

Logic and the set theory

Lecture 16: Relations in How to Prove It.

S. Choi

Department of Mathematical Science
KAIST, Daejeon, South Korea

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About this lecture

- Ordered pairs and Cartesian products

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- Relations

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- More about relations

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- Equivalence relations

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- Course homepages: <http://mathsci.kaist.ac.kr/~schoi/logic.html>
and the moodle page <http://moodle.kaist.ac.kr>

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and the moodle page <http://moodle.kaist.ac.kr>
- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.

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- Introduction to set theory, Hrbacek and Jech, CRC Press. (Chapter 2)

Ordering relations

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- R is a *total order* on A if it is a partial order and $\forall x \in A \forall y \in A (xRy \vee yRx)$.

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- The subset relation is a partial order but not a total order.
- $D = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid x \text{ divides } y\}$.
- $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \geq y\}$.

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- Which is a stronger concept?

Example

- $L = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$ which is a total order on \mathbb{R} . $B = \{x \in \mathbb{R} \mid x \geq 7\}$.
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- $S = \{(x, y) \in P(\mathbb{Z}^+) \times P(\mathbb{Z}^+) \mid x \subset y\}$. $\mathcal{F} = \{x \in P(\mathbb{Z}^+) \mid 2 \in X \wedge 3 \in X\}$.

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Theorem

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- If B has a smallest element, then the smallest element is unique.
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- If R is a total order and b is a minimal element of B , then b is the smallest element of B . (not proved)

Proof of 1



Given	Goal
$\exists b(\forall x \in B(bRx))$	$\exists! b \forall x(bRx)$

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$\forall x \in B(b_0Rx)$	$\forall x(cRx) \rightarrow c = b_0$

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| Given | Goal |
| $\forall x \in B(b_0Rx)$ | $c = b_0$ |
| $\forall x (cRx)$ | |
| cRb_0, b_0Rc | |

Proof of 2

- Divide goal. b is *minimal* and b is uniquely minimal.

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Given	Goal
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Given	Goal
$(\forall x \in B(bRx))$	$\forall x \in B \neg (xRb \wedge x \neq b)$

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Given	Goal
$(\forall x \in B(bRx))$	$x = b$
$x \in B, xRb$	

Proof of 2 continued

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Given		Goal	
$b(\forall x \in B(bRx))$	$\forall c \in B((\forall x \in B(xRc) \rightarrow x = c))$	\rightarrow	$b = c$

Proof of 2 continued

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$c \in B$	
$\forall x \in B(xRc \rightarrow x = c)$	

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- Divide goal. b is minimal and b is *uniquely minimal*.

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- ▶ maximal element: $\neg \exists x \in B(bRx \wedge b \neq x)$.

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Definition

- ▶ $B \subset A$. a is a *lower bound* of B if $\forall x \in B(aRx)$.
- ▶ $a \in A$ is an *upper bound* of B if $\forall x \in B(xRa)$.
- ▶ Let U be the set of upper bounds for B and let L be the set of lower bounds for B .
- ▶ If U has a smallest element, this smallest element is said to be the *least upper bound* (lub, supremum).
- ▶ If L has a greatest element, this element is said to be the *greatest lower bound* (glb, infimum).

- ▶ Largest elements: $B \subset A$. $\forall x \in B(xRb)$
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- These elements may not equal the smallest, minimal (greatest, maximal) element of B ...

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The reflexive closure of a strict partial order (resp. strict total order) is a partial order (resp. total order).

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- $<$ a strict total order.
- An ordered set is *dense* if it has at least two elements and if for all $a, b \in X$, $a < b$ implies there exists $x \in X$ such that $a < x < b$.
- Let $(P, <)$ be a dense linearly (totally) ordered set. P is *complete* if every nonempty subset S bounded above has a supremum.

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- The real number system is the unique complete linearly ordered set without endpoints that has a countable subset dense in it.

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- ▶ ● The real number system is the completion of \mathbb{Q} .
- The real number system is the unique complete linearly ordered set without endpoints that has a countable subset dense in it.
- Conway, Knuth invented surreal numbers...

Reflexive closures

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- In other words, S is such that $R \subset S$, S is reflexive, for every $T \subset A \times A$ and if $R \subset T$ and T is reflexive, then $S \subset T$.

Theorem

(4.5.2) *Suppose that S is a relation on A . Then R has a reflexive closure.*

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Proof.

Let $S = R \cup i_A$. Properties 1, 2 are obvious. For 3, $R \subset T$. Since T is reflexive, $i_A \subset T$. Thus $S = R \cup i_A \subset T$. \square

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Let R be a relation on A . The *symmetric closure* of R is the smallest set $S \subset A \times A$ such that $R \subset S$ and S is symmetric. This is equivalent to.

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- $R \subset S$.
- S is symmetric.
- For any $T \subset A \times A$ and $R \subset T$ and T is symmetric imply that $S \subset T$.

Definition

Let R be a relation on A . The *transitive closure* of R is the smallest set $S \subset A \times A$ such that $R \subset S$ and S is transitive. This is equivalent to.

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Example

See Figures 1,2,3 in pages 206-207 in HTP.

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Proof.

hint: $R \cup R^{-1}$. □

Theorem

Suppose that R is a relation on A . Then R has a transitive closure.

Proof.

hint: Take intersections of all transitive relations containing R . □

Equivalence relations

Definition

Suppose that A is a set and $P(A)$ its power set. $\mathcal{F} \subset P(A)$ is pairwise disjoint if

$$\forall X \in \mathcal{F} \forall Y \in \mathcal{F} (X \neq Y \rightarrow X \cap Y = \emptyset).$$

The family \mathcal{F} is a *partition* of A if $\bigcup \mathcal{F} = A$, $\forall X \in \mathcal{F} (X \neq \emptyset)$, and \mathcal{F} is pairwise disjoint.

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Main aim

An equivalence relation \leftrightarrow a partition of a set.

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The set of all equivalence classes is denoted A/R ($A \bmod R$)

$$A/R := \{[x]_R \mid x \in A\} = \{X \subset A \mid \exists x \in A (X = [x]_R)\}$$

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$$A/R := \{[x]_R \mid x \in A\} = \{X \subset A \mid \exists x \in A (X = [x]_R)\}$$

Theorem

(4.6.5) Suppose that R is an equivalence relation on A . Then for

- For all $x \in A$, $x \in [x]_R$.
- For all $x \in A$ and $y \in A$, $y \in [x]_R \leftrightarrow [y]_R = [x]_R$.

proof

- 1. $x \in A$. Then xRx by reflexivity. Thus $x \in [x]_R$.

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Given	Goal
$y \in [x]_R$	$[y]_R = [x]_R$

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$$\begin{array}{ll} \text{Given} & \text{Goal} \\ y \in [x]_R & [y]_R = [x]_R \end{array}$$

- To show:

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Given	Goal
$y \in [x]_R$	zRx
$z \in [y]_R, yRx, zRy$	

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| Given | Goal |
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Equivalence relation \rightarrow Partition

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(4.6.6) *Let A be a set. \mathcal{F} a partition of A . Then there exists an equivalence relation R on a set A such that $\mathcal{F} = A/R$.*

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- ② The proof: we verify the three properties of equivalence relations.
- ③ We prove the transitivity: xRy, yRz . $(x, y) \in X \times X$ and $(y, z) \in Y \times Y$. Then $X \cap Y \ni y$. Thus, $X = Y$. Thus, $(x, z) \in X \times X$ and xRz .

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(4.6.8) *Let A be a set. \mathcal{F} a partition of A . Let R be the equivalence relation determined by \mathcal{F} . Suppose $X \in \mathcal{F}$ and $x \in X$. Then $[x]_R = X$.*

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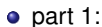
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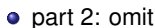
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Given	Goal
$X = [x]_R, x \in A$	$X \in \mathcal{F}$
$x \in Y$ for some $Y \in \mathcal{F}$	
$Y = [x]_R$ by 4.6.8	
$Y = X$	

Proof of Theorem 4.6.6

- part 2: \leftarrow .

Given	Goal
$X \in \mathcal{F}$	$X \in A/R$
$X \neq \emptyset, x \in X$	
$X = [x]_R \in A/R$ by 4.6.8	