A survey of projective geometric structures on 2-,3-manifolds

S. Choi

Department of Mathematical Science
KAIST, Daejeon, South Korea

Tokyo Institute of Technology
Survey: Classical geometries

- Euclidean geometry: Babylonians, Egyptians, Greeks, Chinese (Euclid’s axiomatic methods under Plato’s philosophy)
- Spherical geometry: Greek astronomy, Gauss, Riemann
- Hyperbolic geometry: Bolyai, Lobachevsky, Gauss, Beltrami, Klein, Poincaré
- Conformal geometry (Mobius geometry or circle geometry)
- Projective geometry
- Erlanger program, Cartan connections, Ehresmann connections
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  - Manifolds with geometric structures.
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  - Projective surfaces and gauge theory.
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Euclidean geometry consists of some notions such as lines, points, lengths, angle, and their interplay in some space called a plane...

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- D. Hilbert, and many others made a modern foundation so that the Euclidean geometry was reduced to logic.
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- From astronomical viewpoint, it is nice to view the sky as a unit sphere $S^2$ in the Euclidean 3-space. (See spherical.cdy)
- The great circles replaced lines and angles are measured in the tangential sense. Lengths are measured by taking arcs in the great circles.
- Geometric objects such as triangles behave a little bit different.
- Higher dimensional spherical geometry $S^n$ can be easily constructed.
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Hyperbolic geometry.

- Lobachevsky and Bolyai tried to build a geometry that did not satisfy the fifth axiom of Euclid. (See hyperbolic.cdy)

- Their attempts were justified by Beltrami-Klein model which is a disk and lines were replaced by chords and lengths were given by the logarithms of cross ratios. See Beltrami-Klein model. Later other models such as Poincare half space model and Poincare disk model were developed. Poincare model (Inst. figuring).

- Here the group of rigid motions is the Lie group $SL(2, \mathbb{R})$.

- Higher-dimensional hyperbolic spaces were later constructed. Actually, an upper part of a hyperboloid in the Lorentzian space would be a model and $PO(1, n)$ forms the group of rigid motions.

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Conformal geometry

- Suppose that we use to study circles and spheres only... We allow all transformations that preserves circles...
- This geometry looses a notion of lengths but has a notion of angles. There are no lines or geodesics but there are circles.
- The Euclidean plane is compactifed by adding a unique point as an infinity. The group of motions is generated by inversions in circles. The group is called the Mobius transformation group. That is, the group of transformations of form

  \[ z \rightarrow \frac{az + b}{cz + d}, \quad \frac{a\bar{z} + b}{c\bar{z} + d} \]

- The space itself is considered as a complex sphere, i.e., the complex plane with the infinity added.
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- Projective geometry naturally arose in fine art drawing perspectives during Renascence.

Perspective drawings

- Desargues, Kepler first tried to add infinite points corresponding to each direction that a line in a plane can take. Thus, a plane with infinite point for each direction form a projective plane.

- Transformations are ones generated by change of perspectives. In fact, when we are taking x-rays or other scans.

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- Hyperbolic geometry: $H^n$ imbeds in $\mathbb{R}P^n$ and $PO(n, 1)$ in $PGL(n + 1, \mathbb{R})$ in canonical way.
- Spherical, euclidean geometry.
- The affine geometry $G = SL(n, \mathbb{R}) \cdot \mathbb{R}^n$ and $X = \mathbb{R}^n$.
- Six of eight 3-dimensional geometries: euclidean, spherical, hyperbolic, nil, sol, $\tilde{SL}(2, \mathbb{R})$ (B. Thiel)
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  - Spherical, Euclidean geometry.
  - The affine geometry $G = SL(n, \mathbb{R}) \cdot \mathbb{R}^n$ and $X = \mathbb{R}^n$.
  - Six of eight 3-dimensional geometries: Euclidean, spherical, hyperbolic, nil, sol, $\tilde{SL}(2, \mathbb{R})$ (B. Thiel)
  - $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$. (almost)
  - Some symmetric spaces and matrix groups...

- Recently, some graphics applications.... image superposition,..
Erlangen program of Klein

A survey of projective geometric structures on 2-, 3-manifolds

S. Choi

Outline

Classical geometries
  Euclidean geometry
  Spherical geometry

Manifolds with geometric structures: manifolds need some canonical descriptions..

Manifolds with (real) projective structures

- A Lie group is a set of symmetries of some object forming a manifold.
- Klein worked out a general scheme to study almost all geometries...
- Klein proposed that "geometry" is actually a space with a Lie group acting on it transitively.
- Essentially, the transformation group actually defines the geometry by determining which properties are preserved.
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We extract properties from the pair:

- For each pair \((X, G)\) where \(G\) is a Lie group and \(X\) is a space.
- Thus, Euclidean geometry is given by \(X = \mathbb{R}^n\) and \(G = \text{Isom}(\mathbb{R}^n) = O(n) \cdot \mathbb{R}^n\).
- The spherical geometry: \(G = O(n + 1, \mathbb{R})\) and \(X = S^n\).
- The hyperbolic geometry: \(G = PO(n, 1)\) and \(X\) upper part of the hyperboloid \(t^2 - x_1^2 - \cdots - x_n^2 = 1\). For \(n = 2\), we also have \(G = PSL(2, \mathbb{R})\) and \(X\) the upper half-plane. For \(n = 3\), we also have \(G = PSL(2, \mathbb{C})\) and \(X\) the upper half-space in \(\mathbb{R}^3\).
- Lorentzian space-times: de Sitter, anti-de-Sitter,...
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- Cartan proposed that for each pair \((X, G)\), we can impart their properties to at each point of the manifolds so that they vary also. The flatness implies that the \((X, G)\)-geometry is actually recovered.

- Ehresmann introduced the most general notion of connections by generalizing Cartan connections.

- For example, for euclidean geometry, a Cartan connection gives us Riemannian geometry.

- For projective geometry, a Cartan connection corresponds to a projectively flat torsion-free affine connections and conversely.

- Wilson Stothers’ Geometry Pages

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Manifolds with geometric structures:

Topology of manifolds

- Manifolds come up in many areas of technology and sciences.
- Studying the topological structures of manifolds is complicated by the fact that there is no uniform way to describe many topologically important features and provides useful coordinates.
- We would like to find some good descriptions and perhaps even classify collections of manifolds.
- Of course these are for pure-mathematical uses for now....
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An \((X, G)\)-geometric structure on a manifold \(M\) is given by an atlas of charts to \(X\) where the transition maps are in \(G\).

This equips \(M\) with all of the local \((X, G)\)-geometrical notions.

If the geometry admits notions such as geodesic, length, angle, cross ratio, then \(M\) now has such notions...

So the central question is: which manifolds admit which structures and how many and if geometric structures do not exists, why not?
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Deformation spaces of geometric structures

- In major cases, $M = X/\Gamma$ for a discrete subgroup $\Gamma$ in the Lie group $G$.

- Thus, $X$ provides a global coordinate system and the classification of discrete subgroups of $G$ provides classifications of manifolds like $M$.

- Here $\Gamma$ tends to be the fundamental group of $M$. Thus, a geometric structure can be considered a discrete representation of $\pi_1(M)$ in $G$.

- Often, there are cases when $\Gamma$ is unique up to conjugations and there is a unique $(X, G)$-structure. (Rigidity)

- Given an $(X, G)$-manifold (orbifold) $M$, the deformation space $D_{(X,G)}(M)$ is locally homeomorphic to the $G$-representation space $\text{Hom}(\pi_1(M), G)/G$ of $\pi_1(M)$.
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Examples

- Closed surfaces have either a spherical, euclidean, or hyperbolic structures depending on genus. We can classify these to form deformation spaces such as Teichmüller spaces.
  - A hyperbolic surface equals $H^2/\Gamma$ for the image $\Gamma$ of the representation $\pi \to \text{PSL}(2, \mathbb{R})$.
  - A Teichmüller space can be identified with a component of the space $\text{Hom}(\pi, \text{PSL}(2, \mathbb{R}))/\sim$ of conjugacy classes of representations. The component consists of discrete faithful representations.
- For closed 3-manifolds, it is recently proved that they decompose into pieces admitting one of eight geometrical structures including hyperbolic, euclidean, spherical ones... Thus, these manifolds are now being classified... (Proof of the geometrization conjecture by Perelman).
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- By the **Klein-Beltrami model** of hyperbolic space, we can consider $H^n$ as a unit ball $B$ in an affine subspace of $\mathbb{R}P^n$ and $PO(n + 1, \mathbb{R})$ as a subgroup of $PGL(n + 1, \mathbb{R})$ acting on $B$. Thus, a (complete) hyperbolic manifold has a projective structure. (In fact any closed surface has one.)

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- A question arises whether these projective structures can be deformed to purely projective structures.

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Using this, we have a complete classification of projective structures on closed surfaces (even constructive one).

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A survey of projective geometric structures on 2-,3-manifolds

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Outline

Classical geometries
  Euclidean geometry
  Spherical geometry

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The map is in fact a homeomorphism onto Hitchin-Teichmüller component (Goldman, Choi). The main idea for proof is to show that the image of the map is closed.

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- Labourie recently generalized this to $n \geq 2, 3$: That the component of representations that can be deformed to the Fuchsian representation acts on a hyperconvex curve in $\mathbb{R}P^{n-1}$.

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- Then $M$ is convex if and only if $M$ admits an affine sphere structure in $\mathbb{R}^{n+1}$.
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- In particular, this shows that the deformation space $D(\Sigma)$ of convex projective structures on $\Sigma$ admits a complex structure, which is preserved under the moduli group actions. (Is it Kahler?)
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- Euclidean geometry
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