A survey of projective geometric structures on 2-,3-manifolds

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Outline

- Survey: Classical geometries
  - Euclidean geometry: Babylonians, Egyptians, Greeks, Chinese (Euclid's axiomatic methods under Plato's philosphy)
  - Spherical geometry: Greek astronomy, Gauss, Riemann
  - Hyperbolic geometry: Bolyai, Lobachevsky, Gauss, Beltrami,Klein, Poincare
  - Conformal geometry (Mobius geometry or circle geometry)
  - Projective geometry
  - Erlanger program, Cartan connections, Ehresmann connections

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- Projective manifolds: how many?
- Projective surfaces
  - Projective surfaces and gauge theory.
  - Labourie's generalization
  - Projective surfaces and affine differential geometry.
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  - The Euclid had five axioms
  - D. Hilbert, and many others made a modern foundation so that the Euclidean geometry was reduced to logic.
  - Euclidean geometry has a notion of rigid transformations which made the space homogeneous. They form a group called a group of rigid motions. They preserve lines, length, angles, and every geometric statements.
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Spherical geometry.

- Greeks: astronomical, navigational... Arabs,..
- From astronomical viewpoint, it is nice to view the sky as a unit sphere $\mathbf{S}^2$ in the Euclidean 3-space. (See spherical.cdy)
- The great circles replaced lines and angles are measured in the tangential sense. Lengths are measured by taking arcs in the great circles.
- Geometric objects such as triangles behave a little bit different.
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- Lobachevsky and Bolyai tried to build a geometry that did not satisfy the fifth axiom of Euclid. (See hyperbolic.cdy)
- Their attempts were justified by Beltrami-Klein model which is a disk and lines were replaced by chords and lengths were given by the logarithms of cross ratios. See Beltrami-Klein model. Later other models such as Poincare half space model and Poincare disk model were developed. Poincare model (Inst. figuring).
- Here the group of rigid motions is the Lie group $SL(2, \mathbb{R})$.
- Higher-dimensional hyperbolic spaces were later constructed. Actually, an upper part of a hyperboloid in the Lorentzian space would be a model and $PO(1, n)$ forms the group of rigid motions. Minkowsky model

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Conformal geometry

- Suppose that we use to study circles and spheres only... We allow all transformations that preserves circles...
- This geometry looses a notion of lengths but has a notion of angles. There are no lines or geodesics but there are circles.
- The Euclidean plane is compactifed by adding a unique point as an infinity. The group of motions is generated by inversions in circles. The group is called the Mobius transformation group. That is, the group of transformations of form

$$z \to \frac{az+b}{cz+d}, \frac{a\bar{z}+b}{c\bar{z}+d}$$

- The space itself is considered as a complex sphere, i.e., the complex plane with the infinity added.

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- Projective geometry naturally arose in fine art drawing perspectives during Renascence. Perspective drawings
- Desargues, Kepler first tried to add infinite points corresponding to each direction that a line in a plane can take. Thus, a plane with infinite point for each direction form a projective plane.
- Transformations are ones generated by change of perspectives. In fact, when we are taking x-rays or other scans.
- The notions such as lengths, angles lose meaning. But notions of lines or geodesics are preserved. The Greeks discovered that the cross ratios, i.e., ratios of ratios, are preserved.
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  - Hyperbolic geometry: H^n imbeds in $\mathbb{R}P^n$ and $PO(n,1)$ in $PGL(n+1,\mathbb{R})$ in canonical way.
  - spherical, euclidean geometry.
  - The affine geometry $G = SL(n,\mathbb{R}) \cdot \mathbb{R}^n$ and $X = \mathbb{R}^n$.
  - Six of eight 3-dimensional geometries: euclidean, spherical, hyperbolic, nil, sol, $\tilde{S}L(2,\mathbb{R})$ (B. Thiel)
  - $\mathbf{S}^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$. (almost)
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Cartan, Ehresmann connections: Most general geometry possible

- Cartan proposed that for each pair (X, G), we can impart their properties to at each point of the manifolds so that they vary also. The flatness implies that the (X, G)-geometry is actually recovered.
- Ehresmann introduced the most general notion of connections by generalizing Cartan connections.
- For example, for euclidean geometry, a Cartan connection gives us Riemannian geometry.
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Manifolds with geometric structures:

Topology of manifolds

- Manifolds come up in many areas of technology and sciences.
- Studying the topological structures of manifolds is complicated by the fact that there is no uniform way to describe many topologically important features and provides useful coordinates.
- We would like to find some good descriptions and perhaps even classify collections of manifolds.
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- This equips M with all of the local (X, G)-geometrical notions.
- If the geometry admits notions such as geodesic, length, angle, cross ratio, then M now has such notions...
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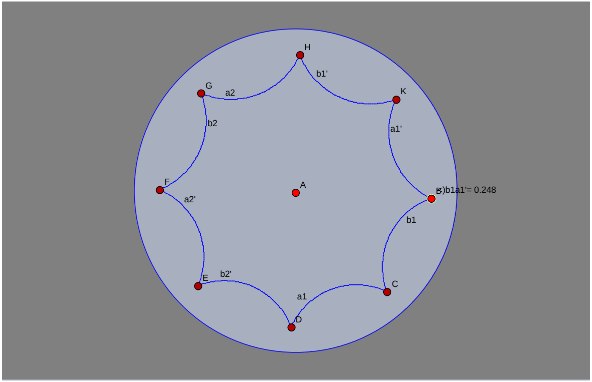
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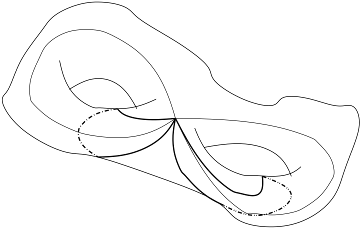

Deformation spaces of geometric structures

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- Often, there are cases when Γ is unique up to conjugations and there is a unique (X, G)-structure. (Rigidity)
- Given an (X, G)-manifold (orbifold) M, the deformation space $D_{(X,G)}(M)$ is locally homeomorphic to the G-representation space $Hom(\pi_1(M), G)/G$ of $\pi_1(M)$.

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Examples

- Closed surfaces have either a spherical, euclidean, or hyperbolic structures depending on genus. We can classify these to form deformation spaces such as Teichmuller spaces.
  - A hyperbolic surface equals H^2/Γ for the image Γ of the representation $\pi \rightarrow \mathrm{PSL}(2, \mathbb{R})$.
  - A Teichmuller space can be identified with a component of the space $\mathrm{Hom}(\pi, PSL(2, \mathbb{R}))/\sim$ of conjugacy classes of representations. The component consists of discrete faithful representations.
- For closed 3-manifolds, it is recently proved that they decompose into pieces admitting one of eight geometrical structures including hyperbolic, euclidean, spherical ones... Thus, these manifolds are now being classified... (Proof of the geometrization conjecture by Perelman).
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Manifolds with (real) projective structures

- Cartan first considered projective structures on surfaces as projectively flat torsion-free connections on surfaces.
- Chern worked on general type of projective structures in the differential geometry point of view.
- The study of affine manifolds precede that of projective manifolds.
  - There were extensive work on affine manifolds, and affine Lie groups by Chern, Auslander, Goldman, Fried, Smillie, Nagano, Yagi, Shima, Carriere, Margulis,...
  - There are outstanding questions such as the Chern conjecture, Auslander conjecture,...

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Projective manifolds: how many?

- By the Klein-Beltrami model of hyperbolic space, we can consider H^n as a unit ball B in an affine subspace of $\mathbb{R}P^n$ and $\mathrm{PO}(n+1,\mathbb{R})$ as a subgroup of $\mathrm{PGL}(n+1,\mathbb{R})$ acting on B. Thus, a (complete) hyperbolic manifold has a projective structure. (In fact any closed surface has one.)
- In fact, 3-manifolds with six types of geometric structures have real projective structures. (In fact, up to coverings of order two, 3-manifolds with eight geometric structures have real projective structures.)
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Projective 3-manifolds and deformations

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