1 Introduction

About this lecture

- Proof strategies
- Proofs involving negations and conditionals.
- Proofs involving quantifiers
- Proofs involving conjunctions and biconditionals
- Proofs involving disjunctions
- Existence and uniqueness proof
- More examples of proofs..
- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://moodle.kaist.ac.kr`
- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.
- A mathematical introduction to logic, H. Enderton, Academic Press.
- `http://plato.stanford.edu/contents.html` has much resource.
- Introduction to set theory, Hrbacek and Jech, CRC Press.
- Thinking about Mathematics: The Philosophy of Mathematics, S. Shapiro, Oxford. 2000.

Some helpful references

- `http://en.wikipedia.org/wiki/Truth_table`,
- `http://logik.phl.univie.ac.at/~chris/gateway/formular-uk-zentral.html`, complete (i.e. has all the steps)
- `http://svn.oriontransfer.org/TruthTable/index.rhtml`, has xor, complete.

2 Proofs involving disjunctions

Proofs involving disjunctions

- To use a given of form $P \vee Q$.
- First method is to divide into cases:
- For case 1, assume P and derive something.
- For case 2, assume Q and derive something, preferably same as above.
- This is the same as disjuction elimination $\vee E$ in Nolt.
- Example in the book $A \subset C, B \subset C, \vdash A \cup B \subset C$.
- The second method: Given $\neg P$ or can show P false, then we can assume Q ony.

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Given	Goal
$P \vee Q$	$----$
$----$	

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Given	Goal
Case 1: P	$----$
$----$	
Case 2: Q	$----$
$----$	

Example

- Example: $A - (B - C) \subset (A - B) \cup C$.
- $\forall x(x \in A - (B - C) \to x \in (A - B) \cup C)$.
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Given	Goal
$----$	$\forall x(x \in A - (B - C) \to x \in (A - B) \cup C)$.

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Given	Goal
x arbitrary	$x \in (A - B) \cup C)$
$x \in A - (B - C)$	

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Given	Goal
$x \in A \wedge \neg(x \in B \wedge x \notin C)$	$(x \in A \wedge x \notin B) \vee x \in C$

Given	Goal
$x \in A$	$(x \in A \wedge x \notin B) \vee x \in C$
$x \notin B \vee x \in C$	

Given	Goal
$x \in A$	$(x \in A \wedge x \notin B) \vee x \in C$
$Case1 : x \notin B$	
$Case2 : x \in C$	

- Case 1 gives $x \in A - B$. Case 2 gives $x \in C$.

To prove a goal of form $P \vee Q$.

- First method: break into cases and prove P and prove Q for each cases.
- Example above $(A - B) - C \subset (A - B) \cup C$.
- Second method: Assume $\neg Q$ and prove P.
Given	Goal
$- - --$	$P \vee Q$
$- - --$	

- Change to

Given	Goal
$- - --$	P
$- - --$	
$\neg Q$	

- Suppose that m and n are integers. If mn is even, then either m is even or n is even.
Given	Goal
mn is even	m is even $\vee$ n is even

Given	Goal
mn is even	n is even
m is odd	

- $$mn = 2k, (2j+1)n = 2k, 2jn + n = 2k, n = 2k - 2jn = 2(k - jn)$$

  Thus, n is even.

3 The existence and uniqueness proof

The existence and uniqueness proof.

- $\exists! x P(x) \leftrightarrow \exists x(P(x) \wedge \neg(\exists y(P(y) \wedge y \neq x)))$.
- Using equivalences we obtain $\neg(\exists y(P(y) \wedge y \neq x)) \leftrightarrow \forall y(P(y) \rightarrow y = x)$.
- $\exists! x P(x) \leftrightarrow \exists x(P(x) \wedge \forall y(P(y) \rightarrow y = x))$.
- The following are equivalent
  - $\exists x(P(x) \wedge \forall y(P(y) \rightarrow y = x))$.
  - $\exists x \forall y(P(y) \leftrightarrow y = x)$.
  - $\exists x P(x) \wedge \forall y \forall z((P(y) \wedge P(z) \rightarrow y = z)$.

Example

- We will prove by proving $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$.
- $1 \rightarrow 2$.
Given	Goal
$\exists x(P(x) \wedge \forall y(P(y) \rightarrow y = x))$	$\exists x \forall y(P(y) \leftrightarrow y = x)$
Given	Goal
$P(x_0)$	$\exists x \forall y(P(y) \leftrightarrow y = x)$
$\forall y(P(y) \rightarrow y = x_0)$	
Given	Goal
$P(x_0)$	$\forall y(P(y) \leftrightarrow y = x)$
$\forall y(P(y) \rightarrow y = x_0)$	
$x = x_0$	
- $\rightarrow$ is clear in the Goal side. $\leftarrow$ is clear also.

Example

- $2 \to 3$.
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Given	Goal
$\exists x \forall y (P(y) \leftrightarrow y = x)$	$\exists x P(x) \wedge \forall y \forall z ((P(y) \wedge P(z) \to y = z)$

- First goal

Given	Goal
$\forall y (P(y) \to y = x_0)$	$\exists x P(x)$
$P(x_0)$	

- Second goal

Given	Goal
$\forall y (P(y) \to y = x_0)$	$\forall y \forall z (P(y) \wedge P(z) \to y = z)$
$P(x_0)$	

-

Given	Goal
$\forall y (P(y) \to y = x_0)$	$(P(y) \wedge P(z)) \to y = z$
$P(x_0)$	
y arbitrary	
z arbitrary	

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Given	Goal
$\forall y (P(y) \to y = x_0)$	$y = z$
$P(x_0)$	
$P(y)$	
$P(z)$	

- Then $y = x_0$ and $z = x_0$. Hence the conclusion.

Example

- Finally $3 \to 1$.
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Given	Goal
$\exists x P(x) \wedge \forall y \forall z ((P(y) \wedge P(z)) \to y = z)$	$\exists x (P(x) \wedge \forall y (P(y) \to y = x))$

-

Given	Goal
$P(x_0)$	$P(x_0) \wedge \forall y (P(y) \to y = x_0)$
$\forall y \forall z ((P(y) \wedge P(z)) \to y = z)$	

- $$\begin{array}{cc} \text{Given} & \text{Goal} \\ P(x_0) & y = x_0 \\ \forall y \forall z((P(y) \wedge P(z)) \to y = z) & \\ P(y) & \end{array}$$

- Since $P(x_0), P(y)$, we have $x_0 = y$. Done.

- To prove a goal of form $\exists! x P(x)$
- Prove $\exists x P(x)$ (existence) and $\forall y \forall z (P(y) \wedge P(z) \to y = z)$. (uniqueness)
- To use a given of form $\exists! x P(x)$.
- Treat as two assumptions $\exists x P(x)$ (existence) and $\forall y \forall z (P(y) \wedge P(z) \to y = z)$. (uniqueness)

Example

- A, B, C are sets. $A \cap B \neq \emptyset, A \cap C \neq \emptyset$. A has a unique element. Then prove $B \cap C \neq \emptyset$.

- $$\begin{array}{cc} \text{Given} & \text{Goal} \\ A \cap B \neq \emptyset & B \cap C \neq \emptyset \\ A \cap C \neq \emptyset & \\ \exists! x (x \in A) & \end{array}$$

- $$\begin{array}{cc} \text{Given} & \text{Goal} \\ \exists x (x \in A \wedge x \in B) & \exists x (x \in B \wedge x \in C) \\ \exists x (x \in A \wedge x \in C) & \\ \exists x (x \in A) & \\ \forall y \forall z (y \in A \wedge z \in A \to y = z) & \end{array}$$

- $$\begin{array}{cc} \text{Given} & \text{Goal} \\ b \in A \wedge b \in B & \exists x (x \in B \wedge x \in C) \\ c \in A \wedge c \in C & \\ a \in A & \\ \forall y \forall z ((y \in A \wedge z \in A) \to y = z) & \end{array}$$

- $b = a$ and $c = a$. Thus $a \in B \wedge a \in C$.

4 More examples of proofs

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- This illustrates the existence proof:
- $\lim_{x\to 2} x^2 + 2 = 6$.
- Cauchy's defintion: $\forall \epsilon > 0(\exists \delta > 0(\forall x(|x-2| < \delta \rightarrow |x^2+2-6| < \epsilon)))$.
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Given	Goal
$\epsilon > 0$ arbitrary	$\exists \delta > 0(\forall x(\|x-2\| < \delta \rightarrow \|x^2+2-6\| < \epsilon))$

-

Given	Goal
$\epsilon > 0$ arbitrary	$\|x^2+2-6\| < \epsilon$
$\delta = \delta_0$ must find	
$\|x-2\| < \delta$	

- Guess work: Find conditions on δ. Assume $|x-2| < \delta, \delta < 1$. Then we obtain $|x-2| < 1, 3 < |x+2| < 5, |x+2| < 5$. Thus $|x^2-4| = |x-2||x+2| < 5\delta$. Thus, choose $\delta = min\{1/2, \epsilon/5\}$. Then
- $\epsilon/5 \leq 1/2$ case: $|x-2| < \epsilon/5 \rightarrow |(x-2)(x+2)| < \epsilon$.
- $1/2 \leq \epsilon/5$ case: $|x-2| < 1/2 \rightarrow |(x-2)(x+2)| < 5(1/2) \leq 5\epsilon/5 < \epsilon$.

- Theorem $\lim_{x\to 2} x^2 + 2 = 6$.
- Proof: Suppose that $\epsilon > 0$. Let $\delta = min\{1/2, \epsilon/5\}$.
- Then $|x-2| < 1/2$ and $|x-2| < \epsilon/5$.
- $|x+2| < 5$ by above and $|(x-2)(x+2)| < \epsilon/5 \cdot 5 < \epsilon$.
- Thus, $|x^2+2-6| < \epsilon$.
- $\forall \epsilon > 0$, if $\delta = min\{1/2, \epsilon/5\}$, then $\forall x(|x-2| < \delta \rightarrow |x^2+2-6| < \epsilon)$. $\square$

- $\lim_{x\to c}\sqrt{x} = \sqrt{c}$, $(x > 0, c > 0)$.
- $\forall c > 0(\forall\epsilon > 0\exists\delta > 0(\forall x > 0(|x-c| < \delta \to |\sqrt{x}-\sqrt{c}| < \epsilon)))$.
-

Given	Goal
$\epsilon > 0$ arbitrary	$\exists\delta > 0(\forall x > 0(\lvert x-c\rvert < \delta \to \lvert\sqrt{x}-\sqrt{c}\rvert < \epsilon))$

-

Given	Goal
$\epsilon > 0$ arbitrary	$\lvert\sqrt{x}-\sqrt{c}\rvert < \epsilon$
$\delta = \delta_0$	
$\lvert x-c\rvert < \delta, x > 0$	

- Guess work:
- $|\sqrt{x}-\sqrt{c}| = |x-c|/(\sqrt{x}+\sqrt{c}) \le |x-c|/\sqrt{c}$.
- Let $\delta = \sqrt{c}\epsilon$.
- Then $|x-c| < \sqrt{c}\epsilon \to |\sqrt{x}-\sqrt{c}| < \epsilon$.
-

Given	Goal
$\epsilon > 0$ arbitrary	$\lvert\sqrt{x}-\sqrt{c}\rvert < \epsilon$
$\delta = \sqrt{c}\epsilon$	
$\lvert x-c\rvert < \delta, x > 0$	

- Theorem: $\lim_{x\to c}\sqrt{x} = \sqrt{c}$. Assume $x, c > 0$.
- Proof: Let $\epsilon > 0$ be arbitrary. Let $\delta = \sqrt{c}\epsilon$. Then $|x-c| < \sqrt{c}\epsilon$.
- $|\sqrt{x}-\sqrt{c}| = |x-c|/(\sqrt{x}+\sqrt{c}) \le |x-c|/\sqrt{c} < \epsilon$.
- $\forall x, x > 0, |x-c| < \sqrt{c}\epsilon \to |\sqrt{x}-\sqrt{c}| < \epsilon$.
- $\forall c > 0(\forall\epsilon > 0\exists\delta > 0(\forall x > 0(|x-c| < \delta \to |\sqrt{x}-\sqrt{c}| < \epsilon)))$.