Logic and the set theory
Lecture 14: Proofs in How to Prove It.

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About this lecture

- Proof strategies

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- Proofs involving negations and conditionals.

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- More examples of proofs..

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- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

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- Example in the book $A \subset C, B \subset C, \vdash A \cup B \subset C$.
- The second method: Given $\neg P$ or can show P false, then we can assume Q ony.

Given	Goal
$P \vee Q$	$\_\ \_\ \_\_$
$\_\ \_\ \_\_$	

-

Given	Goal
$P \vee Q$	$- - --$
$- - --$	

-

Given	Goal
Case 1: P	$- - --$
$- - --$	
Case 2: Q	$- - --$
$- - --$	

Example

- Example: $A - (B - C) \subset (A - B) \cup C$.

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$x \in A \wedge \neg(x \in B \wedge x \notin C)$	$(x \in A \wedge x \notin B) \vee x \in C$

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- $$\begin{array}{cc} \text{Given} & \text{Goal} \\ x \in A \wedge \neg(x \in B \wedge x \notin C) & (x \in A \wedge x \notin B) \vee x \in C \end{array}$$

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- Case 1 gives $x \in A - B$. Case 2 gives $x \in C$.

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Given	Goal
$- - --$	$P \vee Q$
$- - --$	

- Change to

Given	Goal
$- - --$	P
$- - --$	
$\neg Q$	

- Suppose that m and n are integers. If mn is even, then either m is even or n is even.

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$$mn = 2k, (2j+1)n = 2k, 2jn + n = 2k, n = 2k - 2jn = 2(k - jn)$$

Thus, n is even.

The existence and uniqueness proof.

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  - $\exists x \forall y(P(y) \leftrightarrow y = x)$.
  - $\exists x P(x) \wedge \forall y \forall z((P(y) \wedge P(z) \rightarrow y = z)$.

Example

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Example

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- $1 \to 2$.

Example

- We will prove by proving $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$.
- $1 \rightarrow 2$.
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Given	Goal
$\exists x(P(x) \wedge \forall y(P(y) \rightarrow y = x))$	$\exists x \forall y(P(y) \leftrightarrow y = x)$

Example

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Given	Goal
$\exists x(P(x) \wedge \forall y(P(y) \to y = x))$	$\exists x \forall y(P(y) \leftrightarrow y = x)$

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Given	Goal
$P(x_0)$	$\exists x \forall y(P(y) \leftrightarrow y = x)$
$\forall y(P(y) \to y = x_0)$	

Example

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$P(x_0)$	$\forall y(P(y) \leftrightarrow y = x)$
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$x = x_0$	

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Given	Goal
$P(x_0)$	$\forall y(P(y) \leftrightarrow y = x)$
$\forall y(P(y) \to y = x_0)$	
$x = x_0$	

- $\to$ is clear in the Goal side. $\leftarrow$ is clear also.

Example

- $2 \to 3$.

Example

- $2 \rightarrow 3$.
-

Given	Goal
$\exists x \forall y (P(y) \leftrightarrow y = x)$	$\exists x P(x) \wedge \forall y \forall z ((P(y) \wedge P(z) \rightarrow y = z)$

Example

- $2 \to 3.$
-

Given	Goal
$\exists x \forall y (P(y) \leftrightarrow y = x)$	$\exists x P(x) \wedge \forall y \forall z ((P(y) \wedge P(z) \to y = z)$

- First goal

Given	Goal
$\forall y (P(y) \to y = x_0)$	$\exists x P(x)$
$P(x_0)$	

Example

- $2 \to 3$.
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Given	Goal
$\exists x \forall y (P(y) \leftrightarrow y = x)$	$\exists x P(x) \wedge \forall y \forall z ((P(y) \wedge P(z) \to y = z)$

- First goal

Given	Goal
$\forall y (P(y) \to y = x_0)$	$\exists x P(x)$
$P(x_0)$	

- Second goal

Given	Goal
$\forall y (P(y) \to y = x_0)$	$\forall y \forall z (P(y) \wedge P(z) \to y = z)$
$P(x_0)$	

Given	Goal
$\forall y(P(y) \to y = x_0)$	$(P(y) \wedge P(z)) \to y = z$
$P(x_0)$	
y arbitrary	
z arbitrary	

Given	Goal
$\forall y(P(y) \to y = x_0)$	$(P(y) \land P(z)) \to y = z$
$P(x_0)$	
y arbitrary	
z arbitrary	

Given	Goal
$\forall y(P(y) \to y = x_0)$	$y = z$
$P(x_0)$	
$P(y)$	
$P(z)$	

Given	Goal
$\forall y(P(y) \to y = x_0)$	$(P(y) \wedge P(z)) \to y = z$
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z arbitrary	

Given	Goal
$\forall y(P(y) \to y = x_0)$	$y = z$
$P(x_0)$	
$P(y)$	
$P(z)$	

- Then $y = x_0$ and $z = x_0$. Hence the conclusion.

Example

- Finally $3 \rightarrow 1$.

Example

- Finally 3 $\rightarrow$ 1.
-

Given	Goal
$\exists x P(x) \wedge \forall y \forall z ((P(y) \wedge P(z)) \rightarrow y = z)$	$\exists x (P(x) \wedge \forall y (P(y) \rightarrow y = x))$

Example

- Finally $3 \rightarrow 1$.
Given	Goal
$\exists x P(x) \wedge \forall y \forall z ((P(y) \wedge P(z)) \rightarrow y = z)$	$\exists x (P(x) \wedge \forall y (P(y) \rightarrow y = x))$
Given	Goal
$P(x_0)$ $\forall y \forall z ((P(y) \wedge P(z)) \rightarrow y = z)$	$P(x_0) \wedge \forall y (P(y) \rightarrow y = x_0)$

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$\exists x P(x) \wedge \forall y \forall z((P(y) \wedge P(z)) \to y = z)$	$\exists x(P(x) \wedge \forall y(P(y) \to y = x))$
Given	Goal
$P(x_0)$ $\forall y \forall z((P(y) \wedge P(z)) \to y = z)$	$P(x_0) \wedge \forall y(P(y) \to y = x_0)$
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$P(x_0)$ $\forall y \forall z((P(y) \wedge P(z)) \to y = z)$ $P(y)$	$y = x_0$

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Given	Goal
$P(x_0)$ $\forall y \forall z ((P(y) \wedge P(z)) \rightarrow y = z)$ $P(y)$	$y = x_0$

- Since $P(x_0)$, $P(y)$, we have $x_0 = y$. Done.

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Example

- A, B, C are sets. $A \cap B \neq \emptyset$, $A \cap C \neq \emptyset$. A has a unique element. Then prove $B \cap C \neq \emptyset$.

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$A \cap C \neq \emptyset$	
$\exists! x(x \in A)$	

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$A \cap B \neq \emptyset$	$B \cap C \neq \emptyset$
$A \cap C \neq \emptyset$	
$\exists! x(x \in A)$	

-

Given	Goal
$\exists x(x \in A \wedge x \in B)$	$\exists x(x \in B \wedge x \in C)$
$\exists x(x \in A \wedge x \in C)$	
$\exists x(x \in A)$	
$\forall y \forall z(y \in A \wedge z \in A \to y = z)$	

Given	Goal
$b \in A \wedge b \in B$	$\exists x(x \in B \wedge x \in C)$
$c \in A \wedge c \in C$	
$a \in A$	
$\forall y \forall z((y \in A \wedge z \in A) \rightarrow y = z)$	

Given	Goal
$b \in A \wedge b \in B$	$\exists x(x \in B \wedge x \in C)$
$c \in A \wedge c \in C$	
$a \in A$	
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- $b = a$ and $c = a$. Thus $a \in B \wedge a \in C$.

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Given	Goal
$\epsilon > 0$ arbitrary	$\exists \delta > 0 (\forall x (\|x - 2\| < \delta \to \|x^2 + 2 - 6\| < \epsilon))$

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- $\lim_{x \to 2} x^2 + 2 = 6$.
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Given	Goal
$\epsilon > 0$ arbitrary	$\exists \delta > 0 (\forall x (|x-2| < \delta \to |x^2+2-6| < \epsilon))$

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Given	Goal
$\epsilon > 0$ arbitrary	$|x^2+2-6| < \epsilon$
$\delta = \delta_0$ must find	
$|x-2| < \delta$	

- Guess work: Find conditions on δ. Assume $|x-2|<\delta, \delta<1$. Then we obtain $|x-2|<1, 3<|x+2|<5, |x+2|<5$. Thus $|x^2-4|=|x-2||x+2|<5\delta$. Thus, choose $\delta = min\{1/2, \epsilon/5\}$. Then

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- $\epsilon/5 \leq 1/2$ case: $|x-2| < \epsilon/5 \rightarrow |(x-2)(x+2)| < \epsilon$.

- Guess work: Find conditions on δ. Assume $|x-2|<\delta, \delta<1$. Then we obtain $|x-2|<1, 3<|x+2|<5, |x+2|<5$. Thus $|x^2-4|=|x-2||x+2|<5\delta$. Thus, choose $\delta = min\{1/2, \epsilon/5\}$. Then
- $\epsilon/5 \leq 1/2$ case: $|x-2|<\epsilon/5 \rightarrow |(x-2)(x+2)|<\epsilon$.
- $1/2 \leq \epsilon/5$ case: $|x-2|<1/2 \rightarrow |(x-2)(x+2)|<5(1/2) \leq 5\epsilon/5=\epsilon$.

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- $|x + 2| < 5$ by above and $|(x - 2)(x + 2)| < \epsilon/5 \cdot 5 = \epsilon$.
- Thus, $|x^2 + 2 - 6| < \epsilon$.
- $\forall \epsilon > 0$, if $\delta = min\{1/2, \epsilon/5\}$, then $\forall x(|x - 2| < \delta \to |x^2 + 2 - 6| < \epsilon)$. $\square$

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