Logic and the set theory

Lecture 13: Proofs in How to Prove It.

S. Choi

Department of Mathematical Science KAIST, Daejeon, South Korea

Fall semester, 2011

S. Choi (KAIST)

Logic and set theory

October 5, 2011 1 / 32

() < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < ()

Proof strategies

・ロ・・ 日・・ ヨ・ ・ ヨ・ うらぐ

October 5, 2011 2 / 32

Logic and set theor

S. Choi (KAIST)

- Proof strategies
- Proofs involving negations and conditionals.

- Proof strategies
- Proofs involving negations and conditionals.
- Proofs involving quantifiers

- Proof strategies
- Proofs involving negations and conditionals.
- Proofs involving quantifiers
- Proofs involving conjunctions and biconditionals (up to here in this lecture.)

★ 문 → ★ 문 →

- Proof strategies
- Proofs involving negations and conditionals.
- Proofs involving quantifiers
- Proofs involving conjunctions and biconditionals (up to here in this lecture.)
- Proofs involving disjunctions

- Proof strategies
- Proofs involving negations and conditionals.
- Proofs involving quantifiers
- Proofs involving conjunctions and biconditionals (up to here in this lecture.)
- Proofs involving disjunctions
- Existence and uniqueness proof

(B)

- Proof strategies
- Proofs involving negations and conditionals.
- Proofs involving quantifiers
- Proofs involving conjunctions and biconditionals (up to here in this lecture.)
- Proofs involving disjunctions
- Existence and uniqueness proof
- More examples of proofs..

4 B b 4 B

- Proof strategies
- Proofs involving negations and conditionals.
- Proofs involving quantifiers
- Proofs involving conjunctions and biconditionals (up to here in this lecture.)
- Proofs involving disjunctions
- Existence and uniqueness proof
- More examples of proofs..
- Course homepages:

http://mathsci.kaist.ac.kr/~schoi/logic.html and the moodle page http://moodle.kaist.ac.kr

A 3 5 A 3 5 A

- Proof strategies
- Proofs involving negations and conditionals.
- Proofs involving quantifiers
- Proofs involving conjunctions and biconditionals (up to here in this lecture.)
- Proofs involving disjunctions
- Existence and uniqueness proof
- More examples of proofs..
- Course homepages:

http://mathsci.kaist.ac.kr/~schoi/logic.html and the moodle page http://moodle.kaist.ac.kr

• Grading and so on in the moodle. Ask questions in moodle.

(< E) < E)</p>

• Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.

→ 프 → < 프 →</p>

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.
- A mathematical introduction to logic, H. Enderton, Academic Press.

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.
- A mathematical introduction to logic, H. Enderton, Academic Press.
- http://plato.stanford.edu/contents.html has much resource.

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.
- A mathematical introduction to logic, H. Enderton, Academic Press.
- http://plato.stanford.edu/contents.html has much resource.
- Introduction to set theory, Hrbacek and Jech, CRC Press.

4 3 5 4 3

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.
- A mathematical introduction to logic, H. Enderton, Academic Press.
- http://plato.stanford.edu/contents.html has much resource.
- Introduction to set theory, Hrbacek and Jech, CRC Press.
- Thinking about Mathematics: The Philosophy of Mathematics, S. Shapiro, Oxford. 2000.

• http://en.wikipedia.org/wiki/Truth_table,

- http://en.wikipedia.org/wiki/Truth_table,
- http://logik.phl.univie.ac.at/~chris/gateway/ formular-uk-zentral.html, complete (i.e. has all the steps)

(3) (3) (4) (3)

Image: A matrix

- http://en.wikipedia.org/wiki/Truth_table,
- http://logik.phl.univie.ac.at/~chris/gateway/ formular-uk-zentral.html, complete (i.e. has all the steps)
- http://svn.oriontransfer.org/TruthTable/index.rhtml, has xor, complete.

• A mathematician and/or logicians use many methods to obtain results: These includes guessing, finding examples and counter-examples, experimenting with computations, analogies, physical experiments, and thought experiments (like pictures).

- A mathematician and/or logicians use many methods to obtain results: These includes guessing, finding examples and counter-examples, experimenting with computations, analogies, physical experiments, and thought experiments (like pictures).
- Sometimes proofs involve constructions, i.e., the proof of polynomial root existences by Gauss.

- A mathematician and/or logicians use many methods to obtain results: These includes guessing, finding examples and counter-examples, experimenting with computations, analogies, physical experiments, and thought experiments (like pictures).
- Sometimes proofs involve constructions, i.e., the proof of polynomial root existences by Gauss.
- However, the only results that the mathematicians accept are given by logical deductions from the set theoretical foundations. (This includes finding counter-examples by guessing)

イロト イヨト イヨト イヨト

- A mathematician and/or logicians use many methods to obtain results: These includes guessing, finding examples and counter-examples, experimenting with computations, analogies, physical experiments, and thought experiments (like pictures).
- Sometimes proofs involve constructions, i.e., the proof of polynomial root existences by Gauss.
- However, the only results that the mathematicians accept are given by logical deductions from the set theoretical foundations. (This includes finding counter-examples by guessing)
- There are some controversies as to whether the ZFC is the only foundation.

イロト イヨト イヨト イヨト

- A mathematician and/or logicians use many methods to obtain results: These includes guessing, finding examples and counter-examples, experimenting with computations, analogies, physical experiments, and thought experiments (like pictures).
- Sometimes proofs involve constructions, i.e., the proof of polynomial root existences by Gauss.
- However, the only results that the mathematicians accept are given by logical deductions from the set theoretical foundations. (This includes finding counter-examples by guessing)
- There are some controversies as to whether the ZFC is the only foundation.
- Other fields such as numerical mathematics, physics, and so on have different standards.

・ロト ・回ト ・ヨト ・ヨト

- A mathematician and/or logicians use many methods to obtain results: These includes guessing, finding examples and counter-examples, experimenting with computations, analogies, physical experiments, and thought experiments (like pictures).
- Sometimes proofs involve constructions, i.e., the proof of polynomial root existences by Gauss.
- However, the only results that the mathematicians accept are given by logical deductions from the set theoretical foundations. (This includes finding counter-examples by guessing)
- There are some controversies as to whether the ZFC is the only foundation.
- Other fields such as numerical mathematics, physics, and so on have different standards.
- Because of these differences of standards, it is often very hard to communicate with other fields.

・ロト ・回ト ・ヨト ・ヨト

• Finding proofs are hard: example: Fermat's conjecture...

- Finding proofs are hard: example: Fermat's conjecture...
- Finding a proof is an art. However, there are hints.

- Finding proofs are hard: example: Fermat's conjecture...
- Finding a proof is an art. However, there are hints.
- Most proofs that you have to do have no more than 5-6 steps.

- Finding proofs are hard: example: Fermat's conjecture...
- Finding a proof is an art. However, there are hints.
- Most proofs that you have to do have no more than 5-6 steps.
- In this book, the proof strategies are divided into

- Finding proofs are hard: example: Fermat's conjecture...
- Finding a proof is an art. However, there are hints.
- Most proofs that you have to do have no more than 5-6 steps.
- In this book, the proof strategies are divided into
- for a given of form:

 $\neg P, P \land Q.P \lor Q.P \rightarrow Q, P \leftrightarrow Q, \forall x P(x), \exists x P(x), \exists ! x P(x).$

- Finding proofs are hard: example: Fermat's conjecture...
- Finding a proof is an art. However, there are hints.
- Most proofs that you have to do have no more than 5-6 steps.
- In this book, the proof strategies are divided into
- for a given of form: $\neg P, P \land Q.P \lor Q.P \rightarrow Q, P \leftrightarrow Q, \forall xP(x), \exists xP(x), \exists !xP(x).$

• for a goal of form: $\neg P, P \land Q, P \lor Q, P \rightarrow Q, P \leftrightarrow Q, \forall x P(x), \forall n \in \mathbb{N}P(n), \exists x P(x), \exists ! x P(x).$

< ロ > < 同 > < 回 > < 回 >

• We use a "structural method" in this book. The method is that of divide and conquer or "Top down" approach.

() < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < () < ()

< 口 > < 🗇

- We use a "structural method" in this book. The method is that of divide and conquer or "Top down" approach.
- This means breaking down the proof into smaller and smaller pieces which are easier to prove or already proven by someone else.

3 × 1

- We use a "structural method" in this book. The method is that of divide and conquer or "Top down" approach.
- This means breaking down the proof into smaller and smaller pieces which are easier to prove or already proven by someone else.
- Never assert anything until you can justify it fully using hypothesis or the conclusions reached earlier.

- We use a "structural method" in this book. The method is that of divide and conquer or "Top down" approach.
- This means breaking down the proof into smaller and smaller pieces which are easier to prove or already proven by someone else.
- Never assert anything until you can justify it fully using hypothesis or the conclusions reached earlier.
- The basic assumption we will have in mathematics is the ZFC.

- We use a "structural method" in this book. The method is that of divide and conquer or "Top down" approach.
- This means breaking down the proof into smaller and smaller pieces which are easier to prove or already proven by someone else.
- Never assert anything until you can justify it fully using hypothesis or the conclusions reached earlier.
- The basic assumption we will have in mathematics is the ZFC.
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and \mathbb{R} are the important sets.

To prove the form $P \rightarrow Q$

• First method: Assume *P* and prove *Q*. Or add *P* to the list of hypothesis and prove *Q*.
To prove the form $P \rightarrow Q$

• First method: Assume *P* and prove *Q*. Or add *P* to the list of hypothesis and prove *Q*.

Given	Goal
	P ightarrow Q

→ 프 → < 프 →</p>

To prove the form $P \rightarrow Q$

• First method: Assume *P* and prove *Q*. Or add *P* to the list of hypothesis and prove *Q*.

-	Given Goal
	$P ightarrow Q$
Change to	
J 9	Given Goal
	Q
	Р

October 5, 2011 8 / 32

→ 프 → < 프 →</p>

S. Choi (KAIST)

Logic and set theory

◆□ > ◆□ > ◆臣 > ◆臣 > 善臣 - のへで

S. Choi (KAIST)

۲

$\begin{array}{ll} \mbox{Given} & \mbox{Goal} \\ --- & \mbox{0} < a < b \rightarrow a^2 < b^2 \end{array}$

_ _ _

October 5, 2011 9 / 32

э

<ロ> <同> <同> <同> < 同> < 同>

۲



9/32 October 5, 2011

э

<ロ> <同> <同> <同> < 同> < 同>

S. Choi (KAIST)

٩	$\begin{array}{ccc} {\sf Given} & {\sf Goal} \\ & {\sf 0} < {\sf a} < {\sf b} \rightarrow {\sf a}^2 < {\sf b}^2 \\ & \end{array}$
 Change to 	Given Goal a ² < b ² 0 < a < b
•	$\begin{array}{ll} {\rm Given} & {\rm Goal} \\ 0 < a < b & a^2 < b^2 \\ 0 < a^2 < ab \\ 0 < ab < b^2 \end{array}$

(ロ)

October 5, 2011

9/32

To prove $P \rightarrow Q$

• $P \rightarrow Q \leftrightarrow \neg Q \rightarrow \neg P$.

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

October 5, 2011 10 / 32

Logic and set theory

S. Choi (KAIST)

To prove $P \rightarrow Q$

- $P \rightarrow Q \leftrightarrow \neg Q \rightarrow \neg P$.
- Second method: Assume $\neg Q$ and prove $\neg P$.

3

・ロト ・ 日 ・ ・ ヨ ・ ・ ヨ ・

To prove P ightarrow Q

۲

- $P \rightarrow Q \leftrightarrow \neg Q \rightarrow \neg P$.
- Second method: Assume $\neg Q$ and prove $\neg P$.

Given	Goal
	P ightarrow Q

ヘロト ヘヨト ヘヨト ヘヨト

To prove $P \rightarrow Q$

• $P \rightarrow Q \leftrightarrow \neg Q \rightarrow \neg P$

• Second method: Assume
$$\neg Q$$
 and prove $\neg P$.
• Given Goal
 $----$
• Change to
Given Goal
 $----$
 $\neg P$
 $----$
 $\neg Q$

• Example: Let a > b. Then if $ac \le bc$, then $c \le 0$.

• Example: Let a > b. Then if $ac \le bc$, then $c \le 0$.

$$\begin{array}{cc} \text{Given} & \text{Goal} \\ a,b,c \text{ are real numbers} & (ac \leq bc) \rightarrow (c \leq 0) \\ a > b \end{array}$$

イロト イポト イヨト イヨト

S. Choi (KAIST)

• Example: Let a > b. Then if $ac \le bc$, then $c \le 0$.

 $egin{aligned} & ext{Given} & ext{Goal} \ a,b,c ext{ are real numbers } (ac \leq bc) o (c \leq 0) \ a > b \end{aligned}$

Given Goal a, b, c are real numbers ac > bc a > bc > 0

October 5, 2011 11 / 32

→ 프 → → 프 →

< 🗇 🕨

S. Choi (KAIST)

۲

٢

Logic and set theory

Write this in English

• Theorem: Let a > b. Then if $ac \le bc$, then $c \le 0$.

Write this in English

- Theorem: Let a > b. Then if $ac \le bc$, then $c \le 0$.
- Proof: We will prove this by contrapositives. To prove $ac \le bc \rightarrow c \le 0$. It is sufficient to prove $c > 0 \rightarrow ac > bc$. Suppose c > 0. Then ac > bc by a > b.

.

First method: Try to re-express ¬P in some other form. (in a positive form)

・ロ・・団・・団・・団・ クタク

- First method: Try to re-express ¬P in some other form. (in a positive form)
- Example: Suppose that $A \cap C \subset B$ and $a \in C$. Prove $a \notin A B$.

- First method: Try to re-express ¬P in some other form. (in a positive form)
- Example: Suppose that $A \cap C \subset B$ and $a \in C$. Prove $a \notin A B$.

$$\begin{array}{ll} \text{Given} & \text{Goal} \\ A \cap C \subset B & a \notin A - B \\ a \in C \end{array}$$

۲

- First method: Try to re-express ¬P in some other form. (in a positive form)
- Example: Suppose that $A \cap C \subset B$ and $a \in C$. Prove $a \notin A B$.

$$\begin{array}{ll} \mathsf{Given} & \mathsf{Goal} \\ \mathsf{A} \cap \mathsf{C} \subset \mathsf{B} & a \notin \mathsf{A} - \mathsf{B} \\ a \in \mathsf{C} \end{array}$$

• We change $a \notin A - B$.

۲

- First method: Try to re-express ¬P in some other form. (in a positive form)
- Example: Suppose that $A \cap C \subset B$ and $a \in C$. Prove $a \notin A B$.

Given Goal
$$A \cap C \subset B \quad a \notin A - B$$

 $a \in C$

• We change
$$a \notin A - B$$
.

۲

• $a \notin A - B \leftrightarrow \neg (a \in A \land b \notin B)$. $\leftrightarrow (a \notin A \lor a \in B)$. $\leftrightarrow (a \in A \rightarrow a \in B)$.

- First method: Try to re-express ¬P in some other form. (in a positive form)
- Example: Suppose that $A \cap C \subset B$ and $a \in C$. Prove $a \notin A B$.

Given Goal
$$A \cap C \subset B \quad a \notin A - B$$

 $a \in C$

• We change
$$a \notin A - B$$
.
• $a \notin A - B \leftrightarrow \neg (a \in A \land b \notin B)$. $\leftrightarrow (a \notin A \lor a \in B)$. $\leftrightarrow (a \in A \rightarrow a \in B)$.
• Given Goal
 $A \cap C \subset B$ $a \in A \rightarrow a \in B$
 $a \in C$

۲

GivenGoal $A \cap C \subset B$ $a \in B$ $a \in C$ $a \in A$

October 5, 2011 14 / 32

Logic and set theory

S. Choi (KAIST)

Given Goal $A \cap C \subset B$ $a \in B$ $a \in C$ $a \in A$

• Theorem: Suppose that $A \cap C \subset B$ and $a \in C$. Prove $a \notin A - B$.

۵

・ロト ・回ト ・ヨト ・ヨト

Given Goal $A \cap C \subset B$ $a \in B$ $a \in C$ $a \in A$

- Theorem: Suppose that $A \cap C \subset B$ and $a \in C$. Prove $a \notin A B$.
- Proof: To show a ∉ A − B, it is equivalent to show a ∈ A → a ∈ B. (See above). Assume a ∈ A. Since A ∩ C ⊂ B and a ∈ C, it follows that a ∈ B.

A B K A B K

• Second method: Assume *P* and find a contradiction:

・ロト・日本・日本・日本・日本・今日で

- Second method: Assume *P* and find a contradiction:
- As above: Show $A \cap C \subset B$, $a \in C$. Prove $a \notin A B$.

- Second method: Assume *P* and find a contradiction:
- As above: Show $A \cap C \subset B$, $a \in C$. Prove $a \notin A B$.

 $\begin{array}{ll} \text{Given} & \text{Goal} \\ A \cap C \subset B & a \notin A - B \\ a \in C \end{array}$

イロト イヨト イヨト イヨト

- Second method: Assume P and find a contradiction:
- As above: Show $A \cap C \subset B$, $a \in C$. Prove $a \notin A B$.

 $\begin{array}{ll} \text{Given} & \text{Goal} \\ A \cap C \subset B & a \notin A - B \\ a \in C \end{array}$

Given Goal $A \cap C \subset B$ contradiction $a \in C$ $a \in A - B$

S. Choi (KAIST)

٢

۲

< ロ > < 同 > < 回 > < 回 >

 $\begin{array}{ll} {\rm Given} & {\rm Goal} \\ {\it A} \cap {\it C} \subset {\it B} & {\rm contradiction} \\ {\it a} \in {\it C} \\ {\it a} \in {\it A} - {\it B} \\ {\it a} \in ({\it A} \cap {\it C}) - {\it B} \\ {\it a} \in \emptyset \end{array}$

 Image: boot of the state of the s

<ロ> <四> <四> <三</p>

S. Choi (KAIST)

٢

Logic and set theory

• First method: If we are doing a proof by contradiction, then use *P* as the goal.

• First method: If we are doing a proof by contradiction, then use *P* as the goal.

Given Goal ¬*P contradiction*

イロト イヨト イヨト イヨト

• First method: If we are doing a proof by contradiction, then use *P* as the goal.

•	Given Goal <i>¬P contradiction</i>
 Change to 	Given Goal <i>¬P P</i>

• First method: If we are doing a proof by contradiction, then use *P* as the goal.

•	Given Goal <i>¬P contradiction</i>
 Change to 	
	Given Goal

Second method: re-express in some other form (positive form)

(4) (3) (4) (4) (4)

To use the given of the form $P \rightarrow Q$

• Use modus ponens $P, P \rightarrow Q \vdash Q$.

・ロト・日本・日本・日本・日本・日本

To use the given of the form $P \rightarrow Q$

- Use modus ponens $P, P \rightarrow Q \vdash Q$.
- Use modus tollens $P \rightarrow Q, \neg Q \vdash \neg P$.

To use the given of the form $P \rightarrow Q$

- Use modus ponens $P, P \rightarrow Q \vdash Q$.
- Use modus tollens $P \rightarrow Q, \neg Q \vdash \neg P$.
- Example: Suppose A ⊂ B, a ∈ A, and a and b are not both elements of B.
 Prove b ∉ B.
To use the given of the form $P \rightarrow Q$

- Use modus ponens $P, P \rightarrow Q \vdash Q$.
- Use modus tollens $P \rightarrow Q, \neg Q \vdash \neg P$.
- Example: Suppose A ⊂ B, a ∈ A, and a and b are not both elements of B.
 Prove b ∉ B.

GivenGoal
$$A \subset B$$
 $b \notin B$ $a \in A$ $h(a \in B \land b \in B)$

イロト イヨト イヨト イヨト

To use the given of the form $P \rightarrow Q$

- Use modus ponens $P, P \rightarrow Q \vdash Q$.
- Use modus tollens $P \rightarrow Q, \neg Q \vdash \neg P$.
- Example: Suppose A ⊂ B, a ∈ A, and a and b are not both elements of B.
 Prove b ∉ B.

GivenGoal
$$A \subset B$$
 $b \notin B$ $a \in A$ $\neg (a \in B \land b \in B)$ GivenGoal $A \subset B$ $b \notin B$ $a \in A$ $(a \in B \rightarrow b \notin B)$

S. Choi (KAIST)

۲

October 5, 2011 18 / 32

イロト イヨト イヨト イヨト

$$\begin{array}{ccc} {\rm Given} & {\rm Goal} \\ {\it A} \subset {\it B} & {\it b} \notin {\it B} \\ {\it a} \in {\it A} \\ ({\it a} \in {\it B} \rightarrow {\it b} \notin {\it B}) \\ {\it a} \in {\it B} \end{array}$$

October 5, 2011 19 / 32

9 Q P

(日) (四) (王) (王) (王)

Logic and set theory

S. Choi (KAIST)

$$\begin{array}{ccc} {\rm Given} & {\rm Goal} \\ {\it A} \subset {\it B} & {\it b} \notin {\it B} \\ {\it a} \in {\it A} \\ ({\it a} \in {\it B} \rightarrow {\it b} \notin {\it B}) \\ {\it a} \in {\it B} \end{array}$$

 Theorem: Suppose A ⊂ B, a ∈ A, and a and b are not both elements of B. Then b ∉ B.

(4) (3) (4) (4) (4)

Given Goal

$$A \subset B$$
 $b \notin B$
 $a \in A$
 $a \in B \rightarrow b \notin B$)
 $a \in B$

- Theorem: Suppose A ⊂ B, a ∈ A, and a and b are not both elements of B. Then b ∉ B.
- Proof: Since *a* and *b* are not both elements of *B*, it follows that if *a* is an element of *B*, then *b* is not an element of *B*. Since *a* ∈ *A*, we have *a* ∈ *B*. Thus *b* is not an element of *B*.

To show a goal of the form $\forall x P(x)$

• We introduce some arbitrary variable *x* in the assumption and prove *P*(*x*).

To show a goal of the form $\forall x P(x)$

• We introduce some arbitrary variable x in the assumption and prove P(x).

Given	Goal
	$\forall x P(x)$

S Choi (KAIST	Π١.
3. UNU (INAI3)	

۲

★ E > ★ E

To show a goal of the form $\forall x P(x)$

• We introduce some arbitrary variable *x* in the assumption and prove *P*(*x*).

Given 	Goal $\forall x P(x)$	
Given 		Goal P(x)

x is an arbitrary variable.

IST)	(KAIS	Choi	
------	-------	------	--

۲

۲

4 E > 4

• A, B, C are sets. $A - B \subset C$. Prove $A - C \subset B$.

< ロ > < 回 > < 三 > < 三 > < 三 > < 三 > の < で</p>

۰

• A, B, C are sets. $A - B \subset C$. Prove $A - C \subset B$.

Given Goal
$$A - B \subset C$$
 $A - C \subset B$

3

<ロ> <同> <同> <同> < 同> < 同>

• A, B, C are sets. $A - B \subset C$. Prove $A - C \subset B$. • Given Goal $A - B \subset C$ $A - C \subset B$ • Given Goal $\forall x(x \in A - B \rightarrow x \in C)$ $\forall x(x \in A - C \rightarrow x \in B)$

October 5, 2011 21 / 32

イロト イヨト イヨト イヨト

Logic and set theory

• A, B, C are sets. $A - B \subset C$. Prove $A - C \subset B$. ۲ Given Goal $A-B \subset C$ $A-C \subset B$ ۲ Given Goal $\forall x(x \in A - B \rightarrow x \in C) \quad \forall x(x \in A - C \rightarrow x \in B)$ ۲ Given Goal $\forall x (x \in A - B \rightarrow x \in C) \quad x \in A - C \rightarrow x \in B$ x arbitrary

< ロ > < 同 > < 回 > < 回 >

۲

Given Goal $\forall x (x \in A - B \rightarrow x \in C) \quad x \in B$ x arbitrary $x \in A - C$

> <ロ> <同> <同> <同> < 同> < 同> October 5, 2011 22/32

ъ

۲

۵

 $\begin{array}{ccc} \text{Given} & \text{Goal} \\ \forall x (x \in A - B \rightarrow x \in C) & x \in B \\ x \text{ arbitrary} \\ x \in A - C \end{array}$

$$\begin{array}{ccc} \text{Given} & \text{Goal} \\ \forall x (x \in A - B \rightarrow x \in C) & \text{contradiction} \\ x \in A \\ x \notin C \\ x \notin B \end{array}$$

▶ < ≣ ▶ ≣ ∽ < < October 5, 2011 22 / 32

・ロト ・回ト ・ヨト ・ヨト

S. Choi (KAIST)

Logic and set theory

$\begin{array}{ccc} \text{Given} & \text{Goal} \\ \forall x (x \in A - B \rightarrow x \in C) & x \in C \\ x \in A \\ x \notin C \\ x \notin B \end{array}$

October 5, 2011 23 / 32

ъ

<ロ> (日) (日) (日) (日) (日)

Logic and set theory

S. Choi (KAIST)

$$\begin{array}{ccc} \text{Given} & \text{Goal} \\ \forall x (x \in A - B \rightarrow x \in C) & x \in C \\ & x \in A \\ & x \notin C \\ & x \notin B \end{array}$$

• Read the English proof also.

To prove a goal of form $\exists x P(x)$

• We guess x and show P(x).

To prove a goal of form $\exists x P(x)$

• We guess x and show P(x).

S. Choi (KAIST)

۲

Given	Goal
	$\exists x P(x)$

	10		601	hor	
	JU,				

To prove a goal of form $\exists x P(x)$

۲	We	guess	х	and	show	P(x).
---	----	-------	---	-----	------	-------

٢

٢

Given 	Goal ∃ <i>xP</i> (x)	
Given 		Goal P(x)

x the value you decided

э

イロン イヨン イヨン イヨン

•
$$\exists x, |x^2 - 1| < 1/2.$$

◆□▶ ◆□▶ ◆ ≧▶ ◆ ≧▶ ● ⊇ ● のへぐ

S. Choi (KAIST)

Logic and set theory

October 5, 2011 25 / 32



< ロ > < 回 > < 三 > < 三 > < 三 > < 三 > の < で</p>

S. Choi (KAIST)

Logic and set theory

October 5, 2011 25 / 32

•
$$\exists x, |x^2 - 1| < 1/2.$$

• Given Goal
 $x \in \mathbb{R}$ $\exists x, |x^2 - 1| < 1/2$
• Given Goal

$$egin{array}{lll} x \in \mathbb{R} & \exists x, |x^2 - 1| < 1/2 \ x = 1.1 & (x^2 = 1.21, |x^2 - 1| = 0.21 < 1/2) \end{array}$$

October 5, 2011 25 / 32

9 Q P

(日) (四) (王) (王) (王)

S. Choi (KAIST)

Logic and set theory

• $\exists x P(x)$: Introduce new variable x_0 . $P(x_0)$ is true (existential instantiation)

- $\exists x P(x)$: Introduce new variable x_0 . $P(x_0)$ is true (existential instantiation)
- $\forall x P(x)$: wait until a particular value *a* for *x* to pop-up and use P(a).

イロン イヨン イヨン イヨン

- $\exists x P(x)$: Introduce new variable x_0 . $P(x_0)$ is true (existential instantiation)
- $\forall x P(x)$: wait until a particular value *a* for *x* to pop-up and use P(a).
- Example: \mathcal{F}, \mathcal{G} families of sets. Suppose that $\mathcal{F} \cap \mathcal{G} \neq \emptyset$. Then $\bigcap \mathcal{F} \subset \bigcup \mathcal{G}$.

・ロト ・回ト ・ヨト ・ヨト

- $\exists x P(x)$: Introduce new variable x_0 . $P(x_0)$ is true (existential instantiation)
- $\forall x P(x)$: wait until a particular value *a* for *x* to pop-up and use P(a).
- Example: \mathcal{F}, \mathcal{G} families of sets. Suppose that $\mathcal{F} \cap \mathcal{G} \neq \emptyset$. Then $\bigcap \mathcal{F} \subset \bigcup \mathcal{G}$.

 $\begin{array}{ll} \text{Given} & \text{Goal} \\ \mathcal{F} \cap \mathcal{G} \neq \emptyset & \forall x (x \in \bigcap \mathcal{F} \rightarrow x \in \bigcup \mathcal{G}) \end{array}$

۲

イロト イヨト イヨト イヨト

Given Goal $\mathcal{F} \cap \mathcal{G} \neq \emptyset \quad x \in \bigcup \mathcal{G}$ $x \in \bigcap \mathcal{F}$

October 5, 2011 27/32

<ロ> <四> <四> <三> <三> <三> <三

S. Choi (KAIST)

 $\begin{array}{ll} \mbox{Given} & \mbox{Goal} \\ \mathcal{F} \cap \mathcal{G} \neq \emptyset & x \in \bigcup \mathcal{G} \\ x \in \bigcap \mathcal{F} \end{array}$

 $\begin{array}{ll} \text{Given} & \text{Goal} \\ \exists A (A \in \mathcal{F} \cap \mathcal{G}) & \exists A \in \mathcal{G} (x \in A) \\ \forall A \in \mathcal{F} (x \in A) \end{array}$

Э

・ロト ・回ト ・ヨト ・ヨト

S. Choi (KAIST)

٢

۲

Logic and set theory

$$\begin{array}{ll} \mbox{Given} & \mbox{Goal} \\ \mathcal{F} \cap \mathcal{G} \neq \emptyset & x \in \bigcup \mathcal{G} \\ x \in \bigcap \mathcal{F} \end{array}$$

 $\begin{array}{ll} \text{Given} & \text{Goal} \\ \exists A(A \in \mathcal{F} \cap \mathcal{G}) & \exists A \in \mathcal{G}(x \in A) \\ \forall A \in \mathcal{F}(x \in A) \end{array}$

 $\begin{array}{ll} \begin{array}{ll} \mbox{Given} & \mbox{Goal} \\ A_0 \in \mathcal{F} & \\ A_0 \in \mathcal{G} \\ \forall A \in \mathcal{F}(x \in A) \\ x \in A_0 \end{array} \end{array} \\ \end{array}$

S. Choi (KAIST)

۲

٩

October 5, 2011 27 / 32

< □ > < □ > < □ > < □ > < □ >

$\begin{array}{ll} \mbox{Given} & \mbox{Goal} \\ A_0 \in \mathcal{F} & \exists A \in \mathcal{G}(x \in A) \\ A_0 \in \mathcal{G} & \\ \forall A \in \mathcal{F}(x \in A) \\ x \in A_0 & (\mbox{ Use } A = A_0) \end{array}$

イロト イポト イヨト イヨト

Logic and se

S. Choi (KAIST)

$\begin{array}{ll} \mbox{Given} & \mbox{Goal} \\ A_0 \in \mathcal{F} & \exists A \in \mathcal{G}(x \in A) \\ A_0 \in \mathcal{G} & \\ \forall A \in \mathcal{F}(x \in A) \\ x \in A_0 & (\mbox{Use } A = A_0) \end{array}$

• Theorem: Suppose \mathcal{F} and \mathcal{G} are families of sets. $\mathcal{F} \cap \mathcal{G} = \emptyset$. Then $\bigcap \mathcal{F} \subset \bigcup \mathcal{G}$.

۵

イロン イヨン イヨン イヨン

$\begin{array}{ll} \mbox{Given} & \mbox{Goal} \\ A_0 \in \mathcal{F} & \exists A \in \mathcal{G}(x \in A) \\ A_0 \in \mathcal{G} & \\ \forall A \in \mathcal{F}(x \in A) \\ x \in A_0 & (\mbox{Use } A = A_0) \end{array}$

- Theorem: Suppose \mathcal{F} and \mathcal{G} are families of sets. $\mathcal{F} \cap \mathcal{G} = \emptyset$. Then $\bigcap \mathcal{F} \subset \bigcup \mathcal{G}$.
- Proof: Suppose $x \in \bigcap \mathcal{F}$. Since $\mathcal{F} \cap \mathcal{G} \neq \emptyset$. Let A_0 be the common element. Then $A_0 \in \mathcal{F}$. Thus, $x \in A_0$ as $A_0 \in \mathcal{F}$. Since $A_0 \in \mathcal{G}$, then $x \in \bigcup \mathcal{G}$.

< ロ > < 同 > < 回 > < 回 >

• To prove a goal of the form $P \land Q$: Prove P and Q separately.

- To prove a goal of the form $P \land Q$: Prove P and Q separately.
- To use $P \land Q$: Regard as P and Q.

(4) (3) (4) (4) (4)

- To prove a goal of the form $P \land Q$: Prove P and Q separately.
- To use $P \land Q$: Regard as P and Q.
- To prove a goal $P \leftrightarrow Q$: Prove $P \rightarrow Q$ and $Q \rightarrow P$.

A B K A B K

- To prove a goal of the form $P \land Q$: Prove P and Q separately.
- To use $P \land Q$: Regard as P and Q.
- To prove a goal $P \leftrightarrow Q$: Prove $P \rightarrow Q$ and $Q \rightarrow P$.
- To use $P \leftrightarrow Q$: Treat as two givens $P \rightarrow Q$ and $Q \rightarrow P$.

A B K A B K
• Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.

S. Choi (KAIST)

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove \rightarrow : $\forall x \neg P(x) \rightarrow \neg \exists x P(x)$

۲

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove \rightarrow : $\forall x \neg P(x) \rightarrow \neg \exists x P(x)$

Given Goal $\forall x \neg P(x)$ contradiction $\exists x P(x)$

<ロ> (日) (日) (日) (日) (日)

۲

۲

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove \rightarrow : $\forall x \neg P(x) \rightarrow \neg \exists x P(x)$

Given Goal $\forall x \neg P(x)$ contradiction $\exists x P(x)$

Given Goal $\forall x \neg P(x)$ contradiction $P(x_0)$

S. Choi (KAIST)

Logic and set theory

▶ 《 注 ▶ 注 少 Q () October 5, 2011 30 / 32

ヘロト ヘヨト ヘヨト ヘヨト

۲

۲

٢

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove \rightarrow : $\forall x \neg P(x) \rightarrow \neg \exists x P(x)$

Given Goal $\forall x \neg P(x)$ contradiction $\exists x P(x)$ Given Goal $\forall x \neg P(x)$ contradiction $P(x_0)$

Given $P(x_0)$ $\neg P(x_0)$

Goal $\forall \neg P(x)$ contradiction

S. Choi (KAIST)

3 October 5, 2011 30/32

イロト イヨト イヨト イヨト

• Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ● ●

Logic and set theory

S. Choi (KAIST)

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove \leftarrow : $\neg \exists x P(x) \rightarrow \forall x \neg P(x)$

۲

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove \leftarrow : $\neg \exists x P(x) \rightarrow \forall x \neg P(x)$

Given	Goal
$\neg \exists x P(x)$	$\forall x \neg P(x)$

3

<ロ> (日) (日) (日) (日) (日)

• Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$. • Prove $\leftarrow : \neg \exists x P(x) \rightarrow \forall x \neg P(x)$ • Given Goal $\neg \exists x P(x) \quad \forall x \neg P(x)$ • Given Goal $\neg \exists x P(x) \quad \neg P(x)$ x arbitrary

ヘロト ヘヨト ヘヨト ヘヨト

• Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$. • Prove \leftarrow : $\neg \exists x P(x) \rightarrow \forall x \neg P(x)$ ۲ Given Goal $\neg \exists x P(x) \quad \forall x \neg P(x)$ ۵ Given Goal $\neg \exists x P(x) \quad \neg P(x)$ x arbitrary ۵ Given Goal $\neg \exists x P(x)$ contradiction x arbitrary P(x)

E ► < E ► E </p>
October 5, 2011 31 / 32

イロト イヨト イヨト イヨト

Logic and set theory

S. Choi (KAIST)

Given Goal

$$\neg \exists x P(x) \quad \exists x P(x)$$

x arbitrary
 $P(x)$

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

S. Choi (KAIST)

۲

Logic and set theory

October 5, 2011 32 / 32

Given Goal

$$\neg \exists x P(x) \qquad \exists x P(x)$$

 $x \text{ arbitrary}$
 $P(x)$

• Theorem: $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.

3

<ロ> (日) (日) (日) (日) (日)

Given Goal $\neg \exists x P(x) \quad \exists x P(x)$ *x* arbitrary P(x)

- Theorem: $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Proof: (\rightarrow) Suppose $\forall x \neg P(x)$ and suppose $\exists xP(x)$. We choose x_0 such that $P(x_0)$ is true. Since $\forall x \neg P(x)$, we know $\neg P(x_0)$. This is a contradiction. Thus, $\forall x \neg P(x) \rightarrow \neg \exists xP(x)$.

۵

イロト イヨト イヨト イヨト

Given Goal $\neg \exists x P(x) \quad \exists x P(x)$ *x* arbitrary P(x)

- Theorem: $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Proof: (\rightarrow) Suppose $\forall x \neg P(x)$ and suppose $\exists xP(x)$. We choose x_0 such that $P(x_0)$ is true. Since $\forall x \neg P(x)$, we know $\neg P(x_0)$. This is a contradiction. Thus, $\forall x \neg P(x) \rightarrow \neg \exists xP(x)$.
- Proof: (\leftarrow) Suppose $\neg \exists x P(x)$. Let *x* be arbitrary. Suppose that P(x). Then $\exists x P(x)$. This is a contradiction. Thus $\neg P(x)$ is true. Since *x* was arbitrary, we have $\forall x \neg P(x)$.

۵

イロト イヨト イヨト イヨト