Logic and the set theory

Lecture 16: Relations in How to Prove It.

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About this lecture

- Ordered pairs and Cartesian products

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- More about relations

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- Equivalence relations

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- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://moodle.kaist.ac.kr`

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- More about relations
- Ordering relations
- Closures
- Equivalence relations
- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://moodle.kaist.ac.kr`
- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.

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- Introduction to set theory, Hrbacek and Jech, CRC Press. (Chapter 2)

Ordering relations

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- R is a *partial order* on A if it is reflexive, transitive and antisymmetric.
- R is a *total order* on A if it is a partial order and $\forall x \in A \forall y \in A(xRy \vee yRx)$.

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- The subset relation is a partial order but not a total order.
- $D = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | x \text{divides } y\}$.
- $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \geq y\}$.

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- Which is a stronger concept?

Example

- $L = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \leq y\}$ which is a total order on $\mathbb{R}$. $B = \{x \in \mathbb{R} | x \geq 7\}$. $C = \{x \in \mathbb{R} | x > 7\}$.

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Theorem

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- *Suppose that b is a smallest element of B. Then b is minimal element of B and the unique minimal element of b.*

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- *If B has a smallest element, then the smallest element is unique.*
- *Suppose that b is a smallest element of B. Then b is minimal element of B and the unique minimal element of b.*
- *If R is a total order and b is a minimal element of B, then b is the smallest element of B. (not proved)*

Proof of 1

Given	Goal
$\exists b(\forall x \in B(bRx))$	$\exists! b \forall x(bRx)$

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$\forall x \in B(b_0 Rx)$	$c = b_0$
$\forall x(cRx)$	
$cRb_0, b_0 Rc$	

Proof of 2

- Divide goal. *b is minimal* and *b* is uniquely minimal.

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Given	Goal
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$x \in B, xRb$	

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$b(\forall x \in B(bRx))$	$\forall c \in B((\forall x \in B(xRc \rightarrow x = c)) \rightarrow b = c)$

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Given	Goal
$b(\forall x \in B(bRx))$	$b = c$
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bRc, hence $b = c$	

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  - Largest elements: $B \subset A$. $\forall x \in B(xRb)$
  - maximal element: $\neg \exists x \in B(bRx \wedge b \neq x)$.

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Definition

- $B \subset A$. a is a *lower bound* of B if $\forall x \in B(aRx)$.
- $a \in A$ is an *upper bound* of B if $\forall x \in B(xRa)$.
- Let U be the set of upper bounds for B and let L be the set of lower bounds for B.
- If U has a smallest element, this smallest element is said to be the *least upper bound* (lub, supremum).
- If L has a greatest element, this element is said to be the *greatest lower bound* (glb, infimum).

- - Largest elements: $B \subset A$. $\forall x \in B(xRb)$
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- These elements may not equal the smallest, minimal (greatest, maximal) element of B...

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The reflexive closure of a strict partial order (resp. strict total order) is a partial order (resp. total order).

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- $<$ a strict total order.
- An ordered set is *dense* if it has at least two elements and if for all $a, b \in X$, $a < b$ implies there exists $x \in X$ such that $a < x < b$.
- Let $(P, <)$ be a dense linearly (totally) ordered set. P is *complete* if every nonempty subset S bounded above has a supremum.

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- The real number system is the unique complete linearly ordered set without endpoints that has a countable subset dense in it.

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- The real number system is the unique complete linearly ordered set without endpoints that has a countable subset dense in it.
- Conway, Knuth invented surreal numbers...

Reflexive closures

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Reflexive closures

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- In other words, S is such that $R \subset S$, S is reflexive, for every $T \subset A \times A$ and if $R \subset T$ and T is reflexive, then $S \subset T$.

Theorem

(4.5.2) Suppose that S is a relation on A. Then R has a reflexive closure.

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Proof.

Let $S = R \cup i_A$. Properties 1, 2 are obvious. For 3, $R \subset T$. Since T is reflexive, $i_A \subset T$. Thus $S = R \cup i_A \subset T$. □

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- $R \subset S$.
- S is symmetric.
- For any $T \subset A \times A$ and $R \subset T$ and T is symmetric imply that $S \subset T$.

Definition

Let R be a relation on A. The *transitive closure* of R is the smallest set $S \subset A \times A$ such that $R \subset S$ and S is transitive. This is equivalent to.

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Example

See Figures 1,2,3 in pages 206-207 in HTP.

Theorems

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Suppose that R is a relation on A. Then R has a transitive closure.

Proof.

hint: Take intersections of all transitive relations containing R. □

Equivalence relations

Definition

Suppose that A is a set and $P(A)$ its power set. $\mathcal{F} \subset P(A)$ is pairwise disjoint if

$$\forall X \in \mathcal{F} \forall Y \in \mathcal{F}(X \neq Y \rightarrow X \cap Y = \emptyset).$$

The family $\mathcal{F}$ is a *partitition* of A if $\bigcup \mathcal{F} = A$, $\forall X \in \mathcal{F}(X \neq \emptyset)$, and $\mathcal{F}$ is pairwise disjoint.

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Suppose that R is a relation on A. If R is a reflexive, symmetric, and transtive, then R is an *equivalence relation*.

Main aim

An equivalence relation $\leftrightarrow$ a partition of a set.

Equivalence relations

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The set of all equivalence classes is denoted A/R (A mod R)

$$A/R := \{[x]_R | x \in A\} = \{X \subset A | \exists x \in A(X = [x]_R)\}$$

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$$A/R := \{[x]_R | x \in A\} = \{X \subset A | \exists x \in A (X = [x]_R)\}$$

Theorem

(4.6.5) Suppose that R is an equivalence relation on A. Then for

- *For all* $x \in A$, $x \in [x]_R$.
- *For all* $x \in A$ *and* $y \in A$, $y \in [x]_R \leftrightarrow [y]_R = [x]_R$.

proof

- 1. $x \in A$. Then xRx by reflexivity. Thus $x \in [x]_R$.

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Given	Goal
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proof

Given	Goal
$y \in [x]_R$	zRx
$z \in [y]_R, yRx, zRy$	

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□

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Proof.

- To show A/R is a partition of A, we show that $\bigcup A/R = A$, A/R is pairwise disjoint, and no element of A/R is empty.
- For the first item, $\bigcup A/R \subset A$. We show $A \subset \bigcup A/R$. Suppose $x \in A$. Then $x \in [x]_R$. Thus $x \in \bigcup A/R$.

□

Equivalence relation $\to$ Partition

Theorem

Suppose that R is an equivalence relation on a set A. Then A/R is a partition of A.

Proof.

- To show A/R is a partition of A, we show that $\bigcup A/R = A$, A/R is pairwise disjoint, and no element of A/R is empty.
- For the first item, $\bigcup A/R \subset A$. We show $A \subset \bigcup A/R$. Suppose $x \in A$. Then $x \in [x]_R$. Thus $x \in \bigcup A/R$.
- The pairwise disjointness follows from what?

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- The pairwise disjointness follows from what?
- Suppose $X \in A/R$. Then $X = [x]_R \ni x$ and hence is not empty.

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(4.6.6) Let A be a set. $\mathcal{F}$ a partition of A. Then there exists an equivalence relation R on a set A such that $\mathcal{F} = A/R$.

We need two lemmas to prove this.

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2. The proof: we verify the three properties of equivalence relations.
3. We prove the transitivity: xRy, yRz. $(x, y) \in X \times X$ and $(y, z) \in Y \times Y$. Then $X \cap Y \ni y$. Thus, $X = Y$. Thus, $(x, z) \in X \times X$ and xRz.

Lemma

(4.6.8) Let A be a set. $\mathcal{F}$ *a partition of A. Let R be the equivalence relation determined by* $\mathcal{F}$*. Suppose* $X \in \mathcal{F}$ *and* $x \in X$*. Then* $[x]_R = X$*.*

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Given	Goal
$X = [x]_R, x \in A$	$X \in \mathcal{F}$
$x \in Y$ for some $Y \in \mathcal{F}$	
$Y = [x]_R$ by 4.6.8	
$Y = X$	

Proof of Theorem 4.6.6

- part 2: $\leftarrow$.

Given	Goal
$X \in \mathcal{F}$	$X \in A/R$
$X \neq \emptyset, x \in X$	
$X = [x]_R \in A/R$ by 4.6.8	