existence of the fixing section.

11.1 Introduction

We will allow for these structures that a radial lens-cone end could change to a horospherical type and vice versa, and a totally geodesic lens end could change to a horospherical one and vice versa. However, we will not allow a radial lens-cone end to change to a totally geodesic lens end.

For a strongly tame orbifold $\mathcal{O}$, we recall conditions in Definition 1.4.

(IE) $\mathcal{O}$ or $\pi_1(\mathcal{O})$ satisfies the infinite-index end fundamental group condition (IE) if $[\pi_1(E) : \pi_1(\mathcal{O})] = \infty$ for the end fundamental group $\pi_1(E)$ of each end $E$. 

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(NA) $\mathcal{O}$ or $\pi_1(\mathcal{O})$ satisfies the nonannular property if

$$\pi_1(\tilde{E}_1) \cap \pi_1(\tilde{E}_2)$$

is finite for two distinct p-ends $\tilde{E}_1$ and $\tilde{E}_2$ of $\mathcal{O}$.

The following theorems are to be regarded as examples of the so-called Ehresmann-Thurston-Weil principle.

**Theorem 11.1** Let $\mathcal{O}$ be a strongly tame $n$-orbifold with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends. Assume $\partial \mathcal{O} = \emptyset$. Suppose that $\mathcal{O}$ satisfies (IE) and (NA). Then

1. the subspace of SPC-structures

$$\text{CDef}_{\mathcal{R}, \text{u}, \text{lh}}(\mathcal{O}) \subset \text{Def}_{\mathcal{R}, \text{u}, \text{lh}}(\mathcal{O})$$

is open.

2. Suppose further that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup and $\partial \mathcal{O} = \emptyset$. Then hol maps $\text{CDef}_{\mathcal{R}, \text{u}, \text{lh}}(\mathcal{O})$ homeomorphically to a union of components of

$$\text{rep}_{\mathcal{R}, \text{u}, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})).$$

The proof of Theorem 11.1 and that of following Theorem 11.2 are as follows: The openness follows from Theorem 11.4 by the uniqueness section obtained by Lemma 1.5. Corollary 11.4 proves the closedness. For the first item of Theorem 11.2, we give:

**Remark 11.1** A strongly tame SPC-orbifold $\mathcal{O}$ with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends satisfying (IE) and (NA) is strictly SPC with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends if and only if $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to its end fundamental groups. Corollary 7.7 shows this by Theorems 10.3 and 10.5.

**Theorem 11.2** Let $\mathcal{O}$ be a strongly tame strictly SPC $n$-dimensional orbifold with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. Then

1. $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to its end fundamental groups.

2. The subspace $\text{SDef}_{\mathcal{R}, \text{u}, \text{lh}}(\mathcal{O}) \subset \text{Def}_{\mathcal{R}, \text{u}, \text{lh}}(\mathcal{O})$, of strict SPC-structures with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends is open.

3. Suppose further that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup and $\partial \mathcal{O} = \emptyset$. Then hol maps the deformation space $\text{SDef}_{\mathcal{R}, \text{u}, \text{lh}}(\mathcal{O})$ of strict SPC-structures on $\mathcal{O}$ with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends homeomorphically to a union of components of

$$\text{rep}_{\mathcal{R}, \text{u}, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})).$$
Finally, we use the eigenvector-sections to prove in Section 11.4:

**Theorem 11.3 (Main result of the monograph)** Let $\mathcal{O}$ be a strongly tame $n$-dimensional SPC-orbifold with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. Then

- Suppose that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup and $\partial \mathcal{O} = \emptyset$. Then hol maps the deformation space $CDef_{\epsilon, lh}(\mathcal{O})$ of SPC-structures on $\mathcal{O}$ with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends homeomorphically to a union of components of

  $$\text{rep}_{\epsilon, lh}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).$$

- Suppose that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup and $\partial \mathcal{O} = \emptyset$. Then hol maps the deformation space $SDef_{\epsilon, lh}(\mathcal{O})$ of strict SPC-structures on $\mathcal{O}$ with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends homeomorphically to a union of components of

  $$\text{rep}_{\epsilon, lh}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).$$

For example, these apply to hyperbolic manifolds with torus boundary as in [8].

### 11.2 The openness of the convex structures

In this section also, we will only need $\mathbb{RP}^n$ versions. Given a strongly tame real projective orbifold $\mathcal{O}$ with $e_1 \mathcal{R}$-ends and $e_2 \mathcal{I}$-ends, each end $E_i$, $i = 1, \ldots, e_1, e_1 + 1, \ldots, e_1 + e_2$, has an orbifold structure of dimension $n - 1$ and inherits a real projective structure.

Let $\mathcal{U}$ and $s_{\mathcal{U}} : \mathcal{U} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^n)^{e_2}$ be as in Section 9.2.2.

- We define $\text{Def}_{\epsilon, s_{\mathcal{U}}, lh}(\mathcal{O})$ to be the subspace of $\text{Def}_{\epsilon, lh}(\mathcal{O})$ of real projective structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends determined by $s_{\mathcal{U}}$, and stable irreducible holonomy homomorphisms in $\mathcal{U}$.
- We define $CDef_{\epsilon, s_{\mathcal{U}}, lh}(\mathcal{O})$ to be the subspace consisting of SPC-structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends in $\text{Def}_{\epsilon, s_{\mathcal{U}}, lh}(\mathcal{O})$.
- We define $SDef_{\epsilon, s_{\mathcal{U}}, lh}(\mathcal{O})$ to be the subspace of consisting of strict SPC-structures with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends in $\text{Def}_{\epsilon, s_{\mathcal{U}}, lh}(\mathcal{O})$.

**Theorem 11.4** Let $\mathcal{O}$ be a strongly tame real projective $n$-oribifold with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{I}$-ends and satisfies (IE) and (NA). Assume that $\partial \mathcal{O} = \emptyset$. For a $\text{PGL}(n+1, \mathbb{R})$-conjugation invariant open subset $\mathcal{U}$ of a union of semi-algebraic subsets of

$$\text{Hom}_{\epsilon, lh}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).$$
and a $\operatorname{PGL}(n+1, \mathbb{R})$-equivariant fixing section $s_{\mathcal{U}} : \mathcal{U} \to (\mathbb{R}P^e_1)^1 \times (\mathbb{R}P^e_2)^2$, the following are open subspaces

$$
\begin{align*}
\text{CDef}_{s_{\mathcal{U}}, \mathcal{O}}(\mathcal{O}) & \subset \operatorname{Def}_{s_{\mathcal{U}}, \mathcal{O}}(\mathcal{O}), \\
\text{SDef}_{s_{\mathcal{U}}, \mathcal{O}}(\mathcal{O}) & \subset \operatorname{Def}_{s_{\mathcal{U}}, \mathcal{O}}(\mathcal{O}).
\end{align*}
$$

For orbifolds such as these, the deformation space of convex structures may only be a proper subset of space of the characters.

By Theorem 11.4 and Theorem 9.3, we obtain:

**Corollary 11.1** Let $\mathcal{O}$ be a strongly tame real projective $n$-orbifold with generalized lens-shaped or horospherical $\mathcal{B}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Assume that $\partial \mathcal{O} = \emptyset$. Let $\mathcal{U}$ and $s_{\mathcal{U}}$ be as in Theorem 11.4. Suppose that $\mathcal{U}$ has its image $\mathcal{U}'$ in

$$
\operatorname{rep}^s_{s_{\mathcal{U}}, \mathcal{O}}(\pi_1(\mathcal{O}), \operatorname{PGL}(n+1, \mathbb{R})).
$$

Then

$$
\operatorname{hol} : \text{CDef}_{s_{\mathcal{U}}, \mathcal{O}}(\mathcal{O}) \to \mathcal{U}'
$$

is a local homeomorphism, and so is

$$
\operatorname{hol} : \text{SDef}_{s_{\mathcal{U}}, \mathcal{O}}(\mathcal{O}) \to \mathcal{U}'.
$$

**Proof** Theorem 9.3 shows that the map

$$
\operatorname{hol} : \operatorname{Def}_{s_{\mathcal{U}}, \mathcal{O}}(\mathcal{O}) \to \operatorname{rep}^s_{s_{\mathcal{U}}, \mathcal{O}}(\pi_1(\mathcal{O}), \operatorname{PGL}(n+1, \mathbb{R}))
$$

is an open one when we don’t require “ce” condition. Proposition 11.2 tells us the openness of the images here. Theorem 11.4 completes the proof.

Here, in fact, one needs to prove for every possible continuous section.

Koszul [123] proved these facts for closed affine manifolds and expanded by Goldman [94] for the closed real projective manifolds. See [54], [67] and also Benoist [23].

### 11.2.1 The proof of the openness

The major part of showing the preservation of convexity under deformation is Proposition 11.1 on Hessian function perturbations. (These parts are already explored in [72]; however, we are studying $R$-ends and $T$-ends, and we also have conceived these ideas independently.)

We mention that our approach for openness is slightly different from that of Cooper-Long-Tillman [73] since they are using their canonical invariant hessian metrics for end-neighborhoods. Our hessian metrics for end-neighborhoods are not canonical ones as theirs are.
Recall that a convex open cone $V$ is a convex cone of $\mathbb{R}^{n+1}$ containing the origin $O$ in the boundary. Recall that a properly convex open cone is a convex cone so that its closure does not contain a pair of $v, -v$ for a nonzero vector in $\mathbb{R}^{n+1}$. Equivalently, it does not contain a complete affine line in its interior.

A dual convex cone $V^*$ to a convex open cone is a subset of $\mathbb{R}^{n+1*}$ given by the condition $\phi \in V^*$ if and only if $\phi(v) > 0$ for all $v \in \text{Cl}(V) - \{O\}$.

Recall that $V$ is a properly convex open cone if and only if so is $V^*$ and $(V^*)^* = V$ under the identification $(\mathbb{R}^{n+1*})^* = \mathbb{R}^{n+1}$. Also, if $V \subset W$ for a properly convex open cone, then $V^* \supset W^*$.

For properly convex open subset $\Omega$ of $\mathbb{S}^n$, its dual $\Omega^*$ in $\mathbb{S}^n*$ is given by taking a cone $V$ in $\mathbb{R}^{n+1}$ corresponding to $\Omega$ and taking the dual $V^*$ and projecting it to $\mathbb{S}^n*$. The dual $\Omega^*$ is a properly convex open domain if so was $\Omega$.

Recall the Koszul-Vinberg function for a properly convex cone $V$ and the dual properly convex cone $V^*$

$$f_{V^*} : V \rightarrow \mathbb{R}_+, \text{ defined by } x \in V \mapsto f_{V^*}(x) = \int_{V^*} e^{-\phi(x)} d\phi \quad (11.1)$$

where the integral is over the euclidean measure in $\mathbb{R}^{n+1*}$. This function is strictly convex if $V$ is properly convex. $f_{V^*}$ is homogeneous of degree $-(n+1)$. Writing $D$ as the affine connection, we will write the Hessian $Dd\log(f)$. The hessian is positive definite and norms of unit vectors are strictly bounded below in a compact subset $K$ of $V - \{O\}$. (See Chapter 4 of [91] and [162].) The metric $Dd\log(f)$ is invariant under the group $\text{Aff}(V)$ of affine transformation acting on $V$. (See Theorem 6.4 of [91].) In particular, it is invariant under scalar dilatation maps.

A Hessian metric on an open subset $V$ of an affine space is a metric of form $\partial^2 f / \partial x_i \partial x_j$ for affine coordinates $x_i$ and a function $f : V \rightarrow \mathbb{R}$ with a positive definite Hessian defined on $V$. A Riemannian metric on an affine manifold is a Hessian metric if the manifold is affinely covered by a cone and the metric lifts to a Hessian metric of the cone.

Let $\mathcal{O}$ have an SPC-structure $\mu$ with generalized lens-shaped or horospherical $\mathcal{B}$- or $\mathcal{F}$-ends. Clearly $\mathcal{O}$ is a properly convex open domain. Then an affine suspension of $\mathcal{O}$ has an affine Hessian metric defined by $Dd\phi$ for a function $\phi$ defined on the cone in $\mathbb{R}^{n+1}$ corresponding to $\mathcal{O}$ by above.

A parameter of real projective structures $\mu_t, t \in [0, 1]$ on a strongly tame orbifold $\mathcal{O}$ is a collection so that the restriction $\mu_t|K$ to each compact suborbifold $K$ is a continuous parameter; In other words, the associated developing map $\text{dev}_t : \mathcal{K} \rightarrow \mathbb{S}^n$ (resp. $\mathbb{RP}^n$) for every compact subset $K$ of $\mathcal{O}$ is a family in the $C^r$-topology continuous for the variable $t$. (See Definition 9.4, Choi [54] and Canary [36].)

**Definition 11.1** Let $\mathcal{O}$ be a strongly tame orbifold with ends. Let $U$ be a union of mutually disjoint end neighborhoods, and let $\mu_0$ and $\mu_1$ be two real projective structures on $\mathcal{O}$. Let $\text{dev}_0, \text{dev}_1 : \mathcal{O} \rightarrow \mathbb{S}^n$ be extended developing maps. We say that $\mu_0$ and $\mu_1$ on $\mathcal{O}$ are $\delta$-close in the $C^r$-topology, $r \geq 2$, on the compactification $\mathcal{O}^*$ if for a compact path-connected domain $K$ in $\mathcal{O}$ mapping onto $\mathcal{O}^*$, the associated
developing maps \( g_0 \circ \text{dev}_0 | K \) and \( g_1 \circ \text{dev}_1 | K \) are \( \delta \)-close in \( C^r \)-topology for some \( g_0 \) and \( g_1 \) in \( SL_{\pm} (n+1, \mathbb{R}) \).

**Proposition 11.1** Let \( \mathcal{O} \) be a strongly tame orbifold with ends and satisfies (IE) and (NA). Suppose that \( \mathcal{O} \) has an SPC structures \( \mu_0 \) with generalized lens-shaped or horospherical \( \mathcal{A} \)- or \( \mathcal{T} \)-ends and the affine suspension of \( \mathcal{O} \) with \( \mu_0 \) has a Hessian metric. The ends of \( \mathcal{O} \) are given \( \mathcal{A} \)-type or \( \mathcal{T} \)-types. Suppose that one of the following holds:

- \( \mu_0 \) is SPC, and a \( C^r \)-continuous parameter, \( r \geq 2 \), of real projective structure \( \mu_t \), \( t \in [0,1] \), radial or totally geodesic ends with end holonomy groups of generalized lens-shaped or horospherical \( \mathcal{A} \)- or \( \mathcal{T} \)-ends where the \( \mathcal{A} \)-types or \( \mathcal{T} \)-types of ends are preserved.
- We may also let \( \mu_{t_i} \) to be a sequence for \( t_i \in [0,1] \) with \( \{ \mu_{t_i} \} \to \mu_0 \) as \( t_i \to 0 \) in the \( C^r \)-topology for \( r \geq 0 \).

Then for sufficiently small \( t \), the affine suspension \( C(\breve{\mathcal{O}}) \) for \( \breve{\mathcal{O}} \) with \( \mu_t \) also has a Hessian metric invariant under the group of dilatations.

**Proof** We will prove for \( S^n \). It will be sufficient since we aim to obtain the Hessian metric on \( C(\breve{\mathcal{O}}) \). We will keep \( \breve{\mathcal{O}} \) and the action of the deck transformation group fixed and only change the structures on it. Note that the subsets here remain fixed and the only changes are on the real projective structures, i.e., the atlas of charts to \( S^n \).

Let \( \breve{\mathcal{O}} \) in \( S^n \) denote the universal covering domain corresponding to \( \mu_0 \). Again \( \text{dev}_0 \) being an embedding identifies the first with subsets of \( S^n \) but \( \text{dev}_t \) is not known to be so. We shall prove this below.

We will prove this by steps:

(A) The first step is to understand the deformations of the end-neighborhoods.

(B) We change the Hessian function on the cone associated with the universal covers.

We need to obtain one for the deformed end neighborhoods by Hessian functions from Koszul-Vinberg integrals and another one the outside of the union of end neighborhoods by isotopies and patch the two together.

(A) Let \( \breve{E} \) be a p-end of \( \breve{\mathcal{O}} \), and it corresponds to a p-end of \( \breve{\mathcal{O}}' \) as well. Let \( E \) be the end of \( \mathcal{O} \) corresponding to \( \breve{E} \). There exists a \( C^r \)-parameter of real projective structures \( \mu_t \) with generalized lens-shaped or horospherical \( \mathcal{A} \)- or \( \mathcal{T} \)-ends. We can also find a parameter of developing maps \( \text{dev}_t \) associated with \( \mu_t \) where \( \text{dev}_t | K \) is a continuous with respect to \( t \) for each compact \( K \subset \breve{\mathcal{O}} \). To begin with, we assume that \( \breve{E}' \) keeps being a lens-shaped or horospherical p-end.

Let \( h_t \) denote the holonomy homomorphism associated with \( \text{dev}_t \) for each \( t \). Recall that Theorems 7.1 and 7.2 study the perturbation of lens-shaped R-ends and T-ends. Lemma 9.4 studies the perturbations of a horospherical \( \mathcal{A} \)- or \( \mathcal{T} \)-end to either a R-end or to a T-end.

In this monograph, we do not allow \( \mathcal{A} \)-type ends to change to \( \mathcal{T} \)-type ends and vice versa as this will make us to violate the local injectivity property from the deformation space to a space of characters. (See Theorem 9.1.) Let \( \breve{E} \) be a p-end. Thus, we need to consider only four cases to prove openness:
11.2 The openness of the convex structures

(I) \( h_t(\tilde{E}) \) changes from the holonomy group of a radial \( p \)-end to that of a radial \( p \)-end in the cases:

(a) \( h_t(\tilde{E}) \) changes from the holonomy group of a radial \( p \)-end of generalized lens-shaped becoming that of a generalized lens-shaped radial \( p \)-end.

(b) \( h_t(\tilde{E}) \) changes from the holonomy group of a horospherical \( p \)-end to that of a generalized lens-shaped radial \( p \)-end or a horospherical \( p \)-end.

\[ \square \]

(II) \( h_t(\tilde{E}) \) changes from the holonomy group of totally geodesic ends of lens type or horospherical \( \mathcal{R} \)- or \( \mathcal{T} \)-ends changes to that of themselves here.

(a) \( h_t(\tilde{E}) \) changes from the holonomy group of a lens-shaped totally geodesic \( p \)-end to that of a lens-shaped totally geodesic \( p \)-end.

(b) \( h_t(\tilde{E}) \) changes from the holonomy group of a horospherical \( p \)-end to that of a horospherical \( p \)-end or to a lens-shaped totally geodesic \( p \)-end.

These hold for the corresponding holonomy homomorphisms of the fundamental groups of ends by the premise. (The above happens in actuality as well. See [5],[6], and [8].)

We will now work on one end at a time: Let us fix a \( p \)-end \( \tilde{E} \) of \( \mathcal{R} \)-type of \( \tilde{\mathcal{O}} \). Let \( v \) be the \( p \)-end vertex of \( \tilde{E} \) for \( \mu_0 \) and \( v' \) that for \( \mu_1 \). We denote by \( v = v_0 \) and \( v' = v_1 \).

Assume that \( v_i \) is the \( p \)-end vertex of \( \tilde{E} \) for \( \mu_i \). Let \( \text{dev}_i \) and \( h_i \) denote the developing map and the holonomy homomorphism of \( \mu_i \). Assume first that the corresponding \( p \)-end for \( \mu \) is of radial or horospherical type. By post-composing the developing map by a transformation near the identity, we assume that the perturbed vertex \( v_i \) of the corresponding \( p \)-end \( \tilde{E} \) is mapped to \( v_0 \), i.e., \( v = \text{dev}_i(v_i) \).

(1) Suppose that \( \tilde{E} \) is a generalized lens-shaped radial \( p \)-end or a horospherical \( p \)-end for \( \mu_0 \). Then the holonomy group of \( \tilde{E} \) is that of a generalized lens-shaped radial \( p \)-end or a horospherical \( p \)-end for \( \mu_0 \) under (I).

Let \( \Lambda_0 \) denote the limit set in the tube of the radial \( p \)-end \( \tilde{E} \) for \( \tilde{\mathcal{O}} \) if \( \tilde{E} \) is lens-shaped radial \( p \)-end, or \( \{v_\tilde{E} = v\} \) if \( \tilde{E} \) is a horospherical type for \( \mu_0 \). (See Definition 7.1.)

- Recall that \( R_v(\text{dev}_i(\tilde{\mathcal{O}})) \) denotes the space of directions of segments from \( v \) in \( \text{dev}_i(\tilde{\mathcal{O}}) \).
- \( R_v(\text{dev}_i(A_i)) \) denotes the space of directions of segments from \( v \) of \( \text{dev}_i(\tilde{\mathcal{O}}) \) in \( R_v(\tilde{\mathcal{O}}) \) passing through the set \( \text{dev}_i(A_i) \subset \text{dev}_i(\tilde{\mathcal{O}}) \).

(i) We first find domains \( \Omega_{s_0} \) with smooth boundary approximating \( \tilde{\mathcal{O}} \) on which \( h(\pi_1(\tilde{E})) \) acts. Here \( s_0 \) will be a parameter that we will use to vary \( \Omega_{s_0} \) for fixed holonomy representations. Here, \( \Omega_{s_0} \) is a lens-cone for \( \tilde{\mathcal{O}} \) with \( \mu_0 \) so that \( \partial \Omega_{s_0} = \partial \mathcal{O}_{s_0} \) which is the top boundary component of the lens.

- Suppose that \( \tilde{E} \) is a lens-shaped lens-shaped \( \mathcal{R} \)-\( p \)-end for \( \mu_0 \). Then we obtain a hypersurface \( \partial \mathcal{O}_{s_0} \) as the top boundary component of a CA-lens as obtained by Theorem 6.1. The property of the strict lenses of Theorems 6.7 and 6.8 imply

\[ \text{Cl}(\partial \mathcal{O}_{s_0}) - \partial \mathcal{O}_{s_0} \subset \Lambda_{s_0} \]
for the limit set $\Lambda_{t_0}$ of $\tilde{E}$ since a generalized lens-shaped end also satisfies the uniform middle eigenvalue condition by Theorem 6.6. Also, each radial geodesic is transverse to $\partial \Omega_s \cup S(v)$ bounds a properly convex domain $\Omega_s$.

- Suppose that $\tilde{E}$ is a horospherical $R$-p-end for $\mu_t$. Then we obtain a $\partial \Omega_s$ as a boundary of a convex domain invariant under $h_0(\pi_1(E))$. Again $\Omega_s \cup \{v\}$ bounds a properly convex domain $\Omega_{s_0}$.

Choose some small $\varepsilon > 0$. By Lemma 7.2, if $\tilde{E}$ is a lens-shaped $R$-p-end or a horospherical $R$-p-end, we may choose $\Omega_{s_0}$ that

$$d_H(\Omega_{s_0}, \tilde{\partial}) < \varepsilon.$$  

Suppose that $\tilde{E}$ is merely a generalized lens-shaped $R$-p-end. Then we can still form a lens-cone neighborhood $V$ of in the tube domain corresponding to the directions of the $\mu$-end domain $\tilde{\Sigma}_{\tilde{E}}$ for $\tilde{E}$. We can choose a smooth top boundary component approximates $\partial \tilde{\partial}$ from outside by Proposition 6.6. Hence, any $\varepsilon > 0$, sufficiently large 0 satisfies

$$d_H(\Omega_s, \tilde{\partial}) < \varepsilon$$

still holds. Here, $\partial \Omega_s$ may not be a subset of $\tilde{\partial}$ unless $\tilde{E}$ is lens-shaped and not just generalized lens-shaped. However, this is irrelevant for our purposes.

For each $\varepsilon$, we will choose the parameter $s_0$ so that the above is satisfied. So, $s_0$ is considered to vary for our purposes.

(ii) Now our purpose is to find a convex domain $\Omega_{s_0,t}$ where $h_t(\pi_1(E))$ acts on and approximating $\Omega_{s_0}$, and show that it contains an embedded image of a $p$-end neighborhood. We denote by $\tilde{\Sigma}_{E,t}$ the universal cover of the end orbifold associated with for a $p$-end $\tilde{E}$ of $\tilde{\partial}$ with $\mu_t$. By our end holonomy group condition in the premise, Corollary A.1 shows that $\tilde{\Sigma}_{E,t}$ is again complete affine or properly convex.

For the $R$-type end $E$, $\tilde{\partial}$ has a concave end-neighborhood or a horospherical end-neighborhood for $E$ bounded by a smooth compact end orbifold $S_E$ transverse to the radial rays. Here, $S_E$ is diffeomorphic to $\Sigma_E$ clearly.

For a sufficiently small $t$ in $\mu_t$, we obtain a domain $U_t \subset \partial$ with $U_{s_0} \subset \Omega_{s_0}/h_0(\pi_1(\tilde{E}))$ bounded by an inverse image of a compact orbifold $S_{E,t}$ diffeomorphic to $S_E$ still transverse to radial rays by Propositions 6.4 and 6.5. $S_{E,t}$ is either strictly concave if $S_E$ was strictly convex if $\Sigma_E$ was horospherical. (The strict convexity and the transversality follow since the change of affine connections are small as the argument of Koszul [123].)

Since the change was sufficiently small, we may assume that $S_{E,t}$ still bounds an end-neighborhood $U_t$ of product form by Lemma 11.1.

**Lemma 11.1** Suppose that $\tilde{U}$ is a $R$-p-end neighborhood of $\tilde{\partial}$ covering an end-neighborhood $U$ of an end $E$ in $\tilde{\partial}$. Then for sufficiently small change of real projective structures in $C^r$-sense, $r \geq 2$, in the compact open topology, hypersurface $S_t$ sufficiently close to $S$ in the $C^0$-sense in terms of a parameterizing map still bounds
an end neighborhood of $E$. Letting $S_t$ be a component of the inverse image of $S$ on which $\pi_1(\tilde{E})$ acts, we still have that $S_t$ bounds a p-end neighborhood of $\tilde{E}$.

Proof Straightforward.

Proof (Proof of Proposition 11.1 continued) Let $\mathcal{A}_t$ denote the limit set in $\bigcup S(v)_t$ for generalized radial p-end cases and $\mathcal{A}_t = \{v\}$ for the horospherical case. Let $S(v)_t$ denote the set of maximal segments in the closure of $U_t$ from $v$ corresponding to $\partial \Sigma_t$.

Suppose that $\tilde{E}$ is a lens-shaped R-p-end for $\mu_0$. We showed above that the $C^r$-change $r \geq 2$ of $\mu_0$ be sufficiently small so that we obtain a region $\Omega_{\mu_0,t}$ in $\mathbb{B}_t^0$ with $\partial \Omega_{\mu_0,t}$ strictly convex and transverse to radial rays under $\text{dev}_t$. Here, $\Omega_{\mu_0,0} = \Omega_{\mu_0}$.

Choose a compact domain $F$ in $\partial \Omega_{\mu_0,t}$. Let $F_t$ denote the corresponding deformed set in $\partial \Omega_{\mu_0,t}$. By Theorem 4.1, $\pi_1(\tilde{E})$ is virtually abelian. For sufficiently small $t$, $0 < t < 1$, $\text{dev}_t(F_t)$ is a subset of the tube $B_t$ determined by $\text{dev}_t(U_t)$ since $B_t$ and a parameterization of $\text{dev}_t(F_t)$ depends continuously on $t$ by Corollary A.2.

- By transversality to the segments mapping to ones from $v$ under $\text{dev}_t$, it follows that $\text{dev}_t|\partial \Omega_{\mu_0,t}$ gives us a smooth immersion to a convex domain $\tilde{\Sigma}_{E,t}$ that equals the space of maximal segments in $B_t$ with vertices $v$ and $v_-$.

- By Corollary A.1, the immersion $\text{dev}_t|\partial \Omega_{\mu_0,t}$ to a properly convex domain $\tilde{\Sigma}_{E,t}$ is a diffeomorphism if $\tilde{\Sigma}_{E,t}$ is properly convex. It is also so if $h_t(\pi_1(\tilde{E}))$ is horospherical since this follows from the classical Bieberbach theory using the Euclidean metric on $\tilde{\Sigma}_{E}$ where it develops.

Since $\pi_1(\tilde{E})$ is the fundamental group of a generalized lens-shaped or horospherical p-end, it follows that

$$\text{dev}_t(\text{Cl}(\partial \Omega_{\mu_0,t}) - \partial \Omega_{\mu_0,t}) \subset \text{dev}_t(\mathcal{A}_t)$$

by Theorems 4.2, 6.7, and 6.8.

Suppose that $\tilde{E}$ is a generalized lens-shaped or lens-shaped R-p-end for $\mu_0, t > 0$. Since $\partial \Omega_{\mu_0,t}$ is convex, each point of $\partial \Omega_{\mu_0,t} \cup \bigcup S(v)_t$ has a neighborhood that maps under the completion $\text{dev}_t$ to a convex open ball. Thus, $\text{dev}_t(\Omega_{\mu_0,t} \cup \bigcup S(v)_t)$ bounds a compact ball $\Omega_{\mu_0,t} \cup \bigcup S(v)_t$ by Lemma 2.17 since the local convexity implies the global convexity and they are in $B_t$.

Suppose that $\tilde{E}$ is a horospherical R-p-end. For $t > 0$, $\partial \Omega_{\mu_0,t} \cup \{v\}$ bounds a convex domain $\Omega_{\mu_0,t}$ by the local convexity of the boundary set $\partial \Omega_{\mu_0,t} \cup \{v\}$ and Lemma 2.17.

Proposition 11.2 Assume as in Proposition 11.1, and (I) in the proof. Then for $\mu_t$, for sufficiently small $t$, the end corresponding to $E$ is always generalized lens-shaped R-end or a horospherical R-end. Also, if $\mu'$ is sufficiently $C^r$-close to $\mu$, then the end of $\tilde{E}$ with $\mu'$ is a generalized lens-shaped R-end or a horospherical R-end.

Proof The above arguments prove this since we are studying arbitrary deformations. □
Proof (Proof of Proposition 11.1 continued) (iii) We will show how these regions deform approximating $\Omega_{0,t}$ in the Hausdorff metric sense. We define $\tau_v(K)$ the union of great segments with an endpoint $v$ in directions of $K, K \subset R_v$.

- Let $K$ be a compact convex subset of $\Omega_{0,t}$ with smooth boundary, and $K_t$ the perturbed one in $\Omega_{0,t}$ and $\tilde{E}$ be the corresponding p-end. We can form a compact set inside $\Omega_{0,t}$ consisting of segments from the p-end vertex to $K$ in the set of radial segments. For $\mu_t$ from $\mu_0$ changed by a sufficiently small manner, a compact subset $\tau_v(K) \subset \tau_v(\tilde{E})$ is changed to a compact convex domain $\tau_v(K_t) \subset \tau_v(\tilde{E}_t)$.

We choose $s_0$ large enough so that $K \subset \Omega_{0,t}$. For sufficiently small $t$, $\Omega_{0,t} \cap \tau_v(K_t)$ is a convex domain since $\partial\Omega_{0,t}$ is strictly convex and transverse to great segments from $v$ and hence embeds to a convex domain under $\text{dev}_v$. We may assume that $\Omega_{0,t} \cap \tau_v(K_t)$ is sufficiently close to $\Omega_{0,t} \cap \tau_v(K)$ as we changed the real projective structures sufficiently small in the $C^\infty$-sense. See Definition 11.1.

An $\varepsilon$-thin space is a space which is an $\varepsilon$-neighborhood of its boundary for small $\varepsilon > 0$. By Lemma 1.1 and Corollary A.2, we may assume that $\text{Cl}(\tau_v(\tilde{E}))$ and $\text{Cl}(\tau_v(\tilde{E}))$ are $\varepsilon$-d-close convex domains in the Hausdorff sense for sufficiently small $t$. Thus, given an $\varepsilon > 0$, we can choose $K$ and $K'_t$ and a sufficiently small deformation of the real projective structures so that $\Omega_{0,t} \cap (\tau_v(\tilde{E}) - \tau_v(K))$ is an $\varepsilon$-thin space, and so is $\Omega_{0,t} \cap (\tau_v(\tilde{E}) - \tau_v(K_t))$ for sufficiently small changes of $t$. Moreover,

$$\text{Cl}(\Omega_{0,t}) \cap (\tau_v(\tilde{E}) - \tau_v(K)) \subset N_\varepsilon(\text{Cl}(\Omega_{0,t} \cap \tau_v(K)))$$

$$\text{Cl}(\Omega_{0,t}) \cap (\tau_v(\tilde{E}) - \tau_v(K_t)) \subset N_\varepsilon(\text{Cl}(\Omega_{0,t} \cap \tau_v(K_t))); \quad (11.2)$$

The reason is that the sharply supporting hyperspaces of $\text{Cl}(\Omega_{0,t})$ at points of $\partial\tau_v(K) \cap \text{Cl}(\Omega_{0,t})$ are in arbitrarily small acute angles from geodesics from $v$ and similarly for those of $\text{Cl}(\Omega_{0,t})$ for sufficiently small $t$ by Corollary 6.4.

Therefore we conclude for (I) that for any $\varepsilon > 0$, there exists $\delta, \delta > 0$

$$\text{d}_H(\text{Cl}(\Omega_{0,t}), \text{Cl}(\Omega_{0,t})) < \varepsilon$$

(11.3) provided $|t| < \delta$; that is, we choose $\mu_{0,t}$ sufficiently close to $\mu_0$: First, we choose $K$ and deformation $K_t$ so that it satisfies (11.2) for $t < \delta$ for some $\delta > 0$. Then we choose $t$ sufficiently small so that $\Omega_{0,t} \cap R(K_t)$ is sufficiently close to $\Omega_{0,t} \cap R(K)$.

Also, $\Omega_{0,t}$ contains a concave p-end neighborhood of $\tilde{E}$ for $\mu_t$ for sufficiently small $t > 0$. This can be assured by taking $K$ sufficiently large containing a fundamental domain of $R_t(\tilde{E})$ and sufficiently large $K_t$ containing a fundamental domain $F_t$ of $R_t(\text{dev}_v(\tilde{E}))$ for $h_t(\pi_t(\tilde{E}))$ deformed from $F$ by Proposition 11.2.

(II) Now suppose that $\tilde{E}$ is a lens-shaped T-p-end or horospherical p-end of type $\mathcal{T}$, and we suppose that $h_t(\tilde{E})$ is a lens-shaped T-p-end for $\mu_t$ for $t > 0$. Other cases are similar to (I).

We take $\Omega_{0,t}$ to be the convex domain obtained as in Lemma 7.2 with strictly convex boundary component $\partial_t\Omega_{0,t}$ and totally geodesic one $\tilde{S}_{E,0}$ in $\partial\Omega_{0,t}$. Now,
\( \Omega_{0,t}/h(\pi_1(E)) \) has strictly convex boundary component \( \partial \Omega_{0,t}/h(\pi_1(E)) \) and totally geodesic boundary \( \bar{S}_{E,t}/h(\pi_1(E)) \) when \( E \) is a T-p-end. If \( E \) is a horospherical end, \( \partial \Omega_{0,t}/h(\pi_1(E)) \) still is a strictly convex compact \((n-1)\)-orbifold.

Here, we note \( \partial \Omega = \partial \Omega_{0,t} \cup \text{Cl}(\bar{S}_E) \).

Suppose that \( \mu_t \) is sufficiently close to \( \mu_0 \). Then by Theorem 7.2, we deform the lens-shaped T-end for \( E \):

- we obtain a properly convex domain \( \Omega_{0,t} \) for sufficiently small \( t \) with a strictly convex boundary \( \partial \Omega_{0,t} \) and \( \bar{S}_E \), and
- \( \partial \Omega_{0,t} \) is also cocompact under the \( \pi_1(E) \)-action associated with \( \mu_1 \) and strictly convex.

See Proposition 6.4. We choose \( \Omega_{0,t} \) to be \( L_0 \cap \bar{\theta} \) for a lens in an ambient orbifold containing \( L_0 \). We also have \( \partial \Omega_{0,t} = \partial \Omega_{0,t} \cup \text{Cl}(\bar{S}_{E,t}) \) where \( \bar{S}_{E,t} \) is the ideal boundary component for \( E \) for \( \mu_t \). By Proposition 6.4, \( L_0 \) deforms to a properly convex domain \( L_t \) so that \( L_t \cap \bar{\theta}_t \) is \( \Omega_{0,t} \) where \( \bar{\theta}_t \) is \( \theta \) with a real projective structure \( \mu_t \). We have

\[
\text{Cl}(\partial \Omega_{0,t}) - \partial \Omega_{0,t} = \partial \text{Cl}(\bar{S}_{E,t})
\]

for a totally geodesic ideal boundary component \( \bar{S}_{E,t} \) by Theorem 5.4. Therefore the union of \( \partial \Omega_{0,t} \) and a totally geodesic ideal boundary component \( \text{Cl}(\bar{S}_{E,t}) \) bounds a properly convex compact \( n \)-ball in \( \mathbb{S}^n \). We can find a properly convex lens \( L_0 \) for \( E \) at \( \mu_0 \) with \( L_0 \cap \bar{\theta} \) in a p-end neighborhood \( \Omega_{0,t} \). Since the change of \( \mu_t \) is sufficiently small, \( L_t \cap \bar{\theta}_t \) is still in a p-end neighborhood of \( E \) by Lemma 11.1. We obtain a lens-shaped p-end neighborhood for \( E \) and \( \theta \) with \( \mu_t \) by Theorem 5.4.

We may assume without loss of generality that the hyperspace \( V_E \) containing \( S_{E,t} \) is fixed. Moreover, we may assume by choosing sufficiently large \( \Omega_{0,t} \) without loss of generality that \( d_H(\Omega_{0,t}, \bar{\theta}_t) < \varepsilon \) for any \( \varepsilon > 0 \).

By Lemma 6.4, a sharply supporting hyperspace at a point of \( \text{bd}S_E \) is uniformly bounded away from \( V_E \). A sequence of sharply supporting hyperspaces can converge to a sharply supporting one. Let us choose a sufficiently small \( \varepsilon > 0 \). Let \( B \) be a compact \( \varepsilon \)-neighborhood of \( \partial \Omega_{0,t} \) so that

\[
d_H(\partial \Omega_{0,t} - B, \partial \text{Cl}(\bar{S}_E)) < \varepsilon.
\]

Given a sharply supporting hyperspace \( W_x \) of a point \( x \) of \( \partial \Omega_{0,t} \) of \( \Omega_{0,t} \), there exists a sharply supporting closed hemisphere \( H_x \) bounded by \( W_x \).

We define the shadow \( S \) of \( \partial B \) as the set

\[
\bigcap_{x \in \partial B} H_x \cap V_E.
\]

Then we can choose sufficiently small \( \varepsilon \) so that \( d_H(S, \text{dev}(\text{Cl}(\bar{S}_E))) \leq \varepsilon \). We can also assure that \( W_x \) meets \( V_E \) in angles in \( (\delta, \pi - \delta) \) for some \( \delta > 0 \) by compactness of \( \partial B \) and the continuity of map \( x \mapsto W_x \).

Suppose that we change the structure from \( \mu_0 \) to \( \mu_t \) with a small \( C^2 \)-distance. Then \( B \) will change to \( B_t \) with \( W_x \) change by small amount. The new shadow \( S_t \) will
have the property $d_H(S', \text{Cl}(\hat{S}_{E,t}')) \leq \varepsilon$ for a sufficiently small $C'$-change, $r \geq 2$, of $\mu_t$ from $\mu_0$. Hence, we have that for each $\varepsilon > 0$, there exists $\delta, \hat{\delta} > 0$ so that

$$d_H(\partial \Omega_0, - B', \partial \text{Cl}(\hat{S}_{E,t}')) < \varepsilon$$

(11.4)

provided $|t| < \delta$. Therefore by Corollary A.2 for each $\varepsilon > 0$, there exists $\delta, \hat{\delta} > 0$ so that

$$d_H(\text{Cl}(\hat{S}_{E}), \text{Cl}(\hat{S}_{E,t}')) < \varepsilon.$$

Recall that $\text{bd} \Omega_0 = \partial \Omega_0 \cup \text{Cl} (\hat{S}_{E})$ and $\text{bd} \Omega_0 = \partial \Omega_0 \cup \text{Cl} (\hat{S}_{E,t})$. Combining with (11.4) and the sufficiently small change of $B$ to $B_t$, we obtain that each $\varepsilon > 0$, there exists $\delta, \hat{\delta} > 0$ so that

$$d_H(\text{Cl}(\Omega_0), \text{Cl}(\Omega_{0,t})) < \varepsilon$$

(11.5)

provided $|t| < \delta$.

Suppose that $\hat{E}$ was a horospherical $p$-end of type $\mathcal{T}$. Again, the argument is similar. We start with $\Omega_0$, which is horospherical with strictly convex boundary, and tangent to a hyperspace $P_0$. The deformation gives us strictly convex hypersurface $\partial \Omega_{0,t}$ and a hyperspace $P_t$ where $h_t(\pi_t(\hat{E}))$ acts on.

We assume without loss of generality that $P_t = P$ for small $t > 0$. For sufficiently small $t$, we obtain a domain bounded by $\partial \Omega_{0,t}$ and the closure $\text{Cl}(\hat{S}_{E,t})$ of a totally geodesic ideal boundary component. By taking duals by Corollary 6.3, we have lens-shaped or horospherical $R$-$p$-end $\hat{E}$. Proposition 11.2 shows that we obtain a domain $\Omega_0$, $\Omega_{0,t}$ for sufficiently small $t$. Then $\text{bd} \Omega_{0,t}$ is dual to $\partial \Omega_0$. Then $\Omega_0$ as much as one wishes to by Lemma 2.22.

We show that we have an embedded image of a p-end neighborhood in $\Omega_0$.

The hypersurface $\partial \Omega_0$ is embedded in $\mathcal{O}$ with $\mu_0$. Let $F$ be a compact fundamental domain of $\partial \Omega_0$ by $h(\pi_1(\hat{E}))$. Now, $\text{dev}_{t}^{-1}(\partial \Omega_{0,t})$ contains a compact fundamental domain $F_t$ perturbed from $F$. Let $S_t$ denote a component of the inverse image of $\partial \Omega_{0,t}$ under $\text{dev}_t$ containing perturbed fundamental domain $F_t$ deformed from $F$.

We deduce that $h_t(\pi_t(\hat{E}))$ acts on $S_t$ since $h_t(\pi_t(\hat{E}))$ acts on $\partial \Omega_{0,t}$.

By above, $h_t(\pi_t(\hat{E}))$ acts on $\partial \Omega_{0,t}$ properly and cocompactly giving us a closed orbifold as a quotient. Since $\text{dev}_t|F_t$ is an embedding for sufficiently small $t$, the equivariance tells us that $\text{dev}_t : S_t \to \partial \Omega_{0,t}$ is a diffeomorphism.

Since the change is sufficiently small $S_t$ still bounds a p-end neighborhood of $\mathcal{O}$ by Lemma 11.1.

Before going on with the part (B) of the proof we briefly do a slight generalization.

**Corollary 11.2** We consider all cases (I), (II). For each $\varepsilon > 0$, there exists $\delta, \hat{\delta} > 0$ depending only on $\text{Cl}(\Omega_0)$ so that we can choose a convex domain $\Omega_{0,1}$ where $h_1(\pi_1(\hat{E}))$ acts on for the holonomy homomorphism $h_1$ so that

- $\Omega_{0,1}$ contains a domain that is the embedded image of a p-end neighborhood of $\hat{E}$ by a developing map $\text{dev}_1$ for $\mu_1$ associated with $h_1$ and
- $d_H(\text{Cl}(\Omega_0), \text{Cl}(\Omega_{0,1})) < \varepsilon$ (11.6)
provided \( \mu_0 \) and \( \mu_1 \) are \( \delta \)-close in \( C^r \)-topology, \( r \geq 2 \), on the compact set \( \mathcal{O} - U \) for a union of \( U \) of end neighborhoods of \( \mathcal{O} \).

**Proof** Suppose that this is false. There exists a sequence of real projective structures \( \mu_n \) with \( \mu_n \rightarrow \mu_0 \) in the \( C^2 \)-topology on \( \mathcal{O} - U \). Then letting the associated holonomy of \( \mu_n \) be denoted by \( h_n \), \( \{ h_n \} \) converges to \( h_0 \) for \( \mu_0 \). We can apply the argument of cases (I), (II), to show that (11.6) holds for every \( \varepsilon > 0 \).

**Proof (Proof of Proposition 11.1 continued)** (B) With \( \hat{\mathcal{O}} \) with \( \mu_t \), we obtain a special affine suspension on \( \hat{\mathcal{O}} \times \mathbb{S}^1 \) with the affine structure \( \hat{\mu}_t \). Let \( C(\hat{\mathcal{O}}) \) be the cone over \( \hat{\mathcal{O}} \). Then this covers the special affine suspension. Let \( \hat{\mu}_t \) denote the affine structure on \( C(\hat{\mathcal{O}}) \) corresponding to \( \hat{\mu}_t \). For each \( \mu_t \), it has an affine structure \( \hat{\mu}_t \), different from the induced one from \( \mathbb{R}^{n+1} \) as for \( t = 0 \). We recall the scalar multiplication

\[
s \cdot v = sv, v \in C(\hat{\mathcal{O}}), s \in \mathbb{R}
\]

for any affine structure \( \hat{\mu}_t \). Also, given a subset \( K \) of \( \hat{\mathcal{O}} \), we denote by \( C(K) \) the corresponding set in \( C(\hat{\mathcal{O}}) \). This set is independent of \( \hat{\mu}_t \) but will have different affine structures nearby.

For \( \mu_0 \), \( \hat{\mathcal{O}} \) is a domain in \( \mathbb{S}^n \). Recall the Koszul-Vinberg function \( f : C(\hat{\mathcal{O}}) \rightarrow \mathbb{R}_+ \) homogeneous of degree \( -n - 1 \) as given by (11.1). (See Lemma 11.2.) By our choice above, the Hausdorff distance between \( C(\Omega_{t_0}) \) and \( \hat{\mathcal{O}} \) can be made as small as desired for some choice of 0.

By the proof of Theorem 9.2 in Page 328 constructing the local inverse maps applied to strongly tame orbifolds with boundary, there exists a diffeomorphism

\[
F_t : \Omega_{t_0} / h_0(\pi_1(E)) \rightarrow \Omega_{t_0,t} / h_t(\pi_1(E)) \text{ with a lift } \tilde{F}_t : \Omega_{t_0} \rightarrow \Omega_{t_0,t}
\]

so that \( \tilde{F}_t \rightarrow 1 \) on every compact subset of \( C(\hat{\mathcal{O}}) \) in the \( C^r \)-topology as \( t \rightarrow 0 \). That is, on every compact subset \( K \) of \( C(\hat{\mathcal{O}}) \),

\[
\{ \| D^j \tilde{F}_t - D^j 1 |K| \} \rightarrow 0 \text{ for every multi-index } j, 0 \leq |j| \leq r.
\]

We may assume that \( \tilde{F}_t \) commutes with the radial flow \( \Psi_t : \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}} \) for \( s \in \mathbb{R} \) by restricting \( F_t \) to a cross-section of \( C(\hat{\mathcal{O}}) \) of the radial flow and extending radially. (See the paragraph after Definition 2.4.)

By the third item of Lemma 2.22 and Lemma 11.2, the Hessian functions \( f^t_0 \circ F_t \) defined by (11.1) on the inverse image \( F_t^{-1}(C(\Omega_{n,t})) \) is as close to the original Hessian function \( f \) in any compact subset of \( C(\hat{\mathcal{O}}) \) in the \( C^r \)-topology, \( r \geq 2 \), as we wish provided \( |t - t_0| \) is sufficiently small. By construction, \( f^t_0 \) is homogeneous of degree \( -n - 1 \).

The holonomy groups \( h(\pi_1(\mathcal{O})) \) and \( h_t(\pi_1(\hat{\mathcal{O}})) \) being in \( SL_+(n + 1, \mathbb{R}) \) preserve \( f \) and \( f^t_0 \) under deck transformations respectively.

Now do this for all p-ends. Let \( \mathbb{U}_t \) be the \( \pi_1(\mathcal{O}) \)-invariant mutually disjoint union of p-end neighborhoods of p-ends of \( \hat{\mathcal{O}} \). We construct a function \( f^t_0 \) on \( C(\mathbb{U}_t) \) for \( \mu_t \) and sufficiently small \( |t| \).
Let $U$ be the corresponding $\pi_i(\mathcal{O})$-invariant union of proper $p$-end neighborhoods of $\hat{\mathcal{O}}$ for $\mu_0$. For each component $U_j$ of $U$, we construct $f'_{ij} \circ \hat{F}_i$ on $C(U_j)$ using $\Omega_{\nu_0}$ so that $f'_{ij}$ satisfies the above properties and $\hat{F}_i$ is constructed as above for $U_j$. We call $f'$ the union of these functions.

Let $\mathcal{V}$ be a $\pi_1(\mathcal{O})$-invariant compact neighborhood of the complement of $U$ in $\hat{\mathcal{O}}$.

- Let $\partial_2 \mathcal{V}$ be the image of $\partial \mathcal{V} \times \{s\}$ inside the regular neighborhood of $\partial \mathcal{V}$ in $U$ parameterized as $\partial \mathcal{V} \times [-1, 1]$ for $s \in [-1, 1]$.
- We assign $\partial_2 \mathcal{V} = \partial_0 \mathcal{V}$.
- Let $\partial_{[s_1,s_2]} \mathcal{V}$ denote the image of $\partial \mathcal{V}_i \times \{s_1, s_2\}$ inside the regular neighborhood of $\partial \mathcal{V}$ in $\mathcal{V} \cap \mathcal{U}'$ for a neighborhood $\mathcal{U}'$ of $C(U) \cap \hat{\mathcal{O}}$.

We find an $C^\infty$ map $\phi : C(U') \cap C(\mathcal{V}) \to \mathbb{R}_+$ so that $\phi(|s\mathcal{V}|) = \phi(\mathcal{V})$ for every $s > 0$ and $f'_i(|\mathcal{V}|) = \phi(\mathcal{V})f(\mathcal{V})$ and $\phi$ is very close to the constant value $1$ function. By making $f'_i/f$ near $1$ and the derivatives of $f'_i/f$ up to two near $0$ as possible, we obtain $\phi$ that has derivatives up to order two as close to $0$ in a compact subset as we wish: This is accomplished by taking a partition of unity functions $p_1, p_2$ invariant under the radial flow so that

- $p_1 = 1$ on $C(W)$ for

$$W := \partial_{[0,s_1]} \mathcal{V} \cup (\mathcal{U}' - \mathcal{V}) \text{ for } s_1 < 1,$$

- $p_1 = 0$ on $C(\hat{\mathcal{O}} - N)$ for a neighborhood $N$ of $W$ in $\partial_{[-1,1]} \mathcal{V} \cup (\mathcal{U}' - \mathcal{V})$, and
- $p_1 + p_2 = 1$ identically.

We assume that

$$1 - \varepsilon < f'_i/f < 1 + \varepsilon \text{ in } C(U' \cap \mathcal{V}),$$

and $f'_i/f$ has derivatives up to order two sufficiently close to $0$ by taking $f'_i$ and $f$ sufficiently close in $C(U') \cap C(\mathcal{V})$ by taking sufficiently small $t$. We define

$$\phi_t = (f'_i/f - (1 - \varepsilon))p_1 + \varepsilon p_2 + (1 - \varepsilon), 0 < t < 1$$

as $f'_i$ and $f$ are homogeneous of degree $-n - 1$. Then $1 - \varepsilon < \phi_t < 1 + \varepsilon$ and derivatives of $\phi_t$ up to order two are sufficiently close to $0$ by taking sufficiently small $\varepsilon$ as we can see easily from computations. Thus, using $\phi_t$ we obtain a function $f$ obtained from $f'_i$ and $\phi_t f$ on $C(W)$ and extending them smoothly for sufficiently small $|t|$.

We can check the welded function from $f'_i$ and $\phi_t f$ has the desired Hessian properties for $\mu_i$ for sufficiently small $t$ since the derivatives of $\phi_t$ up to order two can be made sufficiently close to zero. Now we do this for every p-end of $\hat{\mathcal{O}}$.

The $-(n+1)$-homogeneity gives us the invariance of the Hessian metric under the scalar dilatations and the affine lifts of the holonomy groups. (See Chapter 4 of [91].) This completes the proof for Proposition 11.1.

This is a strengthened version of Proposition 11.1.
Corollary 11.3 Let $\mathcal{O}$ be a strongly tame orbifold with ends and satisfies (IE) and (NA). Suppose that $\mathcal{O}$ has an SPC-structures $\mu_0$ with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and the suspension of $\mathcal{O}$ with $\mu_0$ has a Hessian metric. Let $U$ be a union of mutually disjoint end neighborhoods of $\mathcal{O}$. Suppose the following hold:

- Let $\mu_1$ be an SPC-structure with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends so that $\mu_0$ and $\mu_1$ are $\epsilon$-close in $C^r$-topology on $\mathcal{O} - U$ for $r \geq 2$, and
- the $R$-end holonomy group of $\mu_1$ are either lens-type or horospherical type and the $T$-end holonomy group of $\mu_1$ are totally geodesic satisfying the lens conditions.

Then for sufficiently small $\epsilon$, the affine suspension $C(\tilde{\mathcal{O}})$ for $\tilde{\mathcal{O}}$ with $\mu_1$ also has a Hessian metric invariant under dilations and the affine suspensions of the holonomy homomorphism for $\mu_1$.

Proof We use Corollary 11.2 so that (11.6) holds with sufficiently small $\epsilon$. Now we use the step (B) of the proof of Proposition 11.1.

Lemma 11.2 Let $V$ be a properly convex cone, and let $V^*$ be a dual cone. Suppose that a Koszul-Vinberg function $f_{V^*}(x)$ is defined on a compact neighborhood $B$ of $x$ contained in a convex cone $V$. Let $V_1$ be another properly convex cone containing the same neighborhood. Let $\Omega := S(V^*)$ and $\Omega_1 := S(V_1^*)$ for the dual $V_1^*$ of $V_1$. For given any integer $s \geq 1$ and $\epsilon > 0$, there exists $\tilde{\delta} > 0$ so that if the Hausdorff distance between $\Omega$ and $\Omega_1$ is $\tilde{\delta}$-close, then $f_{V^*}(x)$ and $f_{V_1^*}(x)$ are $\epsilon$-close in $B$ in the $C^r$-topology.

Proof We prove for $S^n$. By Lemma 2.22, we have

$$\Omega^* \subset N_\delta(\Omega_1^*), \Omega_1^* \subset N_\delta(\Omega^*),$$

$$(\Omega - N_\delta(\partial \Omega))^* \subset \Omega_1^*, \text{ and}$$

$$(\Omega_1 - N_\delta(\partial \Omega_1))^* \subset \Omega^* \quad (11.7)$$

provided $\delta$ is sufficiently small. We choose sufficiently small $\delta > 0$ so that

$$B \subset \Omega - N_\delta(\partial \Omega), \Omega_1 - N_\delta(\partial \Omega_1).$$

Recall the Koszul-Vinberg integral (11.1). For fixed $\phi \in V^*$ or $\in V_1^*$, the functions $e^{-\phi(x)}$ and the derivatives of $e^{-\phi(x)}$ with respect to $x$ in the domains are uniformly bounded on $B$ since $\phi$ and its derivatives are bounded function on $B$. The integral is computable from an affine hyperspace meeting $V^*$ and $V_1^*$ in bounded precompact convex sets. Also, the integration is with respect to $\phi$. The result follows by (11.7).

(See Section 4.1.2 of [91]).
11.2.2 The proof of Theorem 11.4.

Proof (The proof of Theorem 11.4) Suppose that \( \mathcal{O} \) has an SPC-structure \( \mu \) with generalized lens-shaped or horospherical \( \mathcal{R} \)- or \( \mathcal{S} \)-ends. Let \( \mathbb{U} \) be the union of end neighborhoods of product form with mutually disjoint closures. By premises, the end structures are given.

We assume that \( \mu_0 \) and \( \mu_s \) correspond to elements of \( \mathcal{W} \) in \( \text{Def}^s_{\mathcal{G} \times \mathbb{U}}(\mathcal{O}) \). We show that a structure \( \mu_s \) that has generalized lens-shaped or horospherical \( \mathcal{R} \)- or \( \mathcal{S} \)-ends sufficiently close \( \mu \) in \( \mathcal{O} - \mathbb{U} \) is also SPC.

Let \( h : \pi_1(\mathcal{O}) \to \text{SL}_\mathbb{Z}(n + 1, \mathbb{R}) \) be the lift of the holonomy homomorphism corresponding to \( \mu_0 \) where \( \mathcal{O} \subset \mathbb{S}^n \) is a properly convex domain covering \( \mathcal{O} \). Let \( h_s : \pi_1(\mathcal{O}) \to \text{SL}_\mathbb{Z}(n + 1, \mathbb{R}) \) be the lift of the holonomy homomorphism corresponding to \( \mu_s \) sufficiently close to \( h \) in

\[
\mathcal{W} \subset \text{Hom}^s_{\mathcal{G} \times \mathbb{U}}(\pi_1(\mathcal{O}), \text{SL}_\mathbb{Z}(n + 1, \mathbb{R})).
\]

By Theorem 9.3, it corresponds to a real projective structure \( \mu_s \) on \( \mathcal{O} \). Since \( [\mu_s] \in \text{Def}^s_{\mathcal{G} \times \mathbb{U}}(\mathcal{O}) \), it is sufficient to show that \( \mu_s \) is a properly convex real projective structure.

Let \( \mathcal{O}' := C(\mathcal{O})/h(\pi_1(\mathcal{O})) \) with \( C(\mathcal{O}) \) as the universal cover. Let \( \mathring{\mathcal{O}}_s \) denote the lift of \( \mathcal{O}' \) with \( \mu_s \). One applies special affine suspension to obtain an affine orbifold \( \mathcal{O}' \times \mathbb{S}^1 \).

(See Section 2.2.1.) The universal cover is still \( C(\mathcal{O}) \) and has a corresponding affine structure \( \mathring{\mathcal{O}}_s \). We denote \( C(\mathcal{O}) \) with the lifted affine structure of \( \mathring{\mathcal{O}}_s \) by \( C(\mathcal{O})_s \). Recall the Kuiper completion \( \hat{C}(\mathcal{O})_s \) of \( C(\mathcal{O})_s \). This is a completion of \( C(\mathcal{O})_s \), the path metric induced from the pull-back of the standard Riemannian metric on \( \mathbb{S}^{n+1} \) by the developing map \( \text{dev}_s \) of \( \mathring{\mathcal{O}}_s \). (Here the image is in \( \mathbb{R}^{n+1} \) as an affine subspace of \( \mathbb{S}^{n+1} \).) The developing maps always extend to ones on \( \hat{C}(\mathcal{O})_s \) which we denote by \( \text{dev}_s \) again. (See [48] and [51] for details.)

By Corollary 11.3, an affine suspension \( \mathring{\mathcal{O}}_s \) of \( \mu_s \) also have a Hessian function \( \phi \) since \( \mu_s \) is in a sufficiently small \( C^2 \)-neighborhood of \( \mu \) in \( \mathcal{O}' - \mathbb{U} \). The Hessian metric \( D^2 \phi \) is invariant under affine automorphism groups of \( C(\mathcal{O}) \) by construction. We prove that \( \mathring{\mathcal{O}}_s \) is properly convex, which will show \( \mu_s \) is properly convex:

Suppose that \( \mathring{\mathcal{O}}_s \) is not convex. Then there exists a triangle \( T \) embedded in \( \hat{C}(\mathcal{O})_s \), where a point in the interior of an edge of \( T \) is in the ideal set

\[
\delta_m \mathring{\mathcal{O}}_s := \hat{C}(\mathcal{O})_s - C(\mathcal{O})_s
\]

while \( T' \) and the union \( T'' \) of two other edges are in \( C(\mathcal{O})_s \). We can move the triangle \( T \) so that the interior of an edge \( l \) has a point \( x_m \) in \( \delta_m \mathring{\mathcal{O}}_s \), and \( \text{dev}_s(l) \) does not pass the origin. We form a parameter of geodesics \( l, t \in [0, \varepsilon] \) in \( T \) so that

\[
l_0 = l \text{ and } l_t \subset C(\mathcal{O})_s \text{ with } \partial l_t \subset l'
\]

is close to \( l \) in the triangle. (See Theorem A.2 of [51] for details.)

Let \( p, q \) be the endpoints of \( l \). Then the Hessian metric is \( D^2 \phi \) for a function \( \phi \) defined on \( C(\mathcal{O})_s \). And \( d\phi|p \) and \( d\phi|q \) are bounded, where \( D^2 \) is the affine con-
nction of $\mu_t$. This should be true for $p_t$ and $q_t$ for sufficiently small $t$ uniformly. Let $u, v \in [0, 1]$, be the affine parameter of $l$, i.e., $l(s)$ is a constant speed line in $\mathbb{R}^{n+1}$ when developed. We assume that $u \in (\varepsilon_1, 1 - \varepsilon_2)$ parameterize $l_t$ for sufficiently small $t$ where $\varepsilon_1 \to 0$ as $t \to 0$ and $dl_t/ds = v$ for a parallel vector $v$. The function $D^2_d\phi(l_t(u))$ is uniformly bounded since its integral $d\phi(l_t(u))$ is strictly increasing by the strict convexity and converges to certain values as $u \to \varepsilon_1, 1 - \varepsilon_2$.

Since

$$\int_{\varepsilon_1}^{1-\varepsilon_2} D^2_d\phi(l_t(u))du = d\phi(p_t)(v) - d\phi(q_t)(v),$$

the function $\sqrt{D^2_d\phi(l_t(u))}$ is also integrable by Jensen’s inequality, and the length of $l_t$

$$\int_{\varepsilon_1}^{1-\varepsilon_2} \sqrt{D^2_d\phi(l_t(u))}du$$

under the Hessian metric $Dd\phi$ have an upper bound $\sqrt{d\phi(p_t)(v) - d\phi(q_t)(v)}$ by the same inequality. Since

$$\sqrt{d\phi(p_t)(v) - d\phi(q_t)(v)} \to \sqrt{d\phi(p_0)(v) - d\phi(q_0)(v)}$$

as $t \to 0$, the length of $l_t$ is uniformly bounded.

$U$ corresponds to an inverse image $\bar{U}$ in $\tilde{\mathcal{D}}$ and to $C(\bar{U})$, the inverse image in $C(\tilde{\mathcal{D}})$. The minimum distance between components of $U$ is bounded below since the metric is invariant under scalar dilatations in $C(\tilde{\mathcal{D}})$. Since $(C(\tilde{\mathcal{D}}) - C(\bar{U}))$ is compact, if $l$ meets infinitely many components of $C(\bar{U})$, then the length is infinite.

As $t \to 0$, the number is thus bounded, $l$ can be divided into finite subsections, each of which meets at most one component of $C(\bar{U})$.

Let $\tilde{l}$ be the subsegment of $l$ in $C(\tilde{\mathcal{D}})$ containing $x_m$ in the ideal set of the Kuiper completion of $C(\tilde{\mathcal{D}})$ with respect to $\text{dev}$, and meeting only one component $C(\bar{U})$ of $C(\bar{U})$ with $\text{bd } C(\bar{U})$. Let $\tilde{l}_t$ be the subsegment of $l_t$ so that the parameter of the endpoints of segments of form $\tilde{l}_t$ converges to those of $\tilde{l}$ as $t \to 0$. Let $p'$ and $q'$ be the endpoint of $\tilde{l}$.

Suppose that $C(\bar{U}) \subset C(\tilde{\mathcal{D}})$ corresponds to a lens-shaped or horospherical R-p-end neighborhood $\bar{U}'$ in $\tilde{\mathcal{D}}$, and $x_m$ is on a line corresponding to the p-end vertex of $\bar{U}'$. We project to $\mathbb{S}^n$ from the projection $\Pi': \mathbb{R}^{n+1} - \{O\} \to \mathbb{S}^n$. Then by the proper-convexity of $\Sigma_{E}$ contradicts this when $E$ is a R-p-end of generalized lens-type. When $E$ is a horospherical p-end, the whole segment must be in $\text{bd } \tilde{\mathcal{D}}$ by convexity. Theorem 4.2 contradicts this.

Now suppose that $\Pi'(x_m)$ is in the middle of the radial line from the p-end vertex. Then the interior of the triangle is transverse to the radial lines. Since our p-end orbifold $\Sigma_{E}$ is convex, there cannot be such a line with a single interior point in the ideal set.

If $C(\bar{U}) \subset C(\tilde{\mathcal{D}})$ is the inverse image in $C(\tilde{\mathcal{D}})$ of a generalized lens-shaped T-p-end neighborhood $\bar{U}$ in $\tilde{\mathcal{D}}$, then clearly there is no such a segment $l$ containing an ideal $x_m$ in its interior similarly.
Now suppose that a subsegment $l_1$ of $l$ contains an ideal point in its interior but is disjoint from $\tilde{\Omega}$. There is connected arc in $l_1 \cap C(\tilde{\Omega} - \tilde{\U})$, ending at an ideal point $x_\infty$. This is an arc never in a compact subset of $C(\tilde{\Omega})$. However, we showed above that the Hessian length of $l_1$ is bounded. Since for a subarc $l_{1,t}$ of $l_t$, the parameter $\{l_{1,t}\}$ converges to $l_1$ as $t \to 0$. Thus, the Hessian length of $l_1$ is also finite. Since $C(\tilde{\Omega} - \tilde{\U})$ covers a compact orbifold that is the affine suspended over $\tilde{\Omega} - \tilde{U}$, the Hessian metric is compatible with any Riemannian metric. Since $l_1$ is in a compact orbifold, it cannot have a finite Riemannian length.

This is again a contradiction. Therefore, $\tilde{\Omega}_s$ is convex.

Finally, for sufficiently small deformations, the convex real projective structures are properly convex. Suppose not. Then there is a sequence $\{\mu_s\}$ of sufficiently small deformed convex real projective structures which are not properly convex. By Proposition 2.5, there exists a unique great sphere $S^0$ in the boundary of the nonproperly convex set. By uniqueness, the holonomy $h_s$ acts on $S^0$.

The sequence of structures converges to the beginning $\mu$ in $\tilde{\Omega} - U$. By taking limits, the original holonomy has to be reducible.

Suppose now that $\tilde{\Omega}$ with $\mu$ is strictly SPC with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends. The relative hyperbolicity of $\tilde{\Omega}$ with respect to the $p$-ends is stable under small deformations since it is a metric property invariant under quasi-isometries by Theorem 10.3.

The irreducibility and the stability follow since these are open conditions in

$$\text{Hom}(\pi_1(\tilde{\Omega}), SL_\pm(n+1, \mathbb{R}))$$

Also, the ends are lens-shaped or horospherical.

By Theorem 1.2, the holonomy is not in a parabolic group. This completes the proof of Theorem 11.4.

\[S^0S\]

### 11.3 The closedness of convex real projective structures

We recall $\text{rep}_s^e(\pi_1(\tilde{\Omega}), \text{PGL}(n+1, \mathbb{R}))$ the subspace of stable irreducible characters of $\text{rep}_s^e(\pi_1(\tilde{\Omega}), \text{PGL}(n+1, \mathbb{R}))$ which is shown to be an open subset of a semialgebraic set in Section 1.4.1, and denote by $\text{rep}_s^e,\text{lh}(\pi_1(\tilde{\Omega}), \text{PGL}(n+1, \mathbb{R}))$ the subspace of stable irreducible characters of $\text{rep}_s^e,\text{lh}(\pi_1(\tilde{\Omega}), \text{PGL}(n+1, \mathbb{R}))$, an open subset of a semialgebraic set.

#### 11.3.1 Preliminary of the section

Recall the definition of compatible end-compactification from Sections 1.3.2 and 1.3.3.
Lemma 11.3 Let $h_i, h \in \text{Hom}_{\emptyset, \mathbb{H}}(\pi_1(\mathcal{O})), \text{SL}_+(n + 1, \mathbb{R})$

(resp. $\in \text{Hom}_{\emptyset, \mathbb{H}}(\pi_1(\mathcal{O})), \text{PGL}(n + 1, \mathbb{R}))$.

Suppose that the following hold:

- Let $\mathcal{O}$ be a strongly tame real projective orbifold with ends assigned types $\mathcal{B}$ and $\mathcal{T}$ satisfying $(IE)$ and $(NA)$ with a compatible end compactification.
- Let $\Omega$ be a properly convex open domain in $\mathbb{S}^n$ (resp. $\mathbb{R}^n$).
- Suppose that $\Omega_i/h_i(\pi_i(\mathcal{O}))$ is an $n$-dimensional noncompact strongly tame SPC-orbifold with generalized lens-shaped or horospherical $\mathcal{B}$- or $\mathcal{T}$-ends.
- Assume that each $p$-end holonomy group $h_i(\pi_1(E_j))$ of $h_i(\pi_1(\mathcal{O}))$ of type $\mathcal{B}$ has a $p$-end vertex $v_i^j$ corresponding to the $p$-end structure where $\{v_i^j\}$ forms a convergent sequence as $i \to \infty$. We assume that $h_i \mapsto v_i^j$ extends to an analytic function near $h$.
- Assume that each $p$-end holonomy group $h_i(\pi_1(E_j))$ of $h_i(\pi_1(\mathcal{O}))$ of type $\mathcal{T}$ has a hyperplane $P_i^j$ containing the $p$-ideal boundary component where $\{P_i^j\}$ forms a convergent sequence as $i \to \infty$. We assume that $h_i \mapsto P_i^j$ extends to an analytic function near $h$.
- Suppose that $\{h_i\} \to h$ algebraically where $h$ is discrete and faithful.
- $\text{Cl}(\Omega_i) \to K$ for a compact properly convex domain $K \subset \mathbb{S}^n$, $K^o \neq \emptyset$.

Then the following holds:

1. $\mathcal{O}_h := K^o/h(\pi_i(\mathcal{O}))$ is a strongly tame SPC-orbifold with generalized lens-shaped or horospherical $\mathcal{B}$- or $\mathcal{T}$-ends to be denoted $\mathcal{O}_h$ diffeomorphic to $\mathcal{O}$.
2. For each $p$-end $E$ of the universal cover $\tilde{\mathcal{O}_h}$ of $\mathcal{O}_h$, $K^o$ has a subgroup $h(\pi_1(\tilde{E}))$ acting on a $h(\pi_1(\tilde{E}))$-invariant open set $U_{\tilde{E}}$ where $U_{\tilde{E}}/h(\pi_1(\tilde{E}))$ is an end neighborhood that is one of the following:
   - a horospherical or lens-shaped totally geodesic end neighborhood provided $E$ is a $\mathcal{T}$-p-end,
   - a horospherical or concave end neighborhood provided $E$ is a $\mathcal{B}$-p-end.
3. Finally, suppose that there is a fixed strongly tame properly orbifold $\mathcal{O}'$ with an ideal boundary structure and diffeomorphism $f_i : \mathcal{O}' \to \Omega_i/h_i(\pi_i(\mathcal{O}))$ for sufficiently large $i$ extending to a diffeomorphism of an end-compactification $\mathcal{O}'$ to the end compactification of $\Omega_i/h_i(\pi_i(\mathcal{O}))$ compatible with $R$-end and $T$-end structures given by $v_i^j$ and $P_i^j$. Then $K'/h(\pi_1(\mathcal{O}))$ is an orbifold with a diffeomorphism $f$ from $\mathcal{O}'$ extending to a diffeomorphism from $\mathcal{O}'$ to an end compactification of $K'/h(\pi_1(\mathcal{O}))$ with the above $R$-end and $T$-end structures.

Proof: Again, we prove for $\mathbb{S}^n$. The holonomy group $h(\pi_1(\mathcal{O}))$ acts on $K^o$ with a Hilbert metric. Hence, $K^o/h(\pi_i(\mathcal{O}))$ is an orbifold to be denoted $\mathcal{O}_h$. (See Lemma 1 of [59].)

Since $h \in \text{Hom}_{\emptyset, \mathbb{H}}(\pi_1(\mathcal{O})), \text{SL}_+(n + 1, \mathbb{R})$ holds, each $p$-end holonomy group $h(\pi_1(\tilde{E}))$ acts on a horoball $H \subset \mathbb{S}^n$, a generalized lens-cone, or a totally geodesic hypersurface $\tilde{S}_E$ with a CA-lens $L$. In the first case, we can choose a sufficiently
small horoball $U$ inside $K^o$ and in $H$ since the sharply supporting hyperspaces at the vertex of $H$ must coincide by the invariance under $h(\pi_1(\tilde{E}))$ by a limiting argument. By Lemma 6.10, $U$ component of $K^o - \text{bd}U$ is a p-end neighborhood.

Now, we consider the second case. Let $\nu$ be a limit of the sequence $\{v_{\nu,i}\}$ of the fixed p-end vertices of $h(\pi_1(\tilde{E}))$. We obtain $\nu \in K$. Also, $\nu \notin K^o$ since otherwise the elements fixing it has to be of finite order by the proper discontinuity of the Hilbert isometric action of $h(\pi_1(\tilde{E}))$. For each $i$, $h_i(\pi_1(\tilde{E}))$ acts on a lens-cone $L_i \ast \{v_{\nu,i}\}$. We may assume without loss of generality that $v_{\nu,i}$ is constant by changing $\text{dev}_V$ by a convergent sequence $\{g_i\}$ of elements of $\text{SL}_+(n+1,\mathbb{R})$. We may assume that $h_i$ in a continuous parameter converging to $h$ since there are finitely many components in the above real algebraic set. By Corollary A.2 and the condition "ce", we may assume that

$$\{\text{Cl}(R_{v_{\nu,i}}(\Omega_i))\} \to K_\nu$$

for a properly convex domain $K_\nu$ on which $h(\pi_1(\tilde{E}))$ acts. Since $\Omega_i$ is a subset of a tube domain for $R_{v_{\nu,i}}(\Omega_i)$, we deduce that $\Omega$ is a subset of a tube domain for $K_\nu^o$. Since $K_\nu^o / h(\pi_1(\tilde{E}))$ is a closed properly convex orbifold, and $R_{v_{\nu,i}}(\Omega_i) \subset K_\nu^o$, it follows that they are equal by Lemma 2.19. Hence, $R_{v_{\nu,i}}(\Omega_i)$ is properly convex.

By the Hilbert metric on this domain, $h(\pi_1(\tilde{E}))$ acts properly discontinuously on it. Since $h(\pi_1(\tilde{E}))$ satisfies the uniform middle eigenvalue condition by premise, Theorem 6.6 shows that the action is distanced in a tubular domain corresponding to $R_{v_{\nu,i}}(K^o)$. Hence, $h(\pi_1(\tilde{E}))$ acts properly and cocompactly on a generalized lens $L$ in $K$. The group $h(\pi_1(\tilde{E}))$ acts on an open set $U_L := L \ast \{v_{\nu}\} - L$. We may choose one with sufficiently large $L$ so that the lower boundary component $\partial_- L$ is a subset in $K^o$ since we can make $U_L \cap \mathbb{F}(F)$ be as small as we wish for any compact fundamental domain $F$ for $R_{v_{\nu,i}}(K^o)$ and $h(\pi_1(\tilde{E}))$. By Lemma 6.10, $U_L$ is a p-end neighborhood of $\tilde{E}$.

In the third case, we can find a CA-lens neighborhood of a totally geodesic domain $\tilde{S}_{E,i} \subset \text{Cl}(\Omega_i) \cap P_i$ in a hypersurface $P_i$ on which $h_i(\pi_1(\tilde{E}))$ acts. We may assume without loss of generality that $P_i$ is constant by changing $\text{dev}_V$ by a convergent sequence $g_i$ in $\text{SL}_+(n+1,\mathbb{R})$. We may assume by taking subsequences that $\{\text{Cl}(\tilde{S}_{E,i})\} \to D$ for a properly convex domain $D$ by Corollary A.2.

By Corollary 2.3, $D^o / h(\pi_1(\tilde{E}))$ is a closed orbifold homotopy equivalent to $S_{E,i}$ up to finite manifold covers. By Theorem 6.9, $h(\pi_1(\tilde{E}))$ satisfies the uniform middle eigenvalue condition with respect to the hyperspace containing $D$. By Theorem 5.4, the group $h(\pi_1(\tilde{E}))$ acts on a component $L_i$ of $L-\partial E$ is in $K^o$ for a lens $L$. Then $L^o_i$ is a p-end neighborhood of $\tilde{E}$ by Lemma 6.10. Hence, we constructed R-end and T-end structures for each end for $K^o / \pi_1(\Theta)$.

We apply Theorem 1.2 to show that the end structure is SPC.

The end compactification $\tilde{\partial}_E$ of $K^o / h(\pi_1(\Theta))$ is given by attaching the cover of ideal boundary component for each p-T-end and attaching $\Sigma_E \times [1,0)$ to an end-neighborhood of an R-end $E$ by a diffeomorphism restricting to a proper map as in Section 9.1.3.
11.3 The closedness of convex real projective structures

For the final part, recall from Section 1.4.1 that $\text{Hom}_{\mathbb{R}}^\ast (\pi_1(\mathcal{O}), \text{SL}_+(n + 1, \mathbb{R}))$ is Zariski dense. The homomorphism $h$ is in it by Proposition 6.8. We choose p-end vertices the hyperspace containing the p-ideal boundary components be obtained by respective limits of the corresponding sequence of corresponding objects of $h_i$. By premise on the analytic extension, we can build a fixed section $s_W$ on a Zariski open subset $\mathcal{U}$ containing $h$ by Proposition 6.8. By Theorem 9.3, we obtain a sub-space of the parameter of real projective structures on a strongly tame orbifold $\mathcal{O}'$ with end structures determined by $s_W$ for each point of $\mathcal{U}$.

By premise, the p-end vertices of $\Omega_i$ and the hypersurfaces containing the p-ideal boundary components are determined also by $s_W$ for sufficiently large $i$. Now, $\mathcal{O}' := K^o/h(\pi_1(\mathcal{O}))$ is realized as a convex real projective orbifold with ends determined by $s_W$ also. By Theorem 9.3, there is a neighborhood $\mathcal{U}' \subset \mathcal{U}$ where every holonomy is realized by a convex real projective structure with end structures determined by $s_W$. Since $h_i$ may be assumed to be in $\mathcal{U}'$ except for finitely many $i$, a structure $\mu_i$ on $\mathcal{O}'$ has has holonomy $h_i$. By Theorem 11.5, $\Omega_i/h_i(\pi_1(\mathcal{O}))$ is projectively diffeomorphic to $\mathcal{O}'$ with a convex real projective structure $\mu_i$ with identical R-end and T-end structures for sufficiently large $i$.

By Theorem 11.5, for $\mu_i$ with holonomy in $\mathcal{U}'$, $\Omega_i/h_i(\pi_1(\mathcal{O}))$ has an end compactification $\tilde{\mathcal{O}}'_i$. Since this end compactification is compatible with the R-end and T-end structures also, $f_i$ extends to a diffeomorphism $\tilde{\mathcal{O}}' \to \tilde{\mathcal{O}}'_i$. Since $h_i \in \mathcal{U}'$ deformed from $h$, $\tilde{\mathcal{O}}'_i$ is isotopic to $\tilde{\mathcal{O}}_h$.

By premise, the diffeomorphism $f_i : \mathcal{O}' \to \Omega_i/h_i(\pi_1(\mathcal{O}))$ extends smoothly as a diffeomorphism from $\mathcal{O}'$ to $\tilde{\mathcal{O}}'_i$. Hence, $\mathcal{O}'$ and $\mathcal{O}'$ are diffeomorphic with end structures preserved. By Corollary 9.1, the respective end compactifications of $\mathcal{O}'$ and $\mathcal{O}'$ are diffeomorphic.

Again the $\mathbb{R}P^p$-version follows by Proposition 2.13.

Theorem 11.5 (Uniqueness of domains) Let $\Gamma$ be a discrete projective automorphism group of a properly convex open domain $\Omega \subset \mathbb{S}^n$. Suppose that $\Omega/\Gamma$ is a strongly tame SPC $n$-orbifold with generalized lens-shaped or horospherical $\mathcal{B}$- or $\mathcal{T}$-ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. Suppose that for each $v_{E} \in \partial \Omega$ for each R-p-end $E$ is specified up to $\mathcal{A}$ in $\mathbb{S}^n$ and so is each hyperplane for each T-p-end $E$ meeting $\partial \mathcal{O}$. Then $\Omega$ is a unique domain with these properties up to the antipodal map $\mathcal{A}$.

Proof Suppose that $\Omega_1$ and $\Omega_2$ are distinct open domains in $\mathbb{S}^n$ satisfying the above properties. For this, we assume that $\Gamma$ is torsion-free by taking a finite-index subgroup by Theorem 2.3. We claim that $\Omega_1$ and $\Omega_2$ are disjoint:

Suppose that $\Omega' := \Omega_1 \cap \Omega_2$ is a nonempty open set. Since $\Omega_1, \Omega_2$, and $\Omega'$ are all $n$-cells, the set of p-ends of $\Omega_1$, the set of those of $\Omega_2$, in one-to-one correspondences by considering their p-end fundamental groups. The types are also preserved by the premise.

Suppose that $E_1$ and $E_2$ are corresponding R-p-ends of generalized lens-type. The p-end vertex $v_{E_j}$ of a generalized lens-shaped R-p-end $E_j$ of $\Omega_j$, $j = 1, 2$, is determined up to $\mathcal{A}$ by the premise.
Suppose that \( v_{E_1} \) and \( v_{E_2} \) are antipodal. The interior of \( \Omega' \cap \{ v_{E_j} \} \), \( j = 1, 2 \), are in \( \Omega_j \) by the convexity of \( \text{Cl}(\Omega_j) \). Also, \( \Omega_1 \cup \Omega_2 \) is in a convex tube \( \mathcal{T}_{E_j}(\Sigma_{E_j}) \) with the vertices \( v_{E_1} \) and its antipode \( v_{E_2} \) in the direction of \( \Sigma_{E_j} \). Also, the convex hull of \( \text{Cl}(\Omega_1) \cup \text{Cl}(\Omega_2) \) equals \( \mathcal{T}_{E_j}(\Sigma_{E_j}) \). \( h(\pi_1(\mathcal{O})) \) acts on the unique pair of antipodal points \( \{ v_{E_1}, v_{E_2} \} \). Hence, \( h(\pi_1(\mathcal{O})) \) is reducible contradicting the premise.

Suppose that \( v_{E_1} = v_{E_2} \). Then \( R_{v_{E_j}}(\Omega_1) \) and \( R_{v_{E_j}}(\Omega_2) \) are not disjoint since otherwise \( \Omega_1 \) and \( \Omega_2 \) are disjoint. Lemma 2.19 shows that they are equal. The generalized lens-cone p-end neighborhood \( U_1 \) in \( \Omega_1 \) and one \( U_2 \) in \( \Omega_2 \) must intersect. Hence, by Lemma 6.10, the intersection of a generalized lens-cone p-end neighborhood of \( \Omega_1 \) and that of \( \Omega_2 \) is one for \( \Omega' \).

Suppose that \( \tilde{E}_j \) is a horospherical p-end of \( \Omega_j \), \( j = 1, 2 \). Then p-end vertices \( v_{E_j} \), \( j = 1, 2 \), are either equal or antipodal since there is a unique antipodal pair of fixed points for the cusp group \( \tilde{\Gamma}_{E_j} \), \( j = 1, 2 \). Since the fixed point in \( v_{E_j} \) is the unique limit point of \( \{ \gamma^r(p) \} \) as \( n \to \infty \) for any \( p \in \Omega_j \), it follows that \( v_{E_j} = v_{E_2} \). We can verify that \( \Omega_1, \Omega_2 \), and \( \Omega' \) share a horospherical p-end neighborhood from this by Lemma 6.10.

Similarly, consider the ideal boundary component \( \tilde{S}_{E_j} \) for a T-p-end \( \tilde{E}_1 \) of \( \Omega_1 \) and the corresponding \( \tilde{S}_{E_j} \) for a T-p-end \( \tilde{E}_2 \) of \( \Omega_2 \). Since \( \tilde{\Gamma}_{E_j} \) acts on a properly convex domain \( \Omega' \), Theorem 6.9 and Lemma 5.14 show that \( \text{Cl}(\Omega') \cap P \) is a nonempty properly convex set in \( \text{Cl}(\tilde{S}_{E_j}) \).

We claim that a point \( \text{Cl}(\tilde{S}_{E_j}) \) for \( \Omega_2 \) cannot be antipodal to any point of \( \text{Cl}(\tilde{S}_{E_j}) \): Suppose not. Then \( \text{Cl}(\tilde{S}_{E_j}) = R_2(\text{Cl}(\tilde{S}_{E_j})) \) for a projective automorphism \( R_2 \) acting as \( I \) or \( \mathcal{A} \) on a collection of independent subspaces of \( P \) by Lemma 2.19. There exists a pair of extremal points \( p_1 \in \text{Cl}(\tilde{S}_{E_j}) \) and \( p_2 \in \text{Cl}(\tilde{S}_{E_j}) \), antipodal to each other. Here, there exists a point \( c \in \tilde{S}_{E_j} \) and a sequence \( g_i^{(j)} \) such that \( \{ g_i^{(j)}(c) \} \to p_j \) by Lemma 5 of [161]. By Lemma 5.14, \( \{ g_i^{(j)}(d) \} \to p_j \) for a point \( d \in \Omega' \). This implies that \( \Omega' \) is not properly convex, a contradiction.

Thus, we obtain \( \tilde{S}_{E_j} = \tilde{S}_{E_j} \) again by Lemma 2.19. Since \( \Omega' \) is a \( h(\pi_1(\mathcal{E})) \)-invariant open set in one side of \( P \), it follows that \( \Omega' \) contains a one-sided lens neighborhood \( L_1 \) by Lemma 5.2. By Lemma 6.10, \( L_1 \) is a p-end neighborhood of \( \Omega' \).

We have concave p-end neighborhoods for radial p-ends, lens p-end neighborhoods for totally geodesic p-ends, and horoball p-end neighborhoods of p-ends for each of \( \Omega_1, \Omega_2, \) and \( \Omega' \). We verify from above discussions that a p-end neighborhood of \( \Omega_1 \) exists if and only if a p-end neighborhood of \( \Omega_2 \) exists and their intersection is a p-end neighborhood of \( \Omega' \). \( \Omega'/\Gamma \) is a closed submanifold in \( \Omega_1/{\Gamma} \) and in \( \Omega_2/{\Gamma} \). Thus, \( \Omega_1/{\Gamma}, \Omega_2/{\Gamma}, \) and \( \Omega'/\Gamma \) are all homotopy equivalent relative to the union of disjoint end-neighborhoods. The map has to be onto in order for the map to be a homotopy equivalence as we can show using relative homology theories, and hence, \( \Omega' = \Omega_1 = \Omega_2 \).

Suppose that \( \Omega_1 \) and \( \mathcal{A}(\Omega_2) \) meet. Then similarly, \( \Omega_1 = \mathcal{A}(\Omega_2) \).

Suppose now that \( \Omega_1 \cap \Omega_2 = \emptyset \) and \( \Omega_1 \cap \mathcal{A}(\Omega_2) = \emptyset \). Suppose that \( \Omega_1 \) has a p-end \( \tilde{E} \) of type \( \mathcal{A} \). The corresponding pair of the p-end neighborhoods share the p-end vertex or have antipodal p-end vertices by the premise. Since \( \Omega_1 \) and \( \Omega_2 \)
are disjoint, it follows that $\text{Cl}(\Omega_1) \cap \text{Cl}(\Omega_2)$ or $\text{Cl}(\Omega_1) \cap \mathcal{A}(\text{Cl}(\Omega_2))$ is a compact properly convex subset of dimension $< n$ and is not empty since the end vertex of the p-ends are in it. The minimal hyperspace containing it is a proper subspace and is invariant under $\Gamma$. This contradicts the strong irreducibility of $h(\pi_1(\mathcal{O}))$ as can be obtained from Theorem 1.2. This also applies to the case when $\tilde{E}$ is a horospherical end of type $T$.

Suppose that $\Omega_1$ has a p-end $\tilde{E}_1$ of type $T$. Then $\Omega_2$ has a p-end $\tilde{E}_2$ of type $T$. Now, $\Gamma_{\tilde{E}_1} = \Gamma_{\tilde{E}_2}$ acts on a hyperspace $P$ containing the $\tilde{S}_{\tilde{E}_i}/\Gamma_{\tilde{E}_i}$ in the boundary of $\text{bd}(\Omega_i)$ for $i = 1, 2$. Here, $\tilde{S}_{\tilde{E}_i}/\Gamma_{\tilde{E}_i}$ and $\tilde{S}_{\tilde{E}_2}/\Gamma_{\tilde{E}_2}$ are closed $n-1$-orbifolds. Lemma 2.19 shows that their closures always meet or they are antipodal. Hence, up to $\mathcal{A}$, their closures always meet. Again

$$\text{Cl}(\Omega_1) \cap \text{Cl}(\Omega_2) \neq \emptyset \text{ or } \text{Cl}(\Omega_1) \cap \mathcal{A}(\text{Cl}(\Omega_2)) \neq \emptyset$$

while we have $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cap \mathcal{A}(\Omega_2) = \emptyset$. We obtain a lower-dimensional convex subspace fixed by $\Gamma$. This is a contradiction. □

### 11.3.2 The main result for the section

This generalizes Theorem 4.1 of [67] for closed orbifolds which is really due to Benoist [23].

**Theorem 11.6** Let $\mathcal{O}$ be a strongly tame SPC n-orbifold with generalized lens-shaped or horospherical $R$- or $T$-ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. We have an open $\text{PGL}(n+1,R)$-conjugation invariant set $\mathcal{U}$ in a semi-algebraic subset of $\text{Hom}_{\text{F,Epun}}(\pi_1(\mathcal{O}), \text{PGL}(n+1,R))$, and a $\text{PGL}(n+1,R)$-equivariant fixing section $s_{\mathcal{U}} : \mathcal{U} \to (\mathbb{RP}^*)^{e_1} \times (\mathbb{RP}^{*e_2}).$ Let $\mathcal{U}'$ denote the quotient set under $\text{PGL}(n+1,R)$. Assume that every finite index subgroup of $\pi_1(\mathcal{O})$ has no nontrivial nilpotent normal subgroup. Then the following hold:

- The deformation space $\text{CDef}_{\mathcal{O}}(\mathcal{U})$ of SPC-structures on $\mathcal{O}$ with generalized lens-shaped or horospherical $R$- or $T$-ends maps under $\text{hol}$ homeomorphically to a union of components of
  $$\mathcal{U}' \subset \text{rep}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1,R)).$$

- The deformation space $\text{SDef}_{\mathcal{O}}(\mathcal{U})$ of strict SPC-structures on $\mathcal{O}$ with lens-shaped or horospherical $R$- or $T$-ends maps under $\text{hol}$ homeomorphically to the union of components of
  $$\mathcal{U}' \subset \text{rep}_{\mathcal{O}}(\pi_1(\mathcal{O}), \text{PGL}(n+1,R)).$$
Proof Define $\widehat{\text{Def}_{\Gamma,lh}}(\partial')$ to be the inverse image of $\text{Def}_{\Gamma,lh}(\partial')$ in the isotopy-equivalence space $\text{Def}_{\Gamma}(\partial')$ in Definition 9.3. Let $\mathcal{Y}$ denote the points of the inverse image of $\mathcal{Y}$ in $\widehat{\text{Def}_{\Gamma,lh}}(\partial')$ where the vertices of $R$-p-ends and hyperspaces of $T$-p-ends are determined by $s_{\mathcal{Y}}$. Then $\mathcal{Y}$ is an open subset by Theorem 9.3 and Theorem 11.3.

We show that $\text{hol} : \mathcal{Y} \rightarrow \mathcal{Y} \subset \text{Hom}_{E,lh}^{s}(\pi_{1}(\partial'), \text{PGL}(n+1, \mathbb{R}))$ is a homeomorphism onto a union of components. This will imply the results. Theorem 11.5 shows that $\text{hol}$ is injective.

Now, $\text{hol}$ is an open map by Theorems 11.4 and 9.3. To show that the image is of $\text{hol}$ is closed, we show that the subset of $\mathcal{Y}$ corresponding to elements in $\mathcal{Y}$ is closed. Let $(\text{dev}_{i}, h_{i})$ be a sequence of development pairs so that we have $\{h_{i}\} \rightarrow h$ algebraically. Let $\Omega_{i} = \text{dev}_{i}(\partial')$ denote the corresponding properly convex domains for each $i$. The limit $h$ is a discrete representation by Lemma 1.1 of Goldman-Millson [97]. Let $\hat{\Omega}_{i}$ denote the lift of $\Omega_{i}$ in $\mathbb{S}^{n}$ and let $\hat{h}_{i} : \pi_{1}(\partial') \rightarrow \text{SL}_{\pm}(n+1, \mathbb{R})$ be the corresponding lift of $h_{i}$ by Theorem 2.4. The sequence $\{\text{Cl}(\hat{\Omega}_{i})\}$ also geometrically converges to a compact convex set $\hat{\Omega}$ up to choosing a subsequence by Proposition 2.2 where $\hat{h}(\pi_{1}(\partial'))$ acts on as in Lemma 1 of [59]. If $\hat{\Omega}$ have the empty interior, $h$ is reducible, and $h \notin \mathcal{Y}$, contradicting the premise. If $\hat{\Omega}$ is not empty and is properly convex, then the lift of $\Omega$ to $\mathbb{S}^{n}$ contains a maximal great sphere $S_{i}, i \geq 1$, or a unique pair of antipodal points $\{p, p_{-}\}$ by Proposition 2.5. In the both cases, $h$ is reducible. Thus, $\hat{\Omega}$ is not empty and is properly convex. Let $\Omega$ denote the image of $\hat{\Omega}$ under the double covering map. As in [59], since $\Omega$ has a Hilbert metric, $h(\pi_{1}(\partial'))$ acts on $\Omega$ properly discontinuously.

By Lemma 11.3, the condition of the generalized lens or horospherical condition for $T$-ends or lens or horospherical condition for $T$-ends of the holonomy representation is a closed condition in the $\text{Hom}_{E,lh}^{s}(\pi_{1}(\partial'), \text{PGL}(n+1, \mathbb{R}))$ as we defined above. (It is of course redundant to say that it is not a closed condition when we drop the notation “ce” from the above character space.)

Define $\partial' := \Omega / h(\pi_{1}(\partial'))$. We can deform $\partial'$ with holonomy in an open subset of $\mathcal{Y}$ using the openness of $\text{hol}$ by Theorem 11.4. We can find a deformed orbifold $\partial''$ that has a holonomy $h_{i}$ for some large $i$. Now, $\Omega_{i} / h_{i}(\partial')$ is diffeomorphic to $\partial'$ being in the deformation space. $\partial''$ is diffeomorphic to $\partial$ with the corresponding end-compactifications since they share the same open domain as the universal cover by the uniqueness for each holonomy group by Theorem 11.5. By the openness of the map $\text{hol}$ for $\partial'$, $\partial''$ is diffeomorphic to $\partial'$. Hence, $\partial''$ is diffeomorphic to $\partial'$.

Therefore, we conclude that $\mathcal{Y}$ goes to a closed subset of $\mathcal{Y}$. The proof up to here imply the first item.
11.3 The closedness of convex real projective structures

Now, we go to the second item. Define \( \widetilde{S_{\text{Def}, E, I}}(O) \) to be the inverse image of \( S_{\text{Def}, E, I}(O) \) in the isotopy-equivalence space \( \widetilde{\text{Def}}_E(O) \). Let \( \widetilde{U} \) denote the inverse image of \( U \) in \( \widetilde{S_{\text{Def}, E, I}}(O) \). We show that

\[
\text{hol} : \widetilde{\mathcal{U}} \to \mathcal{U}
\]

is a homeomorphism onto a union of components of \( \mathcal{U} \). Theorem 11.4 shows that \( \text{hol} \) is a local homeomorphism to an open set. The injectivity of \( \text{hol} \) follows the same way as in the above item.

We now show the closedness. By Theorem 10.3, \( \pi_1(O) \) is relatively hyperbolic with respect to the end fundamental groups. Let \( h \) be the limit of a sequence of holonomy representations \( \{ h_i : \pi_1(O) \to \text{PGL}(n+1, \mathbb{R}) \} \). As above, we obtain \( \Omega \) as the limit of \( \text{Cl}(\Omega_i) \) where \( \Omega_i \) is the image of the developing map associated with \( h_i \). \( \Omega \) is properly convex and \( \Omega^0 \) is not empty. Since \( h \) is irreducible and acts on \( \Omega^0 \) properly discontinuously, it follows that \( \Omega^0 / h(\pi_1(O)) \) is a strongly tame properly convex orbifold \( O' \) with generalized lens-shaped or horospherical \( A_\ast \) or \( I \)-ends by the above part of the proof. By Theorem 10.5 and Corollary 7.7, \( O' \) is a strict SPC-orbifold with lens-shaped or horospherical \( A_\ast \) or \( I \)-ends. The rest is the same as above. \( \square \)

Remark 11.2 (Thurston’s example) We remark that without the end controls we have, there might be counter-examples as we can see from the examples of geometric limits differing from algebraic limits for sequences of hyperbolic 3-manifolds. (See Anderson-Canary [2].)

11.3.3 Dropping of the superscript \( s \).

We can drop the superscript \( s \) from the above space. Hence, the components consist of stable irreducible characters. This is a stronger result.

Corollary 11.4 Let \( O \) be a noncompact strongly tame SPC \( n \)-dimensional orbifold with lens-shaped or horospherical \( A_\ast \) or \( I \)-ends and satisfies \( (IE) \) and \( (NA) \). Assume \( \partial O = \emptyset \). Assume that no finite-index subgroups \( \pi_1(O) \) has a nontrivial nilpotent normal subgroup. We have a \( \text{PGL}(n+1, \mathbb{R}) \)-conjugation invariant set \( \mathcal{U} \) open in a union of semi-algebraic subsets of

\[
\text{Hom}_{\text{Def}, E, I}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))
\]

and a \( \text{PGL}(n+1, \mathbb{R}) \)-equivariant fixing section \( \mathcal{U} \to (\mathbb{R}P^n \ast_1 \times (\mathbb{R}P^n \ast_2) \). Let \( \mathcal{U}' \) denote the quotient set under \( \text{PGL}(n+1, \mathbb{R}) \). Then the following hold:

- The deformation space \( \text{CDef}_{\text{Def}, E, I}(O) \) of SPC-structures on \( O \) with generalized lens-shaped or horospherical \( A_\ast \) or \( I \)-ends maps under \( \text{hol} \) homeomorphically to a union of components of
\[ \mathcal{U}' \subset \text{rep}_{\delta, \text{lh}}(\pi_1(\mathcal{O})), \text{PGL}(n + 1, \mathbb{R})). \]

- The deformation space \( \text{SDef}_{\delta, \text{SPC}, \text{lh}}(\mathcal{O}) \) of SPC-structures on \( \mathcal{O} \) with lens-shaped or horospherical \( \mathcal{R} \)- or \( \mathcal{T} \)-ends maps under \( \text{hol} \) homeomorphically to the union of components of

\[ \mathcal{U}' \subset \text{rep}_{\delta, \text{lh}}(\pi_1(\mathcal{O})), \text{PGL}(n + 1, \mathbb{R})). \]

Furthermore, \( \mathcal{U}' \) has to be in \( \text{rep}_{\delta, \text{lh}}'(\pi_1(\mathcal{O})), \text{PGL}(n + 1, \mathbb{R})). \]

**Proof** We define \( \widetilde{\text{CDef}}_{\delta, \text{lh}}(\mathcal{O}) \) and \( \widetilde{\text{SDef}}_{\delta, \text{lh}}(\mathcal{O}) \) as above. Let \( \widetilde{\mathcal{U}} \) be the inverse image of \( \mathcal{U} \). We will show that the image of \( \widetilde{\mathcal{U}} \) under \( \text{hol} \) in \( \mathcal{U} \) is closed and consists of stable irreducible characters. Now Theorem 11.6 implies the result.

We will prove by lifting to \( \mathbb{S}^n \). Using Theorem 2.4, let

\[ \{ h_i : \pi_1(\mathcal{O}) \to SL_\pm(n + 1, \mathbb{R}) \} \]

be a sequence of holonomy homomorphisms of real projective structures corresponding to liftings of elements of \( \mathcal{U} \). These are stable and strongly irreducible representations by Theorem 1.2. Let \( \Omega_i \) be the sequence of associated properly convex domains in \( \mathbb{S}^n \), and \( \Omega_i/h_i(\pi_1(\mathcal{O})) \) is diffeomorphic to \( \mathcal{O} \) and has the structure that lifts an element of \( \text{CDef}_{\delta, \text{lh}}(\mathcal{O}) \). We assume that \( \{h_i\} \to h \) algebraically, i.e., for a fixed set of generators \( g_1, \ldots, g_m \) of \( \pi_1(\mathcal{O}) \), \( \{h_i(g_j)\} \to h(g_j) \in SL_\pm(n + 1, \mathbb{R}) \) as \( i \to \infty \). The limit \( h \) is a discrete representation by Lemma 1.1 of Goldman-Millson [97]. We will show that \( h \) is a lifted holonomy homomorphism of an element of \( \text{CDef}_{\delta, \text{lh}}(\mathcal{O}) \), and hence \( h \) is stable and strongly irreducible.

Here we are using the definition of convexity for \( \mathbb{S}^n \) as given in Definition 2.2. Since the Hausdorff metric space \( d_H \) of compact subsets of \( \mathbb{S}^n \) is compact, we may assume that \( \{\text{Cl}(\Omega_i)\} \to K \) for a compact convex set \( K \) by taking a subsequence if necessary as in [59]. We take a dual domain \( \Omega_i^* \subset \mathbb{S}^n \). Then the sequence \( \{\text{Cl}(\Omega_i^*)\} \) also geometrically converges to a convex compact set \( K^* \) by Proposition 2.23. (See Section 2.5.4.)

Recall the classification of compact convex sets in Proposition 2.5. For any 1-form \( \alpha \) positive on the cone \( C_K \), any sufficiently close 1-form is still positive on \( C_K \). If \( K \) has an empty interior and properly convex, then we can easily show that \( K^* \) has a nonempty interior. Also, if \( K^* \) has an empty interior and properly convex, \( K \) has a nonempty interior.

- (I) The first step is to show that at least one of \( K \) and \( K^* \) has nonempty interior. We divide into four cases (i)-(iv) where the types change for R-ends and T-ends.
  - (i) To begin, suppose that there exists a radial p-end \( \widetilde{E} \) for \( \Omega_i \) and \( h_i \) and the type does not become horospherical. We may assume that \( v_{\widetilde{E}, h_i} = v_{\widetilde{E}, h} \) by conjugating \( h_i \) by a bounded sequence of projective automorphisms. Then the mc-p-end neighborhood must be in \( K \) since this is true for all structures in \( \Omega_i \) and holonomy homomorphisms in

\[ \text{Hom}_{\delta, \text{lh}}(\pi_1(\widetilde{E}), SL_\pm(n + 1, \mathbb{R})). \]
For each $i$, $h_i$ acts on a lens-cone $v_{E,h_i} \ast L_i$ in $\Omega_i$ for each $p$-end $E$. By Theorems 6.7 and 6.8, $L_i$ can be chosen to be the convex hull of the closure of the union of attracting fixed sets of elements of $h_i(\pi_1(\hat{E}))$. Hence, $h$ is also in it, and Theorem 6.1 shows that there exists a distanced compact convex set $L$ distanced away from a point $x$, and the lens-cone $\{v_{E,h}\} \ast L - \{v_{E,h}\}$ has a nonempty interior.

Choose an element $g_1 \in \pi_1(\hat{E})$ so that $h(g_1)$ is positive bi-semi-proximal by Theorem 2.7. Then $h_i(g_1)$ is also positive bi-semi-proximal for sufficiently large $i$ since $\{h_i(g_1)\} \to h(g_1)$ as a sequence. We may assume that

$$\{A_{h_i(g_1)}\} \to A'_{h_i(g_1)} \subset A_{h_i(g_1)} \subset \text{bd}L$$

for attracting-fixed-point sets $A_{h_i(g_1)}$ and $A_{h_i(g_1)}$ and a compact subset $A'_{h_i(g_1)}$ since the limits of sequences of eigenvectors are eigenvectors. Since $A'_{h_i(g_1)}$ is a geometric limit of a sequence compact convex sets, it is compact and convex. (See Section 2.3.2.)

We claim that the convex hull

$$\mathcal{C} \mathcal{H} \left( \bigcup_{q \in h(\pi_1(\hat{E}))} h(q)A'(h(g_1)) \cup \{v_E\} \right)$$

has a nonempty interior: Suppose not. Then it is in a proper subspace where $\pi_1(\hat{E})$ acts on. This means that $\pi_1(\hat{E})$ is virtually-factorizable, and $\hat{E}$ is a totally geodesic $R$-end by Theorem 6.8. We choose another $g_2$ with $A(h(g_2))$ is not contained in this subspace by Proposition 2.15. Again, we find $A'(h(g_2)) \subset A(h(g_2))$. Now,

$$\mathcal{C} \mathcal{H} \left( \bigcup_{q \in h(\pi_1(\hat{E}), g=g_1,g_2)} h(q)A'(h(g)) \cup \{v_E\} \right)$$

is in a strictly larger subspace where $\pi_1(\hat{E})$ acts on. By induction, we stop at certain point, and we obtain

$$\mathcal{C} \mathcal{H} \left( \bigcup_{q \in h(\pi_1(\hat{E}), g=g_1,\ldots,g_m)} h(q)A'(h(g)) \cup \{v_E\} \right)$$

that is not contained in a proper subspace.

It is easy to show $h(q)A'(h(g)) = A'(hgq^{-1})$. There exists a finitely many elements $g_1, \ldots, g_m$ in $\pi_1(\hat{E})$ so that the attracting fixed set

$$\mathcal{C} \mathcal{H} \left( \{a_1, \ldots, a_m, v_{E,h_i}\}, a_j \in A'(h(g_j)) \right)$$

has a nonempty interior for some choice of $a_j$.

We have $A(h_j(g_i)) \subset \text{Cl}(L_j)$ for each $j$ and $i$. The sequence $\{A(h_j(g_i))\}$ accumulates only to points of $A'(h(g_j))$ since $\{h_j(g_i)\} \to h(g_i)$. There is a sequence $\{a_{i,j}\}$.
$a_{i,j} \in A(h_j(g_i))$, converging to $a_j \in A'(h(g_j))$ as $i \to \infty$. Lemma 2.4 implies that the sequence
\[
\{C.H'(\{a_{1,j}, \ldots, a_{m,j}, v_{E,h_i}\})\} \text{ in } \Omega_j
\]
converges to $C.H'(\{a_1, \ldots, a_m, v_E\})$ geometrically. Since $\text{Cl}(\Omega_j) \to K$ as $j \to \infty$,
\[
C.H'(\{a_1, \ldots, a_m, v_{E,h_i}\}) \subset K,
\]
by Proposition 2.1; thus, $K$ has nonempty interior in the case. (I) is accomplished.

(ii) Suppose that there is a lens-shaped totally geodesic p-end $\tilde{E}$ for $\mathcal{O}$ and the holonomy group $h(\pi_1(\tilde{E}))$, and the type for $\Omega_l$ and $h_i(\tilde{E})$ do not become horospherical. Then the dual $\Omega^*_l$ and $\mathcal{K}^*$ have a nonempty interior by above arguments since $\Omega^*_l$ has lens-shaped R-p-ends by Corollary 6.3. Hence (I) is accomplished.

(iii) Suppose that there is a lens-shaped R-p-end $\tilde{E}$ for $\mathcal{O}$ and the holonomy group $h(\pi_1(\tilde{E}))$, and the type for $\Omega_l$ and $h_i(\tilde{E})$ becomes horospherical.

This will be sufficient for (I) since when lens-type T-end changes to a horospherical $\mathcal{O}$- or $\mathcal{F}$-end, we can use the duality as in (ii).

We choose a convex fundamental domain $F$ for a complete affine space $\tilde{E}_{E,h} \subset \mathbb{S}^{n-1}_E$ under the action of $h(\pi_1(\tilde{E}))$ by J. Lee [127]. Let $g_1, \ldots, g_m$ be the paring Poincaré transformations of $F$. Suppose that $h_i(\pi_1(\tilde{E}))$ is sufficiently close to $h(\pi_1(\tilde{E}))$ in the algebraic sense, i.e., in term of the images of the generators $h_1(g_1), \ldots, h_m(g_m)$. Then we can choose a fundamental domain $F_i$ close to $F$ for $\tilde{E}_{E,h_i}$ determined by $h_1(g_1), \ldots, h_m(g_m)$ for sufficiently large $i$. We may assume without loss of generality that $\{\text{Cl}(F_i)\} \to \text{Cl}(F)$ geometrically.

Let $h(\pi_1(\tilde{E}))$ be the algebraic limit $h_i(\pi_1(\tilde{E}))$. Then $\mathcal{O} \cap h_1(\pi_1(\tilde{E}))$ is a lattice in a cusp group $\mathcal{O}$. We conjugate $\mathcal{O}$ so that it is a standard unipotent cusp group in $\text{SO}(n,1) \subset \text{SL}_+(n+1,\mathbb{R})$.

We choose any great segment $s_i$ with vertex $v_i$ in a direction of $F_i$. Suppose that
\[
\{d\text{-length}(l_i)\} > \varepsilon \text{ for infinitely many } l_i := \Omega_i \cap s_i
\]
for a uniform $\varepsilon > 0$. Let $l$ be the geometric limit of a subsequence $\{l_{j_i}\}$ of $\{l_i\}$ with a nonzero $d$-length. Then $\{h_{j_i}(g)(l_{j_i})\} \to h(g)(l)$ for $g \in \pi_1(\tilde{E})$. For any finite set $F \subset \pi_1(\tilde{E})$, the set
\[
\{h_{j_i}(g)(l_{j_i})|g \in F\} \to \{h(g)(l)|g \in F\}
\]
geometrically. By Lemma 2.4, we have the geometric convergence of the sequence of convex hulls
\[
\{C.H(\bigcup_{g \in F} h_{j_i}(g)(l_{j_i}))\} \to C.H(\bigcup_{g \in F} h(g)(l)).
\]
Since the later set has a nonempty interior by our assumption on $d$-lengths of $l_i$ and $h(g), g \in F$, is in a group conjugate to a cusp group, the convex hull
\[
C.H(\pi_1(\tilde{E}_{j_i}))(s_{j_i}) \subset \Omega_{j_i}
\]
contains a fixed open ball $B$ for sufficiently large $i$. This means $B \subset K$ showing (I).

Now we suppose

$$\{d\text{-length}(l_i)\} \to 0 \text{ for } l_i := \Omega_i \cap s_i \text{ as } i \to \infty. \quad (11.9)$$

**Lemma 11.4** Let $v, v = (1, 0, \ldots, 0)$ be a fixed point of the standard unipotent cusp group $P$ and let $L$ be a lattice in $P$. Let $H$ be a $P$-invariant hemisphere with $v$ in the boundary, and let $l$ be the maximal line with endpoints $v$ and $v_-$ perpendicular to $\partial H$ with respect to $d$. Then there exists a finite subset $F$ of $L$ so that the following holds:

- for any point $x \in l$ and a $d$-perpendicular hyperspace at $x$ bounding a closed hemisphere $H_x$, 
  $$I_x := \bigcap_{g \in F} g(H_x)$$
  is a properly convex domain, and

- as $x \to v$, the parameter $\{I_x\}$ geometrically converges to $\{v\}$.

**Proof** If $F$ is large enough, then $\{g(H_x) | g \in F\}$ is in a general position. We choose the affine coordinate system of $H^P$ as in Section 8.3.1.1 where we let $i_0 = n$.

The set of outer normal vectors of $\{g(\partial H_x)\}$ in the affine subspace $H_x$ are independent of the choice of $x$. Hence, as $x \to v$, the corresponding intersection set must be contained in any arbitrarily small ball. □

**Proof (Proof of Proposition 11.4 continued)** We may assume $v_{E, h_i} = v_{E, k_i} = v$ without loss of generality by changing the developing map by a sequence of bounded automorphisms $g_i$. Let $H$ denote the $P$-invariant hemisphere containing $K$. We assume that $\Omega_i \subset H$ and recall that radial $p$-end vertices are fixed to be $v$. We assume without loss of generality that the direction of a segment $l$ of the $d$-length $\pi$ is in $F_i$ always.

Let $\epsilon_i$ be the maximum $d$-length of a maximal segment $s'_i$ in $\Omega_i$ from $v$ in direction of $F_i$ for $i \geq I$. Let $F'_i$ denote the set of endpoints of the maximal segments in $\Omega_i$ in a direction of $F_i$. Then $\{\epsilon_i\} \to 0$ by (11.9). A hyperspace perpendicular to $l$ at $x_i \in l$ bounds a closed hemisphere $H'_i$ containing $F'_i$.

For $\delta_i := d(v, x_i)$, we have $\{\delta_i\} \to 0$

since otherwise (11.8) does not hold. By Lemma 11.4, there is a finite set $F \subset \pi_1(E)$ so that

$$\hat{K}_i := \bigcap_{g \in F} h_j(g)(H'_i) \cap H$$

is properly convex for sufficiently large $j$ since $\{h_j(g)\} \to h(g), g \in F$. This set $\hat{K}_i$ contains $\text{Cl}(\Omega_i)$ since $H'_i \supset \text{Cl}(\Omega_i)$.

As $\{x_i\} \to v$ and $\{g(x'_i)\} \to v, g \in F$, it follows that $\{\hat{K}_i\} \to \{v\}$. We just need to show that $\{h_j(g)(\partial H'_i)\}, g \in F, \text{ are uniformly bounded away from that of } \partial H$ and
\( \partial H' \) under the Hausdorff metric \( d_H \). Since \( \{ h_i(g)(\partial H') \} \) for each \( g \in F \) geometrically converges to \( h(g)(\partial H') \), we are done by Lemma 11.4.

Therefore, we conclude that \( K \) is a singleton.

By Proposition 2.23, \( \{ \Omega_i^* \} \) geometrically converges to \( K^* \) dual to \( K \) as \( i \to \infty \).

(See 2.5.4.) In case, \( K \) is a singleton, \( K^* \) must be a hemisphere by Proposition 2.22. We now conclude that \( K \) or its dual \( K^* \) has a nonempty interior.

Thus, by choosing \( h_i^* \) and \( h^* \) if necessary, we may assume without loss of generality that \( K \) has a nonempty interior. We will show that \( K \) is a properly convex domain and this implies that so is \( K^* \).

(II) The second step is to show \( K \) is properly convex.

Assume that \( h(\pi_1(\theta)) \) acts on a convex open domain \( K^o \). We may assume that \( K^o \subset \mathbb{A} \) for an affine subspace \( \mathbb{A} \) and \( \Omega_i \subset \mathbb{A} \) as well by acting by an orthogonal \( \kappa_i \in O(n+1, \mathbb{R}) \), where \( \{ \kappa_i \} \) is converging to \( I \). We can accomplish this by moving \( \Omega \) into \( \mathbb{A} \). Since \( \{ \kappa_i \} \to I \), we still have \( \{ \text{Cl}(\Omega_i) \} \to K \) by Lemma 2.1. Take a ball \( B_C \) of \( d \)-radius \( 2C \), \( C > 0 \), in \( K \). By Lemma 2.2, \( \mathbb{A} \) contains a \( d \)-radius \( C \) ball \( B_C \subset \Omega_i \) for sufficiently large \( i \). Without loss of generality assume \( B_C \subset \Omega_i \) for all \( i \).

Choose the \( d \)-center \( x_0 \) of \( B_C \) as the origin in the affine coordinates.

Let \( g_1, \ldots, g_m \) denote the set of generator of \( \pi_1(\theta) \). Then by extracting subsequences, we may assume without loss of generality that \( \{ h_i(g_j) \} \) converges to \( h(g_j) \) for each \( j = 1, \ldots, m \).

**Lemma 11.5** For each \( g_j, j = 1, \ldots, m \),

\[
d(h_i(g_j)(x_0), \text{bd}\Omega_i) \geq C_0 \text{ for a uniform constant } C_0. \tag{11.10}
\]

**Proof** Suppose not. Then there is a sequence of a \( d \)-length constant \( C \) segment \( s_i, s_i \subset \Omega_i \), with an origin \( x_0 \) is sent to the segment \( h_i(g_j)(s_i) \) in \( \Omega_i \) with endpoint \( h_i(g_j)(x_0) \) and lying on the shortest \( d \)-length segment from \( h_i(g_j)(x_0) \) to \( \text{bd}\Omega_i \). Thus, the sequence of the \( d \)-length of \( h_i(g_j)(s_i) \) is going to zero. This implies that \( h_i(g_j) \) is not in a compact subset of \( \text{SL}_\pm(n+1, \mathbb{R}) \), a contradiction. \( \square \)

**Proof (Proof of Proposition 11.4 continued)** By estimation from (11.10), and the cross-ratio expression of the Hilbert metric, a uniform constant \( C \) satisfies

\[
d_H(x_0, h_i(g_j)(x_0)) < C. \tag{11.11}
\]

By Benzécri [25] (see Proposition 4.3.8 of Goldman [91]), there exists a constant \( R_H > 1 \) and \( \tau_i \in \text{SL}_\pm(n+1, \mathbb{R}) \) so that

\[
B_1 \subset \tau_i(\Omega_i) \subset B_{R_H}.
\]

Now, \( \tau_i h_i(\pi_1(\theta)) \tau_i^{-1} \) acts on \( \tau_i(\Omega_i) \). By Theorem 7.1 of Cooper-Long-Tillmann [73], we obtain that \( \tau_i h_i(g_j) \tau_i^{-1} \) for \( j = 1, \ldots, n \) are in a compact subset of \( \text{SL}_\pm(n+1, \mathbb{R}) \) independent of \( i \).

Therefore, up to choosing subsequences, we have \( \{ \tau_i(\Omega_i) \} \) geometrically converges to a properly convex domain \( K \) in \( B_{R_H} \) containing \( B_1 \) and
11.3 The closedness of convex real projective structures

\[ \{ \tau_i h(\cdot) \tau_i^{-1} : \pi_1(\partial) \to \text{SL}_+(n+1, \mathbb{R}) \} \]

algebraically converges to a holonomy homomorphism

\[ h' : \pi_1(\partial) \to \text{SL}_+(n+1, \mathbb{R}). \]

And the image of \( h \) acts on the interior of the properly convex domain \( \hat{K} \).

Suppose that the sequence \( \{ \tau_i \} \) is not bounded. Then \( \tau_i = k_i d_i k_i' \) where \( d_i \) is diagonal with respect to a standard basis of \( \mathbb{R}^{n+1} \) and \( k_i, k_i' \in O(n+1, \mathbb{R}) \) by the KTK-decomposition of \( \text{SL}_+(n+1, \mathbb{R}) \). Then the sequence of the maximum modulus of the eigenvalues of \( d_i \) are not bounded above. We assume without loss of generality

\[ \{ k_i \} \to k, \{ k_i' \} \to k' \text{ in } O(n+1, \mathbb{R}). \]

Thus, \( \{ k_i' h_i(g_j) k_i^{-1} \} \) converges to \( k'h(g_j) k^{-1} \) for \( k' \in O(n+1, \mathbb{R}) \). Since

\[ \{ k_i d_i k_i' h_i(g_j) k_i'^{-1} d_i^{-1} k_i^{-1} \} \]

is convergent to \( h'(g_j) \), we obtain

\[ \{ d_i k_i' h_i(g_j) k_i'^{-1} d_i^{-1} \} \to k^{-1} h'(g_j) k \text{ for each } j. \]

Thus, \( \{ d_i k_i' h_i(\pi_1(\partial)) k_i'^{-1} d_i^{-1} \} \) converges algebraically to a group \( kh'(\pi_1(\partial)) k^{-1} \) acting on \( k^{-1}(\hat{K}) \).

Since the sequence of the norms of \( d_i \) is divergent, \( kh'(\pi_1(\partial)) k^{-1} \) is reducible: We may assume up to a choice of subsequence and a change of coordinates that the diagonal entries of \( d_i \) satisfy

\[ d_{i,1} \geq d_{i,2} \geq \cdots \geq d_{i,n+1}. \]

Up to a choice of subsequences, there is \( j \) so that \( d_{i,k}/d_{i,j} \geq 1 \) for \( k \leq j \) and \( d_{i,k}/d_{i,j} \to 0 \) for \( k > j \). Then \( \{ d_i k_i' h_i(\pi_1(\partial)) k_i'^{-1} d_i^{-1} \} \) being a bounded sequence converges to a matrix with entries at \( (k+1, \ldots, n+1) \times (1, \ldots, k) \) are identically zero. (Compare to the proof of Lemma 1 of [54].)

By Lemma 11.3, \( k^{-1}(\hat{K})/kh'(\pi_1(\partial))k^{-1} \) is a strongly tame SPC-orbifold with horospherical or generalized lens-shaped ends. By Theorem 1.2, the algebraic limit of

\[ \{ d_i k_i' h_i(\pi_1(\partial)) k_i'^{-1} d_i^{-1} \} \]

cannot be reducible. Therefore the sequence of the norms of \( d_i \) is uniformly bounded. This is a contradiction to the unboundedness of \( \tau_i \).

By Lemma 11.3, we obtain that \( \theta_h := \hat{K}/h(\pi_1(\partial)) \) is a strongly tame SPC-orbifold with generalized lens-shaped or horospherical \( \mathcal{R} \)- or \( \mathcal{T} \)-ends diffeomorphic to \( \partial \). This completes the proof for \( \overline{\mathcal{W}} \subseteq \text{CDef}_{\varepsilon, \mathbb{B}_h}(\partial) \).

To prove for \( \text{SDef}_{\varepsilon, \mathbb{B}_h}(\partial) \), we need additionally Theorems 10.3 and 10.5 as in the last paragraph of the proof of Theorem 11.6. This completes the proof of Corollary 11.4. \( \square \)
11.4 General cases without the uniqueness condition: The proof of Theorem 11.3.

We will construct a section by the following. Let

\[ \text{Hom}_{\mathcal{E}, \text{lh}}(\pi_1(\mathcal{E}), \text{PGL}(n+1, \mathbb{R})) \]  

(resp. \( \text{Hom}_{\mathcal{E}, \text{lh}}(\pi_1(\mathcal{E}), \text{SL}_{\pm}(n+1, \mathbb{R})) \))

denote the space of representations \( h \) fixing a common fixed point \( p \) and acting on a lens \( L \) of a lens-cone of form \( \{ p \} \ast L \) with \( p \not\in \text{Cl}(L) \) or is horospherical with a fixed point \( p \).

Let

\[ \text{Hom}_{\mathcal{E}, \text{lh}}(\pi_1(\mathcal{E}), \text{PGL}(n+1, \mathbb{R})) \]  

(resp. \( \text{Hom}_{\mathcal{E}, \text{lh}}(\pi_1(\mathcal{E}), \text{SL}_{\pm}(n+1, \mathbb{R})) \))

denote the space of representations where \( h(\pi_1(\mathcal{E})) \) for each element \( h \) acts on \( P \) satisfying the lens-condition or acts on a horosphere tangent to \( P \). (See Section 1.4.1.)

We define the sections

\[ s_R : \text{Hom}_{\mathcal{E}, \text{lh}}(\pi_1(\mathcal{E}), \text{PGL}(n+1, \mathbb{R})) \rightarrow \mathbb{R}P^n, \]

\[ s_T : \text{Hom}_{\mathcal{E}, \text{lh}}(\pi_1(\mathcal{E}), \text{PGL}(n+1, \mathbb{R})) \rightarrow \mathbb{R}P^{n*} \]

(resp. \( s_R : \text{Hom}_{\mathcal{E}, \text{lh}}(\pi_1(\mathcal{E}), \text{SL}_{\pm}(n+1, \mathbb{R})) \rightarrow \mathbb{S}^n, \)

\[ s_T : \text{Hom}_{\mathcal{E}, \text{lh}}(\pi_1(\mathcal{E}), \text{SL}_{\pm}(n+1, \mathbb{R})) \rightarrow \mathbb{S}^{n*} \])

by Propositions 6.8 and 6.9.

**Proposition 11.3** The maps \( s_R \) and \( s_T \) for \( \mathbb{R}P^n \) and \( \mathbb{R}P^{n*} \) (resp. \( \mathbb{S}^n \) and \( \mathbb{S}^{n*} \)) are both continuous.

**Proof** We will only prove for \( \mathbb{R}P^n \). The version for \( \mathbb{S}^n \) is obvious. Let \( h \) be an element of \( \text{Hom}_{\mathcal{E}, \text{ce}, p}(\pi_1(\mathcal{E}), \text{PGL}(n+1, \mathbb{R})) \). The vertex of a lens-cone is a common fixed point of all \( h(\pi_1(\mathcal{E})) \). Let \( F \) be the set of generators of \( \pi_1(\mathcal{E}) \) so that \( \{ v \} = \{ w | g(w) = w, g \in F \} \). Otherwise, we will have a line of fixed points for \( \Gamma_\mathcal{E} \) and we obtain a contradiction as in the proof of Proposition 6.8. Hence, the holonomies of elements of \( F \) determine the vertex. The continuity follows by a sequence argument.

For \( s_T \), we take the dual by by Proposition 6.11 and prove the continuity. \( \square \)

**Lemma 11.6** We can construct the uniqueness section of lens-type

\[ s : \text{Hom}_{\mathcal{E}, \text{lh}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \rightarrow (\mathbb{R}P^n)^{e_1} \times (\mathbb{R}P^{n*})^{e_2} \]

**Proof** We can always choose a vertex and the hyperspace by Propositions 6.8 and 6.9. \( s \) is continuous by Proposition 11.3. \( \square \)

**Proof (Proof of Theorem 11.3)** Using the uniqueness section of lens-type, we apply Corollary 11.4. \( \square \)
Appendix A
Projective abelian group actions on convex domains

We will explore some theories of projective abelian group actions on convex domains, which is not necessarily properly convex. In Section A.1.1, we show that a parameter of orbits of free abelian groups geometrically converges to an orbit in Lemma A.3. In Section A.1.2, we show that a free abelian group action decomposes the space into joins. In Section A.1.3, we discuss convex projective orbifolds with free abelian holonomy groups. In Lemma A.8, we will show a decomposition similar to the Benoist decomposition for the divisible projective actions on properly convex domains. In Section A.1.4, we will show that such actions always immediately deform to the abelian group actions on properly convex domains. In Section A.1.5, we prove geometric convergences of convex real projective orbifolds slightly more general than that explored by Benoist. In Section A.2, we give some justification of why we are using the weak middle eigenvalue conditions.

A.1 Convex real projective orbifolds

We will explore a class of convex real projective orbifolds a little bit more general than the properly convex ones. Also see Leitner [129], [128], and [130] for a similar work, where she explores representations of abelian groups; however, these do not act cocompactly on convex domains.

For our purposes in the monograph, we will mostly work on $\mathbb{S}^{n-1}$ but sometimes with $\mathbb{R}P^{n-1}$.

A.1.1 An orbit lemma

Recall the Cartan decomposition $\text{SL}_\pm(n, \mathbb{R}) = KTK$ where $K = O(n, \mathbb{R})$ and $T$ is the group of positive diagonal matrices. Note that the endomorphisms in $M_n(\mathbb{R})$ may have null spaces. Each induced projective endomorphism $f$ for $\mathbb{S}^{n-1}$ may have
a nonempty subspace $V_f$ where it is not defined. A Cartan decomposition $g = k_1Ak_2$ for $k_1, k_2 \in O(n, \mathbb{R})$ and a diagonal matrix $A$ with nonnegative diagonal for each element $g$ of $M_n(\mathbb{R})$ exists since each element is a limit of elements of $GL(n, \mathbb{R})$ admitting a Cartan decomposition. Define

$$C_{K_1, K_2} := \sup_{x \in K_2} d(x, S^{n-1} - K_1 - \alpha f(K_1)).$$

This is a number between $(0, \pi/2)$ when $K_2 \subset K_1^o$.

**Lemma A.1** Suppose that $K_1, K_2$ are compact subsets of $S^{n-1}$ with $K_2 \subset K_1^o$.

- The space of endomorphisms

$$F_{K_1} := \{ f \in S(M_n(\mathbb{R})) | V \text{ is a subspace where } f \text{ is undefined with } V \cap K_1^o = \emptyset \}$$

is compact.

- There exists a constant $C_{K_1, K_2}$ depending only on $C_{K_1, K_2}$ such that $d(f(x), f(y)) \leq C_{K_1, K_2}$ for every $x, y \in K_2$ and $f \in F_{K_1}$.

- Also, as long as $C_{K_1, K_2} \in (0, \pi/2)$ is in a compact subinterval, $C_{K_1, K_2}$ is bounded.

**Proof** Obviously, $F_{K_1}$ is a closed subset of $S(M_n(\mathbb{R}))$. Hence, it is a compact set.

Let $f \in F_{K_1}$. Then $f = k_1A_f k_2$ where $k_1, k_2 \in O(n, \mathbb{R})$ and $A_f$ is diagonal with nonnegative entries by a Cartan decomposition for $M_n(\mathbb{R})$. (See Definition 2.6.) Let $N_f$ be the projectivization of the null space of $A_f$. Then $V_f := k_2^{-1}N_f$ is the undefined subspace of $f$. Let $D_n(\mathbb{R}, N_f)$ denote the set of diagonal matrices with zero entries for directions corresponding to $N_f$.

Since $d(x, V_f) \geq C_{K_1, K_2}$ for $x \in K_2$, we have $d(k_2(x), N_f) \geq C_{K_1, K_2}$. Define $U_{c_{K_1, K_2}, f}$ to be the compact subset of points of $S^{n-1}$ of $d$-distance $\geq C_{K_1, K_2}$ from $V_f$.

Hence, since $x, y \in U_{c_{K_1, K_2}, f}$ for $x, y \in K_2$,

$$d(A_f k_2(x), A_f k_2(y)) \leq C_{K_1, K_2} d(x, y) \text{ for } x, y \in K_2, f \in F_{K_1}$$

holds for $C_{K_1, K_2}$ depending only on $C_{K_1, K_2}$ since $U_{c_{K_1, K_2}, f}$ is a compact set and the set $S(D_n(\mathbb{R}, N_f))$ is compact. Hence, the conclusion follows. (See Section 2.3.3.) \qed

The existence of the following group-invariant metric was first pointed out to me by Thurston in 1984. One can also have discrete group action for this to work. Let $\Omega$ be a convex open domain containing $x_0$ with a projective diagonalizable group $D$ acting transitively and freely on $\Omega$. We define the max metric

$$d_{\text{max}, \Omega}(x, y) := \sup_{g \in D} d(g(x), g(y)),$$

which is shown to be a metric in Proposition A.1.

**Proposition A.1** Let $\Omega$ be a properly convex open domain in $S^{n-1}$ with a projective automorphism group $D$ acting on it. Then $d_{\text{max}, \Omega}$ is a $D$-invariant metric. Also, for
a compact set $K_2$ in $\Omega$, $K_2 \subset \Omega$, there exists a constant $\hat{C}_{\text{Cl}(\Omega), K_2}$ depending only on $K_2$ and on $\Omega$ so that
\[
d(x, y) \leq d_{\text{max}, \Omega}(x, y) \leq \hat{C}_{\text{Cl}(\Omega), K_2}d(x, y) \text{ for } x, y \in K_2.
\]

**Proof** $d_{\text{max}, \Omega}(x, y) > 0$ unless $x = y$ as we are choosing the supremum. The symmetric property is obvious from that of $d$. The triangle inequality holds since the sum of the supremums of the distance occurring for $(g'(x), g'(y)), g' \in D$, and that of $(g''(y), g''(z)), g'' \in D$ should be larger than what occurs for $(g(x), g(z))$ for any $g \in D$ since $g'$ may not equal $g''$.

The $D$-invariance is clear from the definition.

For any sequence $f_i \in S(M_n(\mathbb{R}))$ converging to $f \in S(M_n(\mathbb{R}))$ which is degenerate, there is a subspace $N_f = N_f$ where the sequence of the corresponding diagonal entries of $A_{f_i}$ converges to 0.

Suppose that $f_i \in D$. Let $K_1$ be any compact subset of $\Omega$. Then $k_{2, i}^{-1}(N_f)$ is disjoint from $K_1$ for sufficiently large $i$: If not, $f_i(\Omega)$ has a subsequence $f_i(p), f_i(q)$ for $p, q \in \Omega$ converging to an antipodal pair. This is a contradiction to the proper convexity of $\Omega$.

Assume without loss of generality that $k_{2, i} \to k_{2, \infty}$ in $O(n, \mathbb{R})$. Let $bdD$ denote the boundary of $D$ in $S(M_n(\mathbb{R}))$. For $f \in bdD$, the undefined space $V_f$ does not meet $\Omega$ since $V_f = k_{2, \infty}^{-1}(N_f)$ is a limit of $k_{2, i}^{-1}(N_f)$.

By Lemma A.1, the inequality
\[
d(f(x), f(y)) \leq C_{\text{Cl}(\Omega), K_2}d(x, y), x, y \in K_2,
\]
holds for all $f \in bdD$ and a constant $C_{K_2, \text{Cl}(\Omega)} > 0$. Hence, the inequality holds for a constant $C_{K_2, \text{Cl}(\Omega)} + 1$ for a neighborhood $U$ of $bdD$ in $D$ since the both sides are continuous functions of $f, x, y$.

Since $D - U$ is a compact subset of $S(M_n(\mathbb{R}))$, we are finished with showing the right side of the inequality.

The left part is from the definition. □

**Lemma A.2** Suppose that a sequence of properly convex open simplex domain $\{\Omega_i\}$ in $\mathbb{S}^{n-1}$ geometrically converges to a convex open domain $\Omega$ under $d_H$.

- Suppose that an open $d$-ball $B_1(x_0), x_0 \in \Omega_i$, of radius 1 is contained in $\Omega_i$ for all $i$.
- Let $L$ be an abelian group $\mathbb{R}^n$ acting a positive diagonalizable group acting transitively on $\Omega$ with a right action $\Phi : L \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ and suppose that $\{\Phi_i\}$ converges in uniformly on each compact subset of $L$ to an action $\Phi : L \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$.
- Choose a max metric $d_{i}$ from $x_0$ for each $\Omega_i$.

Then there exists a metric $d_\Omega$ on $\Omega$ so that $\{d_i\}, i > i_0$ uniformly converges to $d_\Omega$ for each compact subset $K$ of $\Omega$ for sufficiently large $i_0$. Furthermore,

- $d_\Omega$ is invariant under the action $\Phi$. 
• for each pair of compact sets $K_2$ in $\Omega$, $K_2 \subset \Omega$, and
• there exists a constant $C_{Cl(\Omega),K_2}$ depending only on $C_{Cl(\Omega),K_2}$

$$d(x,y) \leq d_\Omega(x,y) \leq C_{Cl(\Omega),K_2}d(x,y) \text{ for } x,y \in K_2.$$ (A.1)

**Proof** For a compact subset $K$ in $\Omega$, there exists an integer $I_K$ such that $K \subset \Omega$, for $i > I_K$ by Lemma 2.2. For $x,y \in K$, we define $d_\Omega(x,y) := \sup_{t \in [0,t]} d(t,x,y)$. This gives a metric on $\Omega$ obviously. For $x,y \in K$ for a compact set $K_2 \subset \Omega$ for another compact set $K_1$, the inequality (A.1) holds because it holds for $d_i$.

**Lemma A.3** Let $t_0 \in I$ for an interval $I$. Suppose that we have a parameter of compact convex domains $\Delta_t \subset \mathbb{S}^{n-1}$ for $t < t_0$, $t \in I$, and a transitive group action $\Phi_t : L \times \Delta_t \to \Delta_t$ for a connected free abelian group $L$ of rank $n-1$ for each $t \in I$. Suppose that $\Phi_t$ depends continuously on $t$ and $\Phi_t$ is given by a continuous parameter of homomorphisms $h_t : L \to SL_\mathbb{Z}(n,\mathbb{R})$. Then $\{\Delta_t\} \to \Delta_{t_0}$ geometrically where $\Delta_{t_0}$ is a convex domain, and $L$ acts transitively on $\Delta_{t_0}$.

**Proof** Let $L \cong \mathbb{R}^{n-1}$ have coordinates $(z_1, \ldots, z_{n-1})$. $\Phi_t(g, \cdot) : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is represented as a matrix

$$h_t(g) = \exp(H_l(\sum_{i=1}^{n-1} z_i(g)e_i))$$ (A.2)

where $\{H_l : \mathbb{R}^{n-1} \to sl(n,\mathbb{R})\}$ is a uniformly bounded collection of linear maps.

We may assume without loss of generality that $\bigcap_{t \in I} \Delta_t \neq \emptyset$ by taking a smaller $I$. Choose a generic point $x_0 \in \bigcap_{t \in I} \Delta_t$. Any point $x \in \Delta_{t_0}$ equals $\Phi_0(g,x_0)$ for $g \in L$. Therefore,

$$\{\Phi_t(g,x_0)\} \subset \Delta_{t_0} \to \Phi_0(g,x_0) \text{ as } t \to t_0.$$ Hence, every point of $\Delta_{t_0}$ is the limit of a path $\gamma(t) \in \Delta_{t_0}$ for $t < t_0$.

Now, we show that the geometric limit of $\Delta_t$ as $t \to t_0$ is contained in the closure of a unique open orbit of $L$ under $\Phi_0$. Let $\Delta'_{t_0}$ denote the interior of a geometric limit of $Cl(\Delta_t)$ as $t \to t_0$.

First suppose that $\Delta_t$ is properly convex for $t \in I \setminus \{t_0\}$. Suppose that $\Delta'_{t_0}$ contains more than two open orbits under $\Phi_0$. Then there exists a point $y$ in the interior of $\Delta'_{t_0}$ and in an orbit of dimension $< n$. Here, the stabilizer of $y$ has a dimension $\geq 1$. Since the subgroups of a free abelian group are never compact, the stabilizer group is not compact.

This is contradicting the existence of the metric of Lemma A.2: We can take a $\varepsilon$-neighborhood $B$ in this metric of $y$ so that $bdB$ is in a compact sphere in $\Delta'_{t_0}$. The stabilizer group must be compact since we can choose a finite set $y = x_1$ and $x_2, \ldots, x_{n+1} \in bdB$ corresponding to independent vectors in $\mathbb{R}^{n+1}$. We have $d_\Theta(x_i, V_i) > \varepsilon$ for a uniform $\varepsilon > 0$ where $V_i$ is the subspace spanned by

$$\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}\}.$$
Since this condition is preserved under \( g \) in the stabilizer, any sequence of stabilizing elements has a subsequence converging to a projective automorphism in \( \text{Aut}(S^{n-1}) \). Hence, the stabilizer group must be compact.

Suppose now that \( \Delta_t \) is not properly convex for \( t \) in \( J - \{ t_0 \} \) for some interval \( J \subset I \). Then \( \text{Cl}(\Delta_t) = S_i^{n-1} + K_t \) for a properly convex domain \( K_t \) by Proposition 2.5. \( L \) acts on a great sphere \( S_i^{n-1} \) in the boundary of \( \Delta_t \). Now, \( S_i^{n-1} \) is the common boundary of \( i \)-dimensional affine spaces foliating \( \Delta_t \) by Proposition 2.5 as in Section 8.1.1. We may assume without loss of generality that \( S_i^{n-1} \) is a fixed sphere \( S_i^{n-1} \) acting by an element \( g_t \) where \( \{ g_t \} \) converging to \( I \) as \( t \to t_0 \). There is a projection \( \Pi_{g_i^{-1}} : S^{n-1} \to S^{n-1-i} \).

Then we consider \( \Pi_{g_i^{-1}}(K_t) \subset S^{n-1} \). Now, the discussion reduces to the above by taking a subgroup \( L' \subset L \) acting transitively on the interior of \( \Pi_{g_i^{-1}}(K_t) \) for \( t \in J - \{ t_0 \} \).

\[ \]
Proof Corollary to Theorem 12 of Section 6.8 in [107] implies the first statement. Let \( g_1, \ldots, g_k \) denote the generators of \( A \). We obtain \( C_{i,j} \) for \( g_j \) and taking intersections of the arbitrary collections of \( C_{i,j} \) for all \( i, j \).

A scalar group is a group acting by \( sI \) for \( s \in \mathbb{R} \) and \( s > 0 \). A scalar unipotent group is a subgroup of the product of a scalar group with a unipotent group. Hence, on each \( A C_i \) is a scalar unipotent group for each \( i \).

Lemma A.5 Let \( A \) be a connected free abelian group acting on \( \mathbb{R}^n \) with positive eigenvalues only. Then there exists a decomposition \( \mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_m \) where \( A \) acts as a positive diagonalizable group on \( V_0 \) and acts as a positive scalar unipotent group on each \( V_i \) for \( 1 \leq i \leq m \).

Proof We obtain \( C_1, \ldots, C_m \) by Lemma A.4. On \( C_i \), \( A \) acts as a scalar group acting on a one-dimensional space or a scalar unipotent group since the corresponding factor of the minimal polynomial is \( (x - \lambda_i I)^r \).

Proposition A.2 Suppose that \( \Gamma \) is a discrete free abelian group whose Zariski closure is \( A \). Suppose that elements of \( \Gamma \) have only positive eigenvalues. Then \( A/\Gamma \) is compact.

Proof By Lemma A.5, there is a decomposition \( \mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_m \) where \( A \) acts as a positive diagonalizable group on \( V_0 \) and acts as a positive scalar unipotent group on each \( V_i \) for \( 1 \leq i \leq m \).

Let \( q = \dim V_0 \). Let \( x_1(g), \ldots, x_q(g) \) denote eigenvalues of \( g, g \in A \), for \( V_0 \) and \( x_{q+1}(g), \ldots, x_{q+m}(g) \) denote respective ones for \( V_1, \ldots, V_m \). Let \( D \) be a positive diagonalizable group acting as a scalar group on each \( V_1, \ldots, V_m \) and positive diagonalizable group on \( V_0 \) defined as a subgroup of \( \mathbb{R}^{q+m} \) given by the equation \( x_1(g) \cdots x_{q+m}(g) = 1 \).

There is a homomorphism \( c : A \to D \) given by sending \( g \) to the tuples of eigenvalues respectively associated with \( V_0 \) and \( V_i \) for \( i = 1, \ldots, m \). Then \( \Gamma \) has a cocompact image in \( c(A) \) under \( c \) since \( c(A) \) is a connected diagonalizable group that is a Zariski closure of \( c(\Gamma) \) and \( c \) is continuous.

Let \( K \) be the kernel of \( c : A \to D \) which is an algebraic group. This is a unipotent subgroup of \( A \) and contains \( K \cap \Gamma \). Let \( K_1 \) be the Zariski closure of \( K \cap \Gamma \) in \( K \). Since \( K \cap \Gamma \) is normal in \( \Gamma \), and \( K_1 \) is the minimal algebraic group containing \( K \cap \Gamma \), \( K_1 \) is normalized by \( \Gamma \) and hence by \( A \).

If \( K_1 \) is a proper subgroup of \( K \), then there is a proper algebraic subgroup of \( A \) containing \( \Gamma \) since \( A \) is a product of \( K \) and a group isomorphic to \( c(A) \). This is a contradiction.

Since \( K \) is unipotent, \( K \cap \Gamma \) is cocompact in \( K \). Hence, \( \Gamma \) is cocompact in \( A \).
A.1.3 Convex real projective structures

Recall that for a matrix $A$, we denote by $|A|$ the maximum of the norms of entries of $A$.

We can deform the unipotent abelian representation to diagonalizable ones that are arbitrarily close to the original one.

**Lemma A.6** Let $h : \mathbb{Z}^l \to \text{SL}_+(n, \mathbb{R})$ be a representation to unipotent elements. Let $g_1, \ldots, g_l$ denote the generators. Then given $\varepsilon > 0$ there exists a positive diagonalizable representation $h' : \mathbb{Z}^l \to \text{SL}_+(n, \mathbb{R})$ with matrices satisfying $|h'(g_i) - h(g_i)| < \varepsilon, i = 1, \ldots, l$. Furthermore, we may choose a continuous parameter of $h'$ so that $h_0 = h$ and $h_t$ is positive diagonalizable for $t > 0$.

**Proof** First assume that every $h(g_i), i = 1, \ldots, l$, has matrices that are upper triangular matrices with diagonal elements equal to 1 since the Zariski closure is in a nilpotent Lie group and Theorem 3.5.4 of [160].

Let $\varepsilon > 0$ be given. We will inductively prove that we can find $h'$ as above with eigenvalues of $h'(g_1)$ are all positive and mutually distinct. For $n = 2$, we can simply change the diagonal elements to positive numbers not equal to 1. Then the group embeds in $\text{Aff}(\mathbb{R}^1)$. We choose positive constant $a_i$ so that $|a_i - 1| < \varepsilon$. Let $g_i$ be given as $x \mapsto a_i x + b_i$. The commutativity reduces to equations $a_i b_i = a_j b_j$ for all $i, j$. Then the solution are given by $b_i = a_i^{-1} a_j b_j$ for any given $b_1$. We can construct the diagonalizable representations.

Suppose that the conclusion is true for dimension $k - 1$. We will now consider a unipotent homomorphism $h : \mathbb{Z}^l \to \text{SL}_+(k, \mathbb{R})$. We conjugate so that every $h(g_i)$ is upper-triangular. Since $h(g_i)$ is upper triangular, let $h_1(g_i)$ denote the upper-left $(k - 1) \times (k - 1)$-matrix. By the induction hypothesis, we find a positive diagonalizable representation $h'_1 : \mathbb{Z}^l \to \text{SL}_+(k - 1, \mathbb{R})$. Also assume $|h'_1(g_i) - h_1(g_i)| < \varepsilon/2$ for $i = 1, \ldots, l$. We change

$$h(g_i) = \begin{pmatrix} h_1(g_i) & b(g_i) \\ 0 & 1 \end{pmatrix} \quad \text{to} \quad h'(g_i) = \begin{pmatrix} \frac{1}{\lambda'(g_i)} h'_1(g_i) & b'(g_i) \\ 0 & \lambda'(g_i) \end{pmatrix}$$

for some choice of $h'_1(g), b'(g), \lambda'(g) > 0$ for $i, j = 1, \ldots, l$. For commutativity, we need to solve for $b'(g_i)$ for $i, j = 1, \ldots, l$.

$$\left( \frac{1}{\lambda'(g_i)} h'_1(g_i) - \lambda'(g_j) I \right) b'(g_j) = \left( \frac{1}{\lambda'(g_j)} h'_1(g_j) - \lambda'(g_i) I \right) b'(g_i).$$

We denote by

$$M_i := \left( \frac{1}{\lambda'(g_i)} h'_1(g_i) - \lambda'(g_j) I \right).$$

Note $M_i M_j = M_j M_i$. By generic choice of $\lambda'(g_i)$s, we may assume that $M_i$ are invertible. The solution is given by
\[ b'(g_i) = M(g_i)^{-1}M(g_i)b'(g_1) \]

for an arbitrary choices of \( b'(g_1) \). We choose \( b'(g_1) \) arbitrarily near \( b(g_1) \). Here, \( \lambda_i(g_i) \) has to be chosen generically to make all the eigenvalues distinct and sufficiently near 1 so that \( |h'(g_i) - h(g_i)| < \varepsilon, i = 1, \ldots, l \). We can check the solution by the commutativity. Hence, we complete the induction steps.

To find a parameter denoted \( h'_r \), we simply repeat the induction process building a parameter of \( h'_r \). \( \square \)

**Lemma A.7** Let \( L \) be a connected projective abelian group acting on a properly convex domain \( K \) cocompactly and faithfully. Then \( L \) is positive diagonal and the domain is a simplex.

**Proof** \( L \) contains a cocompact lattice \( L' \). By the Hilbert metric of \( K^o \), \( L' \) acts properly discontinuously on \( K^o \). Proposition 2.15 applies now. Since \( L \) is the Zariski closure of positive diagonalizable \( L' \), we are done. \( \square \)

One can think of the following lemma as a classification of convex real projective orbifolds with abelian fundamental groups. Benoist [16], [19] investigated these in a more general setting.

**Lemma A.8** Let \( \Gamma \) be a free abelian group acting on a convex domain \( \Omega \) of \( \mathbb{S}^{n-1} \) (resp. \( \mathbb{R}P^{n-1} \)) properly and cocompactly. Then the following hold:

(i) the Zariski closure \( L \) of a finite index subgroup \( \Gamma' \) of \( \Gamma \) is so that \( L/\Gamma' \) is compact, and \( L \) has only positive eigenvalues (resp. a lift of \( L \) to \( SL_{\pm}(n, \mathbb{R}) \) has).

(ii) \( \Omega \) is an orbit of the abelian Lie group \( L \) acting properly and freely on it.

(iii) \( \Omega = (A_1 \ast \cdots \ast A_p \ast \{p_1\} \ast \cdots \ast \{p_q\})^o \) for a complete affine subspace \( A_i, i = 1, \ldots, p \), and points \( p_j, j = 1, \ldots, q \). Here, \( \langle A_1 \rangle, \ldots, \langle A_p \rangle, p_1, \ldots, p_q \) are independent.

(iv) \( L \) contains a central Lie subgroup \( Q \) of rank \( p + q - 1 \) acting trivially on \( A_j \) and \( p_k \) for \( j = 1, \ldots, p, k = 1, \ldots, q \).

**Proof** We will prove for the case \( \Omega \subset \mathbb{S}^{n-1} \). The other case is implied by this. If \( \Omega \) is properly convex, then Proposition 2.15 gives us a diagonal matrix group \( L \) acting on a simplex. (i) to (iv) follow in this case.

(i) Assume that \( \Omega \) is not properly convex.

Now, \( \Gamma \) has no invariant lower-dimensional subspace \( P \) meeting \( \Omega \): otherwise, \( \Gamma \) acts on \( P \cap \Omega \) properly so that \( P \cap \Omega / \Gamma \) is virtually homeomorphic to a lower-dimensional manifold homotopy equivalent to a cover of \( \Omega / \Gamma \) by Theorem 2.3. This is a contradiction.

The positivity of the eigenvalue will be proved: Let \( C_1, \ldots, C_{p+q} \) denote the subspaces of \( \Gamma \) obtained by Lemma A.4 where \( \dim C_1, \ldots, \dim C_p \geq 2, \dim C_{p+1} = \cdots = \dim C_{p+q} = 1 \). We also denote \( C_j = \{p_{j-p} \} \) for \( j = p + 1, \ldots, p + q \). Let \( \lambda_1(g), \ldots, \lambda_p(g) \) denote the norms of eigenvalues of each element \( g \) of \( L \) restricted to \( C_1, \ldots, C_p \) of dimension \( \geq 2 \) respectively. The eigenvalues associated with \( C_{p+1}, \ldots, C_{p+q} \) of dimension \( 1 \) are clearly positive.
Define
\[ \hat{S}_j := \mathbb{S}(C_1) \ast \cdots \ast \mathbb{S}(C_j) \ast \mathbb{S}(C_{j+1}) \ast \cdots \ast \mathbb{S}(C_p). \]
Since \( L \) acts transitively on \( \Omega \),
\[ \Omega \cap \hat{S}_j = \emptyset. \]
Let
\[ \Pi_j : \mathbb{S}^{n-1} - \hat{S}_j \to \mathbb{S}(C_j), j = 1, \ldots, p \]
denote the projection. Consider \( \Omega_j \) denote the image under \( \Pi_i \). The image is a convex subset of \( \mathbb{S}(C_j) \) since convex segments go to convex segments or a point under \( \Pi \). The image is open otherwise the dimension of \( \Omega < n \). Now \( \Gamma \) acts cocompactly on \( \Omega_j \), since \( \Gamma \) acts so on \( \Omega \) the domain of \( \Pi_i \). Then \( \Omega \subset \Omega_1 \ast \cdots \ast \Omega_p \).

Since \( \Omega \) is contained in an open hemisphere, the corresponding cone is contained in a half-space in \( \mathbb{R}^n \), and it follows that its image \( \Omega_j \) is contained in a hemisphere and is a convex open domain for each \( j \).

We can consider the action of \( \Gamma \) on \( C_j \) have the norm of the eigenvalue equal to 1 only by multiplying by a representation \( \Gamma \to \mathbb{R}_+ \) and we are working on \( \mathbb{S}^{n-1} \). The action of \( \Gamma \) on \( C_j \) is orthopotent by Theorem 2.6. By Conze-Guivarc'h [68] or Moore [143], there is an orthopotent flag in \( S_j \) and hence a proper \( \Gamma \)-invariant subspace. Let \( \Gamma_j \) denote the image of \( \Gamma \) by the restriction homomorphism to \( \mathbb{S}(C_j) \). Since \( \Gamma_j \) is abelian, \( \Gamma_j \) contains a uniform lattice \( \Gamma'_j \) in the Zariski closure of \( \Gamma_j \). Since \( \Gamma'_j \) is discrete, Theorem 2.6 shows that \( \Gamma'_j \) is virtually unipotent and so is its Zariski closure. Hence \( \Gamma_j \) is virtually unipotent. (See Theorem 3 of Fried [84].) Let \( \Gamma'_j \) be the unipotent subgroup of \( \Gamma_j \) of finite index. We can take \( \Gamma' := \bigcap_{j=1}^p \Pi_j^{-1}(\Gamma_j') \).

The finite index subgroup \( \Gamma' \) of \( \Gamma \) has only positive eigenvalue at \( \mathbb{S}(C_j) \) for each \( j \), \( j = 1, \ldots, p \). Also, the Zariski closure \( Z_j \) of \( \Gamma'_j \) is isomorphic to \( \mathbb{R}^n \) for some \( n_j \).

We assume that \( \Gamma' \) is torsion-free using Šelberg’s lemma. The Zariski closure \( L' \) of \( \Gamma' \) is in \( Z_1 \times \cdots \times Z_p \) and hence is free abelian. \( L'/\Gamma' \) is a closed manifold by Proposition A.2. We take a connected component \( L \) of \( L' \) and let \( \Gamma' = L \cap \Gamma \). Now, \( L'/\Gamma' \) is a manifold, and \( \Omega'/\Gamma' \) is a closed manifold. Since they are both \( K(\Gamma', 1) \)-spaces, it follows that \( \dim L = \dim \Omega = n - 1 \). This proves (i).

(ii) We will now let \( \Gamma \) to be \( \Gamma' \) above without loss of generality. Suppose that \( p = 1 \) and \( q = 0 \), or suppose that the associated eigenvalue of each \( g \in \Gamma \) in \( C_j \) is independent of \( j \). Since \( \Omega \) is a convex domain in an affine subspace in \( \mathbb{S}^{n-1} \), \( \Omega \) is in a complete affine subspace. We can change \( \Gamma \) to be unipotent by changing scalars. A unipotent group acts on a half-space in \( \mathbb{R}^n \) since its dual must fix a point in \( \mathbb{R}^n \) being solvable. Thus \( \Gamma \) acts on an affine subspace \( A^{n-1} \) in \( \mathbb{S}^{n-1} \), and \( \Gamma \) acts as an affine transformation group of \( A^{n-1} \). Proposition 7 of [95] proves our result.

Otherwise, it must be that \( \Omega \) is not complete affine but not properly convex. There exists a great sphere \( S^{i-1} \) in the boundary of \( \Omega \) where \( L \) acts on and is the common boundary of \( i \)-dimensional affine spaces foliating \( \Omega \) by Proposition 2.5 as in Section 8.1.1. There is a projective projection
\[ \Pi_{S^{i-1}} : \mathbb{S}^{n-1} - S^{i-1} \to \mathbb{S}^{n-i-1}. \]
Then the image $\Omega_1$ of $\Omega$ is properly convex, and $\Omega$ is the inverse image $\Pi^{-1}_{\Omega_1}(\Omega_1)$. Since $L$ acts on $\Omega_1$, it follows that $L$ acts on $\Omega$. Since $\dim L = \dim \Omega$ and $\Gamma$ acts properly with a compact fundamental domain, $L$ acts properly and cocompactly on $\Omega$. (See Section 3.5 of [159].)

Let $N$ denote the kernel of $L$ going to a connected Lie group $L_1$ acting on $\Omega_1$ properly and cocompactly.

$$1 \rightarrow N \rightarrow L \rightarrow L_1 \rightarrow 1.$$ 

By Lemma A.7, $\Omega_1$ is a simplex. Hence, $L_1$ is a positive diagonalizable group. Since $\Omega_1/L_1$ is compact, $L_1$ acts simply transitively on $\Omega_1$ by Lemma 2.5 of [21].

$$\dim L_1 = n - i - 1. \text{ Thus, } \dim N = i \text{ and the abelian group } N \text{ acts on each complete affine } i\text{-dimensional affine space } A_1 \text{ that is a leaf. Since the action of } N \text{ is proper, } N \text{ acts on } A_1 \text{ transitively by the proof of Lemma 2.5 of [21]. The action is simple since } \dim A_1 \leq \dim l = i. \text{ Hence, } L \text{ acts transitively on } \Omega. \text{ Since the action of } L \text{ is proper, } L \text{ acts simply transitively by the dimension count.}$$

(iv) By Lemma A.5 and (i), we decompose $\mathbb{R}^n$ into subspaces $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_p \oplus V_0$ where $V_j$ corresponds to $C_j$ for $j = 1, \ldots, p$, $V_0$ corresponds to $C_{p+1} \ast \cdots \ast C_{p+q}$, $L$ acts on $V_0$ as a positive diagonalizable linear group and $L$ acts on $V_j$ as elements of an abelian positive scalar unipotent group for $j = 1, \ldots, p$. (Here $V_0$ can be 0 and $V_i$ for $i \geq 1$ equals $C_i$ for some $j$.)

Since $L$ acts on $V_0$ as a positive diagonalizable group, it fixes points $p_1, \ldots, p_q$ in general position in $\mathbb{S}^{q-1} := \mathbb{S}(V)$ with $q = \dim V_0$. We claim that $L$ contains an abelian Lie subgroup $Q$ of rank $p + q - 1$ acting trivially on each $\Omega_j$, $j = 1, \ldots, p$, and fixing $p_i$, $i = 1, \ldots, q$. Suppose that $\Omega$ is properly convex. Then $\Omega$ is the interior of a simplex. The cocompactness of a lattice of $L$ shows that $L$ contains a discrete free abelian central group of rank $p + q - 1$ by the last part of Proposition 2.15. The central group is contained in $Q$. Since $L$ is the Zariski closure of the lattice, $Q$ is a subgroup of $L$.

Suppose that $\Omega$ is not properly convex. We deform a lattice of $L$ to a diagonalizable one by a generalization of Lemma A.6 to the direct sum of scalar unipotent representations, and use the limit argument.

Actually, $Q$ is the maximal diagonalizable group with over vectors in directions of $p_1, \ldots, p_q$ and eigenspaces $V_j$, $j = 1, \ldots, p$. This again follows by the limit argument. This proves (iv).

(iii) We choose a generic point $\langle \mathbf{x} \rangle$, $\langle \mathbf{x} \rangle \in \Omega$, in the complement of $\mathbb{S}(V_0)$, $\mathbb{S}(V_j)$ for $j = 1, \ldots, m$. Since these are independent spaces, $\mathbf{x} = \mathbf{x}_0 + \sum_{j=1}^{m} x_j$ where $\mathbf{x}_0 \in V_0$, $x_j \in V_j$. We choose a parameter of element $\eta_j$ of $Q$ fixing $V_0$ or $V_j$ for some $j$ with largest norm eigenvalues and $\{ \eta_j \}$ converging to 0-maps on other subspaces as $t \to \infty$. By Theorem 2.8, we obtain a projection to $V_0$ or $V_j$ for each $j$ as a limit in $\mathbb{S}(M_0(\mathbb{R}^n))$, and $\langle \mathbf{x}_0 \rangle$, $\langle \mathbf{x}_j \rangle$ are in the closure of $L(\langle \mathbf{x} \rangle)$.

Since $H_j(L(\mathbf{x})) = L(H_j(\mathbf{x}))$, we obtain

$$L(\langle \mathbf{x} \rangle) \subset L(\langle \mathbf{x}_0 \rangle) \ast L(\langle \mathbf{x}_1 \rangle) \ast \cdots \ast L(\langle \mathbf{x} + m \rangle). \quad (A.4)$$
From the above paragraph, we can show that $L(\langle x_j \rangle)$ is contained in $\text{Cl}(L(\langle x \rangle))$. Hence,

$$\text{Cl}(L(\langle x \rangle)) \supset L(\langle x_0 \rangle) \ast L(\langle x_1 \rangle) \ast \cdots \ast L(\langle x_p \rangle). \quad (A.5)$$

Since $L$ acts transitively on $\Omega$, $L$ acts so on the projection $\Omega_j$ under $\Pi_j$. Hence, $L(\langle x \rangle) = \Omega$ and $L(\langle x_j \rangle) = \Omega_j$. By convexity of the domain $\text{Cl}(\Omega)$, $(\text{Cl}(\Omega))^o = \Omega$. We obtain

$$\Omega = \{p_1 \ast \cdots \ast p_q \ast \Omega_1 \ast \cdots \ast \Omega_p\}^o.$$  

Recall from above that $L/\Omega_j$ is a unipotent abelian group and hence has a distal action. The proof of Theorem 2 of [84] applies here since $L/\Omega_j$ contains a cocompact lattice, and it follows that $L(x_j)$ is a complete affine space in $\mathbb{S}(V_j)$. This proves (iii). [SMT]

### A.1.4 Deforming convex real projective structures

**Lemma A.9** Let $\mu$ be a real projective structure on a closed orbifold $M$ with a developing map $\text{dev}: \tilde{M} \to \mathbb{S}^{n-1}$ (resp. $\mathbb{RP}^{n-1}$) is not injective. Then for any structure $\mu'$ sufficiently close to $\mu$, its developing map $\text{dev}'$ is not injective.

**Proof** We prove for $\mathbb{S}^{n-1}$. We take two open sets $U_1$ and $U_2$ respectively containing two points $x, y \in \tilde{M}$ with $\text{dev}(x) = \text{dev}(y)$ where $\text{dev}|U_i$ is an embedding for each $i = 1, 2$. Then for any developing map $\text{dev}'$ for $\mu'$ perturbed from $\text{dev}$ under the $C^\infty$-topology, $\text{dev}'(U_1) \cap \text{dev}(U_2) \neq \emptyset$. Hence, $\text{dev}'$ is not injective. [SMT]

A convex real projective structure $\mu_0$ on an orbifold $\Sigma$ is **virtually immediately deformable** to a properly convex real structure if there exists a parameter $\mu_t$ of real projective structures on a finite cover $\hat{\Sigma}$ of $\Sigma$ so that $\hat{\Sigma}$ with induced structures $\tilde{\mu}_t$ is properly convex for $t > 0$.

**Proposition A.3** A convex real projective structure on a closed $(n-1)$-orbifold $M$ with virtually free abelian holonomy subgroup of a finite index is always virtually immediately deformable to a properly convex real projective structure.

**Proof** Again, we prove for $\mathbb{S}^{n-1}$. Let $\mathbb{Z}^J$ denote the fundamental group of a finite cover $\tilde{M}'$ of $M$. Let $h \in \text{Hom}(\mathbb{Z}^J, \text{SL}_+(n, \mathbb{R}))$ be the restriction of the holonomy homomorphism to $\mathbb{Z}^J$. Nearby every $h$, there exists a positively diagonalizable holonomy $h': \mathbb{Z}^J \to \text{SL}_+(n, \mathbb{R})$ by Lemmas A.6 and A.8. By the deformation theory of [54], $h''$ is realized as a holonomy of a real projective manifold $M''$ diffeomorphic to $M'$. Also, the universal cover of $M''$ is a union of orbits of an abelian Lie group $L$ by Benoist [16]. Here, $h'(\mathbb{Z}^J)$ is a lattice in $L$ by Lemma A.8.

By premise, $h'$ is deformable to $h''$ where $h''(\mathbb{Z}^J)$ acts on properly convex domain cocompactly. By Lemma A.10, $M''$ is a properly convex real projective orbifold. [SMT]
Let us recall a work of Benoist: Let $M$ be a real projective $(n - 1)$-orbifold with nilpotent holonomy. (Here we are not working with orbifolds.) Let $N$ be the nilpotent identity component of the Zariski closure of the holonomy group, Benoist [16], [19] showed that $\tilde{M}$ decomposes into a union of connected open submanifolds $D_i$, $i \in I$ for an index set $I$, so that $\text{dev} : D_i \to \text{dev}(D_i)$ is a diffeomorphism to an orbit of a nilpotent Lie group $N$. The brick number of $M$ is the number of the $(n - 1)$-dimensional open orbits that map to mutually distinct connected open strata in $M$.

**Lemma A.10** Let $M$ be a closed $(n - 1)$-orbifold. Let $(M, \mu)$ be a convex real projective orbifold with a virtually abelian fundamental group. Suppose $\mu$ is deformed to a continuous parameter $\mu_t$ of real projective structures so that $\mu_0 = \mu$. Let $h_t : \pi_t(M) \to SL_+(n, \mathbb{R})$ (resp. $PGL(n, \mathbb{R})$), $t \in [0, 1] = I$, be a continuous family of the associated holonomy homomorphism with $\text{dev}_t : \tilde{M} \to S^{n-1}$ (resp. $\mathbb{RP}^n$). Suppose $\mu_t$ has the holonomy group $h_t(\pi_t(M))$ with following properties:

(A) it is virtually diagonalizable for $t > 0$ or, more generally, it acts on some proper convex domain $D_t$, or
(B) it acts properly on a complete affine space $D_t$

where $D_t$ has no proper $h_t(\pi_t(M))$-invariant open domain.

Then $(M, \mu_t)$ is properly convex or is complete affine for $t > 0$.

**Proof** Again, we prove for $\mathbb{S}^{n-1}$. For $t = 0$, $\text{dev}_0$ is a diffeomorphism to a complete affine space or a properly convex domain by our conditions.

Define the following sets:

- $A$ is the subset of $t$ satisfying (A) and $\mu_t$ is a properly convex structure, and
- $B$ is the subset of $t$ for (B).

We will decompose $[0, 1] = A \cup B$.

By Lemma A.3, the set $\hat{A}$ of points of $I$ satisfying (A) is open. By Koszul [124], the set $A$ is open and $A \subset \hat{A}$. We have $\hat{A} \cup B = [0, 1]$.

(A) Suppose that we have an open connected subset $U$ with $U \subset \hat{A}$ and $t_0 \in \text{bd}U$ with properly convex or complete affine $\mu_{t_0}$. Then we claim $U \subset A$.

Let $t \in U$. Since the holonomy group $h_t(\pi_t(M))$ is virtually abelian, a finite cover $M'$ of $M$ is octantizable by Proposition 2 of [16]. $\text{dev}_t$ maps to a union of orbits of an abelian Zariski closure $\Delta_t$ of a finite index abelian subgroup $H_t$ of $h(\pi_t(M'))$ in $\mathbb{S}^{n-1}$ by [19]. He also shows that $\Delta_t$ acts on $\tilde{M}$.

By our assumption for $h_t$, $h_t(\pi_t(M'))$ acts on a properly convex domain in $\mathbb{S}^{n-1}$. $\Delta_t$ is positive diagonalizable by Lemma A.7. Now orbits of $\Delta_t$ are convex simplexes in $\mathbb{S}^{n-1}$ by Section 3.1 of Benoist [19] where he explains the classification of such structures by Smillie [154] and [155].

But $\text{dev}_t$ cannot map to a more than one orbit of $\Delta_t$ since $(M, \mu_t)$ is $C^2$-close to a finite cover of a complete closed affine orbifold $\tilde{M}/h_t(\pi_t(M))$ where $\tilde{M}$ was an orbit of the Zariski closure of $h_t(\pi_t(M))$: Suppose not. We take a finite cover $M''$ of $M$ so that $M''$ with induced $\mu_t$ has a brick number $> 1$. We can find a parameter $\{\mu_t\}$ for induced real projective structures of $\mu_t$ on $M''$ converging to a real projective structure $\mu_0$ on $M'$ in the $C^*$-sense, and a compact fundamental domain $F_t$ of $\tilde{M}$.
obtained by perturbing a compact fundamental domain $F$ of $\hat{M}$ for $\mu_0$. Since $F$ maps into an open orbit, $F_t$ maps into an open orbit for sufficiently small $t$ by Lemma A.3. However, this means that $F_t$ is in an orbit of $\hat{M}$. Since $\hat{M}$ is a union of images of $F_t$, it follows that $\hat{M}$ is in an open orbit, which is a contradiction. (In other words, a sequence of real projective structures with more than one bricks cannot converge to a properly convex structure or a complete affine structure.) Since the orbits of $\Delta_0$ on $S^{n-1}$ are properly convex, we conclude that $\mu_t \in A$ for a sufficiently small $|t - t_0|$ by Theorem 2.14. Hence, $U \cap A$ is a nonempty open set.

Also, we claim that $U$ is in $A$: for any sequence $\{t_i\}$ converging to $t_0$ in $U$, choose a point $x_0$ in the developing image of $\text{dev}_{t_i}$ for sufficiently large $i$ since $\mu_i$ are sufficiently $C^\prime$-close. Then $\Delta_i(x_0) \to \Delta_0(x_0)$ by Lemma A.2. Since $\hat{M}$ with $\mu_i$ develops into $\Delta_i(x_0)$, $M$ with $\mu_0$ develops into $\Delta_0(x_0)$. The developing map is a diffeomorphism to $\Delta_0(x_0)$ by Theorem 2.14. Hence, $t_0 \in A$ also. Thus, $U \cap A$ is open and closed and hence $U \subset A$.

Hence, for connected open $U$, $U \subset \hat{A}$, with $t_0 \in \text{bd}\hat{A}$ for properly convex or complete affine $\mu_0$, we have $U \subset A$.

(B) Let $I'$ denote the subset of $I$ in $[0, 1]$ consisting of $t$ with $\mu_t$ satisfying conclusions of the lemma. Since $\mu_0$ is properly convex or complete affine, $I'$ is not empty. Also, $I'$ contains all components of $\hat{A}$ meeting it by the above argument. Also, $A = I' \cap \hat{A}$ is open.

We will show that $I'$ is open and closed in $[0, 1]$ and hence $I' = [0, 1]$:

First, we show that $I'$ is open. Also, for $t_0 \in I' \cap B$, we claim that a neighborhood is in $I'$: Otherwise, there is a sequence $\{t_i\}$ for $t_i \notin I'$ converging to $t_0$ with $t_i \in I$ and $\text{dev}_{t_i}$ is not a diffeomorphism to a complete affine space where $\pi_t$ acts on.

- If $t_i \in \hat{A}$ for infinitely many $i$, then $t_i \in A \subset I'$ for sufficiently large $i$ by the same argument as the fourth paragraph above. This is a contradiction.

- Assume now $t_i \in B$ for sufficiently large $i$. Proposition 2 of [16] shows that $\hat{M}$ decomposes into orbits of an abelian group that is the Zariski closure of $h_0(\pi_t(M))$. By our condition, open orbits are open hemispheres. Since $\text{dev}_{t_i}$ is not a diffeomorphism to an orbit, $\hat{M}$ has more than two open orbits. Also, $\hat{M}$ does not have a compact fundamental domain contained in an orbit since otherwise by $\text{dev}_{t_i}$, $\hat{M}$ must map into an orbit. We choose a compact fundamental domain $F$ for $\text{dev}_{t_0}$, which can be perturbed to a compact fundamental domain $F_t$ which is inside an open orbit. Since $\hat{M}$ is a union of images of $F_t$, $\hat{M}$ is inside one orbit. This is a contradiction.

Hence, we showed that $I'$ is an open subset of $I$.

Now, we show that $I'$ is closed: For any boundary point $t_0$ of $I'$ in $\hat{A}$, we have $t_0 \in I'$ since $I'$ contains a component of $\hat{A}$ it meets.

For any boundary point $t_0$ of $I'$ that is in $B$, we have $\text{dev}_{t_0}$ must be injective by Lemma A.9 since $\text{dev}_{t}$ is injective for $t < t_0$ or $t > t_0$ for $t$ in an open interval with boundary point $t_0$. Hence, the image of $\text{dev}_{t_0}$ meets a complete affine subspace invariant under the action of $h_0(\pi_t(M))$. Since the image of $\text{dev}_{t}$ for $t \in I'$ is in a hemisphere or a compact properly convex domain $D_t$, the image of $\text{dev}_{t_0}$ is in a
hemisphere or a properly convex domain that is the geometric limit of $D_i$ up to a choice of subsequences. Hence, $t_0 \in I'$.

Thus, $I'$ is open and closed. This completes the proof. 

**Lemma A.11** Let $\Sigma$ be a properly convex closed orbifold with a structure $\mu_t$, $t \in [0, 1]$. Let $\text{dev}_t$ be a continuous parameter of developing maps for $\mu_t$. Then

$$\text{Cl}(\text{dev}_t(\Sigma)) = S_1 \ast \cdots \ast S_m$$

for properly convex domains $S_1, \ldots, S_m$, where each $S_{j,t}$ span a subspace $P_{j,t}$. The finite-index subgroup of $h_t(\pi_1(\Sigma))$ acting on $P_{j,t}$ acts strongly irreducible for each $t$. Furthermore, $m_t, \dim S_{j,t}$ are constant and $P_{j,t}, j = 1, \ldots, m_t$ are always independent, and $P_{j,t}$ forms a continuous parameter in the Grassmannian spaces $G(n, \dim P_{j,t})$ up to reordering.

**Proof** The decomposition follows from Proposition 2.15. There is a virtual center $Z$, a free abelian group of rank $m_1$, mapping to a positive diagonalizable group $Z_t$ acting trivially on each $S_{j,t}$. By Theorem 1.1 of [21], an infinite-order virtually central element cannot have nontrivial action on $S_{j,t}$ since otherwise the Zariski closure of the subgroup of $h_t(\pi_1(\Sigma))$ acting on $P_{j,t}$ cannot be simple as claimed immediately after that theorem. Hence, any infinite-order virtually central elements are in a maximal free abelian group of rank $m_1$. Hence, for each $t$, there are subspaces $S_{j,t}$ for $j = 1, \ldots, m_1$ so that the above decomposition hold. Now, we need to show that the dimensions are constant.

We decompose $I$ into mutually disjoint subsets $I_{n_1, \ldots, n_{m_1}}$ where

$$\dim S_{j,t} = n_j$$

where $n_1 + \cdots + n_{m_1} + m_1 = n + 1$ for $n_1 \leq n_2 \leq \cdots \leq n_{m_1}$

by reordering the indices. Then each of these sets is closed as we can see from a sequence argument since the above rank argument shows that there cannot be further decomposition in the limit increasing the rank of the virtual center. Since $I$ is connected, there is only one such set equal to $I$. Now, the conclusion follows up to reordering. $\Box$

We generalize Proposition A.10:

**Corollary A.1** Suppose that a real projective orbifold $\Sigma$ is a closed $(n-1)$-orbifold with the structure $\mu$. Let $\mu_t, t \in [0, 1]$, be a parameter of projective structures on $\Sigma$ so that $\mu_0$ is properly convex or complete affine and $\mu_1 = \mu$. Let $h_t$ denote the associated holonomy homomorphisms.

- Suppose that the holonomy group $h_t(\pi_1(\Sigma))$ in $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}_+^+(n, \mathbb{R})$) acts on a properly convex domain or a complete affine subspace $D_t$.
- Suppose $D_t$ is the minimal $h_t(\pi_1(\Sigma))$-invariant convex open domain.
- We require that $\pi_1(\Sigma)$ to be virtually abelian if $D_{t_0}$ is complete affine for at least one $t_0$.

Then $\Sigma$ is also properly convex or complete affine where the following hold:
A.1 Convex real projective orbifolds

- A developing map $\text{dev}_t$ is a diffeomorphism to $R_t(D_t)$ for every $t \in [0,1]$
- Where $R_t$ is a uniformly bounded projective automorphism for each $t$ and is a composition of reflections commuting with one another.
- Furthermore, $\Sigma$ is properly convex if so is $D_t$.

**Proof** Again, we prove in $S^{n-1}$. If $\pi_1(\Sigma)$ is virtually abelian, then it follows from Proposition A.10.

Now, suppose that $\pi_1(\Sigma)$ is not virtually abelian. Then $D_t$ is properly convex for every $t, t \in I$, by the premise.

The set $A$ where $\mu_t$ is properly convex is open by Koszul [123]. Let $t_0$ be the supremum of the connected component $A'$ of $A$ containing 0. There is a developing map $\text{dev}_t : \tilde{\Sigma} \to S^{n-1}$ for $t \in A$ is a diffeomorphism to a properly convex domain $D'_t$ where $D'_t = R_t(D_t)$ for a projective automorphism $R_t$ by Lemma 2.19.

Since associated developing map $\text{dev}_t$ maps into $D'_{t_0}$ for $t < t_0$, $\text{dev}_t|K$ for a compact fundamental domain $F \subset \tilde{\Sigma}$ maps into a compact subset of $D'_{t_0}$ for $t < t_0$. Since $\text{dev}_t : \tilde{\Sigma}$ is injective, $\text{dev}_{t_0} : \tilde{\Sigma} \to S^{n-1}$ is also injective by Lemma A.9. By the injectivity and the invariance of domain, $\text{dev}_{t_0}$ is a diffeomorphism to an open domain $\Omega$. Since every point of the image of $\text{dev}_{t_0}$ is approximated by the points in the image $D'_{t_0}$ of $\text{dev}_t$ for $t < t_0$. Hence, $\text{Cl}(\Omega)$ is contained in the geometric limit of a convergent subsequence of any sequence $\text{Cl}(D'_{t_i})$ by Proposition 2.1. By Lemma 2.19, $\Omega$ is a properly convex domain since the holonomy group acts on a properly convex domain $D_{t_0}$ and $\Omega = R_{t_0}(D_{t_0})$ for a projective automorphism $R_{t_0}$ that is a composition of reflections commuting with one another.

Hence, $A$ is also closed, and $A = [0,1]$. By Lemma A.11 the uniform boundedness of $R_t$ follows since the subspaces $P_{j,t}$ are continuous and $R_t$ are either I or $\mathcal{A}$ on it. [$S^nT$]

A.1.5 Geometric convergence of convex real projective orbifolds

Note that the third item of the premise below is automatically true by Theorem 4.1 if $\Sigma$ is an end-orbifold of a properly convex affine $n$-orbifold for any $t$.

**Corollary A.2** Suppose that $\Sigma$ is a closed $(n-1)$-orbifold. We are given a path $\mu_t, t \in [0,1]$, of convex $\mathbb{RP}^{n-1}$-structures on $\Sigma$ equipped with the $C^r$-topology, $r \geq 2$. Suppose that $\mu_0$ is properly convex or complete affine.

- Suppose that the holonomy group $h_t(\pi_1(\Sigma))$ in $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}_\pm(n, \mathbb{R})$) acts on a properly convex domain or a complete affine subspace $D_t$.
- Suppose $D_t$ is the minimal holonomy invariant domain.
- We require that if $\mu_t$ is complete affine for at least one $t$, then the holonomy group is virtually abelian.

Then the following holds:
We can find a family of developing maps $\text{dev}_t$ to $\mathbb{R}P^{n-1}$ (resp. in $\mathbb{S}^{n-1}$) continuous in the $C^\infty$-topology and a continuous family of holonomy homomorphisms $h_t : \Gamma \to \Gamma_t$ so that $K_t := \text{Cl}(\text{dev}_t(\Sigma))$ is a continuous family of convex domains in $\mathbb{R}P^{n-1}$ (resp. in $\mathbb{S}^{n-1}$) under the Hausdorff metric topology of the space of closed subsets of $\mathbb{R}P^{n-1}$ (resp. $\mathbb{S}^{n-1}$).

In other words, given $0 < \varepsilon < 1/2$ and $t_0, t_1 \in [0, 1]$, we can find $\delta > 0$ such that if $|t_0 - t_1| < \delta$, then $K_{t_1} \subset N_{\varepsilon}(K_{t_0})$ and $K_{t_0} \subset N_{\varepsilon}(K_{t_1})$.

Also, given $0 < \varepsilon < 1/2$ and $t_0, t_1 \in [0, 1]$, we can find $\delta > 0$ such that if $|t_0 - t_1| < \delta$, then $\partial K_{t_1} \subset N_{\varepsilon}(\partial K_{t_0})$ and $\partial K_{t_0} \subset N_{\varepsilon}(\partial K_{t_1})$.

Finally, $\mu_t$ is always virtually immediately deformable to a properly convex structure.

**Proof** We will prove for $\mathbb{S}^{n-1}$. Suppose first that $\pi_t(\Sigma)$ is not virtually abelian. Then $D_t$ is never a complete affine space and hence is always properly convex by the premise.

First, for any sequence $\{t_i\}$ converging to $t_0$, we can choose a subsequence $\{t_{i_j}\}$ so that $\{K_{t_{i_j}}\}$ converges to a compact convex set $K_\infty$ in the Hausdorff metric. $h_{t_0}(\pi_1(\Sigma))$ acts on $K_\infty$ by Corollary 2.3.

By Lemma A.11, we define $K_{1,t_0} := P_{1,t_0} \cap K_t$ where $h_t(\pi_1(\Sigma))$ acts strongly irreducibly on $P_{1,t_0}$, and $K_t = K_{1,t_0} \ast \cdots \ast K_{m,t_0}$. We obtain $K_{1,t_0} \to K_t$ as $j \to \infty$ for a compact convex set $K_t$ in a subspace $P_{1,t_0}$ where a finite-index subgroup of $h_{t_0}(\pi_1(\Sigma))$ acts on by Lemma A.11 and Proposition 2.1. Proposition 2.15 shows that the action on $P_{1,t_0}$ by $\pi_1(\Sigma')$ is irreducible. Hence, $K_t$ must be properly convex by Proposition 2.12. We have $K_\infty \subset K_1 \ast \cdots \ast K_m$ by Proposition 2.1 since $K_{t_{i_j}} \subset K_{1,t_{i_j}} \ast \cdots \ast K_{m,t_{i_j}}$ for each $j$. Hence, $K_\infty$ is properly convex.

Now, for any sequence $\{t'_i\}$ covering $t_0$, suppose that a convergent subsequence $\{K_{t'_{i_j}}\}$ converges to $K'_\infty$. Then we claim that $K'_\infty = K_\infty$. Here, $K'_\infty$ is properly convex also. Choose a torsion-free finite-index subgroup $\Gamma'$ of $h_{t_0}(\pi_1(\Sigma))$ by Theorem 2.3. $K'_\infty/\Gamma'$ and $K_\infty/\Gamma'$ are homotopy equivalent. Since $\text{dev}_{t_i}$ and $\text{dev}_{t'_i}$ are close, we may assume that $K'_\infty \cap K_\infty \neq \emptyset$. Lemma 2.19 shows that $K'_\infty = K_\infty$. This implies the first item for this case.

Suppose now that $\Gamma$ is virtually abelian. Then $\Omega_t$ is determined by the generators of the free abelian subgroup $\Gamma'$ of a finite index with only positive eigenvalues by Lemma A.8. $\Gamma'$ determines the connected abelian Lie group $\Delta_t$ containing $h_t(\Gamma')$ and $\Omega_t$ is an orbit of $\Delta_t$ by Lemma A.8. Now Lemma A.3 implies the first item.

The second item follows from the first one. The third one can be deduced by Theorem 2.1. The fourth item follows by Proposition A.3.

**Remark A.1** Of course, we wish to generalize Lemma A.10 and Corollaries A.1 and A.2 for fully general cases without the restriction on the domains of actions starting from any properly convex projective orbifold and show the similar results. Then we can allow NPNC-ends into the discussions. We leave this as a question of whether one can achieve such results.
The justification for weak middle eigenvalue conditions

**Theorem A.1** Let $\mathcal{O}$ be a properly convex real projective orbifold with ends. Let $\Sigma_\mathcal{E}$ be an end orbifold of an R-p-end $\tilde{\mathcal{E}}$ of $\mathcal{O}$ with the virtually abelian end-fundamental group $\pi_1(\tilde{\mathcal{E}})$.

- Suppose that $\mu_i$ be a sequence of properly convex structure on an R-p-end neighborhood $U_\mathcal{E}$ of $\tilde{\mathcal{E}}$ corresponding to a generalized lens-shaped R-p-ends satisfying the uniform middle eigenvalue conditions.
- Suppose that $\mu_i$ limits to $\mu_\infty$ in the $C^r$-topology, $r \geq 2$.

Then $\mu_\infty$ satisfies the weak middle eigenvalue condition for $\tilde{\mathcal{E}}$. Furthermore, the holonomy group for $\mu_\infty$ virtually satisfies the transverse weak middle eigenvalue condition for $\tilde{\mathcal{E}}$ if it is NPNC and $\pi_1(\tilde{\mathcal{E}})$ is virtually abelian.

**Proof** Assume first that $\hat{\varnothing} \subset S^n$. We may assume that the p-end vertex $v_\mathcal{E}$ is independent of $\mu_i$ by conjugation of the holonomy homomorphism. Let $h_i : \pi_1(\Sigma_\mathcal{E}) \to \text{SL}_n(\mathbb{R})$. Since these satisfy the uniform middle eigenvalue conditions, we have

$$
C^{-1}\text{cwl}(g) < \log \left( \frac{\lambda_1(h_i(g))}{\lambda_{n-1}(h_i(g))} \right) < C\text{cwl}(g), C > 1, g \in \pi_1(\Sigma_\mathcal{E})
$$

where $C$ is a constant which may depend on $h_i$. Let $h_\infty$ denote the holonomy homomorphism for $\mu_\infty$, which is an algebraic limit of $h_i$. By taking limits, we obtain that $h_\infty$ satisfies the weak middle eigenvalue condition.

Suppose now that $\mu_\infty$ is NPNC. For convenience, we may assume without loss of generality that $\pi_1(\tilde{\mathcal{E}})$ is free abelian. Let $\Gamma$ denote $h_\infty(\pi_1(\tilde{\mathcal{E}}))$.

By Lemma A.8, $\Sigma_\mathcal{E}$ is the interior of a strict join of hemispheres and a properly convex domain

$$
H_1 \ast \cdots \ast H_m \ast K_0 \subset S^{n-1}_\mathcal{E}
$$

where

- $\Gamma \lhd H_i$, $i = 1, \ldots, m$, has the Zariski closure a unipotent Lie group for a finite index subgroup $\Gamma$ of $h_\infty(\pi_1(\tilde{\mathcal{E}}))$.
- $\Gamma \lhd K_0$ is a diagonalizable group acting so, and
- $\Gamma$ has a center $Q$ of rank $m + \dim K_0 - 1$ acting trivially on each $H_i$, $i = 1, \ldots, m$ and fixing the vertices of $K_0$.

Given any $i$-dimensional hemisphere $V$ of $S^{n-1}_\mathcal{E}$ for $0 \leq i \leq n - 1$, there exists unique $i + 1$-dimensional hemisphere $\hat{V}$ in $S^n$ in the direction of $V$ from $v_\mathcal{E}$ and containing $v_\mathcal{E}$ in $\partial \hat{V}$.

We denote by $\hat{H}_i$ the hemispheres in $S^n$ corresponding to the directions of $H_i$ for $i = 1, \ldots, m$ and $\hat{p}_i$ the great segments in $S^n$ corresponding to the directions vertices $p_1, \ldots, p_k$ of $K_0$. Let $g \in h(\pi_1(\tilde{\mathcal{E}}))$. Since a CA-lens for $h_i(\pi_1(\tilde{\mathcal{E}}))$ contains the points affiliated with the largest norm of eigenvalues for $h_j(g)$ for each $g \in \pi_1(\tilde{\mathcal{E}})$, a limiting argument shows that points in $\hat{H}_i$ or $\hat{p}_i$ must be affiliated with the
largest norm $\lambda_1(h(g))$ of the eigenvalues. (Of course, these are not all such points necessarily)

By Proposition 2.5, the maximal dimensional great sphere $S_{n_0-1}^{n_0}$ in $\text{bd} \tilde{\Sigma}_E \subset S_{n_E}^{n-1}$ corresponding the boundary of complete affine leaves in $\tilde{\Sigma}_E$ equals $\partial \hat{H}_1 \cdots \partial \hat{H}_m$. Since these points are not in the directions of $\partial \hat{H}_1 \cdots \partial \hat{H}_m$, the desired inequality $\lambda_{\nu_E}^T(g) \geq \lambda_{\nu_E}(g)$ holds. [Sy'T]
Index of Notations

These are not the definitions. Please see the pages to find precise definitions.

\( \mathbb{R} \) The real number field
\( \mathbb{R}_+ \) The set of positive real numbers
\( \mathbb{C} \) The complex number field
\( \mathcal{A} \) The antipodal map \( S^n \to S^n \).
\( d \) The Fubini-Study metric on \( S^n \). 38
\( \text{Hom}(G,H) \) The set of homomorphisms \( G \to H \) for two groups \( G, H \).
\( \text{rep}(G,H) \) The set of conjugacy classes of homomorphisms \( G \to H \) for two groups \( G, H \).
\( \| \cdot \| \) The maximal norm of the entries of a matrix. p.65
\( \| \cdot \|_{\text{fiber}} \) A fiberwise metric on a vector bundle over \( \mathbb{R} \), (also we use \( \| \cdot \|_E \) for emphasis) p.61
\( \pi_1(\cdot) \) The orbifold fundamental group of an orbifold
\( \mathcal{Z}(\cdot) \) The Zariski closure of a group p.71
\( \mathcal{Z}(\cdot) \) The center of a group p.71
\( \text{Aut}(K) \) The group of projective automorphisms of a set \( K \). p.45
\( \mathbb{R}P^n \) The \( n \)-dimensional projective space. p.5
\( \mathbb{R}P^{n\ast} \) The dual \( n \)-dimensional projective space. p.6
\( \mathcal{A} \) The \( n \)-dimensional affine space p.6
\( S^n \) The sphericalization p.7
\( p_{S^n} \) The double covering map \( S^n \to \mathbb{R}P^n \). 7
\( S^{n\ast} \) The sphericalization p.73
\( \Pi \) \( \mathbb{R}^{n+1} - \{O\} \to \mathbb{R}P^n \) projection p.43
\( \Pi' \) \( \mathbb{R}^{n+1} - \{O\} \to S^n \) projection p.43
\( \text{bd}S^n = \Omega \)
\( \text{bd}^{A\ast} = \{ (x,H) | x \in \text{bd}\Omega, x \in H \} \)
\( H \) is an oriented sharply supporting hyperspace of \( \Omega \} \subset S^n \times S^{n\ast} \). p. 78 (A.6)
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\[ \Pi_{\mathbb{A}^g} \text{ projection } \Pi_{\mathbb{A}^g} : \text{bd}^g\Omega \to \text{bd}\Omega \text{ given by } (x, H) \mapsto x. \text{ p. 78} \]

\[ \partial M \text{ manifold or orbifold boundary of a manifold or orbifold } M \text{ p.38} \]

\[ \text{bd}X \text{ topological boundary of } X \text{ in an ambient space p.38} \]

\[ \text{bd}_X Y \text{ topological boundary of } Y \text{ in an ambient space } X \text{ p.38} \]

\[ \Omega^\alpha \text{ the manifold or orbifold interior of a manifold or orbifold } M \text{ or the relative interior of a convex domain in a projective or affine subspace. p.38} \]

\[ \mathbb{P}(V) \text{ the projectivization of a vector space } V \text{. p. 5} \]

\[ \mathbb{S}(V) \text{ the sphericalization of a vector space } V \text{. p. 46} \]

\[ \overline{pq} \text{ the line segment connecting } p \text{ and } q \text{ not antipodal to } p \text{. 38} \]

\[ \overline{p\zeta q} \text{ the minor arc connecting } p \text{ and } q \text{ antipodal to } p \text{ and passing } \zeta. \text{ 38} \]

\[ (\cdot)^* \text{ Duals of vector spaces or convex sets or linear groups. 6, 78} \]

\[ (\cdot)^\dagger \text{ The proper-subspace dual of a properly convex domain in a subspace. 85} \]

\[ (\cdot)_c \text{ The subscript denotes that the representation space or the character space is restricted by the condition that each end holonomy group to have a fixed point for } R\text{-ends or to have a holonomy group invariant hyperspace satisfying the lens-condition for } T\text{-ends. 22} \]

\[ (\cdot)_e \text{ The subscript denotes that the representation space or the character space or the deformation space is restricted by the condition that the ends be lens-shaped } R\text{-ends or lens-shaped } T\text{-ends only or the corresponding condition for the end holonomy groups. 27} \]

\[ (\cdot)_u \text{ The subscript denotes that the representation space or the character space or the deformation space is restricted by the end holonomy group having a unique fixed point for } R\text{-ends or having a unique end holonomy invariant hyperplane satisfying the lens-condition for } T\text{-ends. 23} \]

\[ (\cdot)^s \text{ The superscript denotes that the representation space or the character space or the deformation space is restricted by the stability condition. 27} \]

\[ (\cdot)^{sy} \text{ The subscript denotes that the deformation has holonomy in an open subset } \gamma \text{ of the character space and the end is determined by the fixing section } s_{\gamma}. \text{ 33} \]
References


References


References for the northeast third section of the document follow:

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Glossary

convergence sequence An unbounded sequence \( \{g_i\}, g_i \in \text{SL}_\pm(n+1, \mathbb{R}) \), so that \( \{g_i\} \) is convergent in \( S(M_{n+1}(\mathbb{R})) \) is called a convergence sequence. An unbounded sequence \( \{g_i\}, g_i \in \text{PGL}(n+1, \mathbb{R}) \) so that \( \{|g_i|\} \) is convergent in \( P(M_{n+1}(\mathbb{R})) \) is called a convergence sequence. 64

Ehresmann-Thurston-Weil principle the principle where the some important part of the deformation space is identifiable with a subspace of the character varieties. 3, 9

middle eigenvalue condition the condition that the maximal norm of the eigenvalues is strictly greater than that of the R-end vertex or the hyperplane corresponding to T-end. 118

orbifold A space with Hausdorff 2nd countable topology locally modelled with quotients of open sets by finite group actions. v

orthopotent A subgroup \( G \) of \( \text{SL}_\pm(n+1, \mathbb{R}) \) is orthopotent if there is a flag of subspaces \( 0 = Y_0 \subset Y_1 \subset \cdots \subset Y_m = \mathbb{R}^{n+1} \) preserved by \( G \) so that \( G \) acts as an orthogonal group on \( Y_{j+1}/Y_j \) for each \( j = 0, \ldots, m - 1 \). 58

uniform middle eigenvalue condition the condition that the eigenvalue ratios between the maximal ones and the ones for the R-end vertex or the hyperplanes corresponding to T-end structures are proportioanl to word lengths. vii, 3, 133, 165

weak middle eigenvalue condition the condition between the maximal norm of the eigenvalues and that of the R-end vertex or the hyperplane corresponding to T-end. 118
Acronyms

\textit{R-end} end of type \textit{R}. 21, 22
\textit{T-end} end of type \textit{T}. 21, 22

CA-lens cocompactly acted lens. 14, 19
generalized CA-lens cocompactly acted generalized lens. 19

IE infinite-index end-fundamental-group condition. 29

NA non-annular property. 29
NS the central normal-nilpotent-subgroup condition. 101

p-end pseudo-end. 12

R-end radial end. 4, 15
R-p-end radial pseudo-end. 15

SPC-structure stable properly convex structure. 29
strict SPC-structure stable properly and strictly convex structure. 30

T-end totally geodesic end. 4, 14
T-p-end totally geodesic pseudo-end. 14
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