1 Introduction

About this lecture

- Reasoning in predicate calculus
- Inference rules for the universal quantifiers
- Inference rules for the existential quantifiers
- Theorems and quantifier equivalence rules
- Inference rules for the identity predicate
- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://moodle.kaist.ac.kr`
- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.
- A mathematical introduction to logic, H. Enderton, Academic Press.
- `http://plato.stanford.edu/contents.html` has much resource.
- `http://ocw.mit.edu/OcwWeb/Linguistics-and-Philosophy/24-241Fall-2005/CourseHome/` See "Monadic Predicate Calculus" and MPC completeness, Predicate Calculus, Predicate Calculus Deriviations
- `http://philosophy.hku.hk/think/pl/`. See Module: Predicate Logic.
- `http://logic.philosophy.ox.ac.uk/`. See "Predicate Calculus" in Tutorial.

Some helpful references

- `http://en.wikipedia.org/wiki/Truth_table`,
- `http://logik.phl.univie.ac.at/~chris/gateway/formular-uk-zentral.html`, complete (i.e. has all the steps)
- `http://svn.oriontransfer.org/TruthTable/index.rhtml`, has xor, complete.

2 Inference rules for the universal quantifiers

Inference rules for the universal quantifiers

- The inferences rules of propositional calculus hold here also.
- Here we use $\vdash$ since this is an inference without models.
- Universal elimination or universal instantiation $\forall E$:
- Given a wff $\forall\beta\phi$, we may infer $\phi^{\alpha/\beta}$ replacing each occurence of β with some name letter α. (could be a new one without assumptions.)
- Example: $\forall x(M_1x \to M_2x), M_1S \vdash M_2S$.
- 1. $\forall x(M_1x \to M_2x)$. A, 2. M_1S A, 3. $M_1S \to M_2S$. 4. M_2S.

Example

- $\neg Fa \vdash \neg\forall xFx$.
- 1. $\neg Fa$ A.
- 2.: $\forall xFx$ H.
- 3.: Fa. 2.
- 4.: $Fa \wedge \neg Fa$. 1.3
- 5. $\neg\forall xFx$. 1-4

Universal Introduction

- ϕ containing a name letter α without any conditions on it. We replace by $\forall\beta\phi^{\beta/\alpha}$.
- Here $\phi^{\beta/\alpha}$ is the result of replacing all occurance of α with β.
- Some restrictions:
  - The name letter α may not appear in any assumptions.
  - The name letter α may not appear in any hypothesis in effect at the line where ϕ occurs.
  - $\phi^{\beta/\alpha}$ here is the result of replacing every occurance of α with β.
  - We can introduce one quantifier at a time.

Examples

- Fa A. $\vdash \forall x Ax$. This is incorrect.
- $\forall x(Fx \to Gx)$, $Fa \to Ga$, : Fa : Ga, : $\forall x Gx$, $Fa \to \forall x Gx$. This is incorrect.
- $\forall x Lxx$ A., Laa, $\forall z Lza$. This is incorrect.
- $\forall x(Fx \land Gx) \vdash \forall x Fx \land \forall x Gx$.
- Proof: 1. $\forall x(Gx \land Fx)$ A.
- 2. $Fa \land Ga$ from 1.
- 3. Fa.
- 4. Gb.
- 5. $\forall x Fx$.
- 6. $\forall x Gx$.
- 7. $\forall x Fx \land \forall x Gx$.
- Is the converse true also. How does one prove it?

Examples

- $\forall x(Fx \to (Gx \lor Hx)), \forall x \neg Gx \vdash \forall x Fx \to \forall x Hx$.
- 1. $\forall x(Fx \to (Gx \lor Hx))$ A.
- 2. $\forall x \neg Gx$ A.
- 3. $Fa \to (Ga \lor Ha)$.
- 4. $\neg Ga$.
- 5.: $\forall x Fx$ H.
- 6.: Fa.
- 7.: $Ga \lor Ha$. from 3.
- 8.: Ha. 4 7. Disjunctive Syllogism
- 9.: $\forall x Hx$.
- 10. $\forall x Fx \to \forall x Hx$.

Interchangiblility proof:

- $\forall x \forall y$ interchangible to $\forall y \forall x$.
- $\exists x \exists y$ interchangible to $\exists y \exists x$.
- The first one can be done.
- $\forall x \forall y \phi(x, y) \leftrightarrow \forall y \forall x \phi(x, y)$.
- Proof follows from the universal instantiations and introductions.
- The second one is similar and proved after the inference rules for existential quantifiers.

3 Inference rules for existential quantifiers

Inference rules for existential quantifiers

- Existential introduction $\exists I$.
- ϕ contains α. Replace with $\exists \beta \phi^{\beta/\alpha}$. Here we replace some occurrences of α with β.
- The previous occurrence of α does not matter.
- We can introduce one quantifier at a time. (This is also true in $\forall I$.)

Inference rules for existential quantifiers

- Example: $\neg \exists x F x \vdash \forall x \neg F x$.
- 1. $\neg \exists x F x$. A.
- 2.: Fa H. (You can do that ...)
- 3.: $\exists x F x$. from 2.
- 4.: $\exists x F x \wedge \neg \exists x F x$.
- 5. $\neg F a$.
- 6. $\forall x \neg F x$.
- The converse can be proven also.

Example

- Example $\neg\forall xFx \vdash \exists x\neg Fx$.
- 1. $\neg\forall xFx$.
- 2.: $\neg\exists x\neg Fx$. (H)
- 3.:: $\neg Fa$ (H) (note this also.)
- 4.:: $\exists x\neg Fx$. 3 ($\exists I$).
- 5.:: $\exists x\neg Fx \wedge \neg\exists x\neg Fx$ 2.4.
- 6.: Fa. 3-5.
- 7.: $\forall xFx$. 6. ($\forall I$).
- 8.: $\forall xFx \wedge \neg\forall xFx$. 1. 7.
- 9. $\exists x\neg Fx$. 2-8.

Existential Elimination

- Existential Elimination $\exists E$:
- $\exists\beta\phi$. We derive $\phi^{\alpha/\beta} \rightarrow \psi$. Discharge ϕ and assert ψ.
  - The name letter α may not have occurred earlier.
  - α may not occur at ψ.
  - α may not occur in assumptions.
  - α may not occur in hypothesis in effect.

Example

- Example: $\forall x(Fx \rightarrow Gx), \exists xFx \vdash \exists xGx$.
- 1. $\forall x(Fx \rightarrow Gx)$ A
- 2. $\exists xFx$. A.
- 3.: Fa. H for $\exists E$.
- 4.: $Fa \rightarrow Ga$. 1,3.
- 5.: Ga.
- 6.: $\exists xGx$.
- 7. $\exists xGx$.

Example

- Example: $\forall x \neg Fx \vdash \neg \exists x Fx$.
- 1. $\forall x \neg Fx$. A.
- 2.: $\exists x Fx$. H for $\neg I$.
- 3.:: Fa. for $\exists E$.
- 4.:: $\neg Fa$. 1,3.
- 5.:: $Fa \wedge \neg Fa$.
- 6.:: $\neg \exists x Fx$. (CON)
- 7.: $\neg \exists x Fx$. 3-6 $\exists E$.
- 8. $\exists x Fx \rightarrow \neg \exists x Fx$. 2-7
- 9. $\neg \exists x Fx$. (Use. $\neg P \rightarrow P, \vdash P$.)

Example

- $\exists x \neg Fx \vdash \neg \forall x Fx$.
- 1. $\exists x \neg Fx$.
- 2.: $\forall x Fx$. (H)
- 3.:: $\neg Fa$. 1. (for $\exists E$).
- 4.:: Fa 2.
- 5.:: $Fa \wedge \neg Fa$. 3.4.
- 6.:: $\neg \forall x Fx$. (CON)
- 7.: $\neg \forall x Fx$. 3-7 ($\exists E$).
- 8. $\forall x Fx \rightarrow \neg \forall x Fx$. 2-6
- 9. $\neg \forall x Fx$.

Some strategies

- 1. Be careful about where the quantifiers apply. Notice paranthesis well.
- 2. To prove
  - $\exists xFx$: We prove Fa.
  - $\forall x(Fx \to Gx)$: We prove $Fa \to Ga$.
  - $\forall x \neg Fx$: We prove $\neg Fa$.
  - $\forall x \exists y Fxy$: We prove $\exists y Fay$.
  - $\exists y Fay$: We prove Fab.
  - $\exists x Fxx$: We prove Faa.
- 3. To prove the forms in negation, conjunction, disjunction, conditional, or biconditional, then use propositional calculus methods.... (Similar to $(\forall xP) \to (\forall xQ)$).

3.1 Theorems and quantifier equivalence relations

Theorems

- The truth that follows from no assumptions.
- These hold in every model.
- These are called theorems.
- Example: $\vdash \neg(\forall xFx \wedge \exists x \neg Fx)$
- 1.: $(\forall xFx \wedge \exists x \neg Fx)$. H.
- 2.: $\forall xFx$.
- 3.: $\exists x \neg Fx$.
- 4.:: $\neg Fa$. H. (for $\exists E$).
- 5.:: Fa 2.
- 6.:: $\neg Fa \wedge Fa$.
- 7.: $\neg Fa \wedge Fa$.
- 8. $\neg(\forall xFx \wedge \exists x \neg Fx)$

Examples

- $\vdash \forall x Fx \vee \exists x \neg Fx$.
- 1.: $\neg \forall x Fx$. H for $\rightarrow I$.
- 2.:: $\neg \exists x \neg Fx$. H for $\neg I$.
- 3.::: $\neg Fa$ H. for $\neg I$.
- 4.::: $\exists x \neg Fx$
- 5.::: $\exists x \neg Fx \wedge \neg \exists x \neg Fx$.
- 6.:: $\neg \neg Fa$.
- 7.:: Fa.
- 8.:: $\forall x Fx$.
- 9.:: $\forall x Fx \wedge \neg \forall x Fx$.
- 10.: $\neg \neg \exists x \neg Fx$.
- 11.: $\exists x \neg Fx$.
- 12. $\neg \forall x Fx \rightarrow \exists x \neg Fx$.
- 13. $\neg \neg \forall x Fx \vee \exists x \neg Fx$. MI
- 14. $\forall x Fx \vee \exists x Fx$. DN.
- There is a way using equivalences.

Examples

- The rule $\vdash \forall x P \rightarrow \exists x P$.
- We apply this rule to obtain:
- $\vdash \forall x \forall y P \rightarrow \forall x \exists y P$.
- $\vdash \forall x \forall y P \rightarrow \exists x \forall y P$.
- $\vdash \forall x \forall y P \rightarrow \exists x \exists y P$.
- $\vdash \forall x \exists y P \rightarrow \exists x \exists y P$.
- In fact, we can use a theorem to generate many more theorems....

Equivalences

- We studied equivalences called interchanges: $\exists x \exists y \leftrightarrow \exists y \exists x$ and $\forall x \forall y \leftrightarrow \forall y \forall x$.
- $\vdash \neg \forall x \neg Fx \leftrightarrow \exists x Fx$.
- $\vdash \neg \forall x Fx \leftrightarrow \exists x \neg Fx$. This was proved above.
- $\vdash \forall x \neg Fx \leftrightarrow \neg \exists x Fx$. This was proved above.
- $\vdash \forall x Fx \leftrightarrow \neg \exists x \neg Fx$.
- The first and the fourth items are consequence of items two and three.

Quantifier exchanges

- Using the above equivalences, we obtain the quantifier exchange rules.
- $\neg \forall \beta \neg \phi$, $\exists \beta \phi$.
- $\neg \forall \beta \phi$, $\exists \beta \neg \phi$.
- $\forall \beta \neg \phi$, $\neg \exists \beta \phi$.
- $\forall \beta \phi$, $\neg \exists \beta \neg \phi$.
- Using this many predicate calculus results are simply the consequences of propositional caculus results.

Some other equivalences (Repeated)

- How would one prove? :
- $\exists x f \leftrightarrow f$ if x is not a free variable of f.
- $\forall x f \leftrightarrow f$ if x is not a free variable of f.
- $\exists x(f \vee g) \leftrightarrow \exists x f \vee \exists x g$.
- $\forall x(f \wedge g) \leftrightarrow \forall x f \wedge \forall y g$.
- $\exists x(f \wedge g) \leftrightarrow (\exists x f) \wedge g$ if x does not occur as a free variable of g. And also $\exists x(f \vee g) \leftrightarrow (\exists x f) \vee g$
- $\forall x(f \vee g) \leftrightarrow (\forall x f) \vee g$ if x does not occur as a free variable of g. And also $\forall x(f \wedge g) \leftrightarrow (\forall x f) \wedge g$
- $\exists y f(x_1, ..., x_n, y) \leftrightarrow \exists z f(x_1, .., x_n, z)$ if neither y, z are part of $x_1, ..., x_n$.
- $\forall y f(x_1, ..., x_n, y) \leftrightarrow \forall z f(x_1, .., x_n, z)$ if neither y, z are part of $x_1, ..., x_n$.

Inference rules of $=$

- Identity introduction ($= I$):
- For any name letter α, assert $\alpha = \alpha$ at any line.
- Example $\vdash \exists x, a = x$.
- 1. $a = a$ ($= I$).
- 2. $\exists x, a = x$ 1. ($\exists I$).

- Identity elimination ($= E$):
- A wff ϕ containing α. We have $\alpha = \beta$ or $\beta = \alpha$. Then infer $\phi^{\beta/\alpha}$.
- Example: $\vdash \forall x \forall y (x = y \rightarrow y = x)$.
- 1.: $a = b$ H for $\rightarrow I$.
- 2.: $a = a$.
- 3.: $b = a$. ($= E$).
- 4. $a = b \rightarrow b = a$. 1-3
- $\forall y (a = y \rightarrow y = a)$. ($\forall I$).
- $\forall x \forall y (x = y \rightarrow y = x)$. ($\forall I$).